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Hole Equilibria in Vlasov-Poisson Systems: a Challenger to Wave Theories of Ideal Plasmas

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Hole equilibria in Vlasov–Poisson systems: A challenge to wave theories of ideal plasmas

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A unified description of weak hole equilibria in collisionless plasmas is given. Two approaches, relying on the potential method rather than on the Bernstein, Greene, Kruskal method and associated with electron and ion holes, respectively, are shown to be equivalent. A traveling wave solution is thereby uniquely characterized by the nonlinear dispersion relation and the "classical" potential $V(\phi)$, which determine the phase velocity and the spectral decomposition of the wave structure, respectively. A new energy expression for a hole carrying plasma is found. It is dominated by a trapped particle contribution occurring one order earlier in the expansion scheme than the leading term in conventional schemes based on a truncation of Vlasov's equation. Linear wave theory—reconsidered by taking the infinitesimal amplitude limit—is found to be deficient, as well. Neither Landau nor van Kampen modes and their general superpositions can adequately describe these trapped particle modes due to an incorrect treatment of resonant particles for phase velocities in the thermal range. It is therefore concluded that wave theories in their present form, dictated by linearity, are not yet properly shaped to describe the dynamics of ideal plasmas (and fluids) correctly. © *2000 American Institute of Physics.* [S1070-664X(00)00412-2]

I. INTRODUCTION

Since the beginning of plasma research and research on gas discharges, the study of electrostatic phenomena has attracted the attention of physicists. Whereas the focus in the early stage was on macroscopic fluid like phenomena in bounded and unbounded plasmas,[1–3] research on dilute and hot plasmas concentrated later on microscopic effects, since details of observations such as Landau damping could no longer be understood macroscopically. Soon after the foundation of a plasma kinetic theory was laid down,[4–6] a controversy between Landau and Vlasov arose as to how to describe linear waves properly, a circumstance which overshadowed to some extent the development of kinetic theory.[7] The excitement has, however, gone too far, since, as we know nowadays, both standpoints can claim validity provided that the appropriate limits are taken (or fail together, see the following). Mathematically, linear wave theory of Vlasov–Poisson plasmas was completed by van Kampen[8] and developed further by Case,[9] showing essentially that besides a discrete spectrum of normal modes belonging to a time asymptotic dispersion relation a continuum of modes and eigenvalues exists. The origin of this peculiar behavior was easily identified as a wave-particle resonance, and further research focused on a more precise description of this phenomenon by treating resonant particles nonlinearly, resulting in two further highlights in plasma physics, the papers of Bohm and Gross[10] and of Bernstein, Greene and Kruskal (BGK).[11] In the latter, the first mathematically rigorous proof of how to construct electrostatic structures in collisionless plasmas was given. Temporal effects associated with trapping and sloshing of particles in the trough of a coherent electrostatic structure were considered in Refs. 12–14, in order to physically understand effects such as linear and nonlinear Landau damping. Hints that trapping of particles can change the ordering, giving rise to a half power rather than an integer power expansion of the distribution function and of the density in terms of a weak electric potential ϕ, appeared in Refs. 15–17.

The existence of steady-state phase space holes—the topic of this paper—first became evident in numerical simulations,[18,19] and examples of analytical descriptions based on primitive distributions such as waterbags appeared in Refs. 16,20,21.

A different method of constructing equilibrium solutions which allows the incorporation of more physical distributions being as general as the BGK method,[11] was presented in Ref. 22 and developed further in Refs. 23 and 24. Based on this second, earlier alternative method, a variety of hole and double layer solutions could be found, as reviewed in Ref. 25.

Furthermore, thanks to improved diagnostics, experiments in the last two decades revealed the omnipresence of holes in collisionless laboratory[25,26] and space plasmas.[27] Other devices in which kinetic vortex structures can be found are circular particle accelerators and storage rings.[28] It is hence of paramount importance to know more about holes and the other types of phase space structures.

The goal of the present investigation is to deepen the understanding of hole structures from an analytical point of view and to draw some conclusions concerning the present status of wave theory. We shall show how structures of different polarity and phase velocities behave and how their characteristic features can emerge from a universal description. We shall deal with the energy of a hole carrying plasma and take the infinitesimal amplitude limit, to allow a comparison with the linearized wave theory.

We shall use the second method of construction,[22] hereafter called the potential method, which means that we describe the solutions of the Vlasov equations completely in terms of constants of motion and look for the self-consistent potential by solving Poisson's equation in the weak amplitude regime.

In Sec. II the equilibrium distribution function is presented. A procedure suited for ion holes is given in Sec. III and a second one—more useful for electron holes and ion acoustic like perturbations—is developed in Sec. IV. A unified description of both procedures and their equivalence is presented in Sec. V. Section VI deals with the energy of a hole–plasma system, and the infinitesimal wave limit is considered in Sec. VII. The paper terminates with a summary and conclusions in Sec. VIII.

II. THE EQUILIBRIUM DISTRIBUTION FUNCTIONS

The governing system of equations to be explored is the Vlasov–Poisson system of a two component, current-carrying plasma consisting of electrons and single charged ions. It reads in normalized form

$$[\partial_t + v\partial_x + \phi'\partial_v]f_e = 0, \tag{1}$$

$$[\mu\partial_t + u\partial_x - \theta\phi'\partial_u]f_i = 0, \tag{2}$$

$$\phi''(x) = n_e(\phi) - n_i(\phi) \equiv -\partial_\phi V(\phi), \tag{3}$$

where space x, time t and electric potential ϕ are normalized in units of the electron Debye length λ_{De}, the inverse plasma frequency ω_{pe}^{-1}, and T_{ef}/e, where T_{ef} is the temperature (in energy units) of the electrons in the unperturbed state. The electron velocity v in (1) is normalized by $v_{the} = \sqrt{T_{ef}/m_e}$ and the ion velocity u in (2) by $v_{thi} = \sqrt{T_{if}/m_i}$. The two parameters characterizing the systems are defined by $\theta := T_{ef}/T_{if}$ and $\mu := (m_i T_{ef}/m_e T_{if})^{1/2} \equiv (\theta/\delta)^{1/2}$, which is a quantity typically much larger than unity and where $\delta = m_e/m_i$ is the mass ratio.

The unperturbed distributions are assumed to be shifted Maxwellians formulated in a frame moving with v_0 in the electron phase space and with u_0 in the ion phase space. They are given by

$$f_e = \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}(v - \tilde{v}_D)^2\right], \tag{4a}$$

$$f_i = \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}(u + u_0)^2\right]. \tag{4b}$$

It holds $u_0 = \mu v_0$, and $\tilde{v}_D$ is defined by $\tilde{v}_D := v_D - v_0$, where v_D describes a given constant drift between electrons and ions existing already in the unperturbed state. In the perturbed state the corresponding equilibrium distributions as solutions of the Vlasov equation (in wave frame) are found to be[22,29]

$$f_e(x,v) = \frac{1+K}{\sqrt{2\pi}}\begin{cases}\exp\left[-\frac{1}{2}(\sigma_e\sqrt{2\epsilon_e} - \tilde{v}_D)^2\right], & \epsilon_e > 0\\ \exp(-\tilde{v}_D^2/2)\exp[-\beta\epsilon_e], & \epsilon_e \leq 0,\end{cases} \tag{5a}$$

$$f_i(x,u) = \frac{1+A}{\sqrt{2\pi}}\begin{cases}\exp\left[-\frac{1}{2}(\sigma_i\sqrt{2\epsilon_i} + u_0)^2\right], & \epsilon_i > 0\\ \exp(-u_0^2/2)\exp[-\alpha\epsilon_i], & \epsilon_i \leq 0,\end{cases} \tag{5b}$$

where K and A are normalization constants which disappear in the limit of a vanishing amplitude of the perturbation $\psi \to 0$. Because they are different for ion, respectively, electron holes, we introduce later on subscript quantities K_i, A_i for ion and K_e, A_e for electron holes. The other quantities like $\alpha, \beta, \epsilon_s, \sigma_s, s = e, i$ are explained further in the following.

III. FIRST PROCEDURE: ION HOLE EQUILIBRIA

Concentrating first on ion hole equilibria, the energy expressions in (5a) and (5b) are given by the definitions

$$\epsilon_e := \frac{v^2}{2} - (\phi + \psi), \quad \epsilon_i := \frac{u^2}{2} + \theta\phi \tag{6}$$

representing invariants of the electron, respectively, ion motion. Equations (5a) and (5b) satisfy (1) and (2), respectively, and reduce to (4a) and (4b) (modified now by different normalization constants) in the limit $\phi \to -\psi$, respectively, $\phi \to 0$, which are the spatial points of no trapped electrons and ions, respectively. $\epsilon_{e,i} = 0$ describe the separatrices between trapped ($\epsilon_{e,i} \leq 0$) and untrapped or free ($\epsilon_{e,i} > 0$) particles. The distribution (state) of trapped particles is controlled by the trapping parameters β, respectively, α, and $\sigma_e := sgn v$, $\sigma_i := sgn u$ are constants of motion of free particles and turn out to be necessary for the description of propagating structures. The distributions are continuous at the separatrix and are characterized by a hole in the trapped region $\epsilon_{e,i} \leq 0$ in cases where β, respectively, α, are negative. The normalization constant A_i will be specified later whereas K_i is given by $K_i \equiv -k_{0-}^2\psi/2$, where k_{0-} is related to the wave number k of the ion hole structure. Velocity integration of (5a) and (5b) yields in the small amplitude limit $\psi \ll 1$:[22–24]

$$n_e(\phi) = \left(1 - \frac{k_{0-}^2}{2}\psi\right)\left[1 - \frac{1}{2}Z_r'\left(\frac{\tilde{v}_D}{\sqrt{2}}\right)(\phi + \psi) - \frac{4}{3}b(\beta, \tilde{v}_D)(\phi + \psi)^{3/2} + \cdots\right], \tag{7a}$$

$$n_i(\phi) = (1 + A_i)\left[1 - \frac{1}{2}Z_r'\left(\frac{u_0}{\sqrt{2}}\right)(-\theta\phi) - \frac{4}{3}b(\alpha, u_0)(-\theta\phi)^{3/2} + \cdots\right], \tag{7b}$$

where $Z_r(x)$ is the real part of the plasma dispersion function given by $Z_r(x) = -2w(x)$, where $w(x)$ is Dawson's integral, $w(x) := \exp(-x^2)\int_0^x dt\, \exp(t^2)$ [see also later (73)]. The function $b(\alpha, u_0)$ is defined by

$$b(\alpha, u_0) := \frac{1}{\sqrt{\pi}}(1 - \alpha - u_0^2)\exp(-u_0^2/2) \tag{8}$$

and an analogous expression holds for $b(\beta,\tilde{v}_D)$.[22,25] The parameter A_i in (5) and (7) is found in the solitary wave limit ($k_{0-}\rightarrow 0$) by equating the densities at $\phi=0$, i.e., at infinity, corresponding to $\phi''=0$. We obtain

$$A_i=-\frac{1}{2}Z_r'\left(\frac{\tilde{v}_D}{\sqrt{2}}\right)\psi-\frac{4}{3}b(\beta,\tilde{v}_D)\psi^{3/2}. \tag{9}$$

With this A_i we get for a finite wavelength perturbation at $\phi=0$: $n_e(0)-n_i(0)=-(k_{0-}^2/2)\,\psi$ which corresponds to a negative curvature at potential maximum, as it should. Multiplication of (3) with ϕ' and integration we obtain the "energy law" $\phi'(x)^2/2+V(\phi)=0$ and a further integration gives $-V(\phi)=\int_0^\phi d\bar{\phi}[n_e(\bar{\phi})-n_i(\bar{\phi})]$ which becomes

$$\begin{aligned}-V(\phi)=&(1+K_i)\left\{\phi-\frac{1}{4}Z_r'\left(\frac{\tilde{v}_D}{\sqrt{2}}\right)\right.\\&\times[(\phi+\psi)^2-\psi^2]-\frac{8}{15}b(\beta,\tilde{v}_D)\\&\left.\times[(\phi+\psi)^{5/2}-\psi^{5/2}]+\cdots\right\}\\&-(1+A_i)\left\{\phi+\frac{1}{4}Z_r'\left(\frac{u_0}{\sqrt{2}}\right)\theta\phi^2\right.\\&\left.+\frac{8}{15}b(\alpha,u_0)\theta^{3/2}(-\phi)^{5/2}+\cdots\right\},\end{aligned} \tag{10}$$

where terms of $O(\psi^3)$ have already been neglected. Two further necessary conditions select the time-independent solutions of the Vlasov–Poisson system (1)–(3) in a current-carrying plasma, namely

$$V(-\psi)=0, \tag{11a}$$

$$V(\phi)<0$$

in

$$-\psi<\phi<0. \tag{11b}$$

(11a) yields the nonlinear dispersion relation (NDR), and (11b) restricts the amount of particles trapped in the potential well. Exploiting (11a) we find

$$\begin{aligned}&k_{0-}^2-\frac{1}{2}Z_r'(\tilde{v}_D/\sqrt{2})-\frac{\theta}{2}Z_r'(u_0/\sqrt{2})\\&=\frac{16}{15}\left[b(\alpha,u_0)\theta^{3/2}+\frac{3}{2}b(\beta,\tilde{v}_D)\right]\psi^{1/2}.\end{aligned} \tag{12}$$

It determines u_0 (respectively, v_0) in terms of v_D, $k_{0-},\theta,\psi,\alpha$, and β. Of interest here are wave solutions with phase velocities in the ion thermal range, i.e., $u_0\leq O(1)$. Then $v_0=\mu^{-1}u_0\leq O(\mu^{-1})$, which is a small quantity. Hence, if the electron drift v_D does not exceed the ion sound velocity appreciably, $v_D\leq O(\sqrt{\delta})$, then $\tilde{v}_D$ is a small quantity, too, and $-\frac{1}{2}Z_r'(\tilde{v}_D/\sqrt{2})$ in (12) can be replaced by $+1$. Assuming furthermore that the electrons behave isothermally (i.e., $\beta=1$), then the last term in (12) is negligible and (12) reduces to

$$-\frac{1}{2}Z_r'\left(\frac{u_0}{\sqrt{2}}\right)=-\frac{(1+k_{0-}^2)}{\theta}+\frac{16}{15}b(\alpha,u_0)\sqrt{\theta\psi}\equiv D_-. \tag{13}$$

The NDR (13) determines the possible phase velocities u_0 in terms of ψ,k_{0-},α and θ. Note that in the solitary wave limit ($k_{0-}\rightarrow 0$) (13) coincides with the NDR of Ref. 30 Eq. (6). In this paper, presumably the first proper ion hole paper, ψ has to be replaced by $(\theta\psi)$ because of the different normalization of ϕ. (13) also corresponds to the NDR of Ref. 31, Eq. (14) or Ref. 32, Eq. (19). Equation (13) in the solitary wave limit $k_{0-}\rightarrow 0$ is also identical with the NDR (3) of Ref. 33.

Equation (13) can be utilized to simplify the potential $V(\phi)$ [the more general form of $V(\phi)$ being based on (12) rather than on (13) will be presented later in Sec. 5, formula (42)]. We obtain

$$\begin{aligned}-V(\phi)=&-\frac{k_{0-}^2}{2}\phi(\psi+\phi)+\frac{8}{15}b(\alpha,u_0)\theta^{3/2}\phi^2\\&\times(\sqrt{\psi}-\sqrt{-\phi})\\\equiv&\frac{8}{15}b(\alpha,u_0)\theta^{3/2}\\&\times[4S_-^{-1}(-\phi)(\psi+\phi)\psi^{1/2}+\phi^2\\&\times(\sqrt{\psi}-\sqrt{-\phi})],\end{aligned} \tag{14}$$

where in the second step the steepening (or anharmonicity) parameter S_- has been introduced.[22] It is defined by

$$S_-:=\frac{64b(\alpha,u_0)\theta^{3/2}\psi^{1/2}}{15k_{0-}^2} \tag{15}$$

and lies in the interval[22,34]

$$-8\leq S_-\leq\infty. \tag{16}$$

It generally describes cnoidal waves given by Jacobian elliptic functions which become harmonic in the limit $S_-\rightarrow 0$. Of special interest are the other two limits $S_-\rightarrow\infty$ and $S_-\rightarrow -8$ because in this case the structure becomes a solitary one, as illustrated in Fig. 1. Figure 1(a) shows the situation for $S_-\rightarrow\infty$ and displays an ordinary solitary ion hole structure for which holds $k_{0-}=0$. As seen from (11b) and (14) $b(\alpha,u_0)$ must be positive and the parameter D_- introduced in (13) becomes $D_-=-1/\theta+\frac{16}{15}b(\alpha,u_0)\sqrt{\theta\psi}$. If $|D_-|$ is small, we can use an expansion of Z_r' in the vicinity of zero [see later (26b)]:

$$-\tfrac{1}{2}Z_r'(u_0/\sqrt{2})=1-u_0/1.307 \tag{17}$$

to get the velocity $u_0=1.307[1-D_-]$. A positive D_- corresponds to a subsonic velocity, subsonic in comparison with the so-called slow ion acoustic velocity[22,25] defined by $\tilde{C}_s=1.307$. A subsonic u_0 implies in view of (8) and of $b(\alpha,u_0)>0$ that the trapping parameter α must be negative ($\alpha<1-u_0^2<0$), corresponding to a notch in the range of

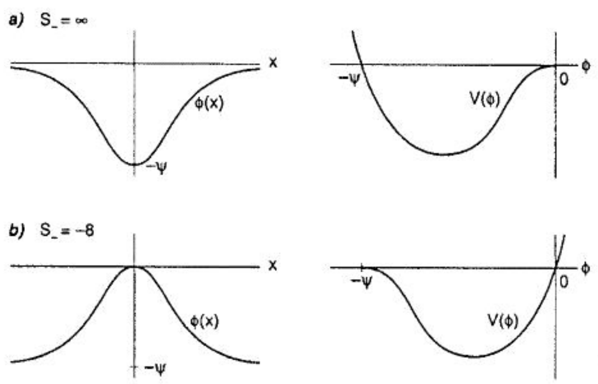

FIG. 1. The two solitary ion hole wave limits of the electric potential $\phi(x)$ and the potential $V(\phi)$ for $S_-=\infty$ (a) and $S_-=-8$ (b).

trapped (resonant) ions (justifying the name ion hole). For this solitary wave the separatrix in ion phase space given by $\epsilon_i=u^2/2+\theta\phi=0$ is of O type.

Figure 1(b), on the other hand, shows the second solitary ionic structure for which holds $8b\theta^{3/2}=-15k_{0-}^2\psi^{-1/2}$. In this case $b(\alpha,u_0)$ is negative and the parameter D_- in (13) becomes $D_-=-(1+3k_{0-}^2)/\theta$, which is negative too. If again $|D_-|$ is small, we get $u_0=1.307[1+|D_-|]$, which is supersonic now. The trapping parameter α in this case is less negative or even positive ($\alpha>1-u_0^2$) resulting in a less pronounced notch or even in a hump in the trapped ion range. Note that the separatrix in ion phase space for this second kind of solitary waves is of X type. It is this structure (more precisely, one half of it) which may typically provide the front in weak electrostatic shocks excited in experiments.[35]

IV. SECOND PROCEDURE: ELECTRON HOLE EQUILIBRIA AND ION ACOUSTIC WAVES

To describe electron holes[36] and nonlinear ion acoustic waves[22] in a convenient manner it is appropriate to change in a sense the polarity of ϕ. This is accomplished by the transformation

$$\phi+\psi\rightarrow\phi, \tag{18}$$

so that the single particle energies in (5) now become

$$\epsilon_e:=v^2/2-\phi, \tag{19a}$$

$$\epsilon_i:=\frac{u^2}{2}+\theta(\phi-\psi) \tag{19b}$$

and it holds $0\leqslant\phi\leqslant\psi$.

Again, $\epsilon_{e,i}=0$ represent the phase space trajectories separating trapped and passing particles. Trapped electrons are absent at $\phi=0$ and trapped ions at $\phi=\psi$. Integrating (5a) and (5b) over the velocity we obtain the densities

$$n_e(\phi)=(1+K_e)\times\Big\{1-\tfrac{1}{2}Z_r'(\tilde{v}_D/\sqrt{2})\phi-\tfrac{4}{3}b(\beta,\tilde{v}_D)\phi^{3/2}+\cdots\Big\}, \tag{20a}$$

$$n_i(\phi)=(1+A_e)\Big\{1-\tfrac{1}{2}Z_r'(u_0/\sqrt{2})\theta(\psi-\phi)-\tfrac{4}{3}b(\alpha,u_0)[\theta(\psi-\phi)]^{3/2}+\cdots\Big\}. \tag{20b}$$

This time, as will be justified in the following, we choose for K the opposite sign and a different proportionality constant k_{0+}: $K_e=k_{0+}^2\psi/2$. The parameter A_e in (5b) and (20b) is then found in the solitary wave limit ($k_{0+}\rightarrow 0$) by equating the densities at $\phi=0$, which is the potential minimum now, corresponding to $\phi''=0$. We obtain

$$A_e=+\tfrac{1}{2}Z_r'(u_0/\sqrt{2})\theta\psi+\tfrac{4}{3}b(\alpha,u_0)(\theta\psi)^{3/2}. \tag{21}$$

With this A_e we get for a finite wavelength perturbation $k_{0+}\neq 0$ at $\phi=0$ for the density difference $n_e(0)-n_i(0)=+(k_{0+}^2/2)\,\psi$, which corresponds to a positive curvature at potential minimum, as it should be.

The classical potential $V(\phi)$ is then found by integration of (3) and becomes

$$\begin{aligned}-V(\phi)=&(1+K_e)\Big\{\phi-\tfrac{1}{4}Z_r'(\tilde{v}_D/\sqrt{2})\phi^2-\tfrac{8}{15}b(\beta,\tilde{v}_D)\phi^{5/2}+\cdots\Big\}-(1+A_e)\\&\times\Big\{\phi-\tfrac{1}{4}Z_r'(u_0/\sqrt{2})\theta[\psi^2-(\psi-\phi)^2]-\tfrac{8}{15}b(\alpha,u_0)\theta^{3/2}[\psi^{5/2}-(\psi-\phi)^{5/2}]+\cdots\Big\}.\end{aligned} \tag{22}$$

Again, the two conditions

$$V(\psi)=0, \tag{23a}$$

$$V(\phi)<0$$

in

$$0<\phi<\psi \tag{23b}$$

guarantee the existence of a solution. The NDR (23a) with A_e given by (21) becomes

$$k_{0+}^2-\frac{1}{2}Z_r'(\tilde{v}_D/\sqrt{2})-\frac{\theta}{2}Z_r'(u_0/\sqrt{2})=\frac{16}{15}\left[b(\beta,\tilde{v}_D)+\frac{3}{2}b(\alpha,u_0)\theta^{3/2}\right]\sqrt{\psi}, \tag{24}$$

which determines the phase velocity u_0 (respectively, $\tilde{v}_0$) in terms of $v_D, k_{0+}, \theta, \psi, \alpha$, and β. In comparison with (12) the right-hand side has changed and essentially differs due to a change in the species being trapped in the potential.

Making use of $K_e=(k_{0+}^2/2)\psi$ and of (21) and (24) the potential $V(\phi)$ in (22) simplifies to

$$\begin{aligned}-V(\phi)=&\frac{k_{0+}^2}{2}\phi(\psi-\phi)+b(\beta,\tilde{v}_D)\frac{8}{15}\phi^2[\sqrt{\psi}-\sqrt{\phi}]\\&+b(\alpha,u_0)\theta^{3/2}\frac{4}{15}\{\phi\sqrt{\psi}(3\phi-5\psi)\\&+2[\psi^{5/2}-(\psi-\phi)^{5/2}]\}.\end{aligned} \tag{25}$$

Again, the potential V determines uniquely the shape or spectral decomposition of the wave structure. Two limits are of special interest in this case: electron holes moving at the electron thermal speed and nonisothermal ion acoustic solitary waves moving at the ion sound speed. We start with the second, the slower structure.

A. Nonisothermal, ion acoustic waves

This class of solutions refers to waves which propagate approximately with ordinary ion sound speed, i.e., $u_0 \approx \sqrt{\theta}$, $v_0 \approx \sqrt{\delta}$. As typical for ion acoustic waves we assume $\theta \gg 1$, so that $u_0 \gg 1$. We also assume a drift velocity not larger than ion acoustic speed, $v_D \approx \sqrt{\delta}$, in which case $|\tilde{v}_D| = |v_D - v_0| \approx \sqrt{\delta} \ll 1$.

To exploit the formulas further, we make use of the following expansions:

$$-\frac{1}{2}Z_r'(x) = \begin{cases} 1-2x^2(1-2x^2/3+\cdots), & |x| \ll 1 \quad (26a) \\ -\frac{1}{x_0}(x-x_0)+(x-x_0)^2, & |x-x_0| \ll 1 \quad (26b) \\ -\frac{1}{2x^2}[1+3/(2x^2)+\cdots], & |x| \gg 1. \quad (26c) \end{cases}$$

In (26b) the quantity $x_0 = 0.924$ marks the zero point of $-\frac{1}{2}Z_r'(x)$ [see also Fig. 3(a), not to mix up with x^0].

It follows

$$-\tfrac{1}{2}Z_r'(\tilde{v}_D/\sqrt{2}) = 1-\tilde{v}_D^2, \tag{27a}$$

$$-\frac{\theta}{2}Z_r'(u_0/\sqrt{2}) = -\frac{\theta}{u_0^2}\left(1+\frac{3}{u_0^2}\right) = -\frac{\delta}{v_0^2}\left(1+\frac{3\delta}{\theta v_0^2}\right). \tag{27b}$$

With this and realizing that $b(\alpha,u_0) \sim \exp(-u_0^2/2)$ is negligible for large values of u_0, we get for the NDR (24):

$$k_{0+}^2 + 1 - \tilde{v}_D^2 - \frac{\delta}{v_0^2}\left(1+\frac{3\delta}{\theta v_0^2}\right) = \frac{16}{15}b(\beta,v_0)\sqrt{\psi}. \tag{28}$$

Equation (28) can be solved by the ansatz $v_0^2 = \delta(1+\epsilon)$ with $\epsilon \ll 1$ provided that k_{0+}^2 and the right-hand side of (28) are small quantities. We get in this limit

$$v_0^2 = \delta\left[1-k_{0+}^2+\frac{3}{\theta}+(v_D-\sqrt{\delta})^2+\frac{16(1-\beta)}{15\sqrt{\pi}}\sqrt{\psi}\right]. \tag{29}$$

Since we have $v_0 \equiv \tilde{v}_0/v_{\text{the}}$, the dimensional phase velocity $\tilde{v}_0 \approx v_{\text{the}}\sqrt{\delta} = \sqrt{T_{ef}/m_i} = C_s$ is of ordinary ion acoustic type. The potential $V(\phi)$ in (25) simplifies in this limit to

$$-V(\phi) = \frac{k_{0+}^2}{2}\phi(\psi-\phi)+\frac{8}{15}\frac{(1-\beta)}{\sqrt{\pi}}\phi^2[\sqrt{\psi}-\sqrt{\phi}]$$
$$= \frac{8}{15}\frac{(1-\beta)}{\sqrt{\pi}}[4S_+^{-1}\phi(\psi-\phi)\sqrt{\psi}+\phi^2(\sqrt{\psi}-\sqrt{\phi})], \tag{30}$$

where

$$S_+ := \frac{64(1-\beta)\sqrt{\psi}}{15\sqrt{\pi}k_{0+}^2}. \tag{31}$$

Equations (29) and (30) (in the limit $v_D \to 0$) agree with the corresponding expressions of Ref. 22. Again, S_+ lies in the interval

$$-8 \leq S_+ \leq \infty \tag{32}$$

and describes cnoidal electrostatic perturbations but moving with ion sound speed now. In the limit $S_+ \to \infty$ ($k_{0+} \to 0$) we get nonisothermal ion acoustic solitons[22,23] in which case the electron trapping parameter β has to satisfy

$$\beta < 1.$$

In particular, if β is negative, this results in a depression (notch) of the electron distribution function in the trapped electron range which is in the bulk of the distribution. The electric potential $\phi(x)$ is bell-shaped. The second solitary wave, on the other hand, is obtained in the limit $S_+ \to -8$ in which case $\phi(x)$ has the shape of a trough approaching $\phi = \psi$ at infinity. We mention that in contrast to the solitary potential hump, where the width Δ scales like $\Delta \sim \psi^{-1/4}$ (see Refs. 22 and 23), the width of the solitary potential dip is amplitude independent. Its shape is given by formula (3.33) in Ref. 34. The ordinary ion acoustic soliton is obtained in the isothermal limit, $\beta \to 1$, in which case the next order contributions, which are of $O(\psi)$ in (28) or (29) and of $O(\psi^3)$ in (30), have to be taken into account.[22,23]

B. Electron holes

The class of electron hole solutions is obtained by demanding phase velocities in the electron thermal range: $v_0 \approx O(1), u_0 \approx \sqrt{\theta/\delta} \gg 1$. In this case the NDR (24) and the potential V (25) simplify to

$$k_{0+}^2 - \tfrac{1}{2}Z_r'(\tilde{v}_D/\sqrt{2}) = \tfrac{16}{15}b(\beta,\tilde{v}_D)\sqrt{\psi}, \tag{33}$$

$$-V(\phi) = \frac{k_{0+}^2}{2}\phi(\psi-\phi)+\frac{8}{15}b(\beta,\tilde{v}_D)\phi^2(\sqrt{\psi}-\sqrt{\phi}), \tag{34}$$

in agreement with Eqs. (3.13) and (3.14) of Ref. 34 which also contain the next order terms.

In the case where the RHS of (33) and k_{0+}^2 are small and where a drift is absent, so that $|\tilde{v}_D|$ in (33) can be replaced by v_0 we obtain by utilizing (26b)

$$v_0 = 1.307\left[1+k_{0+}^2-\tfrac{16}{15}b(\beta,1.307)\sqrt{\psi}\right] \tag{35}$$

an expression found earlier in Ref. 36, Eq. (26), which appears to be the first electron hole paper based on realistic Maxwellian-type distributions.

A typical electron hole velocity lies in the electron thermal range. The corresponding shape is again determined by the anharmonicity parameter S_+ in (31) which satisfies (32). Again the two solitary wave limits represent a potential hump for $S_+ \to \infty$ and a potential dip for $S_+ \to -8$. Note that a potential hump of lower phase velocities is referred in this paper either to a nonisothermal ion acoustic solitary wave, when its phase velocity is ion acoustic like [$v_0 \approx \sqrt{\delta}$, Eq.

(29)], or to a slow ion acoustic solitary wave, when its phase velocity is in the ion thermal range and below $v_0 \leq \sqrt{\delta/\theta}$). For the latter to be true it must hold $S_- = -8$, according to Fig. 1(b). To get a more general solution the isothermal limit $\beta \to 1$, when one goes from (12) to (13), has to be lifted.

V. A UNIFIED DESCRIPTION OF BOTH PROCEDURES AND THEIR EQUIVALENCE

In the previous sections we have derived general expressions for the phase velocity and the shape of hole or hump equilibria. We have distinguished two different procedures depending on whether ion trapping represents a major contribution or not. For phase velocities in the ion thermal range (and below) we arrived at the general NDR (12) and at the (reduced) potential $V(\phi)$ (14) valid for $b(\beta,\tilde{v}_D)=0$. We assumed $-\psi \leq \phi \leq 0$, so that trapped ions are absent only at $\phi=0$, which is the potential maximum.

On the other hand, for hole equilibria propagating with ion sound speed and faster ion trapping becomes less important being dominated by electron trapping. In this case a shift in the electric potential (18) appeared to be appropriate, so that trapped electrons are absent only at $\phi=0$, which is the potential minimum now. The general NDR is within this second procedure given by (24) and the general potential V by (25).

In this section we first show that both general forms of the NDR and of $V(\phi)$ can be unified in single expressions, and in a second step we prove their equivalence.

A. Unified description

For this purpose, we define

$$t_\pm := b(\beta,\tilde{v}_D) \pm \theta^{3/2} b(\alpha;u_0), \tag{36}$$

$$A_\pm(s) := \tfrac{4}{15}(t_+ \pm s t_-), \tag{37}$$

where $s=-1$ for ion hole solutions, as defined in Sec. III, and $s=+1$ for electron hole and ion acoustic solutions, as defined in Sec. IV.

We postulate, that the general NDR is given by

$$k_{0s}^2 - \frac{1}{2} Z_r'(\tilde{v}_D/\sqrt{2}) - \frac{\theta}{2} Z_r'(u_0/\sqrt{2}) = [2A_+(s) + 3A_-(s)]\sqrt{\psi} \tag{38}$$

and the general potential $V(\phi)$ by

$$-V(\phi) = \frac{k_{0s}^2}{2} s\phi(\psi - s\phi) + A_+(s)\phi^2(\sqrt{\psi} - \sqrt{s\phi}) + A_-(s)\{\psi^{5/2} - (\psi - s\phi)^{5/2} - s\phi\sqrt{\psi}(5\psi - 3s\phi)/2\} \tag{39}$$

and show its correctness by considering the two cases $s = \pm 1$ separately.

If $s=-1$, we get

$$A_\pm(-1) = \frac{8}{15}\begin{cases} \theta^{3/2} b(\alpha,u_0) & + \\ b(\beta,\tilde{v}_D) & - \end{cases} \tag{40}$$

so that the NDR (38) becomes

$$k_{0-}^2 - \frac{1}{2} Z_r'(\tilde{v}_D/\sqrt{2}) - \frac{\theta}{2} Z_r'(u_0/\sqrt{2}) = \frac{16}{15}\left[b(\alpha,u_0)\theta^{3/2} + \frac{3}{2} b(\beta,\tilde{v}_D)\right]\sqrt{\psi}, \tag{41}$$

which is identical to (12). For $V(\phi)$, we get from (39) in this limit

$$-V(\phi) = -\frac{k_{0-}^2}{2}\phi(\psi+\phi) + \frac{8}{15} b(\alpha,u_0)\theta^{3/2}\phi^2(\sqrt{\psi} - \sqrt{-\phi}) + \frac{8}{15} b(\beta,\tilde{v}_D)\{\psi^{5/2} - (\psi+\phi)^{5/2} + \phi\sqrt{\psi}(5\psi + 3\phi)/2\}. \tag{42}$$

It agrees in the limit $b(\beta,\tilde{v}_D) \to 0$ with (14), as expected.

If $s=+1$, we get

$$A_\pm(+1) = \frac{8}{15}\begin{cases} b(\beta,\tilde{v}_D) & + \\ \theta^{3/2} b(\alpha,u_0) & - \end{cases} \tag{43}$$

so that the NDR (38) becomes

$$k_{0+}^2 - \frac{1}{2} Z_r'(\tilde{v}_D/\sqrt{2}) - \frac{\theta}{2} Z_r'(u_0/\sqrt{2}) = \frac{16}{15}\left[b(\beta,\tilde{v}_D) + \frac{3}{2}\theta^{3/2} b(\alpha,u_0)\right]\sqrt{\psi}, \tag{44}$$

which is identical to (24). For $V(\phi)$, we get from (39) using (43)

$$-V(\phi) = \frac{k_{0+}^2}{2}\phi(\psi-\phi) + \frac{8}{15} b(\beta,\tilde{v}_D)\phi^2(\sqrt{\psi} - \sqrt{\phi}) + \frac{8}{15}\theta^{3/2} b(\alpha,u_0)\{\psi^{5/2} - (\psi-\phi)^{5/2} - \phi\sqrt{\psi}(5\psi - 3\phi)/2\}, \tag{45}$$

which is identical to (25).

We conclude, that (41), (42), and (44), (45) represent the most general expressions of the NDR and of $V(\phi)$ describing phase space vortices or hole-type equilibria of the Vlasov–Poisson system (1)–(3) in current-carrying plasmas and being unified in the expressions (38), (39).

B. Equivalence of both procedures

To show the equivalence of both procedures, we lift the (ion hole) potential ϕ of Sec. III by ψ, i.e., we define

$$\hat{\phi} = \phi + \psi, \tag{46}$$

rewrite the general expressions in terms of $\hat{\phi}$, for which holds $0 \leq \hat{\phi} \leq \psi$, and show that they agree with the general expressions of Sec. IV.

In terms of $\hat{\phi}$ the general potential V of Sec. III, given by (42), becomes

$$-V(\hat{\phi})=\left[\frac{k_{0-}^2}{2}-\frac{4}{15}b(e)\sqrt{\psi}\right]\hat{\phi}(\psi-\hat{\phi})+\frac{8}{15}b(e)\hat{\phi}^2(\sqrt{\psi}-\sqrt{\hat{\phi}})+\frac{8}{15}b(i)\theta^{3/2}\times(\psi-\hat{\phi})^2[\sqrt{\psi}-\sqrt{\psi-\hat{\phi}}], \quad (47)$$

where we introduced the abbreviations

$$b(e)\equiv b(\beta,\tilde{v}_D); \quad b(i)\equiv b(\alpha,u_0).$$

If we now switch from k_{0-} to k_{0+} by the following relation:

$$k_{0-}^2=k_{0+}^2+\tfrac{8}{15}\sqrt{\psi}[b(e)-\theta^{3/2}b(i)], \quad (48)$$

we easily get from (47)

$$-V(\hat{\phi})=\frac{k_{0+}^2}{2}\hat{\phi}(\psi-\hat{\phi})+\frac{8}{15}b(e)\hat{\phi}^2(\sqrt{\psi}-\sqrt{\hat{\phi}})+\frac{8}{15}b(i)\theta^{3/2}\{\psi^{5/2}-(\psi-\hat{\phi})^{5/2}-\hat{\phi}\sqrt{\psi}\times(5\psi-3\hat{\phi})/2\}, \quad (49)$$

which is identical with (45) [or (25)].

Note that in the limit of $b(e)\rightarrow 0$ and $S_-\rightarrow -8$, when the structure represents a potential hump and moves with approximately ion thermal velocity [Fig. 1(b)], (48) becomes $k_{0+}=0$. The shape of the structure is hence given by the third expression on the RHS of (49).

The general potential hump solution is, therefore, best described by (49) [or (45)] in the limit $k_{0+}^2\rightarrow 0$ for $0\leqslant\phi\leqslant\psi$, whereas the general potential dip solution is most easily described by (42) in the limit $k_{0-}\rightarrow 0$ for $-\psi\leqslant\phi\leqslant 0$ irrespective of the phase velocity.

This was the main reason why we introduced two procedures in deriving the NDR and the potential $V(\phi)$ although one procedure alone would have done the job also.

The equivalence is complete if also the NDRs (41) and (44) correspond to each other. But this is easily seen by insertion of (48) into (41) which then becomes (44).

VI. ENERGY OF HOLE PLASMA SYSTEM

Next, we focus attention on the question which amount of energy is associated with a hole structure. We assume a periodic (cnoidal) wave of period $2L$ and calculate the total energy W per wavelength by making use of the energy law of the Vlasov–Poisson plasma (1)–(3):

$$W=\int_{-L}^{+L}dx\left[\int dv\frac{v^2}{2}f_e+\frac{1}{\theta}\int du\frac{u^2}{2}f_i+\frac{1}{2}(\phi')^2\right], \quad (50)$$

which is the true total energy normalized by $NT_{ef}/2$, where N is the average number of particles per wavelength. If no structure is excited at all ($\psi\equiv 0$), so that the distributions are given simply by the shifted Maxwellians [(4a) and (4b)] we obtain in the limit of no electron drift $v_D=0$:

$$W_{00}=L\left[1+v_0^2+\frac{(1+u_0^2)}{\theta}\right]=L[1+\theta^{-1}+u_0^2(1+\delta)\theta^{-1}], \quad (51)$$

where we used $v_0^2=\delta u_0^2/\theta$. The last term in (51) being proportional to u_0^2 represents the kinetic energy stemming from the remaining drift since we are working in the frame moving with v_0 in the electron and with u_0 in the ion phase space. This term vanishes if we go over to the lab frame in which case W_{00} reduces to $L[1+\theta^{-1}]$.

In the presence of a structure, we have to take into account the full distribution (5a) and (5b) with $K=K_e=k_{0+}^2\psi/2$ and A_e given by (21) (we prefer here the second procedure, given by Sec. IV).

First, we calculate the quantity

$$w_e(\phi):=\int dv\,v^2f_e(v^2-2\phi), \quad (52a)$$

which is twice the electron kinetic energy density. In (52a) we explicitly pointed out the dependence of $f_e(2\epsilon_e)$ with ϵ_e given by (19a). Differentiating $w_e(\phi)$ we find

$$w_e'(\phi)=-2\int dv v^2f_e'(v^2-2\phi)=-\int dv v\frac{d}{dv}f_e=n_e(\phi)$$

and similarly for the ion kinetic expression [with ϵ_i given by (19b)] we get

$$w_i'(\phi)=-\theta n_i(\phi). \quad (52b)$$

Hence, both quantities $w_{e,i}$ can be found by a ϕ integration of the densities

$$w_e(\phi)=\int_0^{\phi}n_e(\phi)d\phi+w_e(0), \quad (53a)$$

$$w_i(\phi)=-\theta\int_0^{\phi}n_i(\phi)d\phi+w_i(0)\equiv\theta\int_{\phi}^{\psi}n_i(\phi)d\phi+w_i(\psi). \quad (53b)$$

Making use of the "energy law" related to Poisson's equation (3),

$$\frac{\phi'(x)^2}{2}=-V(\phi)=\int_0^{\phi}d\phi[n_e(\phi)-n_i(\phi)]=w_e(\phi)+\frac{w_i(\phi)}{\theta}-w_e(0)-\frac{w_i(0)}{\theta},$$

we can rewrite (50) and get

$$W=\frac{1}{2}\int_{-L}^{+L}dx[w_e(\phi)+w_i(\phi)/\theta-2V(\phi)]=L[w_e(0)+w_i(0)/\theta]-\frac{1}{2}\int_{-L}^{+L}dx[V(\phi)+2V(\phi)]\equiv W_0+\frac{3}{2}\int_{-L}^{+L}dx[-V(\phi)], \quad (54)$$

where W_0 is defined by the last equality sign. It furthermore holds

$$w_e(0)=(1+K_e)(1+v_0^2) \tag{55a}$$

and with (53b) we obtain

$$\theta^{-1}w_i(0)=\theta^{-1}(1+A_e)(1+u_0^2)+\int_0^{\psi}n_i(\phi)d\phi, \tag{55b}$$

where the first expression on the right-hand side (RHS) comes from $\theta^{-1}w_i(\psi)$. W_0 in (54) then becomes

$$W_0=L\left[(1+K_e)(1+v_0^2)+\theta^{-1}(1+A_e)(1+u_0^2)+\int_0^{\psi}n_i(\phi)d\phi\right]. \tag{56}$$

Again, in the limit $\psi\to 0$ it reduces to W_{00} and the terms with v_0^2 and u_0^2 disappear if the energy is formulated in the lab frame. The first expression in (56) represents the kinetic electron energy at the position where trapped electrons are absent, i.e., at $\phi=0$, and where f_e reduces to a shifted Maxwellian. The factor $(1+K_e)$ reflects the change in the normalization of (5a) with respect to (4a). Analogously, the first ion term in (56), namely $\theta^{-1}w_i(\psi)$, represents the kinetic ion energy at the position where trapped ions are absent, i.e., at $\phi=\psi$, and where f_i reduces to a shifted Maxwellian, modified by the normalization constant $(1+A_e)$. The last term in (56), as seen from (55b), is the difference of the ion kinetic energy densities between the two states at $\phi=0$ and $\phi=\psi$ and reflects the circumstance that at $\phi=0$ the ion distribution has maximum distortion in the resonant region since the trapped ion velocity range is largest there.

W_0 in (56) refers to $\phi\equiv 0$ and already represents a prepared state. To prepare this state one has to provide an energy per wavelength (in lab frame)

$$\Delta W_0\equiv(W_0-W_{00})=L\left[K_e+\theta^{-1}A_e+\int_0^{\psi}n_i(\phi)d\phi\right]. \tag{57}$$

To this expression one has to add the last term in (54) which is denoted by $\tilde{W}$. It represents the energy due to the spatial variation of $\phi(x)$ and hence corresponds to the wave energy in an ordinary derivation of the energy on the basis of a linearization procedure. Using $\phi'(x)=\pm\sqrt{-2V(\phi)}$ it can be transformed to

$$\tilde{W}=\frac{3}{2}\int_0^{\psi}d\phi\sqrt{-2V(\phi)}. \tag{58}$$

As seen from the second line in (54), one-third of this term stems from the particle (electrostatic) energy and two-thirds from the field (wave) energy, as is true for ordinary linear waves.[37] The full difference in energy per wavelength between the excited state and the unperturbed state ($\psi\to 0$) in lab frame is therefore given by

$$\Delta W=L\left[K_e+\theta^{-1}A_e+\int_0^{\psi}n_i(\phi)d\phi\right]+\tilde{W}, \tag{59a}$$

an expression which holds for arbitrary amplitudes ψ. In the small amplitude limit, $\psi\ll 1$, it becomes by insertion of $K_e=(k_{0+}^2/2)\psi$ and of A_e from (21) and by integration

$$\Delta W=L\psi\left[1+\frac{k_{0+}^2}{2}+\frac{1}{2}Z_r'(u_0/\sqrt{2})+\frac{4}{3}b(\alpha,u_0)\sqrt{\theta\psi}+O(\psi)\right]+\tilde{W}, \tag{59b}$$

where the unity in the brackets comes from the density integral and where u_0 satisfies the NDR (24). By inspection it can be seen that the "wave" term $\tilde{W}$ is $O(\psi^2)$ and therefore negligible. As said before it corresponds to the usual quadratic energy expressions derived in the framework of a linearized treatment.[38–41] Note that in the harmonic wave limit, when $-V(\phi)=(k_{0+}^2/2)\phi(\psi-\phi)$, $\tilde{W}$ becomes $(3\pi/16)k_{0+}\psi^2$.

Equation (59b) tells us that for the excitation of a vortex structure in phase space most of the energy resides in the particle energy needed for the preparation of the underlying ground state (namely that for $\phi\equiv 0$).

We conclude that the phenomenon of trapping forces us to reconsider the standard wave energy expressions resulting in a term that dominates the bilinear contributions.

A further interesting point should be mentioned. Since, in principle, $b(\alpha,u_0)$ can be negative, we might find situations where ΔW is negative, that means that the perturbed state would be energetically lower than the unperturbed one: namely a negative energy state.

To see whether this is possible, let us evaluate the leading term in (59b) further for holes (or humps) propagating with ion sound speed or slower, $u_0\leq O(\sqrt{\theta})$. Since $v_0\leq O(\sqrt{\delta})$ we can replace v_0^2 by zero, neglecting terms of $O(m_e/m_i)$. The NDR (24) with $v_D=0$ reduces in this case to

$$\tfrac{1}{2}Z_r'(u_0/\sqrt{2})=\theta^{-1}(1+k_{0+}^2)-\tfrac{16}{15}\left[\theta^{-1}b(\beta,0)+\tfrac{3}{2}b(\alpha,u_0)\theta^{1/2}\right]\psi^{1/2},$$

which generalizes (13) (note the different use of k_0).

Inserting this expression into (59b), we get

$$\Delta W=L\psi\left[1+\frac{1}{\theta}+\frac{k_{0+}^2}{2}\left(1+\frac{2}{\theta}\right)-\frac{16}{15}\frac{b(\beta,0)}{\theta}\sqrt{\psi}-\frac{4}{15}b(\alpha,u_0)\sqrt{\theta\psi}\right] \tag{59c}$$

and by insertion of k_{0-} instead of k_{0+} from (48) we get

$$\Delta W=L\psi\left[1+\frac{1}{\theta}+\frac{k_{0-}^2}{2}\left(1+\frac{2}{\theta}\right)-\frac{4}{15}\left(1+\frac{6}{\theta}\right)\times b(\beta,0)\sqrt{\psi}+\frac{4}{15}\left(1+\frac{1}{\theta}\right)b(\alpha,u_0)\theta^{3/2}\psi^{1/2}\right]. \tag{59d}$$

Next, we assume like in Sec. III isothermal electrons, $\beta=1$, in which case the electron trapping term vanishes and we obtain

$$\Delta W = L\psi\left[1+\frac{1}{\theta}+\frac{k_{0-}^2}{2}\left(1+\frac{2}{\theta}\right)\right.$$
$$\left.+\frac{4}{15}\left(1+\frac{1}{\theta}\right)b(\alpha,u_0)\theta^{3/2}\psi^{1/2}\right].$$

Inserting S_- from (15) we find

$$\Delta W = L\psi\left[\left(1+\frac{1}{\theta}\right)+\frac{k_{0-}^2}{2}\left(1+\frac{2}{\theta}\right)+\frac{k_{0-}^2}{16}\left(1+\frac{1}{\theta}\right)S_-\right].$$

From S_- we know by Eq. (16) that it lies in the interval $-8\leq S_-\leq\infty$. Hence ΔW must satisfy the inequality

$$\Delta W \geq L\psi\left[\left(1+\frac{1}{\theta}\right)+\frac{k_{0-}^2}{2\theta}\right]>0.$$

At least in this situation (namely for isothermal electrons) a negative energy state is *not* possible.

However, if we admit a notch in the electron distribution corresponding to $-\beta>0$, we see that the electron trapping term in (59d) is negative, since $b(\beta,0)=(1/\sqrt{\pi})(1-\beta)>0$. Furthermore, if b scales like $b>O(\psi^{-1/2})$ this term becomes of order unity and can overcome the positive terms in the brackets of (59d).

A more detailed analysis taking into account the existence conditions of the structure via $V(\phi)$ from (10) rather then from (14) is needed to decide whether this can indeed happen. Physically we expect that this is possible because a notch in f_e principally lowers the kinetic energy of electrons.

The crucial question is, therefore, whether $b\sqrt{\psi}\geq O(1)$ is possible, even if $\psi\ll 1$. To see this explicitly, we perform (and repeat in some sense) in more detail the underlying analysis in the *infinitesimal* amplitude limit, $\psi\rightarrow 0^+$.

VII. THE INFINITESIMAL WAVE LIMIT

A. The harmonic wave limit and a first comparison with linear theory

Let us finally discuss some more details of these wave solutions and make a comparison with the conventional wave theory. We prefer the second procedure and describe the marginal hole solutions by the NDR (24) and the potential (25).

In the limit of negligible terms

$$b(\beta,\tilde{v}_D)\sqrt{\psi}=0,\quad b(\alpha,u_0)\theta^{3/2}\sqrt{\psi}=0, \tag{60}$$

later on called the harmonic wave limit, the RHS of (24) vanishes and we get for a nondrifting electron component ($v_D=0$)

$$k^2-\frac{1}{2}Z_r'(\omega_r/k\sqrt{2})-\frac{\theta}{2}Z_r'(\omega_r\mu/k\sqrt{2})=0, \tag{61}$$

where we replaced k_{0+} by k and v_0 by $v_0=\omega_r/k$, the real phase velocity in the electron frame. The potential $V(\phi)$, on the other hand, becomes

$$-V(\phi)=\frac{k^2}{2}\phi(\psi-\phi), \tag{62}$$

which yields by a quadrature of the "energy law" $\phi'(x)^2/2+V(\phi)=0$,

$$\phi(x)=\frac{\psi}{2}(1+\cos kx) \tag{63}$$

representing a purely harmonic wave which propagates in a stationary manner through the plasma. We contrast this marginal, small amplitude mode with the corresponding linear eigenmode governed by the linear Landau dispersion relation

$$k^2-\frac{1}{2}Z'(\omega/k\sqrt{2})-\frac{\theta}{2}Z'(\omega\mu/k\sqrt{2})=0 \tag{64}$$

valid in the time asymptotic limit of an initial wave problem which can be found in any plasma textbook. The difference to (61) is that ω_r is replaced by $\omega:=\omega_r+i\gamma$, the complex wave frequency, and that the full plasma dispersion function $Z(z)=Z_r(z)+iZ_i(z)$ rather than the real part Z_r is taken into account. A well-known fact is that (64), derived for a thermal plasma, has only damped solutions, the least damped modes being the Langmuir waves with large phase velocities $\omega_r/k\gg 1$ and the ion acoustic waves, assuming $\theta\gg 1$, with phase velocities in the ion acoustic range $\omega_r/k=\sqrt{m_e/m_i}$. For these two modes the Landau damping rate $|\gamma|$ is small, and if these modes exist sufficiently long, they can affect the evolution of the plasma described by a weak nonlinear analysis, as in the weak turbulence regime, taking into account mode coupling, nonlinear wave–particle resonances, parametric interactions, etc. The reason for a weak Landau damping is that $|\partial_v f_0|$ is small for these two linear modes. There are many other complex solutions ω of (64) but all of them represent strongly damped "waves" which die out due to phase mixing long before any influence has taken place.

Equation (61), on the other hand, which is the real part of (64) and involves only real arguments, has solutions in the thermal range where $|\partial_v f_0|$ is no longer small.

Let us for demonstration concentrate in the following on periodic electron hole equilibria propagating at electron thermal velocity in which case the last term in (61) vanishes (see also Sec. IV B). In the long wave length limit $k^2\ll 1$ a solution of (61) [or (33)] is given by [see (35)]

$$\omega_r=1.307k[1+k^2], \tag{65}$$

representing a true solution of the dispersion relation (61). Hence, this marginal wave solution emerges under the constraints (60). One may argue that these constraints can always be achieved by assuming the infinitesimal wave limit, $\psi\rightarrow 0$, in which case (61) would be nothing else but Vlasov's dispersion relation[8] (where the Landau contour in the definition of Z is replaced by the principal value) or a special case of the van Kampen continuum (namely $\lambda=0$) given by the van Kampen relation

$$k^2-\tfrac{1}{2}Z_r'(\omega_r/k\sqrt{2})=\lambda(k,\omega_r/k). \tag{66}$$

The latter results from a solution of the linearized Vlasov equation for the perturbed distribution f_1 in an extended class involving singular distributions

$$f_1=\left[P\frac{-f_0'(v)}{v-\omega_r/k}-\lambda\,\delta(v-\omega_r/k)\right]\phi, \tag{67}$$

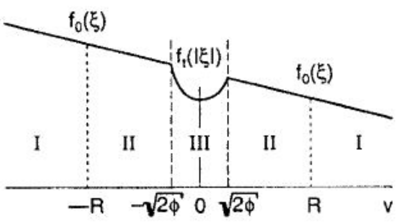

FIG. 2. The distribution function (68) as a function of the velocity v. The trapped range is denoted by III, and the range II (I) refers to untrapped resonant (nonresonant) particles. The quantity ξ is given by (69).

where P stands for the principal value.

This is, however, not the complete story, since at least in van Kampen's theory $\lambda \neq 0$ is admitted. To approve a relationship also in this case let us show how the first constraint in (60) can be lifted even in the infinitesimal wave limit.

B. Derivation of density in a more general context

To understand the appearance and consequence of a nonvanishing term $\frac{16}{15}b(\beta,v_0)\sqrt{\psi}$ in (24), which can appear as we will see even in the infinitesimal wave limit, let us shortly repeat the derivation of the electron density expression (20a)[24,34] in this limit. We first allow for arbitrary functions and then take the Maxwellian case limit. A solution of the time-independent Vlasov equation is given, like Eq. (5a), by

$$f_e(x,v)=(1+K_e)\begin{cases} f_0(\xi), & v^2 \geqslant 2\phi \\ f_t(|\xi|), & v^2 \leqslant 2\phi, \end{cases} \tag{68}$$

where we defined

$$\xi := sgn v |v^2 - 2\phi|^{1/2} \tag{69}$$

and where $f_e(x,v)$ is assumed to be continuous at the separatrix $v^2=2\phi$ and $f_0(v)$ to be normalized to unit density. In the limit of a vanishing perturbation, $\psi=0$, we have $f_e(x,v) \to f_0(v)$ which represents the unperturbed distribution function.

A central role in the density expression is played by the resonance region, which consists of two parts (see Fig. 2)

$\sqrt{2\phi} < |v| \leqslant R$ region II,

$|v| \leqslant \sqrt{2\phi}$ region III,

where R is small, say $R \geqslant O(\sqrt{2\psi})$. In region II, which may be termed the free resonant region, the following Taylor expansion holds

$$f_0(\xi)=f_0(0)+\xi f_0'(0)+\frac{\xi^2}{2}f_0''(0)+\cdots, \tag{70a}$$

whereas in the nonresonant region I, $R \leqslant |v|$, the expansion

$$f_0(\xi)=f_0(v)-\phi\frac{1}{v}f_0'(v)+\cdots \tag{70b}$$

is meaningful. [Note that the correct size of R is of less importance, as we could equally well choose $R \geqslant O(\psi^{1/4})$, and that the following expressions can be obtained also by integrating first over the full nonlinear expressions and then take the small ψ limit.[22]] In the trapped region III, the expansion

$$f_t(|\xi|)=f_t(0)+\frac{\xi^2}{2}f_t''(0)+\cdots \tag{70c}$$

holds, assuming that the term $f_t'(0)$ vanishes. We, however, can equally well use the algebraic equation (70c) as the full trapped electron distribution in which case $f_t''(0)$ would not be subject to any restriction (except, perhaps, the mild condition that f_t must be positive in III). With this we can find the density as follows [we drop for the next three formulas the factor $(1+K_e)$]:

$$n_e(\phi)=\int_{\mathrm{I}} f_0(\xi)dv+\int_{\mathrm{II}} f_0(\xi)dv+\int_{\mathrm{III}} f_t(|\xi|)dv.$$

Inserting (70b) in the first and (70a) in the second integral and using for f_t an expansion similar to (70a) in the third integral we obtain

$$n_e(\phi)=\int_{\mathrm{I}}\left[f_0(v)-\frac{1}{v}f_0'(v)\phi\right]dv+\int_{\mathrm{II}}\left[f_0(0)+\frac{1}{2}f_0''(0)\xi^2\right]dv+\int_{\mathrm{III}}\left[f_t(0)+\frac{1}{2}f_t''(0)\xi^2\right]dv,$$

where the terms linear in ξ dropped out because of symmetry reasons. Extending the first integral to $v=0$ and subtracting the overdue terms in II and III we get

$$\begin{aligned} n_e(\phi)=&P\int_{-\infty}^{+\infty}\left[f_0(v)-\phi\frac{1}{v}f_0'(v)\right]dv \\ &+\int_{\mathrm{II}}\left[f_0(0)+\frac{1}{2}f_0''(0)\xi^2-f_0(v)+\phi\frac{1}{v}f_0'(v)\right]dv \\ &+\int_{\mathrm{III}}\left[f_t(0)+\frac{1}{2}f_t''(0)\xi^2-f_0(v)+\phi P\frac{1}{v}f_0'(v)\right]dv. \end{aligned}$$

Due to the continuity of f_e at $v^2=2\phi$, several terms of the second integral cancel with terms of the third integral and we finally get[24]

$$n_e(\phi)=(1+K_e)\times\left[1-P\int\frac{f_0'(v)}{v}dv\,\phi+\frac{4\sqrt{2}}{3}[f_0''(0)+f_t''(0)]\phi^{3/2}\right]. \tag{71}$$

[A more general expression can be found in Ref. 34, Appendix A, Eq. (A6).] We see that due to the last term in (71) the density in the infinitesimal wave limit cannot be specified by the unperturbed distribution f_0 alone. Doing so, part of the solution has already been thrown away. Moreover, $f_t''(0)$ can be large and dominate the expression $f_0''(0)$, stemming from the unperturbed distribution, without being in conflict with the expansion scheme, e.g., when $|f_t''(0)| \approx (\psi^{-1/2})$. In order to get (20a) we merely have to set ($v_D=0$)

$$f_0(v)=\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}(v+v_0)^2\right], \tag{72a}$$

$$f_r(\xi)=\frac{1}{\sqrt{2\pi}}\exp(-v_0^2/2)\left(1+\frac{\beta}{2}\xi^2\right), \tag{72b}$$

where (72b) coincides with the second expression of (5a) in the small amplitude limit. With this and using

$$Z_r(x)=\frac{1}{\sqrt{\pi}}P\int dt\,\frac{\exp(-t^2)}{t-x} \tag{73}$$

we finally get

$$n_e(\phi)=(1+K_e)\left[1-\frac{1}{2}Z_r'(v_0/\sqrt{2})\phi+\frac{4}{3\sqrt{\pi}}(v_0^2-1+\beta)e^{-v_0^2/2}\phi^{3/2}+\cdots\right], \tag{74}$$

which agrees with (20a).

C. Nonlinear cnoidal waves as a specific superposition of linear van Kampen modes

Let us discuss next the corresponding NDR and potential $V(\phi)$,

$$k^2-\frac{1}{2}Z_r'(v_0/\sqrt{2})=\frac{16}{15\sqrt{\pi}}(1-v_0^2-\beta)e^{-v_0^2/2}\sqrt{\psi}\equiv k^2S, \tag{75}$$

$$-V(\phi)=\frac{k^2}{2}\phi(\psi-\phi)+\frac{8}{15}b(\beta,v_0)\phi^2(\sqrt{\psi}-\sqrt{\phi})$$
$$=\frac{k^2\psi^2}{2}[\varphi(1-\varphi)+S\varphi^2(1-\sqrt{\varphi})], \tag{76}$$

where $\varphi:=\phi/\psi$ and S is defined in (75) and corresponds in the appropriate limit to the steepening or anharmonicity parameter $S_+/4$ given in (31). The use of k instead of k_{0+} will be justified later in the small S limit.

In Fig. 3(a) we compare the NDR (75), denoted by N, with the van Kampen relation (66) denoted by L. Shown is the limit $\lambda=0=S$ and the solution for small k given by $\omega_r^0/k=1.307[1+k^2]\equiv\sqrt{2}x^0$ which lies in the thermal range of the electron distribution function, as shown in Fig. 3(b), 3(c) for $\lambda=0$ and $S=0$, respectively. If λ and S become positive (negative) and increase (decrease) the phase velocity decreases (increases).

To see the relationship between our and van Kampen's approach, also in the nonlinear case of $S\neq0$ and $\lambda\neq0$ let us first present as an example the solution for the electrostatic potential for the case $0\leq S<1/4$ by integrating the "energy law" $\phi'^2(x)/2+V(\phi)\equiv0$, using (76). The solution becomes [Ref. 34, (3.28), (3.25)]

$$\phi(x)=\psi[1-K_-sn^2(u|m)]^2 \tag{77}$$

with

$$K_\mp=1+[1\mp\sqrt{1-4S}]/2S,$$
$$m=K_-/K_+,\quad u=\sqrt{SK_+}kx/4, \tag{78}$$

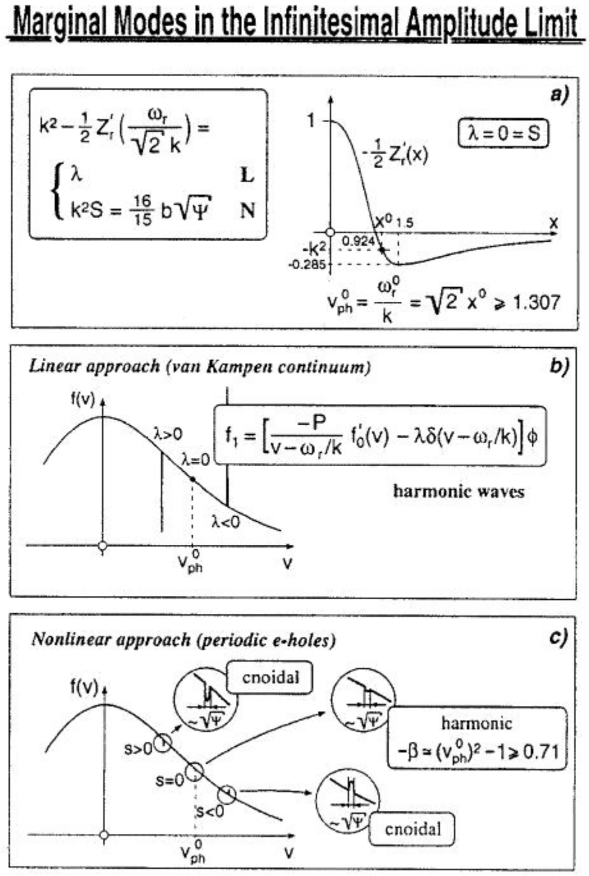

FIG. 3. The two cases of marginal modes in the infinitesimal amplitude limit. (a) The function $-\frac{1}{2}Z_r'(x)$ as a function of x for $\lambda=0=S$; the inset shows the dispersion relation for van Kampen modes (L) and for electron holes (N), respectively, its solution being given by v_{ph}^0. (b) The distribution function $f=f_0+f_1$ for harmonic van Kampen modes as a function of v for three values of λ; the inset shows the perturbation f_1 analytically. (c) The distribution function (68) for anharmonic e-hole equilibria as a function of v; the insets at three values of S show on an enlarged scale that f is well-behaved; for the harmonic case $S=0$ the expression of $-\beta$ is given analytically in a further inset.

where sn is a Jacobian elliptic function. It represents a cnoidal wave involving a descending, albeit infinite series of harmonics. This already holds in the limit of small S as shown next. Expanding the above-mentioned expressions with respect to small S, $0\leq S\ll1$, we get up to $O(S)$

$$K_-=2+S,\quad K_+=1/S,\quad m=2S,\quad u=kx/4. \tag{79}$$

Furthermore using the expansion of $\mathrm{sn}(u|m)$ for small m (Ref. 42, 16.13.1)

$$\mathrm{sn}(u|m)=\sin u-\frac{m}{4}(u-\sin u\cos u)\cos u \tag{80}$$

we obtain

$$\phi(x)=\psi\left\{\frac{1+\cos kx}{2}+S\left[\frac{1}{2}\cos\frac{kx}{2}\times\left(\cos kx+2\cos\frac{kx}{2}-3\right)+\frac{1}{4}kx\sin kx\right]\right\}. \quad (81)$$

For $S=0$ we get as expected the harmonic wave solution (63). Whereas the first expression in $O(S)$ involves a finite number of half harmonics $k/2$, the last expression corresponds to a slowly decreasing alternating *infinite* series of higher harmonics $k_n:=nk;n=0,\pm 1,\pm 2,\ldots$. The latter is true because of

$$kx\sin kx=1-\frac{1}{2}\cos kx-2\left[\frac{\cos 2kx}{2^2-1}-\frac{\cos 3kx}{3^2-1}+\ldots(-1)^n\frac{\cos nkx}{n^2-1}+\cdots\right]. \quad (82)$$

This implies that already in the simplest and weakest nonlinear solution (i.e., when $0<S\ll 1$) *all* harmonics are involved, a peculiarity of hole equilibria which is absent for standard weak nonlinear wave theories. This effect is even more apparent if we consider the electron density, $n_e(x)=\phi''(x)+1$, in the expansion of which the nth harmonic is multiplied by a factor n^2. Hence, the spectrum is extremely slowly decreasing with n, the convergence being guaranteed by the alternating sign of the series only. The consequences can immediately be seen.

First, we realize from expression (81), which is symmetric in x, that ϕ becomes zero at $x=\pm\pi/k$, so that the actual wave number is indeed given by $k=2\pi/\lambda$ where λ is the wavelength now. In this small S limit there is hence no need to distinguish between k_{0+} and k, the difference being of $O(S^2)$.

It then follows that (81), formulated in the lab frame now, can be written as an infinite series of harmonic van Kampen modes

$$\phi(x-v_0t)=\sum_n C_n\exp[i(k_nx-\omega_nt)], \quad (83)$$

each component satisfying the van Kampen relation (66),

$$k_n^2-\frac{1}{2}Z_r'(\omega_n/k_n\sqrt{2})=(n^2-1+S)k^2 \equiv\lambda_n\left(k_n,\frac{\omega_n}{k_n};k,\beta,\psi\right). \quad (84)$$

Hence, to each harmonic $k_n=nk$ we can find in the van Kampen continuum a wave frequency $\omega_n:=k_nv_0$ and a corresponding parameter λ_n such that all selected modes have the same phase velocity $\omega_n/k_n=v_0$. According to (84) the van Kampen parameters λ_n have to be chosen in a very specific manner. They depend on β and ψ through S, which is given by (75), and hence reflect the status of trapping and nonlinearity.

When S becomes of order unity or larger, namely when $|\beta|\geq O(\psi^{-1/2})$, k in (75) has to be replaced by k_{0+} which is a functional[34] of k and λ_n in (84) is then given by $\lambda_n=n^2k^2+(S-1)k_{0+}^2(k)$.

It then follows that only by a sophisticated superposition of an infinite number of linear van Kampen modes, a nonlinear exact solution of the Vlasov–Poisson system can be achieved, which is true already in the weakest nonlinear regime, $0<S\ll 1$.

A single van Kampen mode and all other superpositions cannot be considered as a proper solution of the full nonlinear system, even in the infinitesimal amplitude limit.[43] Although an arbitrary phase velocity can be obtained by selecting λ appropriately [Fig. 3(b)], a smooth and from the perturbation theory permitted small perturbation of the distribution function can only be obtained by going into the nonlinear regime, as it holds for electron holes and related structures [Fig. 3(c)].

There exists only one exceptional case where a van Kampen mode comes closer to a real solution of the full nonlinear system in the $\psi\to 0^+$ limit, namely when $\lambda=0$ in which case the δ-function contribution vanishes. This is, however, nothing else but the harmonic solution of Vlasov's linearized dispersion relation. It also corresponds to $S\to 0$ in the present context. There are yet two different ways to perform this latter limit. One way, we may call the linear one, is to take a finite β and let $\psi\to 0^+$. Another one, we may call the nonlinear regularized one, is to select β appropriately to let b in S be zero already before the $\psi\to 0^+$ limit is taken [see the inset of Fig. 3(c)]. Vlasov's interpretation of the resonance singularity in terms of the principal value is hence justified *a posteriori* and best by the present nonlinear procedure and others,[43] as indicated in Sec. I.

An electron hole with its tiny, seed-like distortion of the distribution function in the trapped range [see Figs. 2 and 3(c)] does not experience Landau damping, as we know already from numerical simulations [see, e.g., Fig. 2(c) of Ref. 34]. Landau theory, being a linearized, time-asymptotic concept, is simply not applicable here. In the framework of specifically superimposed van Kampen modes, phase mixing and hence Landau damping is absent because of their common phase velocity $v_0=\omega_r/k$.

We hence state that a regularized solution with $-\beta\simeq S\psi^{-1/2}$ [see (75)], $-2\leq S\leq\infty$, will remain *nonlinear*, no matter how small ψ is taken. This is in contrast with the usual assumption made in any conventional theory, namely that waves can be described by *linear* equations in the infinitesimal amplitude limit.

It is the presence of the wave–particle resonance that renders the treatment intrinsically nonlinear meaning that the linear wave spectrum generally cannot be used for getting solutions in this domain by superposition. Only by a very delicate superposition of infinitely many modes, a nonlinear solution could be constructed. A description of the Vlasov–Poisson system is therefore incomplete unless hole (and hump) solutions are included as new members in the wave spectrum at least. Van Kampen modes are merely solutions of a truncated system and are hence generally not suited to describe proper solutions of the full nonlinear system.

VIII. SUMMARY AND CONCLUSIONS

In the present paper the problem of wave–particle resonance has been treated rigorously in the weak amplitude formulation. Stationary BGK-like traveling waves have been constructed characterized by a nonlinear dispersion relation giving the phase velocity of the perturbation and a "classical" potential representing the spectral decomposition of the wave. We have evaluated the basic properties of these solutions in various velocity regimes and their dependence on the anharmonicity parameter such as (15) or (31) which comprises the effects of trapping, amplitude, temperature, and periodicity. Two different approaches were shown to be equivalent and a unified description encompassing both approaches has been presented. A new energy expression of a hole carrying plasma has been derived which essentially differs from that relying on linearized treatments. Whereas the latter are quadratic in the amplitude ψ the present expression appears earlier in an expansion scheme in ψ and hence dominates. It is due to the excitation of the "ground state" that involves trapped particles. This will add a novel component to the controversial discussion as how to define the energy of a Vlasov–Poisson plasma appropriately.[44,45] The possibility of having negative energy states is mentioned also. The question as to how linear van Kampen theory and the present exact nonlinear wave theory are interrelated has been answered by taking the infinitesimal wave limit. It was shown that there is some connection between both wave solutions but that this relationship is more sophisticated than expected (and described in the literature). Already the mildest anharmonically distorted wave solution involves a superposition of an infinite number of van Kampen modes, each mode with mode number n being associated with a specific parameter λ_n in front of the delta function ansatz for the distribution function in van Kampen's theory. The way λ_n is determined by the harmonic wave number, phase velocity, status of trapped particles, and wave amplitude has been shown explicitly. We conclude that statements like that found in Refs. 15 and 46, saying that a BGK wave becomes a van Kampen mode in the infinitesimal wave limit, hence must be treated with caution. Only in the above sense, i.e., by a specific superposition of an infinite number of van Kampen modes a proper solution of the full nonlinear Vlasov–Poisson system can be obtained, remaining valid in the infinitesimal amplitude limit.

We emphasize that these solutions could be obtained only by choosing the potential[22] rather than the BGK[11] method for construction. With it we could from the beginning select "physical" distributions and formulate the necessary conditions in phase space rather than in real space. This allows a much broader class of distributions including notches or humps in the resonant region which are topologically different from distributions that are monotonic in the energy, as often used. In this sense, the ordinary BGK method[11] is ill-posed because it does not allow the incorporation of the full information of a phase space distribution function for this entirely kinetic phenomenon. It is almost impossible to guess within the BGK method the correct form of $\phi(x)$ together with the correct phase velocity to come out with well-behaved hole distributions.

In this paper we have treated weak hole solutions only. The extension to finite amplitudes is possible but requires a numerical evaluation of the corresponding equations. Reviews about finite amplitude hole[47] and double layer[48] solutions can be found in Refs. 25, 32 and 36. Another extension is the admittance of weak collisions. In a two component, current-carrying plasma it could be shown[34] that electron hole solutions do survive weak collisions in a modified form provided that ion mobility is taken into account. That opens, as we believe, new horizons for wave theories and anomalous transport descriptions[49,50] inclusively multidimensional, magnetic features.[51] This point definitely deserves and will attract further attention in the near future due to the omnipresence of more or less collisionless plasmas in fusion and space research as well as due to the improved diagnostics,[52] allowing for the first time the experimental approach to kinetic structures without manipulating (or destroying) them.

And finally we mention that the present approach should be applicable to other types of resonances and associated continuous wave spectra too, such as the Rayleigh spectrum[53] for an inviscid, shearing fluid[54] or the spectrum of ideal Alfvén waves propagating in inhomogeneous magnetic fields.[55]

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