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Collisionless Magnetic Field Generation in Relativistic Plasmas

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Collisionless magnetic field generation in relativistic plasmas

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Abstract

At low frequencies and long time scales, magnetic fields emerge as the dominant factor in the plasma dynamics as a consequence of the effective cancellation of the electric forces due to plasma quasi-neutrality.

Conversely, at high frequencies and shorter time scales, magnetic fields play an increasingly important role when the particle velocity approach the speed of light c.

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Outline

A relativistic plasma exhibits new phenomena where the nonlinearity of the relativistic particle kinematics and the nonlinearity of the magnetic part of the Lorentz force become dominant.

In these relativistic regimes extremely large, quasi-stationary magnetic fields can be generated in plasmas, e.g., by high intensity laser pulses. These fields can change the plasma dynamics.

Relativistic regimes of interaction between a plasma and a laser pulse can be characterized in terms of the dimensionless amplitude of the laser pulse by the condition

$$a \equiv (eA/m_e c^2) > 1$$

where A is the amplitude of the pulse vector potential.

I will discuss a fundamental mechanism of magnetic field generation through the onset of a collective plasma instability.

This mechanism applies to high temperature - high energy plasma regimes where collisions are weak in the case where the electron distribution function in momentum space is anisotropic.

In this context I will discuss the main features of the Weibel instability of two counter-streaming electron beams (also called Current Filamentation Instability).

This mechanism of magnetic field generation applies to plasmas ranging from laboratory experiments to space plasma and cosmology.

Plasma dynamics constraints on the magnetic field generation

A direct link between the particle and the magnetic field dynamics in a plasma, which can be applied to a variety of different plasma regimes, is obtained by combining Faraday's law and the mean electron momentum equation

$$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t} \tag{1}$$

$$m_e n \left[\frac{\partial u_e}{\partial t} + (u_e \cdot \nabla) u_e\right] = -\nabla \cdot \Pi_e - ne\left[E + \frac{u_e}{c} \times B\right] + \mathcal{C}, \tag{2}$$

where Π_e is the effective electron "pressure" tensor and $\mathcal{C}$ stands generically for collisional effects (such as electron viscosity and, most important, electron resistivity). For the sake of simplicity we start by using nonrelativistic equations.

As long as the form and dependencies of the effective "pressure" tensor Π_e are not specified, Eq.(2) is general and is not based on any fluid model[1].
Kinetic effects enter the expression of the pressure tensor which is defined in terms of the electron distribution function by

$$\Pi_{e,jk} \equiv \int dv \; f_e (v_j - u_{e,j})(v_k - u_{e,k}), \tag{3}$$

where

$$u_{e,j} \equiv (\int dv \; f_e v_j)/(\int dv \; f_e).$$

In the absence of a fluid closure, the expression of Π_e in Eq.(2) must be determined independently as indicated in Eq.(3) from the electron distribution function $f_e(x, v, t)$ obtained from the solution of the Vlasov equation.

[1]Aside for the dissipative term $\mathcal{C}$, Eq.(2) corresponds to the first velocity moment of Vlasov's equation for the electron distribution function $f_e(x, v, t)$.

In the case of low frequency, large scale phenomena in a magnetized plasma described by the Magnetohydrodynamic equations, we can identify the electron mean velocity u_e with the plasma fluid velocity u and we can assume that the pressure is isotropic, $\Pi_e \to p_e I$, and that it obeys a polytropic closure of the form $p_e = p_e(n)$.
For these low frequency phenomena, the effect of electron inertia on the l.h.s. of Eq.(2) and of electron viscosity in the collisional term $\mathcal{C}$ can be neglected in most cases.

Then, from Eqs.(1,2) we obtain

$$\nabla \times \left[E + \frac{u}{c} \times B \right] = \nabla \times \left(\frac{\eta c}{4\pi} \, \nabla \times B \right), \tag{4}$$

where η is the electric resistivity of the plasma.

In the ideal limit $\eta \to 0$, Eq.(4) reduces to the well known magnetic flux conservation theorem[2]

$$\frac{d\Phi}{dt} = 0, \tag{5}$$

where Φ is the magnetic flux through a surface moving together with the plasma, i.e. with the plasma fluid velocity u. The flux conservation expressed by Eq.(5) is generally referred to as the "freezing" of the magnetic field in the plasma.

In the case of fast phenomena that occur on times scales much shorter then the ion dynamical time, we can assume that the ions remain at rest. Again, if we assume an isotropic electron pressure with a polytropic closure and neglect collisional effects, from Eqs.(1,2) we obtain

$$\nabla \times \left[E_e + \frac{u_e}{c} \times B_e \right] = 0. \tag{6}$$

[2] see, e.g., N. Krall, A. Trivelpiece, *Principles of Plasma Physics*, (McGraw-Hill, New York, 1978), chapter 3.

The "generalized " electric and magnetic fields E_e, B_e include the effect of electron inertia, obey the homogeneous Maxwell's equations and are defined by

$$E_e \equiv E + \frac{m_e}{2e}\nabla u_e^2 + \frac{1}{e}\nabla h_e + \frac{m_e}{e}\frac{\partial u_e}{\partial t} \equiv -\nabla\varphi_e - \frac{1}{c}\frac{\partial A_e}{\partial t}, \qquad (7)$$

$$B_e \equiv B - \frac{m_e c}{e}\nabla \times u_e \equiv \nabla \times A_e. \qquad (8)$$

Here φ_e and A_e are the generalized scalar and vector potentials,
$A_e \equiv A + (m_e c/e)u_e$ is related to the electron canonical momentum, and
$\nabla h_e \equiv \nabla p_e/n$ is the gradient of the electron enthalpy
with p_e the isotropic electron density.

Equation (6) expresses the freezing of the generalized magnetic field B_e (often called "generalized vorticity", in contrast to the standard fluid vorticity $\nabla \times u_e$) in the electron fluid.

In a uniform density plasma the generalized vector potential A_e can be written as $A - d_e^2\nabla^2 A$, where $d_e \equiv c/\omega_{pe}$, is the collisionless electron skin depth and ω_{pe} is the plasma frequency.
Thus, in the case of phenomena characterized by spatial scales larger than d_e, the generalized vector potential A_e and the fields E_e and B_e reduce to A, E and B respectively.
In this limit, if the assumptions mentioned above Eq.(6) apply, the magnetic field B is frozen in the electron fluid[3].

These flux conservation theorems are widely used both in astrophysical and laboratory plasmas as they are very convenient when describing the plasma behaviour on space- and time-scales where dissipative effects are unimportant.

[3]see A.S. Kingsep, K.V. Chukbar, V.V. Yan'kov, 1990, *Reviews of Plasma Physics*, ed. by B. Kadomtsev, (Consultants Bureau, New York, N.Y.) 16, 243.

However these conservation theorems are based on two strong assumptions that, as is well known, can be easily violated in a real plasma in particular when kinetic effects become important:

a) that the effective pressure tensor Π_e in Eq.(2) is isotropic
(we recall that, in the case of an anisotropic pressure tensor, $\nabla \times (\nabla \cdot \Pi)$ does not vanish so that, in general, an anisotropic effective pressure tensor violates magnetic flux conservation)

b) and, if $\Pi = pI$, that the scalar pressure p satisfies a polytropic closure
($\nabla \times [(1/n)(\nabla p)]$ does not vanish unless $p = p(n)$).
This implies that the magnetic flux conservation is violated if the electron temperature gradient is not parallel to the density gradient (baroclynic effect).

Indeed all the mechanisms of magnetic flux generation that have been introduced in the literature or investigated experimentally, can be viewed as violations either of condition a) (non-potential ponderomotive force, electron anisotropy etc.) or of condition b) (baroclynic effect).

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M. Borghesi, A.J. Mackinnon, et. al., Phys. Rev. Lett. 80, 5137 (1998);

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Y. Sentoku. K. Mima, S-i. Kojima, H. Ruhl, Phys. Plasmas, 7, 689 (2000).

An anisotropic electron distribution function leads according to Eq.(3) to an anisotropic pressure tensor Π_e which can generate a magnetic field due to the development of a Weibel-type instability

E.W. Weibel, Phys. Rev. Lett., 2, 83, (1959).

In this lecture the explicit case of the magnetic field generation by the current filamentation instability will be examined.
In this case the anisotropy of the effective pressure tensor Π_e arises from the relative motion of the two counterstreaming (cold) electron populations.

Pressure anisotropy, repulsion of opposite currents and magnetic field generation

I will briefly recall the physical mechanism that is at the basis of the Weibel instability and derive the dispersion relation of the closely related "electromagnetic current filamentation instability" (ECFI) that is of direct interest for explaining the generation of a quasistatic magnetic field in the wake of an ultra short ultraintense laser pulse propagating in a plasma[4]

[4] see G.A. Askar'yan, S.V. Bulanov, F. Pegoraro, A.M. Pukhov, JETP Letters, 60, 240 (1994)
G.A. Askar'yan, S.V. Bulanov, *et. al.*, Comm. Plasma Physics Contr. Fusion, 17, 35, (1995);
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S.V. Bulanov, T.Zh. Esirkepov, *et. al.*, Physica Scripta, T 63, 280 (1996);
F. Pegoraro, S.V. Bulanov, *et. al.*, Physica Scripta, T 63, 262 (1996);
G.A. Askar'yan, S.V. Bulanov, *et. al.*, Plasma Phys. Contr. Fusion, 39, 137 (1997);
S.V. Bulanov, T.Zh. Esirkepov, *et. al.*, Phys. Rev. Lett., 76, 3562, (1996).

The electromagnetic current filamentation instability occurs in the case of two counterstreaming electron populations (with zero net total current) and develops perpendicularly to the direction of the electron streams leading, because opposite currents repel each other, to their spatial separation and to the generation of a magnetic field.

The connection between the Weibel instability and the current filamentation instability can be seen by observing that, in the framework of the mean electron momentum equation (2), the effect of the relative velocity between the two counterstreaming electron populations appears as a contribution to the effective pressure tensor Π_e.

This can be understood by referring, e.g., to an anisotropic electron distribution with temperature T_x along the x-direction larger than the temperature $T_\perp$ in the y, z directions.

By interpreting the portions of this distribution function with positive and with negative velocity along x as corresponding to two different electron populations with non zero, oppositely directed, net stream velocities, we can draw the analogy with a distribution function which consists of two separate populations with isotropic temperature equal to $T_\perp$ and velocity separation $\delta u_x \sim 2(2T_x/m_e)^{1/2}$. Clearly this analogy is meaningful only if T_x is sufficiently larger than $T_\perp$.

In the case of two counter propagating electron populations, the transverse electromagnetic current filamentation instability is coupled to the two stream electrostatic instability that develops along the x direction.

The effect of this latter instability is to transfer momentum from one electron population to the other.

For the sake of illustration we will recall here the linear dispersion relation of these coupled instabilities in the two electron fluid approximation following the analysis of [5]

A. D. Steiger, C. H. Woods, Phys. Rev. A, 5, 1467 (1971)

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F. Califano, F. Pegoraro, S.V. Bulanov, Phys. Rev., E 56, 963 (1997);

F. Califano, R. Prandi, et. al., Phys. Rev. E 58, 7837 (1998)

F. Califano, R. Prandi, et. al., J. Plasma Physics 60, 331 (1998) and ref. therein.

In the analysis of nonlinear phase of these instabilities it has been shown that the saturation mechanism of these coupled instabilities is related to the formation of vortex-like structures in phase space.

F. Califano, F. Pegoraro, S.V. Bulanov, Phys. Rev. Lett. 84, 3602 (2000).

F. Califano, F et. al., 32nd EPS Plasma Physics Conf., Tarragona, Spain O4.023 (2005)

[5] This analysis has been extended to a kinetic treatment in F. Califano, F. Pegoraro, S.V. Bulanov, A. Mangeney, Phys. Rev., E 57, 7048 (1998).

Assuming the ions to be at rest and to provide a uniform neutralizing background, the linear dispersion relation can be obtained by linearizing the relativistic equations for the two counter-streaming cold electron populations together with Maxwell's equations:

$$\frac{\partial n_\alpha}{\partial t} = \nabla \cdot j_\alpha, \qquad \frac{\partial p_\alpha}{\partial t} = -u_\alpha \cdot \nabla p_\alpha - (E + u_\alpha \times B), \tag{9}$$

$$\frac{\partial B}{\partial t} = -\nabla \times E, \quad \frac{\partial E}{\partial t} = \nabla \times B - \sum_\alpha j_\alpha, \tag{10}$$

with $u_\alpha = p_\alpha/(1+p_\alpha^2)^{1/2}$, and $j_\alpha = -n_\alpha u_\alpha, \quad \alpha = 1, 2.$
In Eqs.(9,10) quantities are normalized on a characteristic density $\bar{n}$, on the speed of light c and on the plasma frequency $\bar{\omega}_{pe} = (4\pi\bar{n}e^2/m)^{1/2}$.

We consider a homogeneous plasma with velocities along the x direction $u_{0,\alpha}$, such that the net current density is zero $\sum_\alpha n_{0,\alpha} u_{0,\alpha} = 0$.

We consider a perturbation with frequency ω and wavevector $k = (k_x, k_y)$, such that the perturbed magnetic field is in the z direction.

Defining $\Omega_\alpha = \omega - k_x u_{0,\alpha}$ and $\Gamma_\alpha = (1 - u_{0,\alpha}^2)^{-1/2}$, the linear dispersion relation reads:

$$(1 - \Omega_2^{-2}) \left[k_x^2(1 + \Omega_4^{-2}) - \omega^2(1 - \Omega_1^{-2}) - 2\omega k_x \Omega_3^{-2}\right] \tag{11}$$

$$+k_y^2 \left[(1 - \Omega_1^{-2})(1 + \Omega_4^{-2}) + \Omega_3^{-4}\right] = 0,$$

with

$$\Omega_1^{-2} = \sum_\alpha \frac{n_{0,\alpha}}{\Gamma_\alpha \Omega_\alpha^2}, \quad \Omega_2^{-2} = \sum_\alpha \frac{n_{0,\alpha}}{\Gamma_\alpha^3 \Omega_\alpha^2}, \tag{12}$$

$$\Omega_3^{-2} = \sum_\alpha \frac{n_{0,\alpha} u_{0,\alpha}}{\Gamma_\alpha \Omega_\alpha^2}, \quad \Omega_4^{-2} = \sum_\alpha \frac{n_{0,\alpha} u_{0,\alpha}^2}{\Gamma_\alpha \Omega_\alpha^2}.$$

When the perturbation propagates parallel to the mean electron streams, i.e. $k_y = 0$, the electrostatic two-stream instability amplifies the electric field E_x with a growth rate obtained by solving the equation $1 - \Omega_2^{-2} = 0$.
No magnetic field is produced in this case.

In the opposite limit, $k_x = 0$, the dispersion relation reduces to

$$\omega^2(1 - \Omega_2^{-2})(1 - \Omega_1^{-2}) - k_y^2\left[(1 - \Omega_1^{-2})(1 + \Omega_4^{-2}) + \Omega_3^{-4}\right] = 0, \qquad (13)$$

which contains two oscillatory solutions and one purely growing electromagnetic instability (the current filamentation instability) which amplifies the magnetic field B_z with a growth rate that is linear on k_y for $k_y d_e < 1$ (in dimensional units) and becomes approximately constant and of order ω_{pe} for $k_y d_e > 1$ when the velocity on the two counterstreaming beams is close to the velocity of light.

The fact that in the relativistic case the ECFI growth rate is of the order of the Langmuir frequency indicates that this mechanism of magnetic field generation can indeed be effective in the case of the interaction of an ultrashort, ultraintense laser pulse with a plasma where most phenomena occur on timescales of the order of the electron dynamical time ω_{pe}^{-1}.
In this framework the two counterstreaming electron populations consist of a smaller population of fast (relativistic) electrons, accelerated by the laser pulse interacting with the plasma, and by a larger population of slow electrons that provide the return current needed in order to maintain charge neutrality in the plasma.

It is clear that the linear stability analysis in a homogeneous plasma sketched above is not sufficient in order either to determine the efficiency of the conversion of the kinetic energy of the fast electrons into magnetic energy, which require a nonlinear saturation analysis, or the spatial structure of the magnetic field generated in the wake of a laser pulse.

A rough estimate of the magnitude of the generated magnetic field can be obtained by observing that the ECFI growth rate reaches its maximum value for wavenumbers of the order of the inverse collisionless electron skin depth $d_e^{-1} \equiv \omega_{pe}^{-1}/c$.

Thus we may expect that the characteristic transverse size of the current channels produced by the nonlinear evolution of the ECFI be of the order of d_e.

Since the maximum current density in the current channel is given by

$$J_{max} \sim -enc$$

in the quasistatic approximation we obtain for the maximum dimensionless value of the generated magnetic field

$$eB/(m_e c\omega) \sim \omega_{pe}/\omega.$$

Here ω is the carrier frequency of the laser pulse and the normalization is chosen so as to follow the one generally adopted for the dimensionless amplitude a of the laser pulse $a \equiv |eA/(m_e c^2)| \equiv |eE/(m_e c\omega)|$, where A and E are the amplitudes of the vector potential and of the electric field in the pulse.

In this estimate correction factors arising from the relativistic modification of the electron mass have been disregarded.

For a relativistic laser pulse laser pulse, $a > 1$, with wavelength $\lambda \sim 1\mu m$ propagating in a plasma with density, e.g., half its critical value, the amplitude B of the generated quasistatic magnetic field is extremely large, $\approx 100MG$.

A similar estimate can be obtained from energy considerations, by requiring that the magnetic energy density be at most of the order of the kinetic energy density of the fast electrons. Taking this latter to be roughly of order nm_ec^2, we obtain

$$e^2B^2/(m_e^2c^2) \equiv \Omega_{ce} < \omega_{pe}^2.$$

A more detailed estimate of the magnitude of the magnetic field and of the efficiency of the conversion from kinetic to magnetic energy can be obtained by studying the kinetic saturation of the ECFI [6].

[6] see F. Califano, F. Pegoraro, S.V. Bulanov, A. Mangeney, Phys. Rev., E 57, 7048 (1998)

The overall result is that, in the case of two symmetric oppositely propagating fast beams, the conversion efficiency can be rather large, leading to approximate equipartition between kinetic and magnetic energy.
On the other hand, when the beams are non-symmetric, as is the case where the velocity of the electrons in the return current is much smaller than that of the fast electrons, the conversion efficiency drops significantly below energy equipartition.

In a number of cases of interest the fast electron beam is strongly localized in the plane perpendicular to its direction of propagation and the separation between the fast electron current and the return current is expected to lead to a strongly inhomogeneous magnetic field.

The effect of the finite transverse with of the beam was investigated in 2-D in

F. Califano, F. Pegoraro, S.V. Bulanov, Phys. Rev., E 56, 963 (1997);

The magnetic field inhomogeneity along y is enhanced by the fact that the ECFI has a resonant-type spatial behaviour.

Resonant Weibel Instability

Let us assume that the beams inhomogeneity can be described as one dimensional along the y axis and, for the sake of simplicity, let us consider the symmetric (equal densities) case.

In the case of inhomogeneous stream velocities a singularity occurs in the spatial structure of the Weibel instability.
This is best seen by taking at first one-dimensional perturbations with given growth rate γ of the form $E_x(y,t) = E_{xo}(y)\exp(\gamma t)$. Then, the linearized system of Eqs.(9,10) in the nonrelativistic limit can be cast as a second order differential equation for the inductive electric field E_x.

This differential equation reads

$$\frac{\partial}{\partial y}\left\{[2v_0^2(y)-\gamma^2]\,\frac{\partial E_{xo}(y)}{\partial y}\right\}+\gamma^2(\gamma^2+2)E_{xo}(y)=0, \qquad (14)$$

where $v_0^2(y)=v_{0,1}^2(y)$.

If $\gamma<\gamma_{max}$, where γ_{max} is the maximum growth rate computed for a uniform plasma with the largest value of $v_0^2(y)$, the coefficient of the second order derivative vanishes for purely growing modes.

A local Frobenius analysis[7] of Eq. (14) shows that the solution is singular at the point $\bar{y}$ where $2v_0^2(\bar{y})=\gamma^2$. In the neighborhood this point we find

$$E_{xo} \propto \ln|y-\bar{y}|$$

which leads to

$$B_{zo} \propto 1/(y-\bar{y}).$$

[7] C. M. Bender and S. A. Orszag, Advanced Mathematical Methods for Scientists and Engineers McGraw-Hill, New York, (1978), p. 70.

The logarithmic singularity in E_{xo} is mathematically analogous to the one which is encountered at the Alfvèn resonance in the case of shear-Alfvèn waves propagating in a weakly inhomogeneous plasma and for oscillations in inhomogeneous flows[8].

The singularity in the spatial dependence of B_{zo} indicates that the magnetic field generated by the Weibel instability in a nonuniform plasma is strongly inhomogeneous, and that it is localized in the neighborhood of the resonant point.
Around this point the field reverses its polarity, which corresponds to the formation of a current sheet.

[8]see A. Hasegawa and C. Uberoi, The Alfvèn Wave, DOE Critical Review Series U.S. Dept. of Energy, Washington, DC, (1982);
G. Bertin, G. Einaudi, F. Pegoraro, Comments Plasma Phys. Control. Fusion 17, 35 (1986);
A. V. Timofeev, Usp. Fiz. Nauk 102, 185 (1970) [Sov. Phys. Usp. 13, 632 (1971).

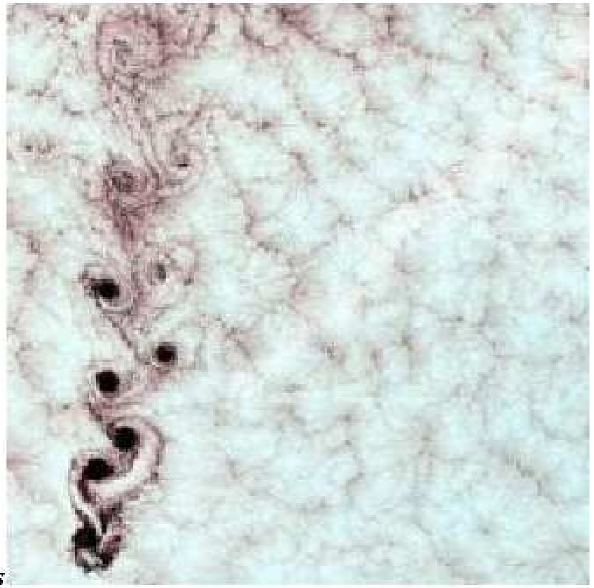
Magnetic Vortices

The long time behaviour of the bipolar quasistatic magnetic field generated by inhomogeneous beams has been shown ito develop vortex structures

S.V. Bulanov, T.Zh. Esirkepov,et. al., Phys. Rev. Lett., **76***, 3562, (1996);*

S.V. Bulanov, T. Zh. Esirkepov, et. al., Plasma Physics Reports **23***, 284, (1997).*

The bipolar magnetic ribbon develops an instability which tends to bend it and to produce an electron velocity pattern similar to the von Karmàn row in hydrodynamics. The resulting configuration of the magnetic field corresponds that of an antisymmetric vortex street in the electron fluid velocity, where the oppositely polarized vortices are shifted, one with respect to the other, along the two chains.



Karman Vortices

These vortices appeared over Alexander Selkirk Island in the southern Pacific Ocean. Rising precipitously from the surrounding waters, the island's highest point is nearly a mile (1.6 km) above sea level. As wind-driven clouds encounter this obstacle, they flow around it to form these large, spinning eddies. These Karman Vortices can be found on Landsat 7 WRS Path 6 Row 83, center: -33.18, -79.99.|Landsat 7

After the ECFI has saturated and the counterstreaming electron populations have "thermalized", the electrons can satisfactorily be described as a single cold population.

In the absence of dissipation and of source terms and for relatively fast phenomena such that the ions can be taken as immobile, the generalized vorticity

$$\nabla \times [p - (e/c)A].$$

i.e. of the rotation of the canonical electron momentum field, is frozen in the (cold) electron flow (see Eqs.(6).

If we further assume that the electron motion that sustains the quasistatic magnetic field is slow compared to the Langmuir time and that its velocity is much smaller than the speed of light c, the electron fluid can be regarded as incompressible and the electron fluid velocity is related to the magnetic field as $v = -(c/4\pi en)\nabla \times B$.

Thus the domains where the magnetic field is stronger correspond to vortices in the electron fluid motion.

In a two dimensional configuration where the plasma currents flow in the x- y plane, taking B to be along the z-axis, we obtain for $B \equiv Be_z$

$$(\partial/\partial t + \hat{z} \times \nabla B \cdot \nabla)\,(\Delta B - B) = 0, \qquad (15)$$

where the time and space units are the inverse cyclotron frequency in the generated magnetic field Ω_{Ce}^{-1}, and $d_e = c/\omega_p$, the collisionless electron skin depth. Equation (15) is known as the Hasegawa-Mima equation in the limit of zero drift velocity[9].

Equation (15) admits point-like vortex solutions[10]. These solutions provide a convenient tool for representing the antisymmetric vortex street, and for showing that its propagation velocity is slow compared to the speed of light c when the distance between the vortices becomes larger than d_e.

[9] A. Hasegawa, K. Mima, Phys. Rev. Lett. **39**, 205 (1997); A. Hasegawa, K. Mima, Phys. Fluids 21, **87** (1978).
[10] W. Horton, A. Hasegawa, *Chaos*, 4, 227 (1994); G.Matsuoka, K.Nosaki, *Phys. Fluids B*,4, 551 (1992).

Vortex dynamics is considered to be important in explaining a wide variety of nonlinear processes in magnetized plasmas and to represent the final stage of the development of turbulence[11].

In a discrete vortex solution the generalized vorticity is localized at the points $r = r_j$:

$$\Omega = \Delta B - B = \sum_j \Gamma_j \delta(r - r_j(t)).$$

Here Γ_j are constants and $r = (x, y)$.

Then we have $B = \sum_j B_j, \quad B_j(r, r_j(t)) = -(\Gamma_j/2\pi)\ K_0(|\ r - r_j(t)\ |)$.
Here and below $K_n(\xi)$ are modified Bessel functions.

[11] V.I.Petviashvili, and O.M.Pokhotelov, *Solitary Waves in Plasmas and in the Atmosphere*. Gordon and Breach Science Publishers (1992); R.L.Stenzel, J.M.Urrutia, and C.L.Rousculp, *Phys. Rev. Lett.* 74, 702, (1995); ‘ W.Horton, and A.Hasegawa, *Chaos*, 4, 227 (1994).

The curves $r_j(t)$ are determined by the characteristics

$$\dot{r}_j = \hat{z} \times \nabla \cdot \sum_{k \neq j} B_k(r_j(t), r_k(t)).$$

From these expressions we obtain the equation of motion of the vortices

$$\dot{x}_j = -\frac{1}{2\pi} \sum_{k \neq j} \Gamma_k \frac{y_j - y_k}{r_{jk}} K_1(r_{jk}), \;\; \dot{y}_j = \frac{1}{2\pi} \sum_{k \neq j} \Gamma_k \frac{x_j - x_k}{r_{jk}} K_1(r_{jk}), \qquad (16)$$

where $r_{jk} =\mid r_j - r_k \mid= [(x_j - x_k)^2 + (y_j - y_k)^2]^{1/2}$.

We will assume that all vortices have the same absolute amplitude and take $|\Gamma_j| = 1$.

We consider the problem of the stability of an infinite vortex chain.
In the initial equilibrium the vortices have coordinates
$x_j^0 = js,\ y_j^0 = 0,\ -\infty < j < +\infty$ and amplitudes $\Gamma_j = 1$.

If the distance s between neighboring vortices is much smaller than one (in dimensional units much smaller than the collisionless skin depth), for $s \ll |y| \ll 1$ the chain separates two subregions, an upper and lower one, with opposite electron velocity along x, $v_x = \mp U = \mp 1/(2s)$.
This is equivalent to a vortex film with uniform surface density of generalized vorticity, $-1/s$. Far from the film, for $|y| \gg 1$, both B and v_x tend to zero exponentially. This structure corresponds to two, oppositely directed, electric current sheets that have a width of order one.

In the analysis of a vortex chain stability we follow the standard approach developed in hydrodynamics[12].

[12] H. Lamb, *Hydrodynamics*, Cambridge University Press (1932).

We consider the motion of the j-th vortex with coordinates $x = js + x_j(t)$ and $y = y_j(t)$. Due to the translational invariance of the initial configuration we seek solutions of Eqs.16, linearized around the equilibrium configuration, of the form $x_j = X \exp[\gamma t + i(j\phi)]$, $y_j = Y \exp[\gamma t + i(j\phi)]$, with $0 < \phi < 2\pi$.
If ϕ is small, the perturbation has the form of a sinusoidal wave with wavelength $\lambda = 2\pi/\kappa = 2\pi s/\phi$, where κ is the wavenumber. The perturbations grow exponentially in time, and the growth rate γ is given by

$$\gamma = \frac{1}{\pi}\left\{ \left[\sum_{j=1}^{\infty} \frac{K_1(js)}{js}(1 - \cos j\phi) \right] \left[\sum_{k=1}^{\infty} (K_2(ks) - \frac{K_1(ks)}{ks})(1 - \cos k\phi) \right] \right\}^{1/2}. \tag{17}$$

If $s \ll 1$ and $\phi \gg 2\pi s$, $\lambda < 1$ and Eq.(17) gives $\gamma = [\phi(2\pi - \phi)]/(4\pi s^2)$. When $\phi \ll 1$, we have $\gamma \approx \phi/(2s^2) = \kappa U$, where $U = 1/(2s)$, which coincides with the growth rate of the Kelvin–Helmholtz instability.

For $s \ll 1$ and $\phi < 2\pi s$, i.e., for wavelengths greater than the width of the current sheet ($\lambda > 1$), Eq.(17) gives $\gamma = \phi^2/(\pi s^3) = \kappa^2 U/(2\pi)$. In the long wavelength limit the instability becomes slow compared to the Kelvin–Helmholtz instability. In the limit $s \gg 1$, when the distance between two neighboring vortices is larger than one, Eq.(17) gives $\gamma \approx (1 - \cos\phi)\exp(-s)/(s\sqrt{2\pi})$ and the instability is exponentially slow.

Let us consider a double chain of opposite vortices in which the coordinates and the amplitudes of point vortices are equal to

$$x_j^0 = js + Ut,\ y_j^0 = \tfrac{1}{2}q,\ \ -\infty < j < +\infty,\ \Gamma_j = -1$$

for the upper chain, and

$$x_k^0 = (k + \sigma)s + Ut,\ y_k^0 = -\tfrac{1}{2}q,\ \ -\infty < k < +\infty,\ \Gamma_k = 1$$

for the lower chain respectively. The distance between neighboring vortices in a chain is s, the distance between the chains in the y-direction is q, and the lower chain is shifted along the x-direction by σs: $\sigma = 0$ and $\sigma = 1/2$ correspond to the symmetrical and to the antisymmetrical configurations respectively.

Here

$$U = \frac{q}{\pi} \sum_{k=0}^{\infty} \frac{K_1(\rho'_k)}{\rho'_k}, \quad \rho'_k = [(k+\sigma)^2 s^2 + q^2]^{1/2} \tag{18}$$

is the global velocity of the double chain in the x-direction. When $s \ll 1$ and $q \ll 1$ we recover known results: $U = (1/2s)\coth(\pi q/s)$ for $\sigma = 0$, and $U = (1/2s)\tanh(\pi q/s)$ for $\sigma = 1/2$. Far from the vortex row the magnetic field and the electron fluid velocity tend to zero exponentially. For $q < 1$ this configuration corresponds to an electron current sheet with thickness q surrounded by two opposite current sheets with thickness of order one.

From Eqs.(16) we can obtain the linearized equation of motion of the vortices. Looking for solutions of the form $x_j = X\exp(\gamma t + i(j\phi))$, $y_j = Y\exp(\gamma t + i(j\phi))$, $x'_k = X'\exp(\gamma t + ik\phi))$, $y'_k = Y'\exp(\gamma t + i(k\phi))$, for the perturbations of the coordinates of vortices from the upper and the lower chain, respectively, we find the dispersion relation.

$$2\pi\gamma = -i\sum_k \frac{(k+\sigma)sq}{\rho'^2_k} K_2(\rho'_k)\sin(k+\sigma)\phi \ \pm \ \Bigg\{\Bigg[\sum_j \frac{K_1(\rho_j)}{\rho_j}(1-\cos j\phi) + \sum_k \left(\frac{K_1(\rho'_k)}{\rho'_k} - \frac{q^2K_2(\rho'_k)}{\rho'^2_k}\right)(1 \mp \cos(k+\sigma)\phi)\Bigg]\Bigg[\sum_j \left(\frac{K_1(\rho_j)}{\rho_j} - K_2(\rho_j)\right)(1-\cos j\phi) + \sum_k \left(\frac{K_1(\rho'_k)}{\rho'_k} - \frac{(k+\sigma)^2 s^2 K_2(\rho'_k)}{\rho'^2_k}\right)(1 \pm \cos(k+\sigma)\phi)\Bigg]\Bigg\}^{1/2}. \qquad (19)$$

The symmetrical, $\sigma = 0$, vortex row is always unstable.
In the limit $s \ll q \ll 1$ and $q \ll 2\pi s/\phi \ll 1$, we recover Rayleigh's result for the growth rate $\mathcal{R}e(\gamma) = \phi U(q\phi)^{1/2}/s^{3/2} = \kappa U(\kappa q)^{1/2}$ of the bending instability of a finite width, fluid stream.
When the perturbation wavelength is larger than one and q ($q < 1 < 2\pi s/\phi$), we can estimate the instability growth rate as $\mathcal{R}e(\gamma) \approx \kappa^2 U(\kappa q)^{1/2}$.

If the distance between neighboring vortices is larger than one, $s > 1$, the growth rate is exponentially small:
$2\pi\mathcal{R}e(\gamma) \approx [2e^{-s/2}/(2\pi s^3)^{1/4}]\ [K_1(q)/q]^{1/2}(1-\cos\phi)^{1/2}$.

In the case of the antisymmetrical vortex row with $\sigma = 1/2$ we expect a more complicated behavior of the perturbations, compared to that of the symmetrical configuration.
In standard hydrodynamics the antisymmetrical von Karman's vortex row is stable for $q/s \approx 0.281$.
In the hydrodynamic case a point vortex is described by $(\Gamma_j/2\pi)\ln|r - r_j(t)|$ instead of the Bessel function $K_0(|r - r_j(t)|)$.
We can see by direct inspection that for large distance between neighboring vortices the antisymmetric vortex row is stable.

Conclusions

Quasistatic magnetic fields and magnetic vortices are a generic feature of relativistic plasmas.

These coherent structures can be studied with analytical tools, as exemplified in this presentation, and with the help of high dimensionality kinetic simulations of the plasma dynamics.

Particle in Cell simulations have shown that the asymptotic evolution of a finite length laser pulse in a plasma corresponds to the excitation of high amplitude electron vortices and of low frequency solitons.

N. M. Naumova, J. Koga, et. al., Phys Plasmas, 8, 4149 (2001).