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Quantum Methodologies in Beam, Fluid and Plasma Physics

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The **quantum methodologies** are useful for describing in a **unified way** several problems of **nonlinear** and **collective** dynamics of **fluids**, **plasmas** and **beams**

They are tools provided by

- **Schrödinger-like equations**
- **Madelung fluid picture**
- **Moyal-Ville-Wigner "quasidistribution"**
- **von Neumann-Weyl formalism**
- **quantum tomography**
- ***n*-waves parametric processes**

GENERAL ROLE OF THE QUANTUM METHODOLOGIES

The quantum methodologies are widely used in almost all branches of nonlinear physics. For example, they are frequently encountered in dispersive media such as

- **laboratory, space and astrophysical plasmas**
- **fluids**
- **Kerr media, optical fibers and electrical transmission lines**
- **many other physical systems including cosmological and biological systems**
- **optical beam physics and charged particle beam physics**

They are intensively applied in all the above branches as result of international collaborations belonging to the frontiers of the physics researches and they are one of the main topics of several important interdisciplinary scientific international conferences

In fact, each of the above physical systems exhibit a behavior that can be described with a <u>quantum formalism</u>

Typically, their evolution in space and time is governed by suitable linear or <u>nonlinear Schrödinger-like equations</u> (NLSE) that are coupled, through an <u>effective potential</u>, with a set of equations describing the <u>interaction system-surroundings</u>

THE DIVERSE ORIGIN OF THE INTERACTION “SYSTEM-SURROUNDINGS” IN PROCESSES DESCRIBED BY NLSEs

in plasmas: harmonic generation and the ponderomotive force

in nonlinear optics: Kerr nonlinear refractive index

in accelerator physics: image charges and image currents of the beam created on the walls of the accelerator vacuum chamber. This interaction is conveniently described in terms of the so-called "coupling impedance", whose imaginary part accounts for both the space charge blow up and the magnetic self attraction, and whose real part accounts for the resistive effects occurring on the walls

in surface gravity wave physics: high values of the wave elevation

DESCRIPTIONS ALTERNATIVE TO THE ONE GIVEN BY LINEAR AND NONLINEAR SCHRÖDINGER EQUATIONS (IN CONFIGURATION SPACE)

- **Madelung fluid equation, obtained with the eikonal representation of the wave function**

- **Moyal-Ville-Wigner kinetic equation or von Neumann-Weyl equation, obtained transiting from configuration spase to phase space by means of the Moyal-Wille-Wigner transform (quasidistribution)**

- **tomographic map which provides a description in terms of a marginal distribution (classical probability function), starting from the quasidistribution**

In this scenario:

- **the study of the quantum methodologies have been recognized as very important for a <u>synergetic development</u> of the above branches of physics with very powerful <u>multidisciplinary</u> as well as <u>interdisciplinary</u> approaches**

- **the intense study on nonlinear and collective effects in the several physical systems have stimulated a number of <u>interdisciplinary approaches</u> and <u>transfer of know how</u> from one discipline to another, obtaining, in turn, a big growing of importance of the methodologies used to investigate very different physical phenomena governed by formally identical equations**

ADVANTAGES OF THIS INTERDISCIPLINARY STRATEGY

- **communities of physicists from different areas are stimulated to collaborate more an more exchanging their own experiences and make available their own expertise**

- **subsequent very rapid improvement of all the methodologies to be used and goals to be reached in each specific discipline. This aspect is connected with the efforts done during the last decades in transferring know how and methodologies from one discipline to another trying to predict new effects as well as to give answers for scientific and technological problems of international expectation**

Remarkably, the applications of the quantum methodologies:

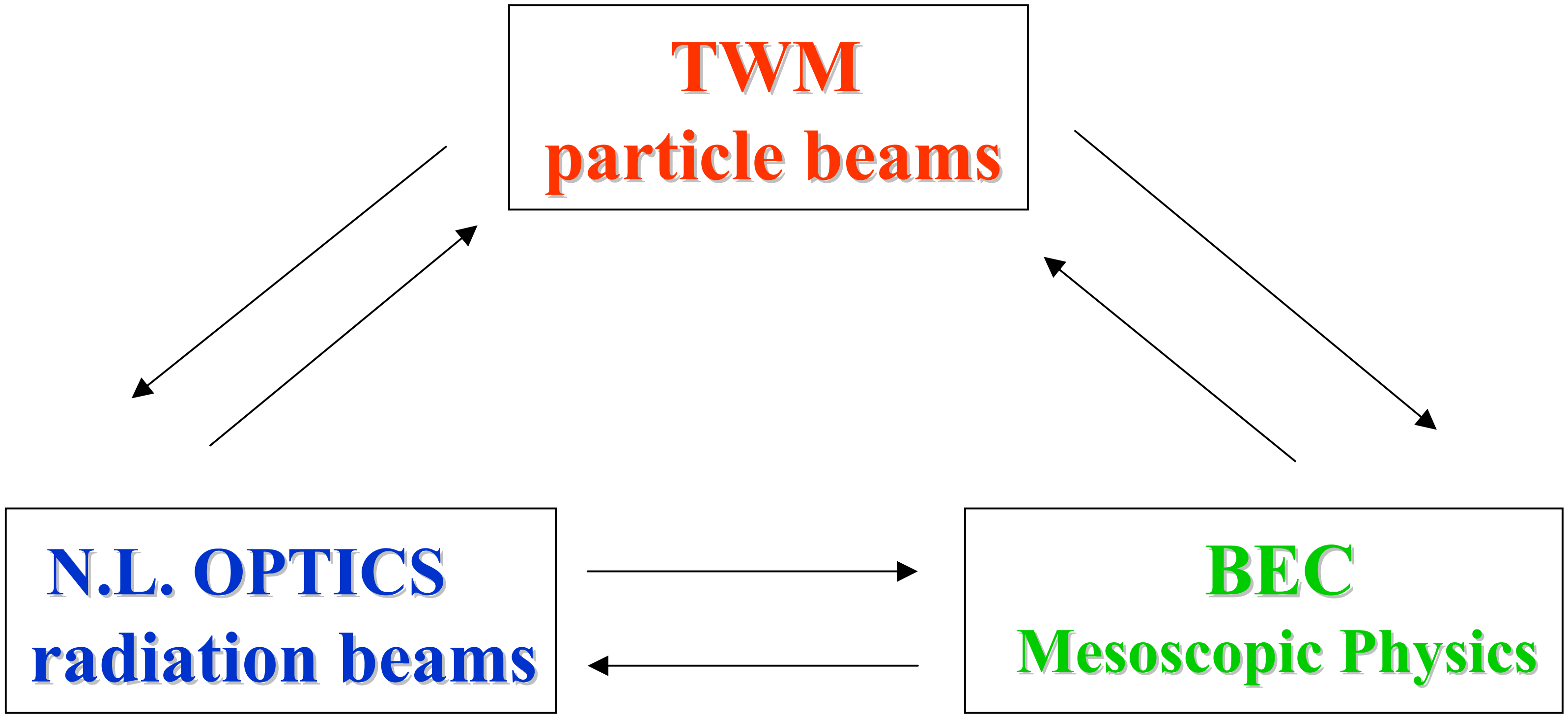
(i) **to gravity ocean waves**, touche the very important and hot problem of the environmental risk due to natural catastrophes, as the one that recently took place in the South-Est of Asia;

(ii) **to beam physics**, open up the possibility to develop an emerging area of physics, called Quantum Beam Physics, which in the limit of very low temperature should provide the realization of non-classical (but collective and nonlinear) states of charged particle beams fully similar to the ones obtained for the light (optical beams) and for Bose-Einstein condensation;

(iii) **to nonlinear optics** (f.i., optical fibers) and electric transmission lines deal with important and modern aspects of telecommunications;

(iv) **to discrete systems**, are relevant for the very recent development of nanotechnologies

QUANTUMLIKE METHODOLOGIES



- <u>Different systems</u> described by the <u>same formalism</u>
- <u>Transfer of know how</u> from one discipline to another one
- Alternative “keys of reading” for each discipline
- New insights and possibility of <u>new predictions</u>

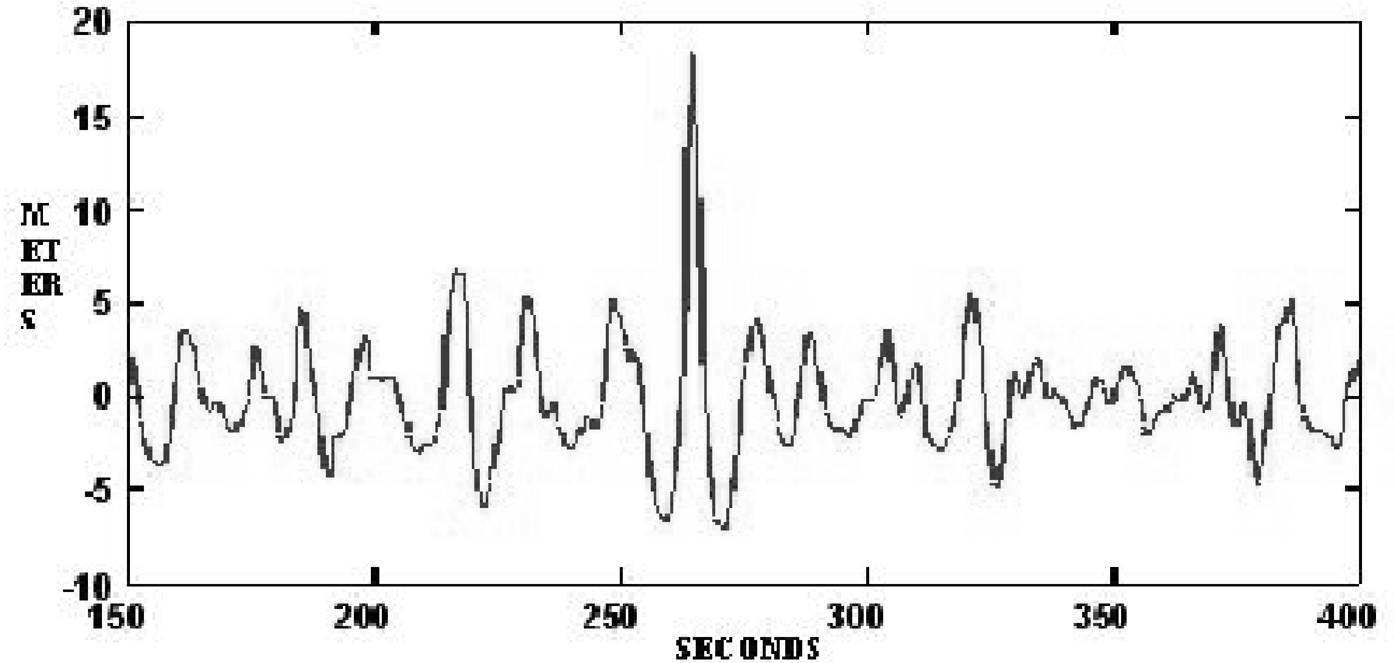
$$i\alpha \frac{\partial \Psi}{\partial s} + \frac{\alpha^2}{2} \frac{\partial^2 \Psi}{\partial x^2} - \mathbf{U}[\,|\Psi|^2]\Psi = 0$$

- **<u>Thermal Wave Model (TWM):</u>** $\alpha = \varepsilon$ (beam emittance)
  $|\Psi|^2$ = density of the particle beam
  U accounts for: self interaction (image charges and image currents) + external potential (optical magnetic devises, RF cavities, kickers, etc.)
- **<u>Optical beams:</u>** $\alpha = \lambda/2\pi$ (wavelength)
  $|\Psi|^2$ = e.m. energy density (density of the photon beam)
  U accounts for: self interaction (dependence of the refractive index on the opical intensity) + external superimposed refractive index variations
- **<u>Bose Einstein Condensates (BEC):</u>** $\alpha = h/2\pi$ (Panck's constant)
  $|\Psi|^2$ = density of the condensates
  U accounts for: mean field collective interaction among atoms + external trapping potential well

SOME ASPECTS OF THE PHENOMENOLOGICAL PLATFORM INVESTIGATED WITH QUANTUM METHODOLOGIES

• **modulational instability (MI), also known as Benjamin-Feir instability**: a general phenomenon encountered when a quasi-monochromatic wave is propagating in a weak nonlinear medium. It has been predicted and experimentally observed in almost all fields of physics where these conditions are present.

For ocean gravity waves: the MI has been discovered independently by Benjamin and Feir and by Zakharov in the Sixties; the instability predicts that in deep water a monochromatic wave is unstable under suitable small perturbations. This phenomenon is well described by the NLSE. In this framework, it has been established that the MI can be responsible for the formation of freak waves



*One of the most spectacular recordings of a freak wave measured in the North Sea the 1st of January 1995 from Draupner platform (Statoil operated platform, Norway). The time series of the surface elevation shows a single wave whose height from crest to through is **26 meters** (a **9 floor building**) in a **10 meters height sea state**. In deep water those large amplitude have been recently attributed to MI*

For plasmas waves: finite amplitude Langmuir waves can be created when some free energy sources, such as electron and laser beams, are available in the system as a result of a nonlinear coupling between high-frequency Langmuir and low-frequency ion-acoustic waves. Under suitable physical conditions, the dynamics can be described by a NLSE and the MI can be analyzed directly with this equation.

For large amplitude e.m.waves: a modification of the refractive index affects the propagation and makes possible the formation of wave envelopes. In the slowly-varying amplitude approximation, this propagation is governed again by suitable NLSEs and the MI plays a very important role.

In electrical transmission lines: the propagation of modulated non-linear waves is governed by discrete equations of the LC circuit which, in turn, can be reduced to single or two coupled NLSEs.

• **localized structures**: MI is responsible of formation of **robust nonlinear excitations** of the medium. In particular, the asymptotic behaviour of MI may be characterized by the formation of very stable **localized solutions**, such as **envelope solitons**, **cavitons**, **holes**, etc., which, in turn, are involved in a long timescale dynamics that have been shown to be of great importance in all nonlinear systems. In general localized solutions are the result of the **interplay** between **nonlinearity** and **dispersion** effects.

IMPACTS PRODUCED BY QUANTUM METHODOLOGY IN NONLINEAR PHYSICS

- **INVERSE SCATTERING**

One of the most relevant example of using quantum methodologies in nonlinear physics is surely given by the **inverse scattering method**

This method construts a connection between the **Korteweg-de Vries equation** (**KdVE**) and the **linear Schrödinger equation** (**LSE**).

KdVE is put in corresponde with LSE in such a way that the soliton of the former plays the role of the (linear) potential of the latter

$$-\frac{\partial^2\psi}{\partial x^2}-\left[\frac{1}{6\beta}\,u\right]\psi=\epsilon\psi$$

$$\frac{\partial u}{\partial t}+u\,\frac{\partial u}{\partial x}+\beta\,\frac{\partial^3 u}{\partial x^3}$$

The problem of solving the KdVE is reduced to a quantumlike problem, i.e., to an inverse eigenvalue problem of the LSE. Very important theorems have been found for this method which has been successfully extended to NLSE. The capability and the richness of similar methods currently applied to nonlinear partial differential equations for solving a number of physical problems have produced an autonomous research activity in mathematical physic called "**inverse problems**".

- **Outside of the inverse scattering method framework**

A correspondence between soliton-like and envelope soliton-like solutions, in the form of travelling waves, of wide families of generalized Korteweg-de Vries equation (gKdVE) and generalized nonlinear Schrödinger equation (gNLSE), respectively, has been constructed within the framework of the Madelung fluid and used to find solitonlike solutions NLSE with nonlinearities more complicated than the cubic one.

OTHER IMPORTANT METHODOLOGICAL TRANSFER AND SUBSEQUENT IMPACTS

•**A transfer of know how** from **nonlinear optics** to **accelerator physics** has allowed to predict, within the context of the TWM, **new results**, such as **soliton density structures** associated with the longitudinal dynamics of a charged-particle bunch in a circular high-energy accelerating machine for the case of **purely reactive impedance** that the conventional approach, based of Vlasov equation, was not yet capable to predict.

• Later on, by including the **resistive part of the coupling impedance**, the resulting integro-differential NLSE was capable to describe the **nonlocal** and **distortion effects**, **non dissipative shock waves** and **wave breaking** on an initially given soliton-like particle beam density profile.

• A **further methodological transfer** from **nonlinear optics** to **accelerator physics** was done with the analysis of **modulational instability** of **macroscopic matter waves** as described by the TWM.

In particular:

(i) The well known **coherent instability** (for instance, **positive or negative mass instability**), described by the Vlasov theory, is nothing but a sort of MI predicted by TWM for macroscopic matter waves with the above integro-differential NLSE;

(ii) The phenomenon of **Landau damping** and its **stabilizing role against the coherent instability** was recovered and then extended in a more general framework by using the the quantum-like kinetic approach;

(iii) **Until few years ago, the MI description in nonlinear optics was not yet capable to include the stabilizing effects**, as in the coherent instability description in accelerator physics. However, the results given by TWM **were soon transferred back to nonlinear optics** to **extend the standard MI theory of optical beams and bunches** to the context of **ensemble of partially incoherent waves** whose dynamics include the **statistical properties of the medium.**

• **At the present time, we can say that two distinct ways to treat MI are possible.**

The deterministic approach (the standard one), where the **linear stability analysis around a carrying wave is considered**. This corresponds to consider the **stability/instability** of **monochromatic wave trains** (system of coherent waves).

The statistical approach, where **the statistical properties of the medium (whether continuum or discrete) are taken into account**. In these physical conditions, **the stability analysis cannot be carried out as in the monochromatic case**. An **ensemble of partially incoherent waves must be taken into account**. This second approach **stimulated very recently a new branch of investigation** devoted to **MI of ensemble of partially incoherent waves with both theoretical and experimental aims.**

It was rapidly applied to Kerr media and soon extended to plasma physics (ensemble of partially incoherent Langmuir wave envelopes) and **physics of lattice vibrations**.

• **New improvements** were also registered in the **statistical formulation of MI for large amplitude surface gravity waves.**

THE MADELUNG FLUID PICTURE:

Hydrodynamical description of a system whose dyanamics is governed by LSE or NLSE

• A very valuable seminal contribution to quantum mechanics was given by de Broglie while developing the pilot wave theory with the concept of "**quantum potential**", but a systematic presentation of this idea came only several years later

•At the beginning of Fifties, Bohm also have considered the concept of quantum potential

• However, the concept was naturally appearing in a hydrodynamical description meanwhile proposed in 1926 by Madelung (first proposal of hydrodynamical model of quantum mechanics)

• A second proposal of a hydrodynamical description came in 1927 by Korn

$$i\alpha \frac{\partial \Psi}{\partial s} + \frac{\alpha^2}{2} \frac{\partial^2 \Psi}{\partial x^2} - \mathbf{U}[\,|\Psi|^2]\Psi = 0$$

$$\Psi = \sqrt{\rho(x,s)} \exp\left[\frac{i}{\alpha}\Theta(x,s)\right]$$

$$\frac{\partial \rho}{\partial s} + \frac{\partial}{\partial x}(\rho V) = 0,$$

$$\left(\frac{\partial}{\partial s} + V\frac{\partial}{\partial x}\right)V = -\frac{\partial U}{\partial x} + \frac{\alpha^2}{2}\frac{\partial}{\partial x}\left[\frac{1}{\rho^{1/2}}\frac{\partial^2 \rho^{1/2}}{\partial x^2}\right],$$

$$V(x,s) = \frac{\partial \Theta(x,s)}{\partial x}.$$

MADELUNG'S FLUID:

$$\frac{\partial \rho}{\partial s} + \frac{\partial}{\partial x}(\rho V) = 0\,,$$

$$\left(\frac{\partial}{\partial s} + V\frac{\partial}{\partial x}\right)V = -\frac{\partial U}{\partial x} + \frac{\alpha^2}{2}\frac{\partial}{\partial x}\left[\frac{1}{\rho^{1/2}}\frac{\partial^2 \rho^{1/2}}{\partial x^2}\right]$$

"DENSITY":

$$\rho = |\Psi|^2$$

"CURRENT VELOCITY":

$$V(x,s) = \frac{\partial \Theta(x,s)}{\partial x}$$

"NONLINEAR POTENTIAL ENERGY":

$$U[|\Psi|^2] = U[\rho]$$

SOLITONS IN THE MADELUNG FLUID DESCRIPTION

- WE WILL *NOT* SOLVE AN INVERSE SCATTERING PROBLEM.

- **PHYSICAL ASSUMPTIONS**:

STATIONARY PROFILE:

$$V(x,t) = V(x - u_0 s)$$

u_0 is a real constant

$$\rho(x,t) = \rho(x - u_0 s)$$

– (i) WE CONSTRUCT A CORRESPONDENCE

between

ENVELOPE SOLITONLIKE SOLUTIONS OF THE GENERALIZED NLSE:

$$i\alpha\frac{\partial\Psi}{\partial s} + \frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2} - U\left[|\Psi|^2\right]\Psi = 0$$

and

THE SOLITONLIKE SOLUTIONS OF THE GENERALIZED KdVE:

$$a\frac{\partial u}{\partial s} - G[u] \frac{\partial u}{\partial x} + \frac{\nu^2}{4} \frac{\partial^3 u}{\partial x^3} = 0$$

TRAVELLING WAVES

- stationary-profile envelope solution of gNLSE:

$$\Psi(x,s) = \sqrt{\rho(x - u_0 s)} \exp\left[i\Theta(x,s)/\alpha\right]$$

- stationary-profile solution of gKdVE:

$$u = u(x - u_0 s)$$

(ii) THIS CORRESPONDENCE SEEMS TO BE HELPFUL FOR FINDING ONE FAMILY OF SOLUTIONS OF THE gNLSE (gKDVE) STARTING FROM THE KNOWLEDGE OF THE gKdVE (gNLSE)

(iii) THE THEORY IS APPLIED TO WIDE CLASSES OF BOTH MODIFIED NLSE AND MODIFIED KdVE.

IN PARTICULAR:

$$U[|\Psi|^2] = q_1 ||\Psi|^2 + q_2 |\Psi|^4$$

or

$$U[|\Psi|^2] = q_0 |\Psi|^{2\beta}$$

WHERE β IS AN ARBITRARY POSITIVE REAL CONSTANT

BRIGHT, GRAY and DARK SOLITONLIKE SOLUTIONS ARE FOUND

BASIC EQUATIONS

$$\frac{\partial \rho}{\partial s} + \frac{\partial}{\partial x}(\rho V) = 0 \quad (1)$$

$$\left(\frac{\partial}{\partial s} + V\frac{\partial}{\partial x}\right) V = -\frac{\partial U}{\partial x} + \frac{\alpha^2}{2}\frac{\partial}{\partial x}\left[\frac{1}{\rho^{1/2}}\frac{\partial^2 \rho^{1/2}}{\partial x^2}\right] \quad (2)$$

- **BY MULTIPLYING (1) BY V:**

$$\rho\left(\frac{\partial}{\partial s} + V\frac{\partial}{\partial x}\right) V = -V\frac{\partial \rho}{\partial s} - V^2\frac{\partial \rho}{\partial x} + \rho\frac{\partial V}{\partial s} \quad (3)$$

- MULTIPLYING (3) BY ρ, COMBINING THE RESULT WITH (2) AND OBSERVING THAT:

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho^{1/2}}\frac{\partial^2 \rho^{1/2}}{\partial x^2}\right) = \frac{1}{\rho}\left(\frac{1}{2}\ \frac{\partial^3 \rho}{\partial x^3} - 4\ \frac{\partial \rho^{1/2}}{\partial x}\ \frac{\partial^2 \rho^{1/2}}{\partial x^2}\right)$$

WE OBTAIN:

$$\rho\left(\frac{\partial}{\partial s} + V\frac{\partial}{\partial x}\right)V =$$

$$= -\frac{\partial U}{\partial x}\rho + \frac{\alpha^2}{4}\ \frac{\partial^3 \rho}{\partial x^3} - 2\alpha^2\ \frac{\partial \rho^{1/2}}{\partial x}\ \frac{\partial^2 \rho^{1/2}}{\partial x^2} \quad (4)$$

- COMBINING (4) WITH (3):

$$-V\frac{\partial\rho}{\partial s} \; - \; V^2\frac{\partial\rho}{\partial x} \; + \; \rho\frac{\partial V}{\partial s} =$$

$$= -\frac{\partial U}{\partial x}\rho + \frac{\alpha^2}{4}\,\frac{\partial^3\rho}{\partial x^3} - 2\alpha^2\,\frac{\partial\rho^{1/2}}{\partial x}\,\frac{\partial^2\rho^{1/2}}{\partial x^2} \qquad (5)$$

- BY INTEGRATING (2) WITH RESPECT TO x AND MULTIPLYING THE RESULTING EQUATION BY

$$\rho^{1/2}\,\left(\partial\rho^{1/2}/\partial x\right)$$

WE HAVE:

$$-2\alpha^2 \, \frac{\partial \rho^{1/2}}{\partial x} \, \frac{\partial^2 \rho^{1/2}}{\partial x^2} = -2\frac{\partial \rho}{\partial x} \int \left(\frac{\partial V}{\partial s}\right) \, dx -$$

$$-V^2\frac{\partial \rho}{\partial x} \; - 2 \; U\frac{\partial \rho}{\partial x} \; + \; 2 \; c_0(s) \; \frac{\partial \rho}{\partial x} \qquad (6)$$

WHERE $c_0(s)$ IS AN ARBITRARY FUNCTION OF s.

- FINALLY, BY COMBINING (5) AND (6) WE OBTAIN THE FOLLOWING EQUATION:

$$-\left(\frac{\partial V}{\partial s}\right)\rho+V\frac{\partial \rho}{\partial s}+2\left[c_0(s)-\int\left(\frac{\partial V}{\partial s}\right)\ dx\right]\frac{\partial \rho}{\partial x}-$$

$$-\left(\frac{\partial U}{\partial x}\rho+2\ U\frac{\partial \rho}{\partial x}\right)+\frac{\alpha^2}{4}\frac{\partial^3 \rho}{\partial x^3}\ =\ 0 \qquad (7)\ .$$

IN PARTICULAR FOR:

$$U[\rho] = q_0 \rho^{\beta}$$

EQ.(7) BECOMES:

$$-\left(\frac{\partial V}{\partial s}\right)\rho + V\frac{\partial \rho}{\partial s} + 2\left[c_0(s) - \int \left(\frac{\partial V}{\partial s}\right)\ dx\right]\frac{\partial \rho}{\partial x} -$$

$$-(\beta + 2)\, q_0\ \rho^{\beta}\frac{\partial \rho}{\partial x} + \frac{\alpha^2}{4}\frac{\partial^3 \rho}{\partial x^3} = 0 \qquad (8)\ .$$

UNDER SUITABLE PHYSICAL ASSUMPTIONS FOR $V(x,s)$, EQ.(8) TAKES THE FORM OF A *MODIFIED* KOTEWEG-de VRIES EQUATION (MKdVE)

FOR DETAILS, SEE:

R. Fedele and H. Schamel, Eur. Phys. J. B **27**, 313 (2002);

R. Fedele, Physica Scripta **65**, 502 (2002)

R. Fedele, H. Schamel and P.K. Shukla,Physica Scripta T98, 18 (2002)

I. gNLSE vs gKDVE

the set of all the stationary-profile envelope solutions of the gNLSE:

$$\mathcal{E} = \{\Psi\}$$

the set of all non-negative stationary-profile solutions of the gKdVE:

$$\mathcal{S} = \{u(\xi) \geq 0\}$$

$\mathcal{E} \rightarrow \mathcal{S}$

If $\Psi \in \mathcal{E}$:

$$\rho = \rho(\xi) \ , \quad \text{and} \quad V = V(\xi) \ .$$

- $c_0(s)$ becomes constant (so that, let us put $c_0(s) \equiv c_0$);

- continuity equation becomes:

$$u_0 \frac{d\rho}{d\xi} = \frac{d}{d\xi}(\rho V) \quad ,$$

which integrated gives:

$$V(\xi) = u_0 + \frac{A_0}{\rho(\xi)} \quad ,$$

where A_0 is an arbitrary constant.

Consequently:

Eq. (7) becomes:

$$\left(u_0^2 + 2c_0\right)\frac{d\rho}{d\xi} \;-\; \mathcal{I}\left[\rho\right]\frac{d\rho}{d\xi} \;+\; \frac{\alpha^2}{4}\frac{d^3\rho}{d\xi^3} \;=\; 0$$

$$\mathcal{I}\left[\rho\right] \;\equiv\; \rho\frac{dU\left[\rho\right]}{d\rho} \;+\; 2U\left[\rho\right] \quad .$$

On the other hand, for $u = u(\xi)$, gKdVE becomes:

$$-u_0 a \, \frac{du}{d\xi} \; - \; G\,[u]\,\frac{du}{d\xi} \; + \; \frac{\nu^2}{4}\frac{d^3 u}{d\xi^3} \; = \; 0 \qquad .$$

Then:

$$a \equiv -\left(u_0^2 + 2c_0\right)/u_0 \, , \quad G\,[u] \equiv \mathcal{I}\,[u] \; , \quad \nu \equiv \alpha$$

$u(\xi)$ is a non-negative stationary-profile solution of ($u_0 \neq 0$):

$$-\frac{u_0^2 + 2c_0}{u_0} \, \frac{\partial \rho}{\partial s} \; - \; \mathcal{I}\,[\rho] \; \frac{\partial \rho}{\partial x} \; + \; \frac{\alpha^2}{4} \, \frac{\partial^3 \rho}{\partial x^3} \; = \; 0$$

WE HAVE CONSTRUCTED THE FOLLOWING CORRESPONDENCE

$$
\begin{aligned}
\mathcal{F} \quad &: \quad \Psi \in \mathcal{E} \rightarrow u \in \mathcal{S} \quad , \\
u \quad &= \quad \mathcal{F}[\Psi] \;=\; |\Psi|^2 \;=\; \rho(\xi) \quad .
\end{aligned}
$$

In particular, $\mathcal{F}$ associates an envelope solitonlike solution of gNLSE with a solitonlike solution of gKdVE.

$\underline{\mathcal{S} \to \mathcal{E}}$

- **Note that**:

$$-\partial\Theta/\partial s = c_0 + u_0 V \ ,$$

which, combined with continuity eq. becomes:

$$\begin{aligned}\Theta(x,s) \quad &= \quad \phi_0 \ - \ \left(c_0 + u_0^2\right) s \ + \ u_0 x \ + \\ &+ \quad A_0 \int \frac{d\xi}{\rho(\xi)} \quad ,\end{aligned}$$

where ϕ_0 is an arbitrary real constant.

- **Let** $u(\xi) \in \mathcal{S}$. Thus, $u(\xi)$ satisfies the following gKdVE

$$a \frac{\partial u}{\partial s} - G[u] \frac{\partial u}{\partial x} + \frac{\nu^2}{4} \frac{\partial^3 u}{\partial x^3} = 0 \quad .$$

for a given functional $G[u]$.

Consequently:

$u(\xi)$ satisfies the following pair of coupled equations:

$$\frac{\partial u}{\partial s} + \frac{\partial}{\partial x}\left(u\tilde{V}\right) = 0\,,$$

$$-\left(\frac{\partial \tilde{V}}{\partial s}\right) u + \tilde{V}\frac{\partial u}{\partial s} + 2\left[c_0(s) - \int \left(\frac{\partial \tilde{V}}{\partial s}\right)\; dx\right]\frac{\partial u}{\partial x} -$$

$$-\left(\frac{\partial \mathcal{U}}{\partial x}u + 2\,\mathcal{U}\frac{\partial u}{\partial x}\right) \;+\; \frac{\nu^2}{4}\frac{\partial^3 u}{\partial x^3} \;=\; 0 \quad ,$$

where

$$\tilde{V} \equiv \frac{\partial \tilde{\Theta}(x,s)}{\partial x}$$

$$\tilde{\Theta}(x,s) \;=\; \phi_0 \;-\; \left(c_0 + u_0^2\right) s \;+\; u_0 x \;+ \;+\; A_0 \int \frac{d\xi}{u(\xi)} \quad ,$$

$$u\,\frac{d\mathcal{U}}{du} \;+\; 2\mathcal{U} \;=\; G\,[u]$$

namely

$$\mathcal{U}\left[u\right] \;=\; \frac{1}{u^2}\left[K_0 \;+\; \int \; G\left[u\right] \;\; u \; du\right]$$

$$\Psi = \sqrt{u(\xi)}\;\; \exp\left[\frac{i}{\alpha}\tilde{\Theta}(x,s)\right]$$

is a stationary-profile envelope solution of the following gNLSE:

$$i\nu\frac{\partial\Psi}{\partial s} \;+\; \frac{\nu^2}{2}\frac{\partial^2\Psi}{\partial x^2} \;-$$

$$-\;\left[\frac{K_0 \;+\; \int \; G\left[|\Psi|^2\right] \;\; |\Psi|^2 \; d|\Psi|^2}{|\Psi|^4}\right]\Psi \;=\; 0$$

EQUATION FOR $u(\xi)$, AFTER SEPARATING REAL AND IMAGINARY PARTS:

$$-\frac{\nu^2}{2}\frac{d^2u^{1/2}}{d\xi^2} \;+\; \frac{K_0}{u^{3/2}}+\frac{1}{u^{3/2}}\int G\left[u\right]\; u\; du =$$
$$= \left(c_0 + \frac{u_0^2}{2}\right) u^{1/2} - \frac{A_0^2}{2u^{3/2}}$$

- K_0, c_0, and A_0 NOT ALL INDENPENDENT

- FOR EACH GIVEN $u \in \mathcal{S}$ AND FOR EACH GIVEN SET OF ϕ_0, c_0, and A_0, MODULUS AND PHASE OF Ψ UNIQUELY DETERMINED

IN CONCLUSION, STARTING FROM THE gKdVE WE HAVE CONSTRUCTED THE FOLLOWING CORRESPONDENCE:

$$\mathcal{H} \;:\; u \in \mathcal{S} \rightarrow \Psi \in \mathcal{E} \quad ,$$

$$\Psi = \mathcal{H}\left[u\right] = \sqrt{u(\xi)}\times$$

$$\times \exp\left\{\frac{i}{\nu}\left[\phi_0 - \left(c_0 + u_0^2\right) s + u_0 x + A_0 \int \frac{d\xi}{u(\xi)}\right]\right\}$$

- AS THE PARAMETERS VARY OVER ALL THEIR ACCESSIBLE RANGES OF VALUES, $\mathcal{H}[u]$ DESCRIBES THE SUBSET OF STATIONARY-PROFILE ENVELOPE SOLUTIONS OF THE ASSOCIATED gNLSE.

- IF $u(\xi)$ IS A LOCALIZED SOLUTION OF gKdVE, THUS $\mathcal{H}[u]$ DESCRIBES THE SUBSET OF ENVELOPE LOCALIZED SOLUTIONS OF THE ASSOCIATED gNLSE

II. SOME IMPORTANT ASPECTS CONCERNING THE ROLE OF THE BOUNDARY CONDITIONS FOR SOLITONLIKE SOLUTIONS

bright solitons

Let $u > 0$ be bright solitonlike solutions satisfying the following boundary conditions:

$$\lim_{\xi \to \pm\infty} u(\xi) = 0 \quad ;$$

Thus ($u = \rho$):

$$A_0 = 0 \;, \qquad \text{and} \qquad V = \tilde{V} = u_0 \quad .$$

Consequently,

$$\Psi = \sqrt{u(\xi)} \exp\left\{\frac{i}{\nu}\left[\phi_0 - \left(c_0 + u_0^2\right) s + u_0 x\right]\right\}$$

Additionally: regular behaviour in such a way that

$$\lim_{\xi \to \pm\infty} \left| \frac{1}{u^2} \int \; G\left[u(\xi)\right] \; u \; du \right| \; < \infty$$

Thus:

$$K_0 \; = \; 0 \quad ,$$

$$-\frac{\alpha^2}{2}\frac{d^2 u^{1/2}}{d\xi^2} + \left[\frac{1}{u^2} \int \; G\left[u(\xi)\right] u du\right] u^{1/2} = E_0 u^{1/2}$$

where $E_0 = c_0 + u_0^2/2$.

IN CONCLUSION

- FOR STANDARD BRIGHT SOLITON-LIKE SOLUTIONS,

$$K_0 = A_0 = 0$$

AND THE PHASE OF Ψ IS LINEAR

- $u(\xi)$ AND THE CONSTANT E_0 PLAY, RESPECTIVELY, THE ROLE OF EIGEN-STATE AND EIGENVALUE OF THE gNLSE

- CONTINUITY EQUATION BECOMES

$$\frac{\partial \rho}{\partial s} + V_0 \frac{\partial \rho}{\partial x} = 0 ,$$

WHICH IMPLIES THAT ρ IS A FUNCTION OF THE COMBINED VARIABLE

$$\xi \equiv x - V_0 s \quad :$$

$$\rho = \rho(\xi) = \rho(x - V_0 s) \ .$$

- UNDER THE ABOVE HYPOTHESIS:

$$c_0(s) \equiv c_0 = \text{const.}$$

1. $U[\rho] \; = \; q_0|\Psi|^{2\beta} \; = \; q_0\rho^{\beta}$

$$2E \; \frac{d\rho}{d\xi} - (\beta+2) \;\; q_0 \;\; \rho^{\beta}\frac{d\rho}{d\xi} + \frac{\alpha^2}{4}\frac{d^3\rho}{d\xi^3} \; = \; 0$$

WHERE $E \; = \; c_0 \; - \; \frac{V_0^2}{2}$.

$$-\frac{\alpha^2}{2}\frac{d^2\rho^{1/2}}{d\xi^2} \; + \; \left(q_0\rho^{\beta}\right)\rho^{1/2} \; = \; E\rho^{1/2}$$

$$\Psi(x,s) \; = \; \rho^{1/2}(x - V_0 s) \;\; \exp\left[ikx - i\omega s\right]$$

WHERE

$$k \equiv V_0/\alpha \;\; , \qquad \textbf{and} \qquad \omega \equiv \left(E + \frac{V_0^2}{2}\right)/\alpha$$

Note that in this case the eikonal is

$$\Theta(x,s) \; = \; V_0 x \;\; - \;\; \left(E + \frac{V_0^2}{2}\right) s$$

1A. BRIGHT SOLITONS

FOR: $0 < \beta < \infty$, THE FAMILY OF MKdVE EQUATIONS:

$$2E\ \frac{d\rho}{d\xi} - (\beta + 2)\, q_0\ \rho^{\beta} \frac{d\rho}{d\xi} + \frac{\alpha^2}{4} \frac{d^3\rho}{d\xi^3}\ =\ 0 \quad ,$$

PROVIDED THAT:

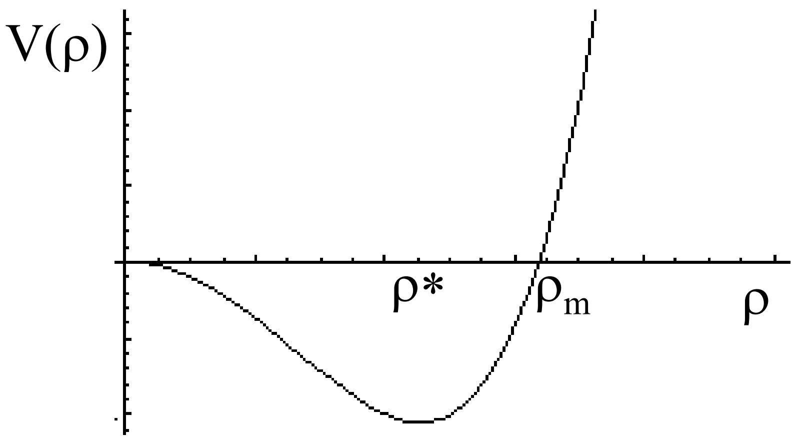
$$q_0 < 0 \ \ \textbf{AND}\ E < 0\ ,$$

ADMITS THE FOLLOWING FAMILY OF BRIGHT SOLITON-LIKE SOLUTIONS:

$$\rho(x - V_0 s)\ =\ \rho_m\ \textbf{sech}^{2/\beta}\left(\frac{x - V_0 s}{\Delta}\right) \quad ,$$

$$\rho_m = \left[\frac{|E|}{|q_0|}(\beta + 1)\right]^{1/\beta} \qquad \Delta = \frac{|\alpha|}{\beta} \frac{1}{\sqrt{2|E|}}$$

THIS RESULT CAN BE EASILY PROVEN BY USING THE SAGDEEV POTENTIAL METHOD



$$V(\rho) = E_0\, \rho^2 - [q_0 / (\beta + 1)]\, \rho^{\beta+2}$$

$\beta > 0$

$$\rho_m = [(|E_0|/|q_0|)(\beta+1)]^{1/\beta}$$

1B. DARK SOLITONS

FOR ARBITRARY β IN THE RANGE: $0 < \beta < \infty$, THE INVESTIGATION OF DARK SOLITONS IS STILL IN PROGRESS.

HOWEVER, THE STANDARD DARK SOLITON OF THE CUBIC NLSE ($\beta = 1$) HAS BEEN ALSO RECOVERED, BY ASSUMING THAT THE FOLLOWING MKdVE:

$$-\frac{2E}{V_0}\frac{\partial \rho}{\partial s} \; - \; 3q_0 \; \rho\frac{\partial \rho}{\partial x} \; + \; \frac{\alpha^2}{4} \; \frac{\partial^3 \rho}{\partial x^3} \; = \; 0$$

HAS A SOLUTION OF THE FORM

$$\rho(\xi) \; = \; \rho_0 \; + \; \rho_1(\xi)$$

with $\rho_0 > 0 \,, \quad \rho_1(\xi) < 0 \,, \quad$ **and** $\quad |\rho_1| \leq \rho_0$

AND WITH THE BOUNDARY CONDINTIONS: $\lim_{\xi \to \pm\infty} \rho_1(\xi) = 0$

$$\rho(x - V_0 s) \;=\; \rho_0 \; \mathbf{tanh}^2 \left[\frac{\sqrt{q_0 \rho_0}}{|\alpha|} (x - V_0 s) \right]$$

$$\Psi(x, s) \;=\; \sqrt{\rho_0} \; \left| \mathbf{tanh} \left[\frac{\sqrt{q_0 \rho_0}}{|\alpha|} (x - V_0 s) \right] \right| \times$$

$$\times \exp \left\{ \frac{i}{\alpha} \left(V_0 x - \left(q_0 \rho_0 + \frac{V_0^2}{2} \right) s \right) \right\}$$

2. $U = a_1|\Psi|^2 + a_2|\Psi|^4 = a_1\rho + a_2\rho^2$

BY USING THE EQUATION:

$$-\left(\frac{\partial V}{\partial s}\right)\rho + V\frac{\partial \rho}{\partial s} + 2\left[c_0(s) - \int\left(\frac{\partial V}{\partial s}\right)\ dx\right]\frac{\partial \rho}{\partial x} -$$

$$-\left(\frac{\partial U}{\partial x}\rho + 2\ U\frac{\partial \rho}{\partial x}\right) + \frac{\alpha^2}{4}\frac{\partial^3 \rho}{\partial x^3} = 0 \qquad (7)$$

WE GET THE FOLLOWING STATIONARY MKdVE:

$$2E'\ \frac{d\rho}{d\xi} - 4a_2\,(\rho - \rho_0)^2\,\frac{d\rho}{d\xi} + \frac{\alpha^2}{4}\frac{d^3\rho}{d\xi^3} = 0$$

WHERE:

$$E' = E + \frac{9}{32}\frac{a_1^2}{a_2}\ ,\ \ \rho_0 = -\frac{3}{8}\frac{a_1}{a_2}$$

DARK SOLITONS $\left(\lim_{\xi\to\pm\infty}\rho(\xi)=\rho_0\right)$

$$\rho(\xi)=\frac{3a_1}{8|a_2|}\left[1-\mathbf{sech}\,(\xi/\Delta)\right]$$

$$\Psi(x,s)=\sqrt{\frac{3a_1}{8|a_2|}}\left[1-\mathbf{sech}\,(\xi/\Delta)\right]^{1/2}\times$$

$$\times\exp\left\{\frac{i}{\alpha}\left(V_0x-\left(U(\rho_0)+\frac{V_0^2}{2}\right)s\right)\right\}$$

WHERE:

$$\Delta\ =\ \sqrt{\frac{8\alpha^2}{3}\frac{|a_2|}{a_1^2}}$$

II. SOLITONS FOR ARBITRARILY LARGE STATIONARY-PROFILE PERTURBATION OF THE CURRENT VELOCITY:

$$V = V_0 + V_1(\xi)$$

where $\xi = x - u_0 s$ (u_0 being a constant)

1. $U = a_1|\Psi|^2 + a_2|\Psi|^4 = a_1\rho + a_2\rho^2$

- **WE GET:**

$$2E'' \frac{d\rho}{d\xi} - 4a_2 (\rho - \rho_0)^2 \frac{d\rho}{d\xi} + \frac{\alpha^2}{4}\frac{d^3\rho}{d\xi^3} = 0$$

where:

$$E'' = \frac{3}{64}\frac{a_1^2}{a_2} + \frac{(u_0 - V_0)^2}{2} \ , \ \rho_0 = -\frac{3}{8}\frac{a_1}{a_2}$$

- **BOUNDARY CONDITIONS:**

$$\lim_{\xi \to \pm\infty} \rho(\xi) \; = \; \rho_0 \; , \qquad \lim_{\xi \to \pm\infty} V_1(\xi) \; = 0$$

- **FROM CONTINUITY EQUATION:**

$$V \; = \; (u_0 - V_0) \; \frac{\rho - \rho_0}{\rho}$$

- **SOLITON-LIKE SOLUTIONS FOR:**

$$a_1 \; > \; 0 \; , \qquad a_2 \; < \; 0$$

$$V_0 \; - \; \frac{a_1}{4}\sqrt{\frac{3}{2|a_2|}} \;\; < u_0 \; < V_0 \; + \; \frac{a_1}{4}\sqrt{\frac{3}{2|a_2|}}$$

- **ASSOCIATED MKdVE SOLITON:**

$$\rho(\xi) = \frac{3}{8}\frac{a_1}{|a_2|} - \sqrt{\frac{9}{64}\frac{a_1^2}{a_2^2} - \frac{3}{2}\frac{(u_0 - V_0)^2}{|a_2|}}\; \mathbf{sech}\,(\xi/\Delta)$$

- ENVELOPE SOLITON:

$$\Psi(x,s) =$$

$$= \sqrt{\frac{3}{8}\frac{a_1}{|a_2|} - \sqrt{\frac{9}{64}\frac{a_1^2}{a_2^2} - \frac{3}{2}\frac{(u_0 - V_0)^2}{|a_2|}}\ \textbf{sech}\,(\xi/\Delta)} \times$$

$$\times \exp\left\{\frac{i}{\alpha}\left[\Theta_0(x,s) \; + \; \Theta_1(\xi)\right]\right\}$$

(a). GRAY SOLITONS FOR:

$$u_0 \neq V_0$$

(b). DARK SOLITONS FOR:

$$u_0 = V_0$$

"BACKGROUND" PHASE:

$$\Theta_0(x,s) \; = \; \int V_0 \; dx \; + \; \phi(s) \; =$$

$$= \; V_0 x - \left(U(\rho_0) + \frac{V_0^2}{2} \right) s$$

"PERTURBATION" PHASE:

$$\Theta_1(\xi) \; = \; \Theta_{10} \; + \; \int V_1(\xi) \; d\xi \; =$$

$$= \Theta_{10} - \mathbf{sign}(u_0 - V_0) A \; \mathrm{arctan}\left[B \;\; \mathrm{tanh}\left(\xi/2\Delta\right)\right]$$

$$A = 2\sqrt{\frac{3}{32}\frac{a_1^2}{|a_2|} + (u_0 - V_0)^2}$$

$$B = \frac{\frac{3}{8}\frac{a_1}{|a_2|} - \sqrt{\frac{9}{64}\frac{a_1^2}{a_2^2} - \frac{3}{2}\frac{(u_0 - V_0)^2}{|a_2|}}}{|u_0 - V_0|\sqrt{\frac{3}{2|a_2|}}}$$

2. $U = q_0|\Psi|^2 = q_0\rho$

A PROCEDURE FULLY SIMILAR TO THE ONE PRESENTED ABOVE ALLOWS ALSO TO RECOVER THE WELL-KNOWN SOLITONS OF THE CUBIC NLSE:
FOR:

$$q_0 > 0$$

$$V_0 - \sqrt{q_0\rho_0} < u_0 < V_0 + \sqrt{q_0\rho_0}$$

- **SOLUTION OF THE ASSOCIATED MKdVE:**

$$\rho(\xi) = \rho_0 \left[1 - C^2 \ \mathbf{sech}^2 \left(\frac{\sqrt{q_0\rho_0 - (u_0 - V_0)^2}}{|\alpha|} \xi \right) \right]$$

- **SOLUTION OF THE NLSE:**

$$\Psi(x,s) = \sqrt{\rho_0 \left[1 - C^2 \ \mathbf{sech}^2 \left(\frac{\sqrt{q_0\rho_0 - u_0^2}}{|\alpha|} \xi \right) \right]} \times$$

$$\times \exp \left\{ \frac{i}{\alpha} \left[\Theta_0(x,s) + \Theta_1(\xi) \right] \right\}$$

$$C^2 = \frac{q_0\rho_0 - (u_0 - V_0)^2}{q_0\rho_0} > 0$$

(a). GRAY SOLITONS FOR:

$$u_0 \neq V_0 \quad (-1 < C < 1)$$

(b). DARK SOLITONS FOR:

$$u_0 = V_0 \quad (C = \pm 1)$$

"BACKGROUND" PHASE:

$$\Theta_0(x, s) \; = \; \int V_0 \; dx \; + \; \phi(s) \; =$$

$$= \; V_0 x - \left(q_0 \rho_0 + \frac{V_0^2}{2} \right) s$$

"PERTURBATION" PHASE:

$$\Theta_1(\xi) \;=\; \Theta_{10} \;-\; \int V_1(\xi)\; d\xi \;=$$

$$= \Theta_{10} - D \;\; \mathrm{arctan} \left[F \,\mathrm{tanh} \left(\frac{|C| \sqrt{q_0 \rho_0}}{|\alpha|} \xi \right) \right]$$

$$D = -\frac{|\alpha|}{|C| \sqrt{q_0 \rho_0}} \; (u_0 - V_0) \; F$$

$$F = \frac{C}{\sqrt{1 - C^2}}$$

EQUIVALENT EXPRESSION FOR $\Theta_1(\xi)$:

$$\Theta_1(\xi) =$$

$$\Theta_{10} - D \ \arcsin\left[\frac{C \tanh\left(\frac{|C|\sqrt{q_0\rho_0}}{|\alpha|}\xi\right)}{\sqrt{1 - C^2 \mathbf{sech}^2\left(\frac{|C|\sqrt{q_0\rho_0}}{|\alpha|}\xi\right)}}\right]$$

3. ENVELOPE SOLITONS OF NLSE WITH AN "ANTI-CUBIC" NON-LINEARITY

The method presented above can be also used to find envelope soliton-like solutions of the following modified NLSE containing, besides the cubic and quintic nonlinearities, an **anti-cubic** nonlinearity (i.e. $|\Psi|^{-4}\Psi$):

$$i\alpha\frac{\partial\Psi}{\partial s} + \frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2} - Q_0 \, |\Psi|^{-4}\Psi + \\ + \left[q_1 \, |\Psi|^2 + q_2 \, |\Psi|^4\right]\Psi = 0$$

where Q_0 is a real constant.

In this case, we have to take the general functional form of $U[u]$, i.e.,

$$U\,[u] \;=\; \frac{1}{u^2}\left[K_0 \;+\; \int\; G\,[u]\;\; u\; du\right],$$

where K_0 is an arbitrary real constant, but $G[u]$ is given by the same functional expression corresponding to the case of cubic-quintic nonlinearity only:

$$G\,[u] \;=\; 3q_1\; u \;+\; 4q_2\; u^2\,.$$

Provided that: $K_0 = Q_0\,,$

$$-\frac{\nu^2}{2}\frac{d^2u^{1/2}}{d\xi^2} + \frac{Q_0}{u^{3/2}} + \frac{1}{u^{3/2}} \int G\left[u\right] \ u \ du \ = $$

$$\left(c_0 \ + \ \frac{u_0^2}{2}\right) u^{1/2} \ - \ \frac{A_0^2}{2u^{3/2}} \quad .$$

Consequently, a family of solitary wave solutions of the cubic-quintic NLSE with the inclusion of the anti-cubic term can be obtained by imposing the following condition

$$A_0 = \pm\sqrt{-2Q_0}$$

which implies that such a kind of family of solutions exists for $Q_0 < 0$. They are nonlinear stationary states of the following NLSE:

$$-\frac{\alpha^2}{2}\frac{d^2 u^{1/2}}{d\xi^2} + \left(q_1 u + q_2 u^2\right) u^{1/2} = E_0 u^{1/2}$$

where $E_0 = c_0 + u_0^2/2$.

It follows that, for any $Q_0 < 0$:

$$\Psi_{\pm}(x,s) = \sqrt{\overline{u}\ \left[1 \ + \ \epsilon \ \mathrm{sech}\left(\frac{\xi}{\Delta}\right)\right]}$$

$$\times \exp\left\{\frac{i}{\alpha}\left[\phi_0 - \left(E_0 + \frac{u_0^2}{2}\right)s + u_0 x\right]\right\}$$

$$\times \exp\left\{\frac{i}{\alpha}\left[\pm\sqrt{2|Q_0|}\ \int\frac{d\xi}{\overline{u}\ \ [1 \ + \ \epsilon\ \mathrm{sech}\,(\xi/\Delta)]}\right]\right\}$$

where, in principle, ϵ should be taken in the following range

$$-1 < \epsilon \le 1 \quad ,$$

which excludes the standard "dark" solitary waves ($\epsilon = -1$).

Actually, the direct substitution of $u = |\Psi|^2$ into the eigenvalue equation allows us to find that:

$$\epsilon = 1$$

$$\overline{u} = -\frac{3q_1}{8q_2}$$

$$q_1 > 0 \, , \quad q_2 < 0$$

$$E_0 = -\frac{15q_1^2}{64q_2}$$

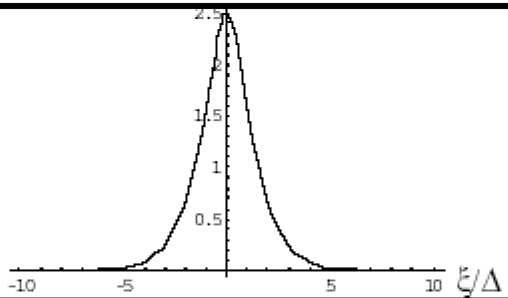
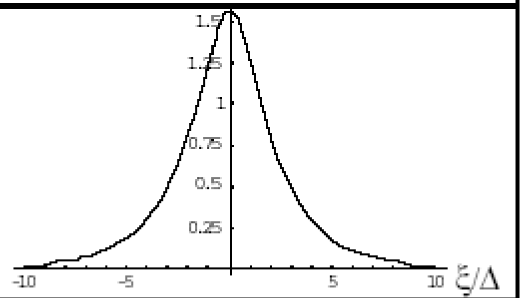
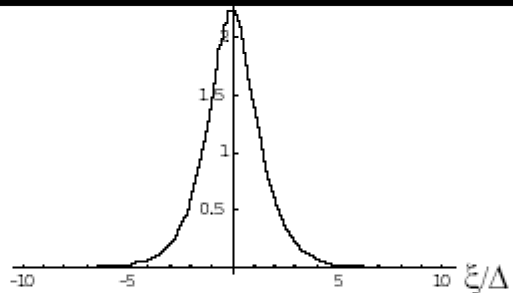
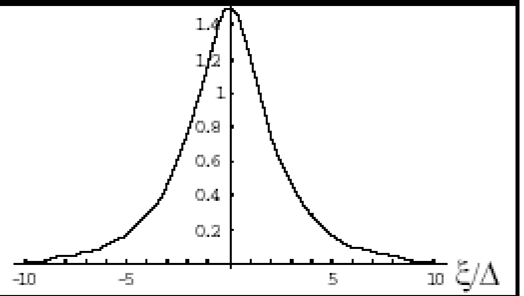
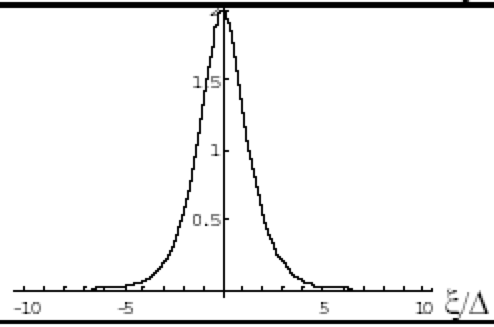
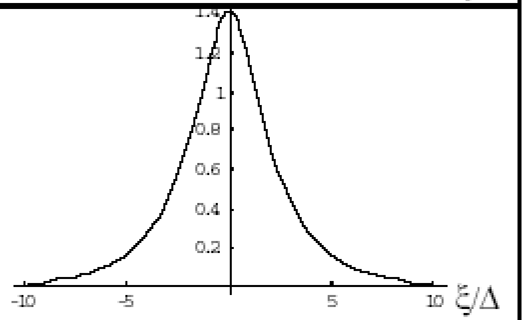
$$\Delta = \frac{2|\alpha|}{q_1}\sqrt{\frac{2|q_2|}{3}}$$

The condition $\epsilon = 1$ corresponds to **upper-shifted** bright solitons.

For details see:
R. Fedele, H. Schamel, V. I. Karpman and P K Shukla
J. Phys. A: Math. Gen. **36** 1169 (2003)

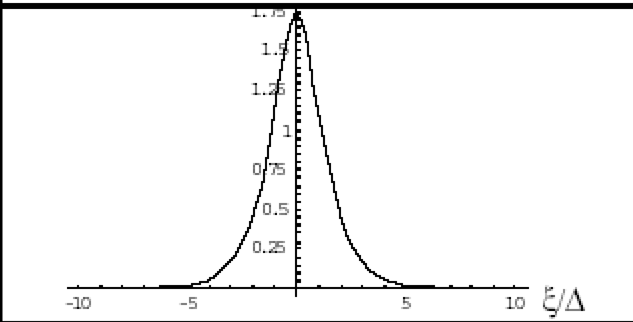
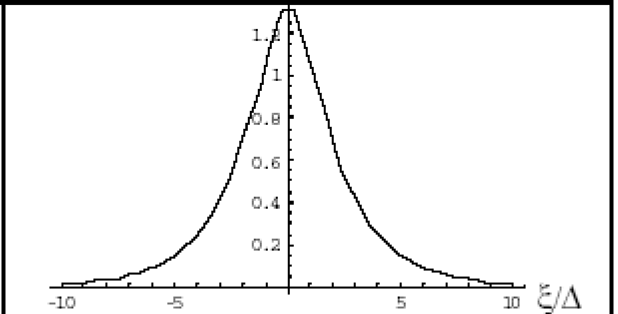
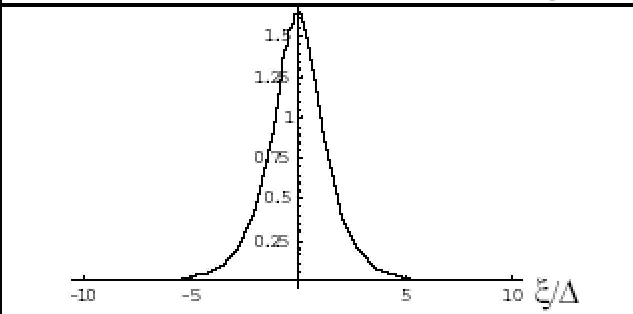
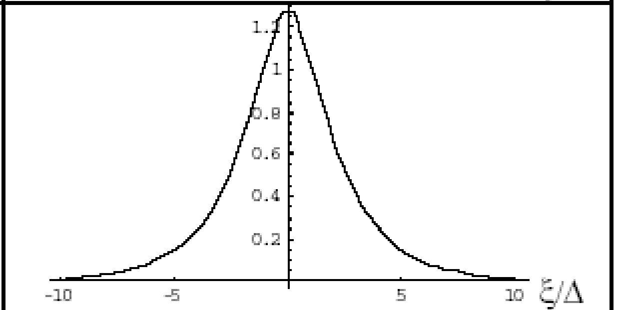
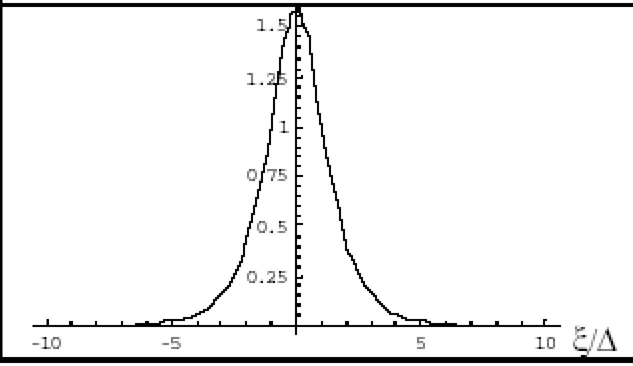
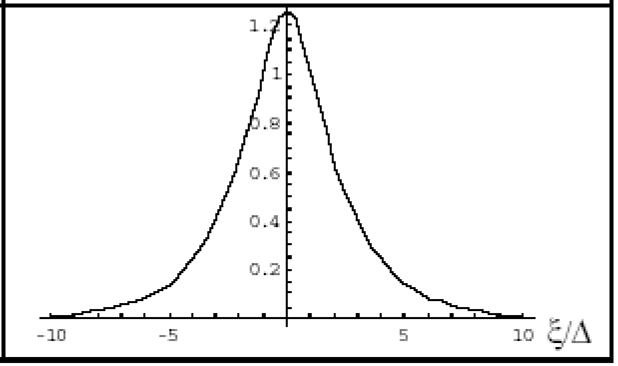
β	MKdVE	u	MNLSE	$\lvert\Psi\rvert$
1/10	$a\frac{\partial u}{\partial s} - \frac{21}{10}q_0 u^{1/10}\frac{\partial u}{\partial x} + \frac{\alpha^2}{4}\ \frac{\partial^3 u}{\partial x^3} = 0$	$\left(\frac{11\lvert E_0\rvert}{10\lvert q_0\rvert}\right)^{10}\operatorname{sech}^{20}\left(\frac{\sqrt{2\lvert E_0\rvert}}{10\lvert\alpha\rvert}\xi\right)$	$i\alpha\frac{\partial\Psi}{\partial s} = -\frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2} - q_0\lvert\Psi\rvert^{1/5}\Psi$	$\left(\frac{11\lvert E_0\rvert}{10\lvert q_0\rvert}\right)^{5}\operatorname{sech}^{10}\left(\frac{\sqrt{2\lvert E_0\rvert}}{10\lvert\alpha\rvert}\xi\right)$
1/8	$a\frac{\partial u}{\partial s} - \frac{17}{8}q_0 u^{1/8}\frac{\partial u}{\partial x} + \frac{\alpha^2}{4}\ \frac{\partial^3 u}{\partial x^3} = 0$	$\left(\frac{9\lvert E_0\rvert}{8\lvert q_0\rvert}\right)^{8}\operatorname{sech}^{16}\left(\frac{\sqrt{2\lvert E_0\rvert}}{8\lvert\alpha\rvert}\xi\right)$	$i\alpha\frac{\partial\Psi}{\partial s} = -\frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2} - q_0\lvert\Psi\rvert^{1/4}\Psi$	$\left(\frac{9\lvert E_0\rvert}{8\lvert q_0\rvert}\right)^{4}\operatorname{sech}^{8}\left(\frac{\sqrt{2\lvert E_0\rvert}}{8\lvert\alpha\rvert}\xi\right)$
1/3	$a\frac{\partial u}{\partial s} - \frac{7}{3}q_0 u^{1/3}\frac{\partial u}{\partial x} + \frac{\alpha^2}{4}\ \frac{\partial^3 u}{\partial x^3} = 0$	$\left(\frac{4\lvert E_0\rvert}{3\lvert q_0\rvert}\right)^{3}\operatorname{sech}^{6}\left(\frac{\sqrt{2\lvert E_0\rvert}}{3\lvert\alpha\rvert}\xi\right)$	$i\alpha\frac{\partial\Psi}{\partial s} = -\frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2} - q_0\lvert\Psi\rvert^{2/3}\Psi$	$\left(\frac{4\lvert E_0\rvert}{3\lvert q_0\rvert}\right)^{3/2}\operatorname{sech}^{3}\left(\frac{\sqrt{2\lvert E_0\rvert}}{3\lvert\alpha\rvert}\xi\right)$
1/2	$a\frac{\partial u}{\partial s} - \frac{5}{2}q_0 u^{1/2}\frac{\partial u}{\partial x} + \frac{\alpha^2}{4}\ \frac{\partial^3 u}{\partial x^3} = 0$	$\left(\frac{3\lvert E_0\rvert}{2\lvert q_0\rvert}\right)^{2}\operatorname{sech}^{4}\left(\frac{\sqrt{2\lvert E_0\rvert}}{2\lvert\alpha\rvert}\xi\right)$	$i\alpha\frac{\partial\Psi}{\partial s} = -\frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2} - q_0\lvert\Psi\rvert\Psi$	$\left(\frac{3\lvert E_0\rvert}{2\lvert q_0\rvert}\right)\operatorname{sech}^{2}\left(\frac{\sqrt{2\lvert E_0\rvert}}{2\lvert\alpha\rvert}\xi\right)$
1	$a\frac{\partial u}{\partial s} - 3q_0 u\frac{\partial u}{\partial x} + \frac{\alpha^2}{4}\ \frac{\partial^3 u}{\partial x^3} = 0$	$\left(\frac{2\lvert E_0\rvert}{\lvert q_0\rvert}\right)\operatorname{sech}^{2}\left(\frac{\sqrt{2\lvert E_0\rvert}}{\lvert\alpha\rvert}\xi\right)$	$i\alpha\frac{\partial\Psi}{\partial s} = -\frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2} - q_0\lvert\Psi\rvert^{2}\Psi$	$\left(\frac{2\lvert E_0\rvert}{\lvert q_0\rvert}\right)^{1/2}\operatorname{sech}\left(\frac{\sqrt{2\lvert E_0\rvert}}{\lvert\alpha\rvert}\xi\right)$
3/2	$a\frac{\partial u}{\partial s} - \frac{7}{2}q_0 u^{3/2}\frac{\partial u}{\partial x} + \frac{\alpha^2}{4}\ \frac{\partial^3 u}{\partial x^3} = 0$	$\left(\frac{5\lvert E_0\rvert}{2\lvert q_0\rvert}\right)^{2/3}\operatorname{sech}^{4/3}\left(\frac{3\sqrt{2\lvert E_0\rvert}}{2\lvert\alpha\rvert}\xi\right)$	$i\alpha\frac{\partial\Psi}{\partial s} = -\frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2} - q_0\lvert\Psi\rvert^{3}\Psi$	$\left(\frac{5\lvert E_0\rvert}{2\lvert q_0\rvert}\right)^{1/3}\operatorname{sech}^{2/3}\left(\frac{3\sqrt{2\lvert E_0\rvert}}{2\lvert\alpha\rvert}\xi\right)$
2	$a\frac{\partial u}{\partial s} - 4q_0 u^{2}\frac{\partial u}{\partial x} + \frac{\alpha^2}{4}\ \frac{\partial^3 u}{\partial x^3} = 0$	$\left(\frac{3\lvert E_0\rvert}{\lvert q_0\rvert}\right)^{1/2}\operatorname{sech}\left(\frac{2\sqrt{2\lvert E_0\rvert}}{\lvert\alpha\rvert}\xi\right)$	$i\alpha\frac{\partial\Psi}{\partial s} = -\frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2} - q_0\lvert\Psi\rvert^{4}\Psi$	$\left(\frac{3\lvert E_0\rvert}{\lvert q_0\rvert}\right)^{1/4}\operatorname{sech}^{1/2}\left(\frac{2\sqrt{2\lvert E_0\rvert}}{\lvert\alpha\rvert}\xi\right)$
5/2	$a\frac{\partial u}{\partial s} - \frac{9}{2}q_0 u^{5/2}\frac{\partial u}{\partial x} + \frac{\alpha^2}{4}\ \frac{\partial^3 u}{\partial x^3} = 0$	$\left(\frac{7\lvert E_0\rvert}{2\lvert q_0\rvert}\right)^{2/5}\operatorname{sech}^{4/5}\left(\frac{5\sqrt{2\lvert E_0\rvert}}{2\lvert\alpha\rvert}\xi\right)$	$i\alpha\frac{\partial\Psi}{\partial s} = -\frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2} - q_0\lvert\Psi\rvert^{5}\Psi$	$\left(\frac{7\lvert E_0\rvert}{2\lvert q_0\rvert}\right)^{1/5}\operatorname{sech}^{2/5}\left(\frac{5\sqrt{2\lvert E_0\rvert}}{2\lvert\alpha\rvert}\xi\right)$
3	$a\frac{\partial u}{\partial s} - 5q_0 u^{3}\frac{\partial u}{\partial x} + \frac{\alpha^2}{4}\ \frac{\partial^3 u}{\partial x^3} = 0$	$\left(\frac{4\lvert E_0\rvert}{\lvert q_0\rvert}\right)^{1/3}\operatorname{sech}^{2/3}\left(\frac{3\sqrt{2\lvert E_0\rvert}}{\lvert\alpha\rvert}\xi\right)$	$i\alpha\frac{\partial\Psi}{\partial s} = -\frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2} - q_0\lvert\Psi\rvert^{6}\Psi$	$\left(\frac{4\lvert E_0\rvert}{\lvert q_0\rvert}\right)^{1/6}\operatorname{sech}^{1/3}\left(\frac{3\sqrt{2\lvert E_0\rvert}}{\lvert\alpha\rvert}\xi\right)$
10	$a\frac{\partial u}{\partial s} - 12q_0 u^{10}\frac{\partial u}{\partial x} + \frac{\alpha^2}{4}\ \frac{\partial^3 u}{\partial x^3} = 0$	$\left(\frac{11\lvert E_0\rvert}{\lvert q_0\rvert}\right)^{1/10}\operatorname{sech}^{1/5}\left(\frac{\sqrt{2\lvert E_0\rvert}}{10\lvert\alpha\rvert}\xi\right)$	$i\alpha\frac{\partial\Psi}{\partial s} = -\frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2} - q_0\lvert\Psi\rvert^{20}\Psi$	$\left(\frac{11\lvert E_0\rvert}{\lvert q_0\rvert}\right)^{1/20}\operatorname{sech}^{1/10}\left(\frac{10\sqrt{2\lvert E_0\rvert}}{\lvert\alpha\rvert}\xi\right)$

TABLE I

β	ρ	$\rho^{1/2}$
1/5		
1/2		
1		

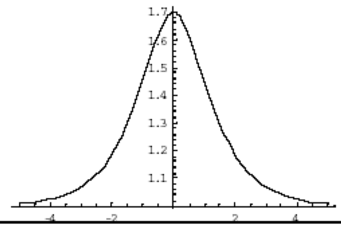
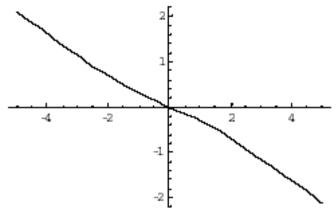
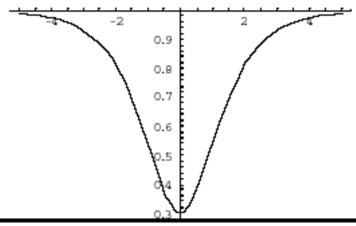
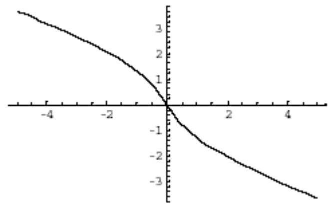
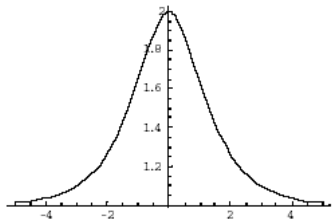
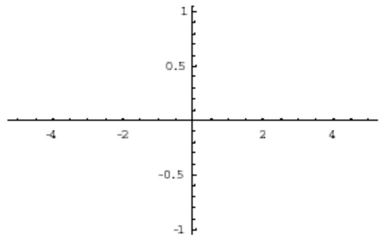
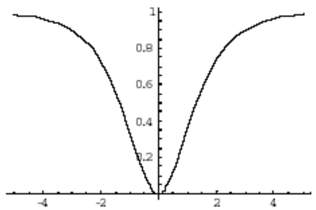
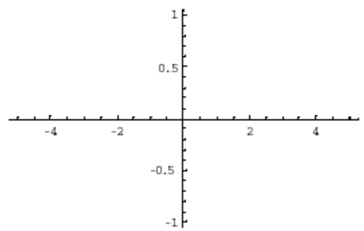
Plots of ρ and $\rho^{1/2}$ (bright solitons) as function of ξ/Δ for $\beta \leq 1$. E_0=-1, q_0=-1, α = 1.

TABLE II

β	ρ	$\rho^{1/2}$
2		
5/2		
3		

Plots of ρ and $\rho^{1/2}$ (bright solitons) as function of ξ/Δ for $\beta>1$. E_0=-1, q_0=-1, α = 1.

TABLE III

ε	**u**	$\boldsymbol{\Theta}_1$	**Soliton's name**
0.7			**up-shifted**
-0.7			**gray**
1		 ($\Theta_1 = 0$)	**upper-shifted**
-1		 ($\Theta_1 = 0$)	**dark**

Plot of solutions u and the nonlinear part of the phase $\Theta_1 \equiv A_0 \int \frac{d\xi}{u(\xi)}$ as function of ξ/Δ for $\overline{u} = 1$, $\varepsilon = \pm .7$ with $u_0 - V_0 = .5$, $\varepsilon = \pm 1$ which corresponds to $u_0 - V_0 = 0$.

$$i\varepsilon \frac{\partial \Psi}{\partial z} = -\frac{\varepsilon^2}{2} \nabla_\perp^{\,2} \Psi$$

$\left|\Psi(x,y,z)\right|^2 \propto$ ***transverse density profile of the beam particles***

$$\Psi(x,y,z) = \sqrt{n(x,y,z)}\exp[\frac{i}{\varepsilon}\theta(x,y,z)]$$

$$\int \left|\Psi(x,y,z)\right|^2 dxdy = \int n(x,y,z)dxdy = 1$$

Fluid interpretation:

$n(x,y,z) =$ ***transverse probability density of the beam particles***

$\mathbf{V}(x,y,z) = \nabla_\perp \theta =$ ***transverse probability current velocity***

Gaussian solution for the BWF

$$\Psi(x,y,z)=\frac{\exp[-x^2/4\sigma_x(z)-y^2/4\sigma_y(z)]}{\sqrt{2\pi\sigma_x(z)\sigma_y(z)}}\exp[\frac{i}{\varepsilon}\theta(x,y,z)]$$

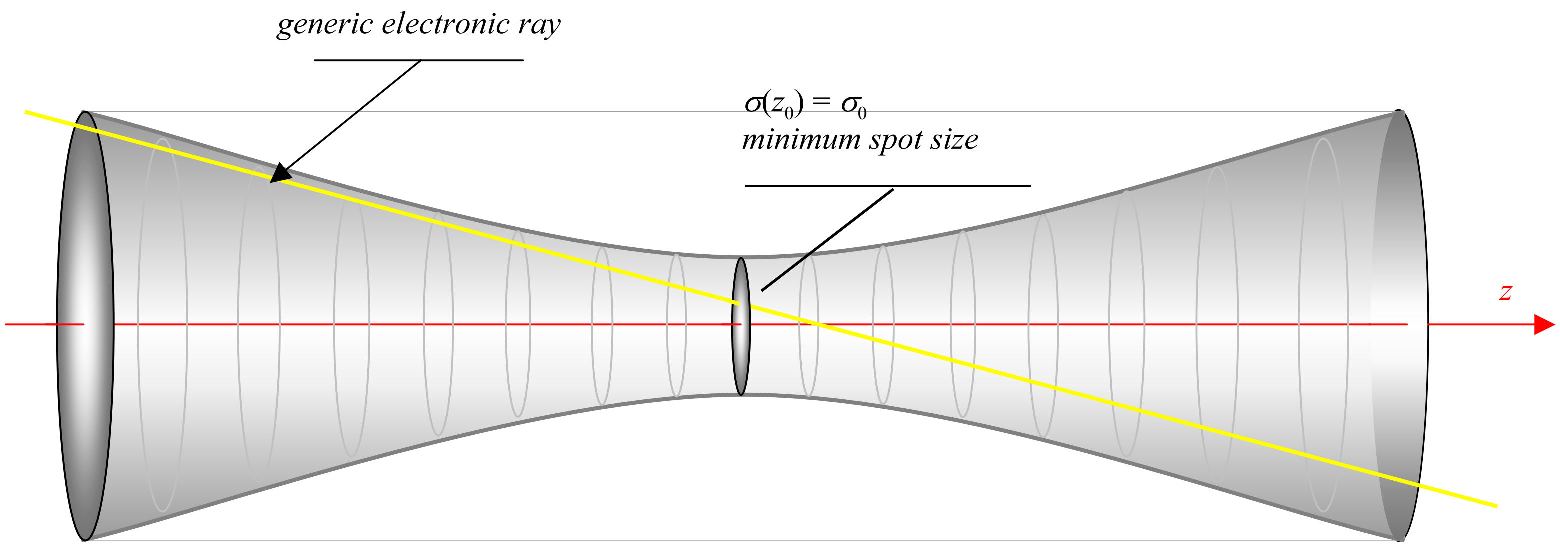
$$\theta(x,y,z)=\frac{x^2}{2R_x(z)}+\frac{y^2}{2R_y(z)}+\phi_x(z)+\phi_y(z)$$

$$\frac{1}{R_j(z)}=\frac{1}{\sigma_j(z)}\frac{d\sigma_j(z)}{dt}$$

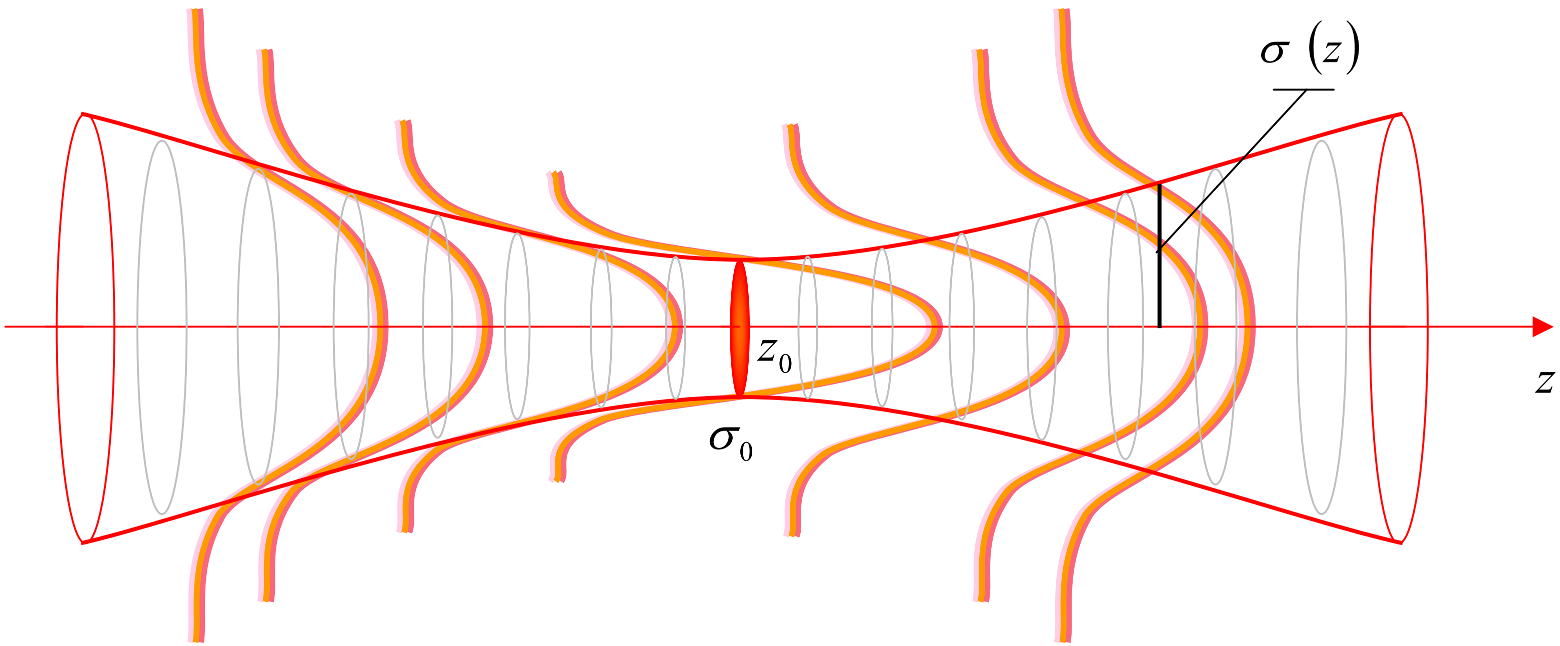
$$\frac{d\phi_{j0}(z)}{dt}=-\frac{\varepsilon}{4\sigma_j^2(z)}$$

$j=x,y$

Qualitative rappresentation of the free envelope motion (paraxial approximation) of a cilindrically-symmetric beam travelling in vacuo.



Qualitative envelope evolution of a cilindrically-symmetric Gaussian beam propagating in vacuo.



$$\sigma(z) = \sqrt{\sigma_0{}^2 + 2E(z - z_0)^2}$$

PURELY TRANSVERSE DYNAMIC OF A CHARGED-PARTICLE BEAM INTERACTING WITH THE SURROUNDINGS

Suppose now:

- **a transverse force field** $\mathbf{F}_\perp(x,y,z)$ **is acting on the beam**
- **this force is due to the <u>total interaction (along the transverse directions) of the beam with the sourrondings</u>**
- $\mathbf{F}_\perp(x,y,z) = \mathbf{F}_\perp^{ext}(x,y,z) + \mathbf{F}_\perp^{coll}(x,y,z)$

$\mathbf{F}_\perp^{ext}(x,y,z) \rightarrow$ **interaction with external devices electric and magnetic lenses and traps, rf or mw cavities, kickers, wigglers, undulators, etc. Some of these external force can be derived by an effective potential,** U_{ext}.

$\mathbf{F}_\perp^{coll}(x,y,z) \rightarrow$ **self interactions as a result of the interaction between the beam and the transverse electromagnetic fields (transverse wake fields) generated by the image charges and the image currents induced by the beam itself on the surrounding bodies (collective effect). They can be derived by a potential energy,** U_{coll} **(wake potential) .**

$$U_{\perp}(x,y,z) = U_{\perp}^{ext}(x,y,z) + U_{\perp}^{coll}\left(|\Psi(x,y,z)|^2\right)$$

dimensionless potential energy (normalized with respect to $m\beta c^2$**)**

TWM ASSUMPTION

the transverse beam dynamics is governed by

$$i\varepsilon \frac{\partial \Psi}{\partial z} = -\frac{\varepsilon^2}{2} \nabla_{\perp}{}^2 \Psi + U(x,y,z)\Psi$$

R. Fedele and G. Miele, *Nuovo Cim. D* **13,** 1527 (1991)

A generalized nonlinear Schrödinger equation (NLSE) is obtained

$$i\varepsilon\frac{\partial\Psi}{\partial z}=-\frac{\varepsilon^2}{2}\nabla_\perp{}^2\Psi+U_\perp^{ext}(x,y,z)\Psi+U_\perp^{coll}\left[|\Psi(x,y,z)|^2\right]\Psi$$

In transverse optics, TWM has been applied to a number of linear and nonlinear problems, such as:

- **Gaussian particle-beam optics and dynamics for a quadrupole-like device** [*Nuovo Cim. D* **13,** 1527 (1991)]

- **luminosity estimates in final focusing stages of linear colliders in the presence of small aberrations** [R. Fedele and G. Miele, *Phys.Rev.A* **46,** 6634 (1992)]

- **the TWM predictions have been compared with tracking-code simulations and a fair agreement has been demonstrated**

- **A self consistent theory of the interaction bewteen a relativistic electron (positron) beam and a cold plasma has been also developed** [R. Fedele and P.K. Shukla, *Phys. Rev. A* **44,** 4045 (1992)]

To give an idea of the capability of the TWM in describing correctly the transverse beam dynamics we briefly present the results of <u>two interesting cases</u>:

- **The motion of a 1D charged particle beam travelling through a linear thin magnetic lens (quadrupole) with sextupole and octupole deviations and therefore subject to the action of external forces**

- **The motion of a cilidrically symmetric relativistic charged particle beam travelling in an overdense plasma an therefore subject to the action of the plasma wake fields (self interaction)**

TRANSVERSE NONLINEAR BEAM DYNAMICS IN A COLD PLASMA

- **A cilindrically symmetric Gaussian relativistic charged particle beam, with transverse rms** R_0 **(initial beam radius) and unperturbed number density** n_{b0}**, is travelling along the z-axis with velocity** βc ($\beta \approx 1$) **and transverse emittance** ε**.**
- **At** $z=0$ **the beam enters a semi- infinite slab of cold unmagnetized plasma with unperturbed number density** n_{p0} **in "overdense condition"** ($n_{b0} << n_{p0}$)**.**

- **The beam length $\sigma_z >> \lambda_p$ (the plasma density perturbation n_1 is produced adiabatically) and therefore**

$$e\, n_1(r,\xi) \approx q\, n_{b0}(r,\xi),$$

r=cilindrical radial coordinate, $\xi = z - \beta ct$

- **According to the theory of plasma wake field excitation:**

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)-k_p^2\right]U_\perp^{coll}(r,\xi)=\frac{4\pi q^2 n_b}{m\gamma\beta^2 c^2}\approx\frac{4\pi q^2 n_b}{m\gamma c^2}$$

$$\Downarrow$$

$$i\varepsilon\frac{\partial\Psi}{\partial\xi}=-\frac{\varepsilon^2}{2}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right)+U_\perp^{coll}\left[\left|\Psi(r,\xi)\right|^2\right]\Psi$$

[R. Fedele and P.K. Shukla, *Phys. Rev. A* **44,** 4045 (1992)]

$$\text{(a).}\quad k_p R >> 1, \qquad i\varepsilon \frac{\partial \Psi}{\partial \xi} = -\frac{\varepsilon^2}{2}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Psi}{\partial r}\right) - \frac{n_{b0}}{n_{p0}\gamma}\left|\Psi\right|^2 \Psi$$

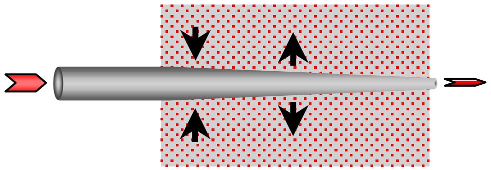
$$\mathcal{A} = \frac{\varepsilon^2}{2}\int_0^\infty \left|\frac{\partial \Psi}{\partial r}\right|^2 r dr - \frac{n_{b0}}{2n_{po}\gamma}\int_0^\infty \left|\Psi\right|^4 r dr = const. \quad \Rightarrow \quad \mathcal{A} = \frac{1}{2}\left[\frac{\varepsilon^2}{R_0^2} - \frac{1}{2}\frac{n_{b0}}{n_{p0}\gamma}\right]$$

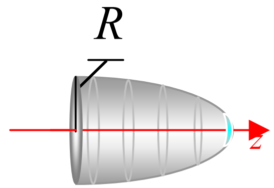
$$R^2(\xi) = R_0^2 + 2\mathcal{A}(\xi - \xi_0)^2$$

$\mathcal{A}<0$: *self-focusing*

$\mathcal{A}>0$: *self-defocusing*

$\mathcal{A}=0$: *stationary solution*





Weibel instability threshold: $\beta_\perp = \frac{v_{th}}{c} \approx 0.7\left(\frac{n_{b0}}{n_{p0}\gamma}\right)^{1/2}$

(b). $k_p R << 1,$ $$\frac{d^2 R}{d\xi^2} = \frac{\varepsilon^2}{R^3} - \frac{2K}{R}\left\langle \int_0^r \left|\Psi(r',\xi)\right|^2 r' dr' \right\rangle$$

$$K = \frac{2\pi e^2 n_{b0}}{m\gamma\beta c^2}$$

$$\frac{dR}{d\xi} = 0$$ $\Rightarrow$ *Bennett self-pinch equilibrium condition:*

$$\frac{\varepsilon^2}{R_0^2} = \frac{1}{2} K R_0^2 \quad \Rightarrow \quad \frac{I^2}{c^2} = \frac{N}{\sigma_z} T$$ (cgs unts)

- Aberrationless approximate solution of NLSE:

$$\frac{d^2 R}{d\xi^2} + KR - \frac{\varepsilon^2}{R^3} = 0$$

[R. Fedele and P.K. Shukla and V.G. Vaccaro, *J. Physique IV, Coll. C6, suppl. Jour. Phys. II*, **5**, C6-119 (1995)]

LONGITUDINAL BEAM DYNAMICS OF A CHARGED PARTICLE BUNCH IN CIRCULAR ACCELERATING MACHINES

$$i\epsilon\frac{\partial\Psi}{\partial s}+\frac{\epsilon^2\eta}{2}\frac{\partial^2\Psi}{\partial x^2}+U(x,s)\Psi=0$$

$\eta=\frac{1}{\gamma_T^2}-\frac{1}{\gamma^2}$ ($1/\eta$ **plays the role of an effective mass**) $\left(\frac{\Delta\omega}{\omega}=\eta\frac{\Delta p}{p}\right)$

$$U[\lambda_1(x,s)]=\frac{q^2\beta c}{E_0}\left(R_0Z_I'\lambda_1(x,s)+Z_R'\int_0^x\lambda_1(x',s)\,dx'\right)$$

$Z'=Z_R'+iZ_I'$ **longitudinal coupling impedance**

$\lambda_1=|\Psi|^2-|\Psi_0|^2$ **arbitrary longitudinal bunch density perturbation**

This equation has been used to describe a number of physical problems involving relatively intense high-energy charged particle coasting beams in accelerating machines:

- **synchrotron oscillations with and without radiation damping and quantum excitation effects**
- **soliton structure predictions** [R. Fedele, G. Milele, L. Palumbo, and V.G. Vaccaro, *Phys. Lett. A* **179**, 407 (1993)]
- **coherent instabilities of coasting beams, with the language of the modulational instability, and the stabilizing role played by the Landau damping** [R. Fedele and D. Anderson, *J. Opt. B: Quant. Semicl. Opt.* **2**, 207 (2000)]
- ***nonlocal* effects of charged-particle beams**

IMAGINARY PART (REACTIVE PART) OF THE LONGITUDINAL COUPLING IMPEDANCE:

$$Z_I' = \frac{1}{2\pi R_0}\left(\frac{g_0 Z_0}{2\beta\gamma^2} - \omega_0 \mathcal{L}\right) \equiv \frac{Z_I}{2\pi R_0}$$

where Z_0 is the vacuum impedance, $\omega_0 = \beta c/R_0$ is the nominal orbital angular frequency of the particles and $\mathcal{L}$ is the total inductance. This way, Z_I represents the total reactance as the difference between the total space charge capacitive reactance, $g_0 Z_0/(2\beta\gamma^2)$, and the total inductive reactance, $\omega_0 \mathcal{L}$.

$$U[\lambda_1] = \frac{q^2 \beta c}{2\pi E_0}\left(\frac{g_0 Z_0}{2\beta\gamma^2} - \omega_0 \mathcal{L}\right)\lambda_1$$

$$U[|\Psi|^2] = -\alpha \left\{ \mathcal{X} \left[|\Psi|^2 - |\Psi_0|^2 \right] + \mathcal{R} \int_0^x \left[|\Psi(x', s)|^2 - |\Psi_0|^2 \right] dx \right\}$$

$$\alpha = \epsilon\eta = \epsilon \left(\gamma^{-2} - \gamma_T^{-2} \right),$$ ($1/\eta$ **plays the role of an effective mass)**

$$\mathcal{X} = \frac{q^2 \beta c R_0}{\epsilon E_0} Z_I',$$

$$\mathcal{R} = \frac{q^2 \beta c}{\epsilon E_0} Z_R'.$$

$$\left(\frac{\Delta\omega}{\omega} = \eta \frac{\Delta p}{p} \right)$$

$$i \frac{\partial \Psi}{\partial s} + \frac{\alpha}{2} \frac{\partial^2 \Psi}{\partial x^2} + \mathcal{X} \left[|\Psi|^2 - |\Psi_0|^2 \right] \Psi + \\ + \mathcal{R}\, \Psi \int_0^x \left[|\Psi(x', s)|^2 - |\Psi_0|^2 \right] dx' = 0$$

$\lambda_1 = |\Psi|^2 - |\Psi_0|^2$ **(arbitrary longitudinal bunch density perturbation)**

DETERMINISTIC APPROACH TO MI

Under the conditions assumed above, let us consider a monochromatic coasting beam travelling in a circular high-energy machine with the unperturbed velocity V_0 and the unperturbed density $\rho_0 = |\psi_0|^2$ (equilibrium state). In these conditions, all the particles of the beam have the same velocity and their collective interaction with the surroundings is absent.

Let us now introduce small perturbations:

$$
\begin{aligned}
V &= V_0 + V_1\,, \quad |V_1| << |V_0|\,,\\
\rho &= \rho_0 + \rho_1\,, \quad |\rho_1| << \rho_0\,.
\end{aligned}
$$

After linearizing the Madelung fluid equations:

$$\frac{\partial \rho_1}{\partial s} + V_0 \frac{\partial \rho_1}{\partial x} + \rho_0 \frac{\partial V_1}{\partial x} = 0 \,,$$
$$\frac{\partial V_1}{\partial s} + V_0 \frac{\partial V_1}{\partial x} = \alpha \mathcal{R} \rho_1 + \alpha \mathcal{X} \frac{\partial \rho_1}{\partial x} + \frac{\alpha^2}{4\rho_0} \frac{\partial^3 \rho_1}{\partial x^3} \,.$$

$$\rho_1(x,s) = \int dk\, d\omega\, \tilde{\rho}_1(k,\omega) e^{ikx - i\omega s} \,,$$
$$V_1(x,s) = \int dk\, d\omega\, \tilde{V}_1(k,\omega) e^{ikx - i\omega s} \,,$$

$$-\rho_0 k \tilde{V}_1 = (kV_0 - \omega)\, \tilde{\rho}_1 \,,$$
$$i\,(kV_0 - \omega)\, \tilde{V}_1 = \left(\alpha \mathcal{R} + i\alpha k \mathcal{X} - i \frac{\alpha^2}{4\rho_0} k^3 \right) \tilde{\rho}_1$$

Linear dispersion relation:

$$\left(\frac{\omega}{k} - V_0\right)^2 = i\alpha\rho_0\left(\frac{\mathcal{Z}}{k}\right) + \frac{\alpha^2 k^2}{4}$$

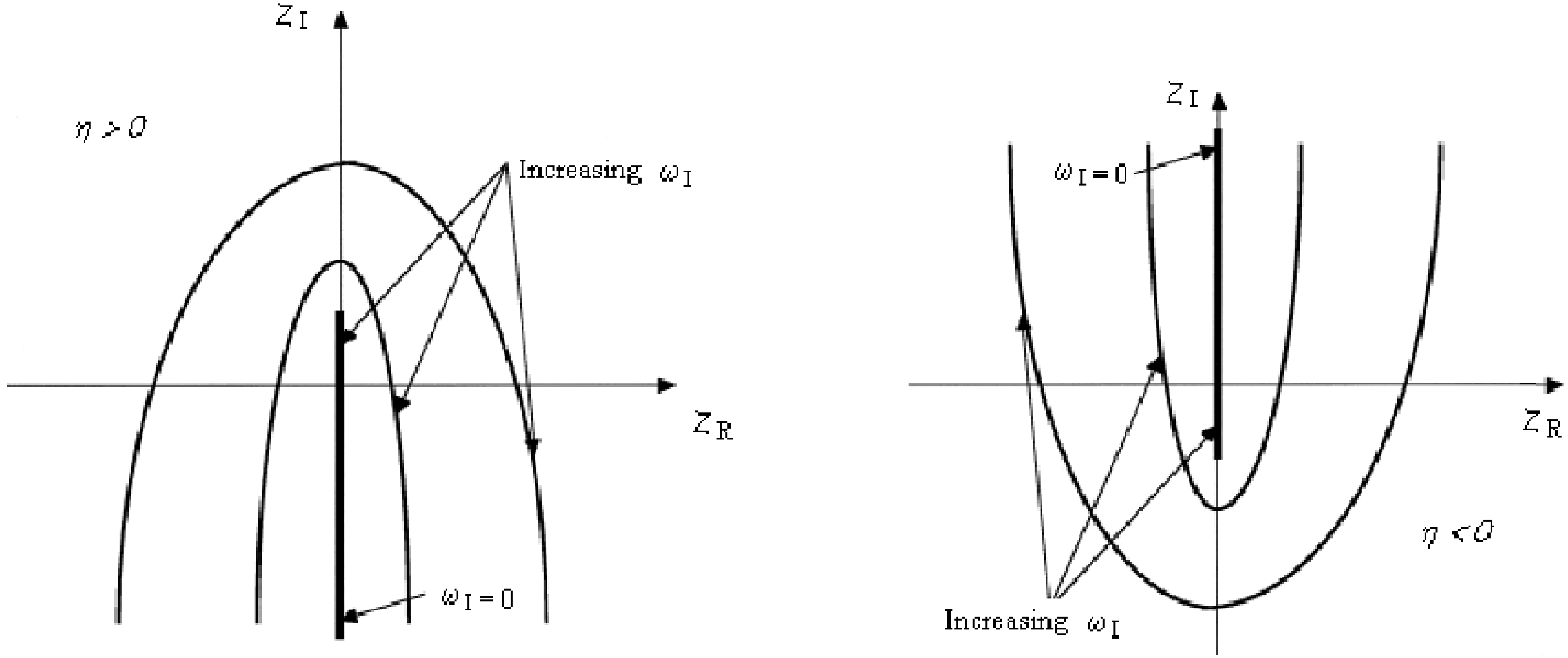
$$\mathcal{Z} = \mathcal{R} + ik\mathcal{X} \equiv \mathcal{Z}_R + i\mathcal{Z}_I$$

In general, ω is a complex quantity:

$$\omega \equiv \omega_R + i\omega_I$$

If $\omega_I \neq 0$, **the modulational instability takes place.**

$$Z_I = -\eta \frac{\epsilon k \rho_0}{4\omega_I^2} Z_R^2 + \frac{1}{\eta} \frac{\omega_I^2}{\epsilon k \rho_0} + \eta \frac{\epsilon k^3}{4\rho_0}$$



[D. Anderson, R. Fedele, V.G. Vaccaro, M. Lisak, A. Berntson and S. Johanson, *Phys. Lett. A* **258**, 244 (1999)]

Purely reactive impedance: $Z_R = 0, Z = iZ_I$

$$\Psi(x,s) = \psi(x,s)\exp\left(i\kappa|\Psi_0|^2 s\right)$$

$$\kappa \equiv -Z_I / k$$

$$i\alpha \frac{\partial \psi}{\partial s} = -\frac{\alpha^2}{2}\frac{\partial^2 \psi}{\partial x^2} + \kappa|\psi|^2\psi$$

Envelope solitons: in principle, the cubic NLSE admits *bright*, *dark* and *grey* solitons. In particular, for

Bright envelope soliton $\kappa < 0, E < 0$

$$\psi(x,s) = \frac{1}{\sqrt{2\sigma_z}}\operatorname{sech}\left(\frac{x - V_0 s}{\sigma_z}\right)\exp\left\{\frac{i}{\alpha}\left[V_0 x - \left(E + \frac{V_0^2}{2}\right)s\right]\right\}$$

$$E = -\frac{\alpha^2}{2\sigma_z^2} = -\frac{\kappa^2}{8\alpha^2}$$

R. Fedele, L. Palumbo and V.G. Vaccaro, Proc. of the Third European Particle Accelerator Conference (EPAC 92), Berlin, 24-28 March, 1992 (Edition Frontieres, Singapore, 1992), p. 762; R. Fedele, G. Miele, L. Palumbo and V.G. Vaccaro, *Phys. Lett.* A **179**, 407 (1993).

- **If N is the total number of particles of the bunch:**

$$\lambda(\xi,\zeta) = \frac{N}{2\sigma_z}\operatorname{sech}^2\left(\frac{\xi - V_0\zeta}{\sigma_z}\right)$$

- **Coherent instability condition for coasting beams (by means of a standard modulational instability analysis):**

$$\frac{\omega_I^2}{k^2} = -\epsilon\eta\rho_0\left(\frac{Z_I}{k}\right) + \frac{\alpha^2 k^2}{4}$$

$$\omega_R = V_0 k$$

⇒ **$\eta Z_I > 0$**
(Lighthill criterion)

	$Z_I < 0$ (inductive impedance)	**$Z_I > 0$** (capacitive impedance)
$\eta < 0$ (above transition energy)	**instability**	**stability**
$\eta > 0$ (below transition energy)	**stability**	**instability**

The above dispersion relation allows us to write an expression for the <u>admittance</u> of the coasting beam

$$\mathcal{Y} \equiv 1/\mathcal{Z} \quad \textbf{(admittance)}$$

$$k\mathcal{Y} = \frac{i\alpha\rho_0}{(\omega/k - V_0)^2 - \alpha^2 k^2/4}$$

MODULATIONAL INSTABILITY OF A "*WHITE*" COASTING BEAM

Let us now consider a *non-monochromatic coasting beam*.

Such a system may be thought as an "ensemble" of elementary "incoherent" coasting beams with different unperturbed velocities (*white beam*).

$f_0(V)$ = distribution function of the velocity at the equilibrium

The subsystem corresponding to a coasting beam collecting the particles having velocities between V and $V + dV$ has an elementary admittance dy.

$$k\,d\mathcal{Y} = \frac{i\alpha\, f_0(V)\, dV}{(V-\omega/k)^2 - \alpha^2 k^2/4}\,.$$

All the elementary coasting beams in which we have divided the system suffer the same electric voltage per unity length along the longitudinal direction.

The total admittance of the system is the sum (i.e., the integral) of the all elementary admittances (system of electric wires connected all in parallel):

$$k\mathcal{Y} = i\alpha \int \frac{f_0(V)\, dV}{(V-\omega/k)^2 - \alpha^2 k^2/4}$$

$\mathcal{Y}$ = ***total admittane***

$$1 = i\alpha \left(\frac{\mathcal{Z}}{k}\right) \int \frac{f_0(V)\, dV}{(V-\omega/k)^2 - \alpha^2 k^2/4}$$

$\mathcal{Z} = 1/\mathcal{Y}$

total impedance

An interesting equivalent form can be obtained:

(i)

$$\frac{1}{(V-\omega/k)^2-\alpha^2k^2/4}=\frac{1}{\alpha k}\left[\frac{1}{(V-\alpha k/2)-\omega/k}-\frac{1}{(V+\alpha k/2)-\omega/k}\right]$$

(ii)

$$1=i\left(\frac{\mathcal{Z}}{k}\right)\frac{1}{k}\left[\int\frac{f_0(V)\,dV}{(V-\alpha k/2)-\omega/k}-\int\frac{f_0(V)\,dV}{(V+\alpha k/2)-\omega/k}\right]$$

(iii)

$$p_1=V-\alpha k/2 \qquad\qquad p_2=V+\alpha k/2$$

$$1=i\left(\frac{\mathcal{Z}}{k}\right)\frac{1}{k}\left[\int\frac{f_0(p_1+\alpha k/2)\,dp_1}{p_1-\omega/k}-\int\frac{f_0(p_2-\alpha k/2)\,dp_2}{p_2-\omega/k}\right]$$

Finally, we arrive to the following dispersion relation:

$$1 = i\alpha \left(\frac{Z}{k}\right) \int \frac{f_0(p + \alpha k/2) - f_0(p - \alpha k/2)}{\alpha k} \frac{dp}{p - \omega/k}$$

(a). The dispersion relation for the case of monochromatic beam is recovered by assuming: $f_0(V) \propto \delta(V - V_0)$

(b). In general, this dispersion relation takes into account the equilibrium velocity (or energy) spread of the beam particles, but it has not obtained with a kinetic treatment. We have only assumed the existence of an equilibrium state, without taking into account any phase-space evolution in terms of a kinetic distribution function.

OUR RESULT HAS BEEN BASICALLY OBTAINED WITHIN THE FRAMEWORK OF MADELUNG FLUID, EXTENDING THE STANDARD MI ANALYSIS TO NON MONOCHROMATIC WAVE PACKETS (STATISTICAL ENSEMBLE OF MONOCHROMATIC COASTING BEAMS).

The above dispersion relation can be also obtained within the kinetic picture provided by the ***Moyal-Ville-Wigner description***, as it has been done for charged-particle beam for the first time in the context of the *TWM* and soon extended to *nonlinear optics, plasma physics, surface gravity waves, lattice vibration physics.*

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A. Visinescu and D. Grecu, Eur. Phys. J. B **34**, 225 (2003); A. Visinescu, D. Grecu AIP Conf. Proc. Vol.729, p. 389 (2004).

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STATISTICAL APPROACH TO MI (SAMI)

The Moyal-Ville Wigner "quasidistribution"

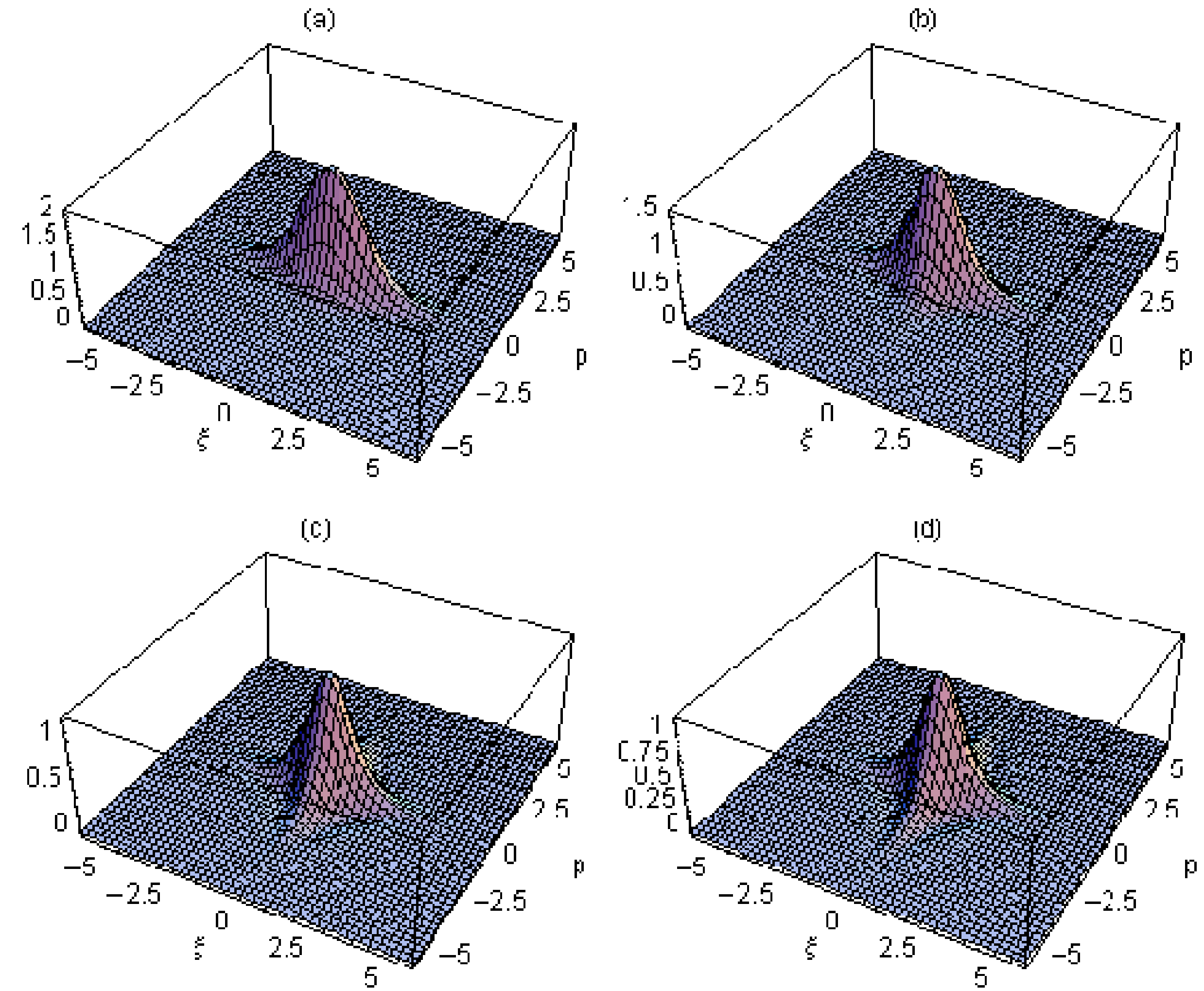
$$w(x,p,s) = \frac{1}{2\pi\alpha}\int_{-\infty}^{\infty} \Psi^*\left(x+\frac{y}{2},s\right)\ \Psi\left(x-\frac{y}{2},s\right)\exp\left(i\frac{py}{\alpha}\right)\ dy$$

$\int\ w(x,p,s)\,dp$ and $\int\ w(x,p,s)\,dx$ are proportional to the probability density in configuration space and momentum space, respectively.

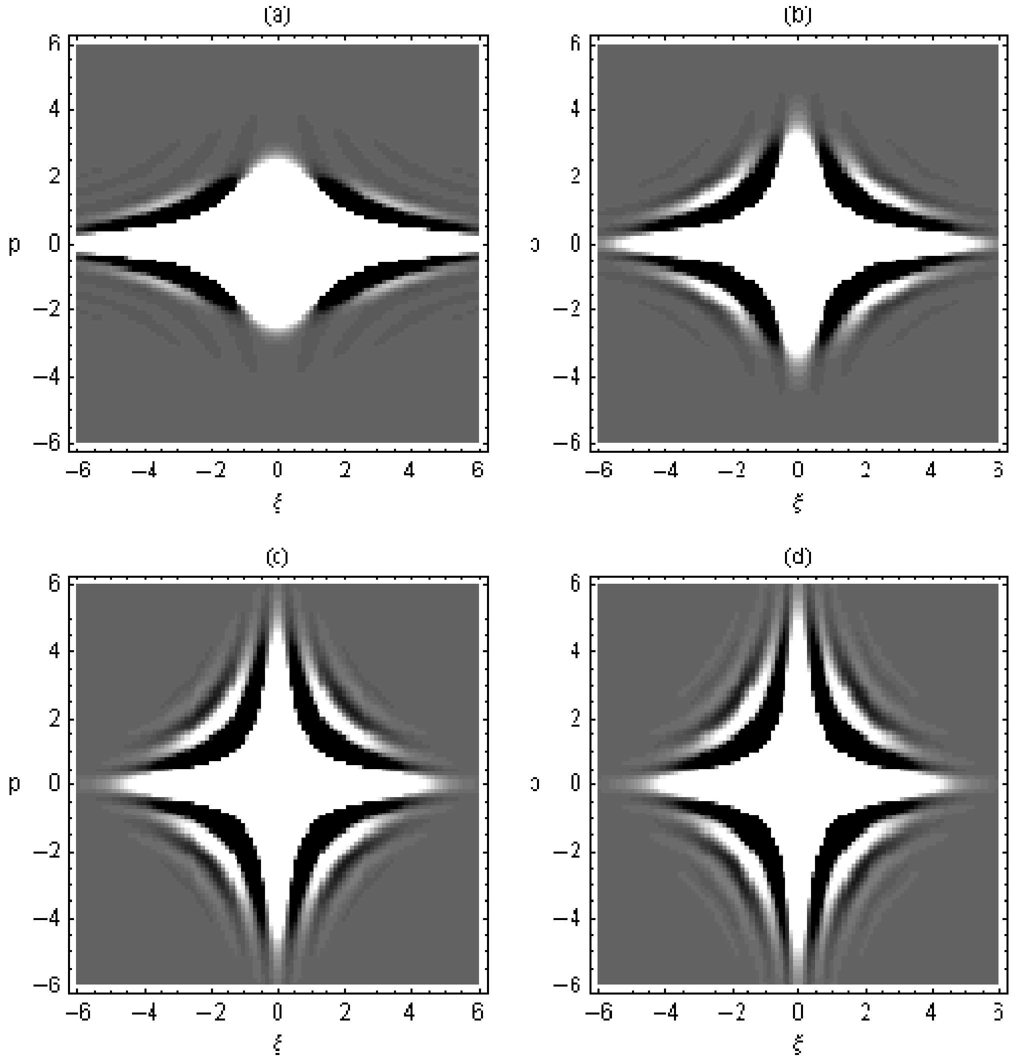
In particular:

$$\rho(x,s) = |\Psi(x,s)|^2\ =\ \int_{-\infty}^{\infty} w(x,p,s)\ dp$$

$$U\ =\ U\left[\int_{-\infty}^{\infty} w\ dp\right]$$

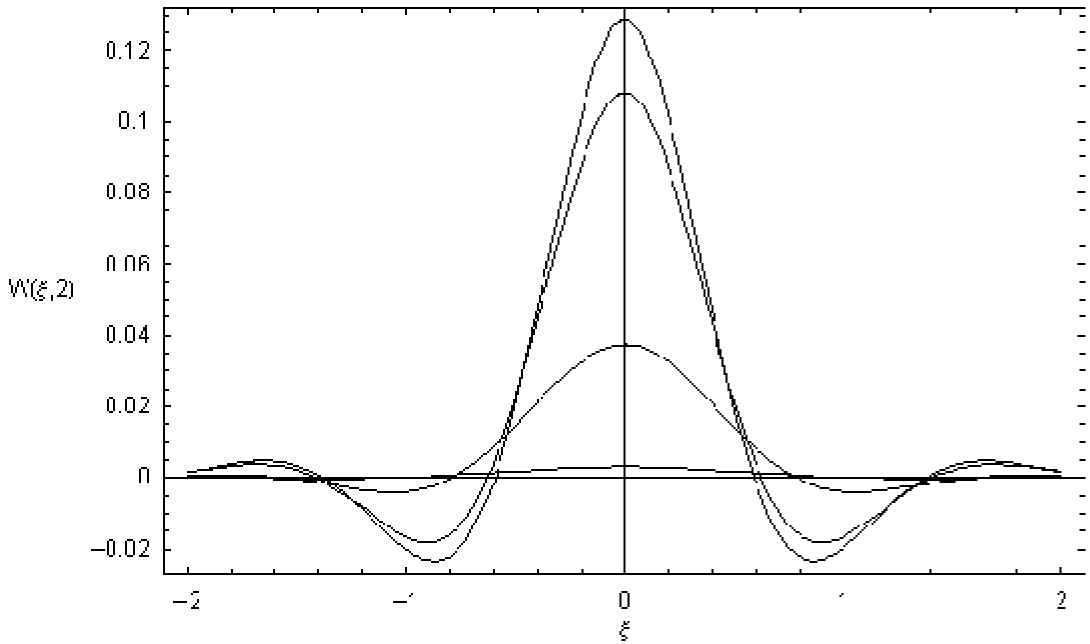


3D plot of the *quasidistribution* of the mNLSE soliton for various β:
(a) β =0.5, (b) β =1;(c) β =2; (d) β =2.5



Density plot of the *quasidistribution* of the mNLSE soliton for various β:

(a) β =0.5, (b) β =1;(c) β =2; (d) β =2.5



Cross section of the *quasidistribution* (at p = 2) of the mNLSE soliton for various β: (a) β =0.5, (b) β =1;(c) β =2; (d) β =2.5. The greater β the greater amplitude.

The nonlinear Wigner-Moyal kinetic equation (nonlinear von Neumann-Weyl equation)

$$\frac{\partial w}{\partial s}+p\frac{\partial w}{\partial x}-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\alpha}{2}\right)^{2n}\frac{\partial^{2n+1}}{\partial x^{2n+1}}\left(U\left[\int_{-\infty}^{\infty} w\ dp\right]\right)\frac{\partial^{2n+1} w}{\partial p^{2n+1}} = 0$$

$$U\left[\int_{-\infty}^{\infty} w\,dp\right] = -\alpha\mathcal{X}\left(\int_{-\infty}^{\infty} w\,dp - |\Psi_0|^2\right)$$

$$- \alpha\mathcal{R}\int_0^x \left(\int_{-\infty}^{\infty} w(x',p,s)\,dp - |\Psi_0|^2\right) dx'$$

we start from the equilibrium state: $w = w_0(p)$

$$\rho_0 = |\Psi_0|^2 = \int_{-\infty}^{\infty} w_0(p)\,dp$$

$$U = U_0 \equiv U\left[\int_{-\infty}^{\infty} w_0\,dp\right] = 0$$

Then, we introduce the following small perturbations in *w* and *U*, respectively:

$$w(x,p,s) = w_0(p) + w_1(x,p,s) \quad ,$$

$$U(x,s) \;=\; U_0 + U_1(x,s) \;=\; U\left[\int_{-\infty}^{\infty} w_1(x,p,s)\, dp\right] \quad ,$$

where $w_1(x, p, s)$ **and** $U_1(x, s)$ **are first-order quantities.**

$$\frac{\partial w_1}{\partial s} + p\frac{\partial w_1}{\partial x} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{\alpha}{2}\right)^{2n} \frac{\partial^{2n+1} U_1}{\partial x^{2n+1}}\, w_0^{(2n+1)} = 0$$

$$w_0^{(2n+1)} \equiv d^{2n+1} w_0 / dp^{2n+1}$$

$$U_1 = -\alpha \mathcal{X} \int_{-\infty}^{\infty} w_1\, dp - \alpha \mathcal{R} \int_0^x \int_{-\infty}^{\infty} w_1(x', p, s)\, dp\; dx'$$

Fourier transform:

$$U_1(x,s) = \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\omega\, \widetilde{U_1}(k,\omega)\ \exp\left(ikx - i\omega s\right) \quad ,$$

$$w_1(x,p,s) = \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\omega\, \widetilde{w_1}(k,p,\omega)\ \exp\left(ikx - i\omega s\right)$$

DISPERSION RELATION

$$1 = i\alpha \left(\frac{Z}{k}\right) \int \frac{f_0(p + \alpha k/2) - f_0(p - \alpha k/2)}{\alpha k} \frac{dp}{p - \omega/k}$$

Case of $\alpha k << 1$

Since $\alpha k << 1$, we have:

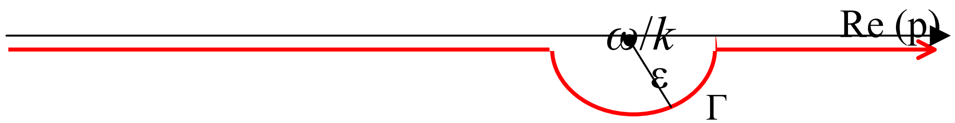
$$\frac{w_0\,(p + \alpha k/2) \;\; - \;\; \rho_0\,(p - \alpha k/2)}{\alpha k} \approx dw_0/dp \equiv w_0^{'}$$

$$1 \;=\; i\alpha\left(\frac{Z}{k}\right) \int_{-\infty}^{\infty} \frac{w_0^{'}}{p - \omega/k}\, dp$$

weak Landau damping, as described in the Vlasov theory of charged-particle beam physics.

WEAK LANDAU DAMPING

$$\omega / k >> 1$$



$$\int_{-\infty}^{+\infty} \frac{\rho_0'}{p - \omega / k} dp \equiv P.V. \left[\int_{-\infty}^{+\infty} \frac{\rho_0'}{p - \omega / k} dp \right] + i\pi\rho_0'(\omega / k)$$

$$\omega_I \propto \rho_0'(\omega / k)$$

ANALOGY WITH PLASMA PHYSICS
(Vlasov-Maxwell system)

$$Z_R = 0$$

$$i\alpha\left(\frac{Z_I}{k}\right) \rightarrow -i\frac{\omega_p^2}{c^2 k^2}$$

$$V_R \; + \; i \; V_I \; \equiv \; \alpha \left(\frac{\mathcal{Z}_R}{k} \; + \; i \; \frac{\mathcal{Z}_I}{k} \right)$$

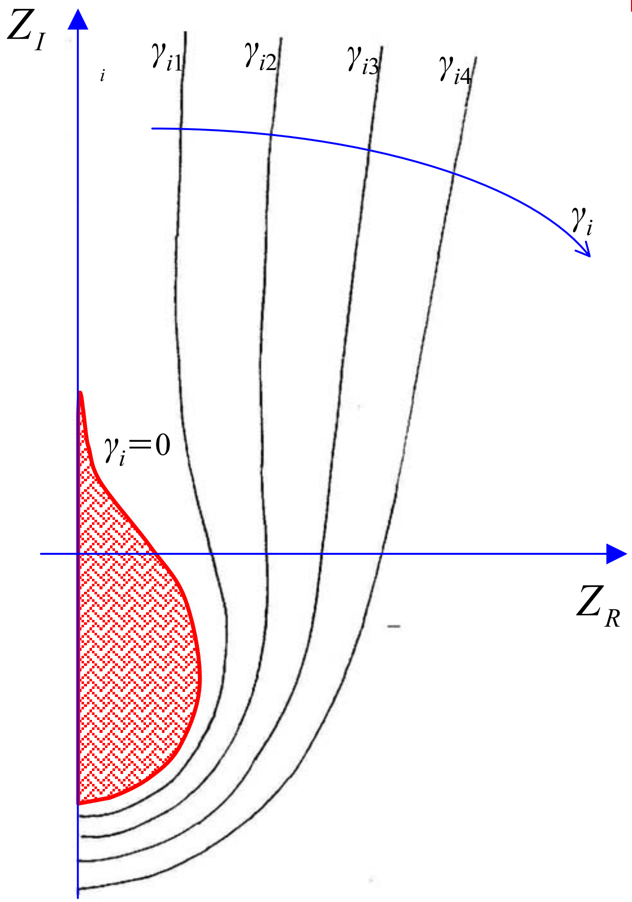
$$V_R \; + \; i \; V_I \;\; = - \left[i \; \int_{PV} \; \frac{w_0'}{p - \beta_{ph}} \; dp \; + \; \pi \; \rho_0'(\beta_{ph}) \right]^{-1}$$

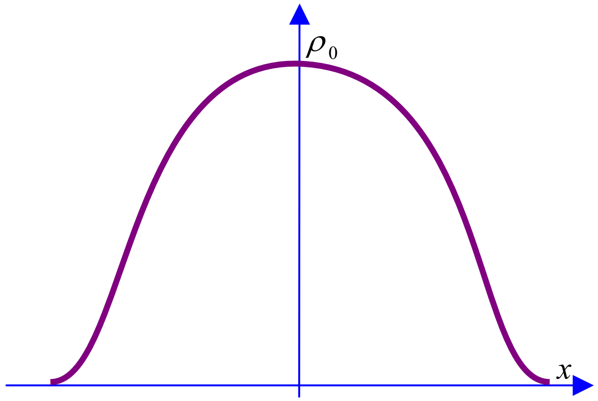
$$\beta_{ph} \; = \; \omega / k \; \equiv \; \gamma_R + i \, \gamma_I$$

Limiting case

$$\alpha k \ll 1$$

$$1 = i\alpha\left(\frac{Z}{k}\right)\int_{-\infty}^{\infty}\frac{\rho_0'(\mathrm{p})}{\mathrm{p}-\omega/k}d\mathrm{p}$$

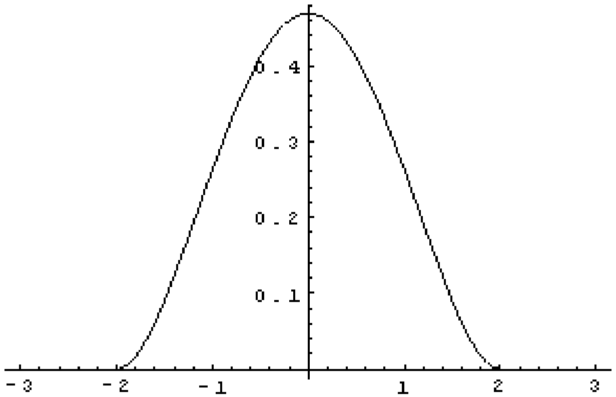


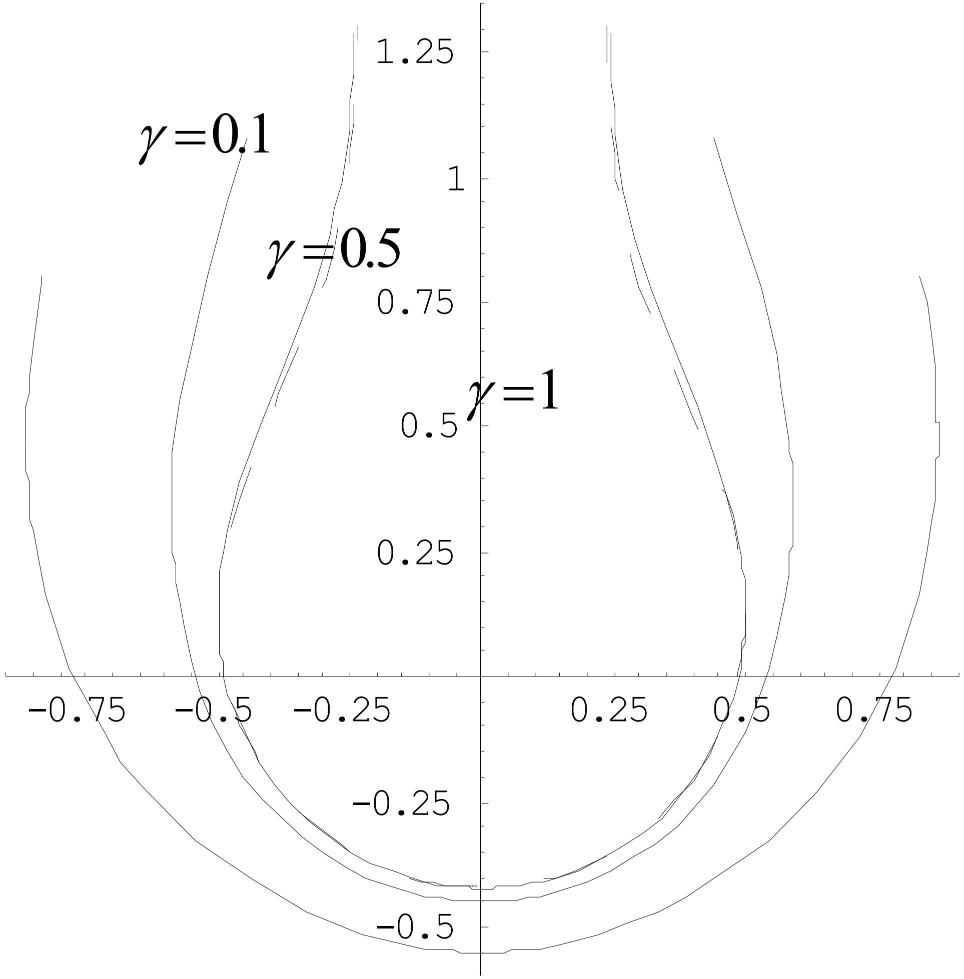


arbitrary values of αk

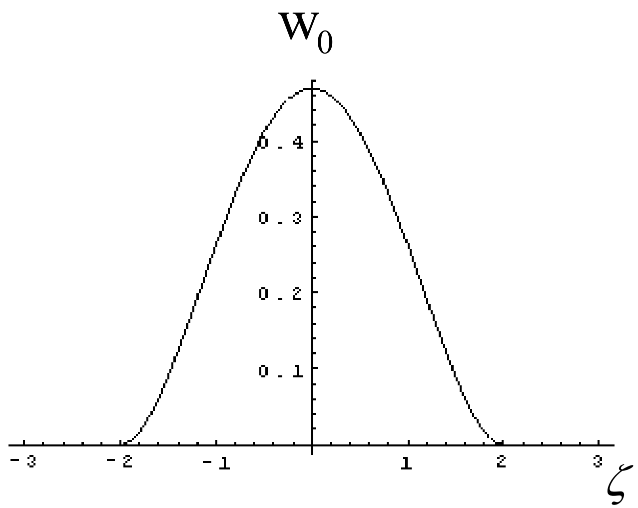
$$\gamma_i = 0.1$$

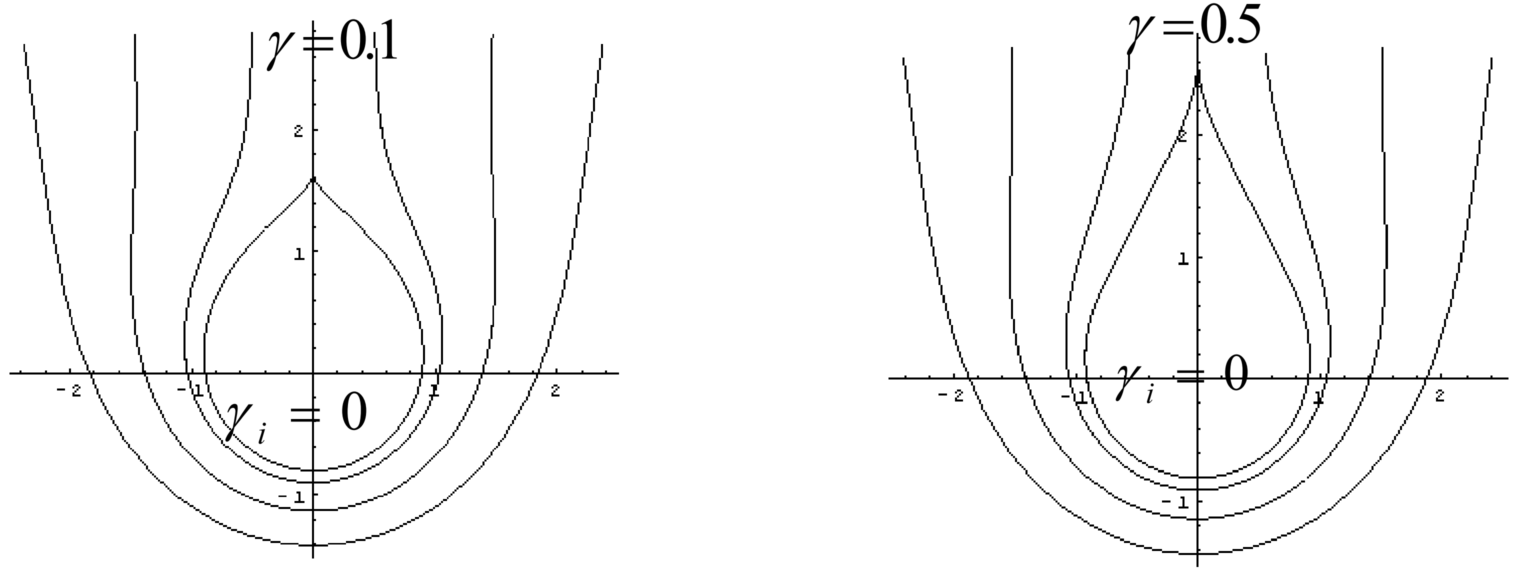
$$\gamma \equiv \alpha k / 2$$





[R. Fedele, S. De Nicola, V.G. Vaccaro, D. Anderson, M. Lisak, Proc. of QABP2000 (P. Chen ed., World Scientific, Singapore, 2002)]





[R. Fedele, S. De Nicola, V.G. Vaccaro, D. Anderson, M. Lisak, Proc. of QABP2000 (P. Chen ed., World Scientific, Singapore, 2002)]

LANDAU-TYPE DAMPING OF PARTIALLY-INCOHERENT LANGMUIR WAVE ENVELOPES

$$i\frac{\partial E}{\partial t} + \frac{3v_{te}^2}{2\omega_{pe}}\frac{\partial^2 E}{\partial x^2} - \omega_{pe}\frac{\delta n}{2n_0}E = 0$$

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2\frac{\partial^2}{\partial x^2}\right)\frac{\delta n}{2n_0} = -\frac{c_s^2}{2}\frac{\partial^2}{\partial x^2}\left(\frac{|E|^2 - |E_0|^2}{4\pi n_0 T_e}\right)$$

The above equations have been obtained by assuming that the electric field of the wave has the envelope form:

$$\mathsf{E} = E(x,t)\exp\left(-i\Omega t\right) \quad ,$$

where $\Omega \leq \omega_{pe}$ and $\left|\,\Omega^{-1}\partial E(x,t)/\partial t\,\right| \ll 1.$

The above system of equations can be cast in the form:

$$i\alpha\frac{\partial\Psi}{\partial s} + \frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2} - U\,\Psi = 0 \quad ,$$

$$\left(\frac{\partial^2}{\partial s^2} - \mu^2\frac{\partial^2}{\partial x^2}\right)U = -\frac{\partial^2}{\partial x^2}\left(\langle|\Psi|^2 - |\Psi_0|^2\rangle\right)$$

the angle brackets account for the statistical ensemble average due to the partial incoherence of the waves

$$s = \sqrt{3}\lambda_{De}\omega_{pe}t \qquad \alpha = \sqrt{3}\lambda_{De}$$

$$U = \delta n/2n_0,\ \mu = \sqrt{m_e/3m_i}$$

$$\Psi = \frac{\mu E}{\sqrt{8\pi n_0 T_e}}$$

$$\rho_0 \equiv |\Psi_0|^2 = \mu^2\frac{\epsilon}{\epsilon_T}$$

$$\Psi_0 = \frac{\mu E_0}{\sqrt{8\pi n_0 T_e}}$$

One can introduce the correlation function (whose corresponding meaning in Quantum Mechanics is nothing but the density matrix for mixed states)

$$\rho(x, x', s) = \langle \Psi^*(x, s)\Psi(x', s)\rangle$$

$$w(x, p, s) = \frac{1}{2\pi\alpha}\int_{-\infty}^{\infty} \rho\left(x + \frac{y}{2},\ x - \frac{y}{2},\ s\right)\ \exp\left(i\frac{py}{\alpha}\right)\ dy$$

⇩

$$\omega^2 - \mu^2 k^2 = k^2 \int_{-\infty}^{\infty} \frac{w_0\,(p + \alpha k/2) - w_0\,(p - \alpha k/2)}{\alpha k}$$

$$w_0(p) = \frac{\rho_0}{\pi}\frac{\Delta}{\Delta^2 + p^2}$$

$$\omega^2 = \left(1 + \frac{3\Delta^2}{\rho_0}\right) k^2 + i\pi k^2 w_0'(\beta)$$

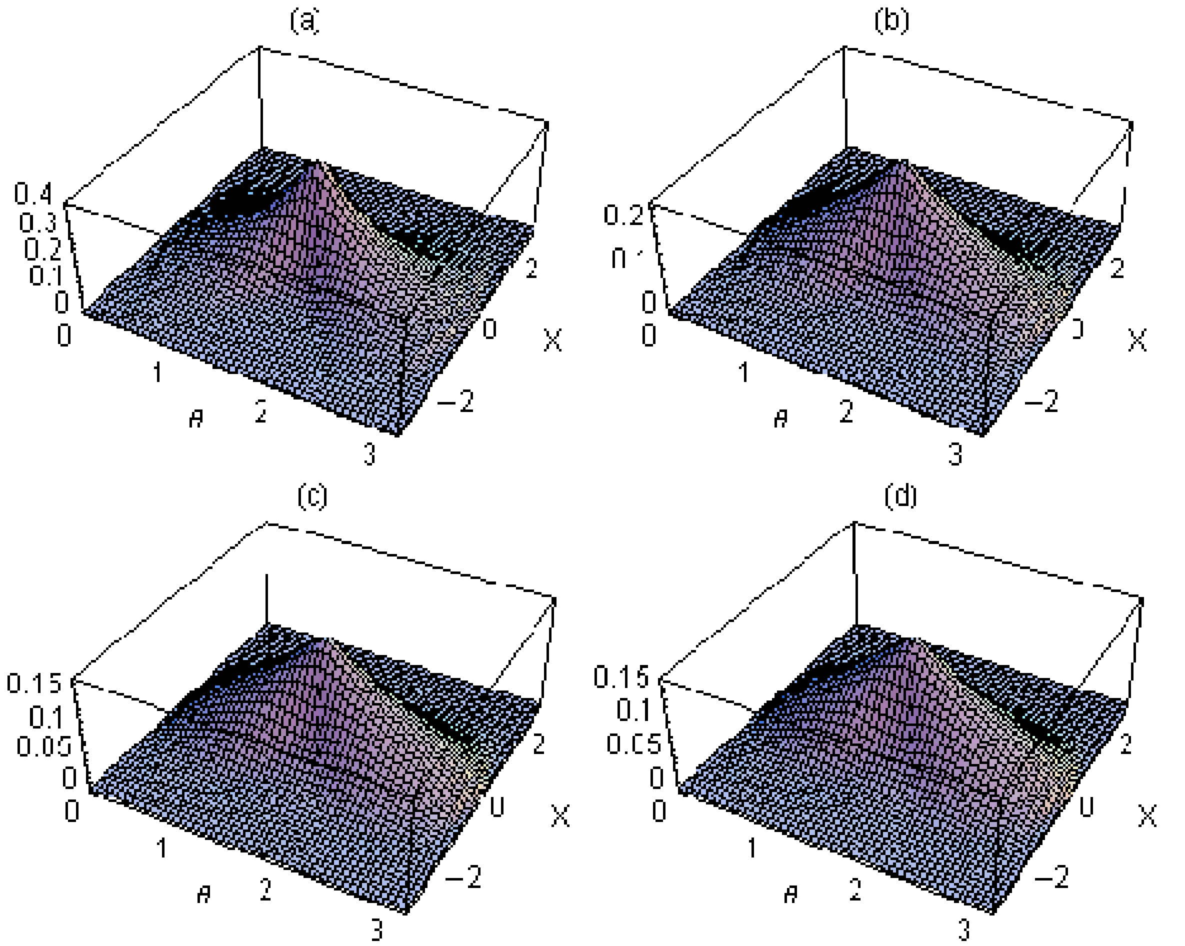
TOMOGRAMS

One of the main reasons to use the tomogram technique is justified by the natural possibility of measuring the states usually described by the complex wave function, in principle solution of LSE or NLSE.

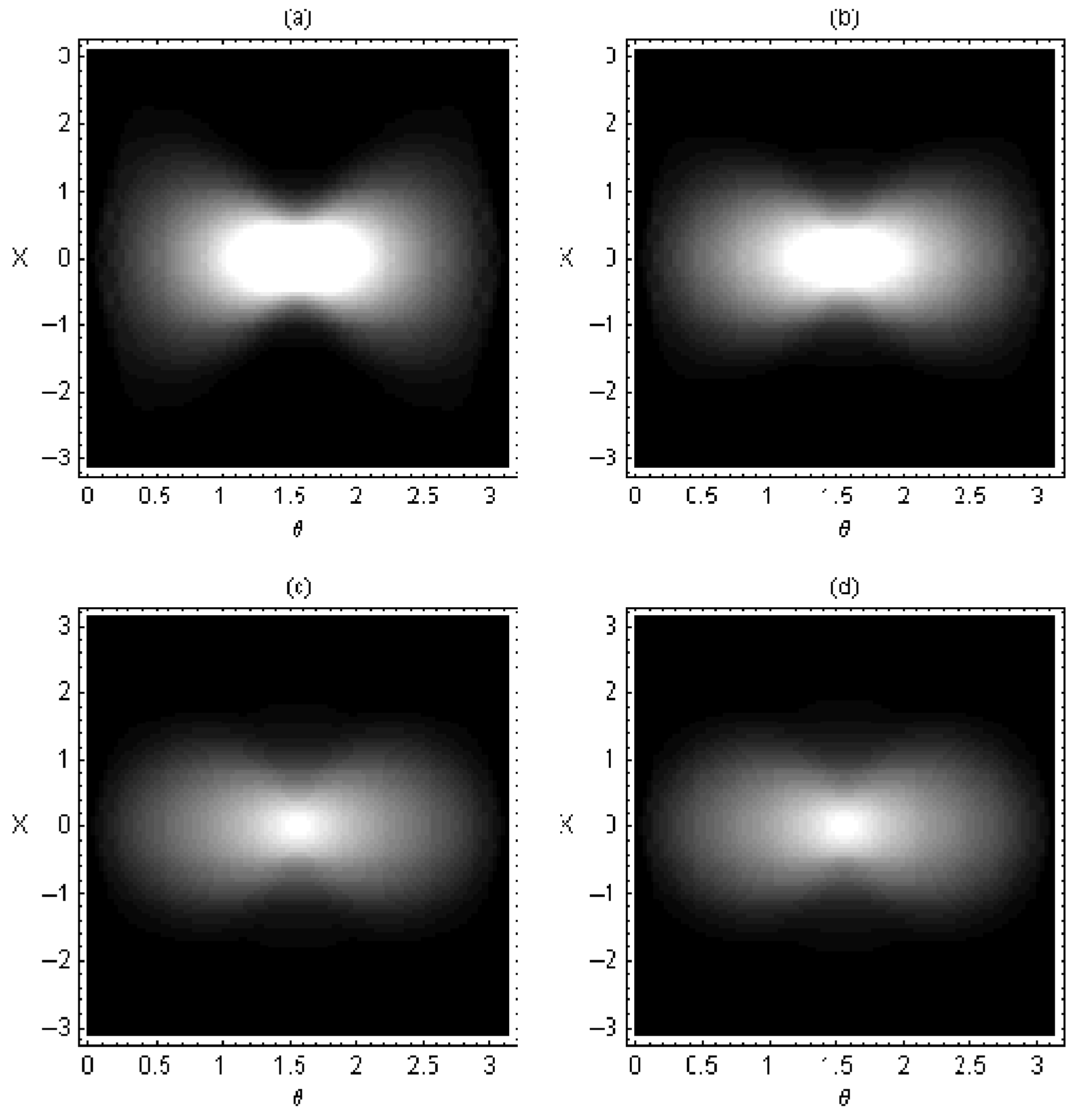
$$F_w(X,\mu,\nu) = \frac{1}{2\pi|\nu|}\left|\int \psi(y)\exp\left(\frac{i\mu}{2\nu}y^2 - \frac{iX}{\nu}y\right)dy\right|^2$$

A relation among the parameters can be in principle assumed. In particular, one can take $\mu = cos\theta$ and $\nu = sin\theta$ and the optical tomogram becomes:

$$F_w(X,\theta) = \frac{1}{2\pi|\sin\theta|}\left|\int \psi(y)\exp\left(\frac{i\cot\theta}{2}y^2 - \frac{iX}{\sin\theta}y\right)dy\right|^2$$



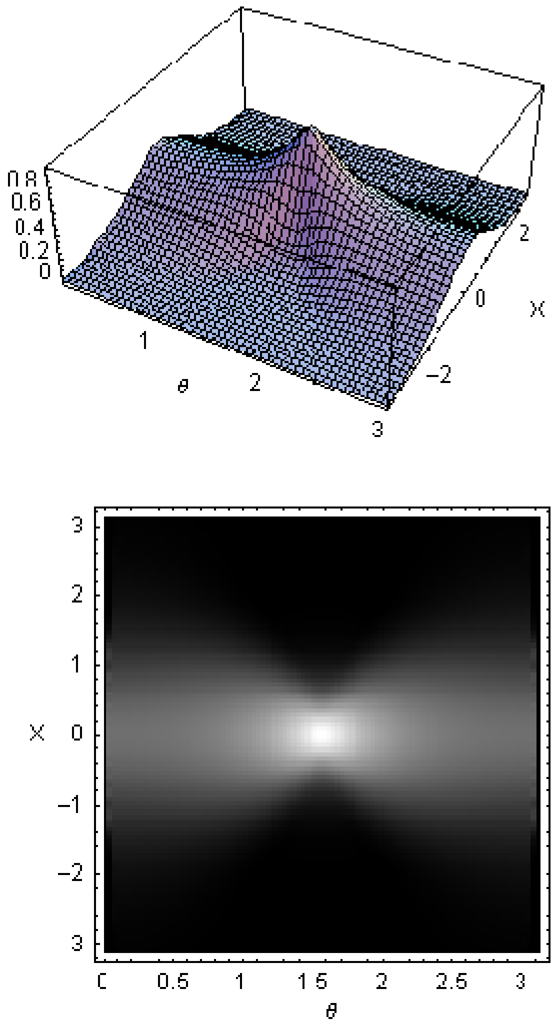
3D plot of the tomogram of the mNLSE soliton for various β:
(a) β=0.5, (b) β=1;(c) β=2; (d) β=2.5



Density plot of the tomogram of the mNLSE soliton for various β: (a) β=0.5, (b) β=1;(c) β=2; (d) β=2.5

(a)

(b)



(a) Tomographic map and (b) the corresponding density plot of a quasi-1D bright soliton of BEC as function of x and θ. The trap size of BEC is L = 1.4 μm **and the soliton width is** L_z = 1.7 μm),