The Abdus Salam
International Centre for Theoretical Physics

United Nations
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SMR 1673/27

AUTUMN COLLEGE ON PLASMA PHYSICS

5 - 30 September 2005

Nonlinear Wave Dynamics in Nonlocal Media

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Nonlinear Wave Dynamics in Nonlocal Media

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Contents

RISØ

- Nonlocal nonlinear media; examples
- Modulational instability of plane waves; NL suppress MI
- Interaction of solitons; dark solitons
- Wave collapse/self-focusing; NL prevents blow-up
- Conclusions

Nonlocal nonlinear interaction

- Spatial nonlocality: response in a point depends on the intensity in its vicinity
- Nonlocality results from a transport process: atom diffusion, heat transfer, drift of charges.
- Generic feature in many nonlinear systems: plasma, atom gasses, liquid crystals, Bose-Einstein condensates
- Stationary solutions in quadratic nonlinear ($\chi^{(2)}$) media are equivalent to stationary solutions in nonlocal Kerr-media

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Model equations

Optical beam propagating along the z-axis of a nonlinear medium;
$\psi(\vec{r}, z)$ is the slowly varying amplitude

$$E(\vec{r}, z) = \psi(\vec{r}, z) \exp(iKz - i\Omega t) + c.c.$$

Model equations

$$E(\vec{r}, z) = \psi(\vec{r}, z) \exp(iKz - i\Omega t) + c.c.$$

Refractive index change $N(I)$; $I(\vec{r}, z) = |\psi(\vec{r}, z)|^2$

$$N(I) = s \int R(\vec{\xi} - \vec{r}) I(\vec{\xi}, z) d\vec{\xi},$$

$s = 1$ ($s = -1$): focusing (defocusing) nonlinearity

$R(\vec{r}) = R(r)$, $r = |\vec{r}|$ real, localized: $\int R(\vec{r}) d\vec{r} = 1$.

Model equations

$$E(\vec{r}, z) = \psi(\vec{r}, z)\exp(iKz - i\Omega t) + c.c.$$

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Nonlocal nonlinear Schrödinger (NLS) equation

$$i\partial_z\psi + \frac{1}{2}\nabla_\perp^2\psi + N(I)\psi = 0,$$

Model equations

$$E(\vec{r}, z) = \psi(\vec{r}, z)\exp(iKz - i\Omega t) + c.c.$$

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$R(\vec{r}) = R(r)$, $r = |\vec{r}|$ real, localized: $\int R(\vec{r})d\vec{r} = 1$.

$$i\partial_z\psi + \frac{1}{2}\nabla_\perp^2\psi + N(I)\psi = 0,$$

Singular response, $R(r) = \delta(|\vec{r}|)$, : $N(I) = sI(\vec{r}, z)$: ,

$$i\partial_z\psi + \frac{1}{2}\nabla_\perp^2\psi + s\psi|\psi|^2 = 0$$

$s = 1$ ($s = -1$): focusing (defocusing) nonlinearity

Degrees of nonlocality

Weakly nonlocal:

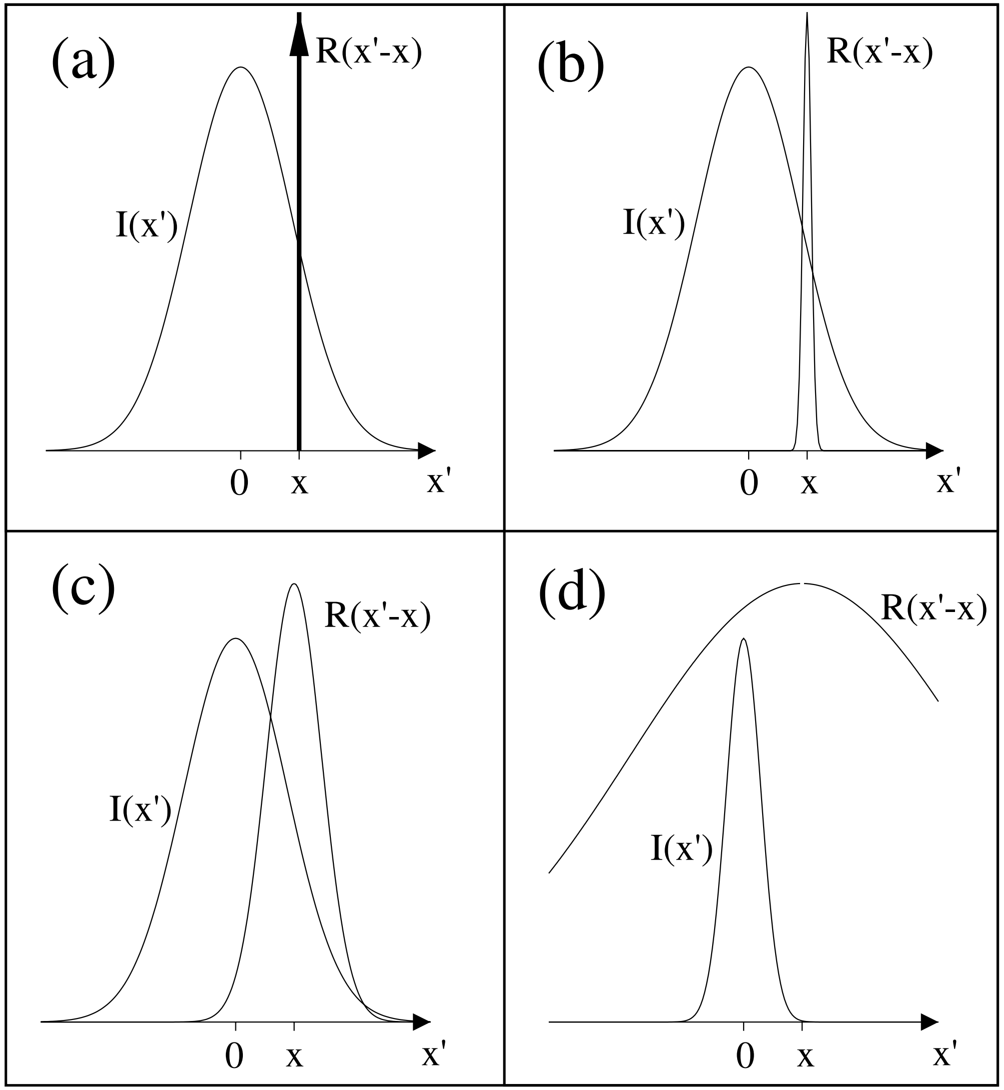
$N(I) = I + \gamma \nabla_\perp^2 I,$

$\gamma = \frac{1}{2} \int r^2 R(r) d\vec{r}$

Strongly nonlocal:

$N(I) = sR(\mathbf{r})P,$

$P = \int I d\vec{r}$



Modulational Instability

Plane wave

$$\psi(\vec{r},z) = \sqrt{\rho_0}\,\exp(i\vec{k_0}\cdot\vec{r} - i\beta z), \quad \rho_0 > 0, \quad \beta = \frac{1}{2}k_0^2 - s\rho_0,$$

Modulational Instability

$$\psi(\vec{r}, z) = \sqrt{\rho_0}\exp(i\vec{k_0}\cdot\vec{r} - i\beta z), \quad \rho_0 > 0, \quad \beta = \frac{1}{2}k_0^2 - s\rho_0,$$

Perturbation to the plane wave solution

$$\psi(\mathbf{r}, z) = [\sqrt{\rho_0} + a_1(\vec{r}, z) + cc]\exp(i(\vec{k_0}\cdot\vec{r} - \beta z)),$$

$$a_1(\vec{r}, z) = \int \widetilde{a}_1(\vec{k}) exp(i\vec{k}\cdot\vec{r} + \lambda z) d\vec{k}$$

Modulational Instability

$$\psi(\vec{r}, z) = \sqrt{\rho_0}\,\exp(i\vec{k_0}\cdot\vec{r} - i\beta z), \quad \rho_0 > 0, \quad \beta = \frac{1}{2}k_0^2 - s\rho_0,$$

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Dispersion relation

$$\lambda^2 = -k^2\rho_0\left[\frac{k^2}{4\rho_0} - s\widehat{R}(\vec{k})\right],$$

Modulational Instability

$$\psi(\vec{r},z)=\sqrt{\rho_0}\exp(i\vec{k_0}\cdot\vec{r}-i\beta z),\quad \rho_0>0,\quad \beta=\frac{1}{2}k_0^2-s\rho_0,$$

$$\psi(\mathbf{r},z)=[\sqrt{\rho_0}+a_1(\vec{r},z)+cc]\exp(i(\vec{k_0}\cdot\vec{r}-\beta z)),$$

$$a_1(\vec{r},z)=\int\widetilde{a}_1(\vec{k})exp(i\vec{k}\cdot\vec{r}+\lambda z)d\vec{k}$$

$$\lambda^2=-k^2\rho_0\left[\frac{k^2}{4\rho_0}-s\widehat{R}(\vec{k})\right],$$

$k=|\vec{k}|$ spatial frequency, $\widehat{R}(\vec{k})=\int R(\vec{r})\exp(i\vec{k}\cdot\vec{r})d\vec{r}$ Fourier spectrum of $R(r)$.

Modulational Instability

$$\psi(\vec{r}, z) = \sqrt{\rho_0}\exp(i\vec{k_0}\cdot\vec{r} - i\beta z), \quad \rho_0 > 0, \quad \beta = \frac{1}{2}k_0^2 - s\rho_0,$$

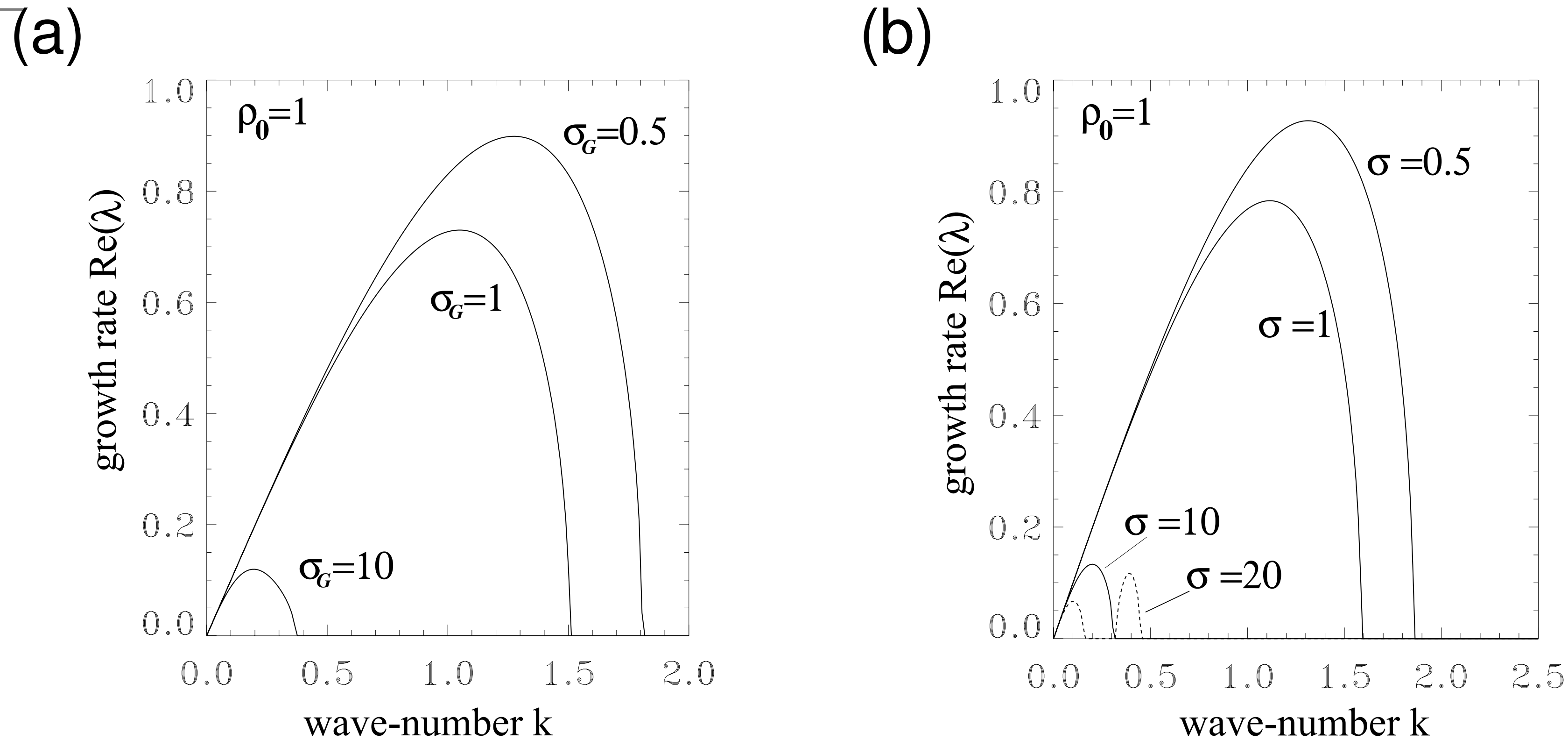
$$\psi(\mathbf{r}, z) = [\sqrt{\rho_0} + a_1(\vec{r}, z) + cc]\exp(i(\vec{k_0}\cdot\vec{r} - \beta z)),$$

$$a_1(\vec{r}, z) = \int \widetilde{a}_1(\vec{k}) exp(i\vec{k}\cdot\vec{r} + \lambda z)d\vec{k}$$

$$\lambda^2 = -k^2\rho_0\left[\frac{k^2}{4\rho_0} - s\widehat{R}(\vec{k})\right],$$

$\widehat{R}(\vec{k})$ positive definite: $s = 1$ MI; $s = -1$ stability
$\widehat{R}(\vec{k})$ sign indefinite: $s = 1$ MI; $s = -1$ possibility for MI!

MI in focusing media



MI gain for $\rho_0 = 1$, $s = 1$

(a) Gaussian: $R(x) = \frac{1}{\sigma\sqrt{\pi}} \exp\left[-\frac{x^2}{\sigma^2}\right]$ with $\widehat{R}(k) = \exp\left[-\frac{1}{4}\sigma^2 k^2\right]$

(b) Rectangular: $R(x) = \frac{1}{2\sigma}$ for $|x| \leq \sigma$ and $R(x) = 0$ for $|x| > \sigma$

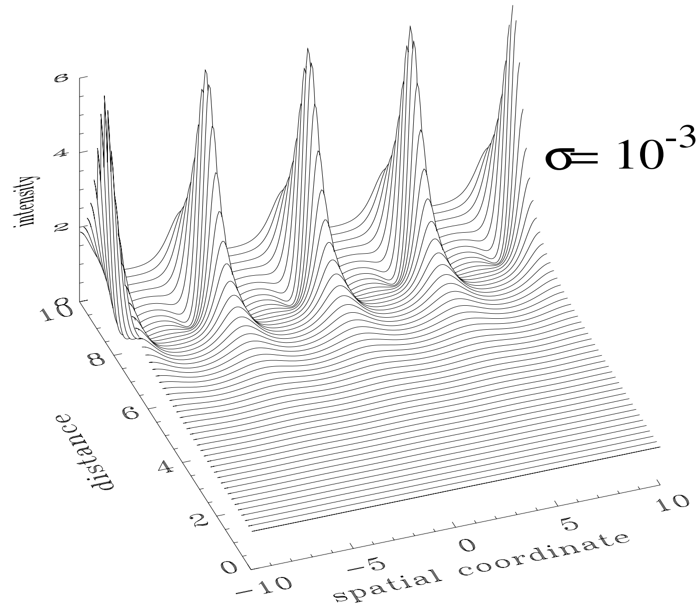
$\widehat{R}(k) = \frac{\sin(k\sigma)}{k\sigma}$

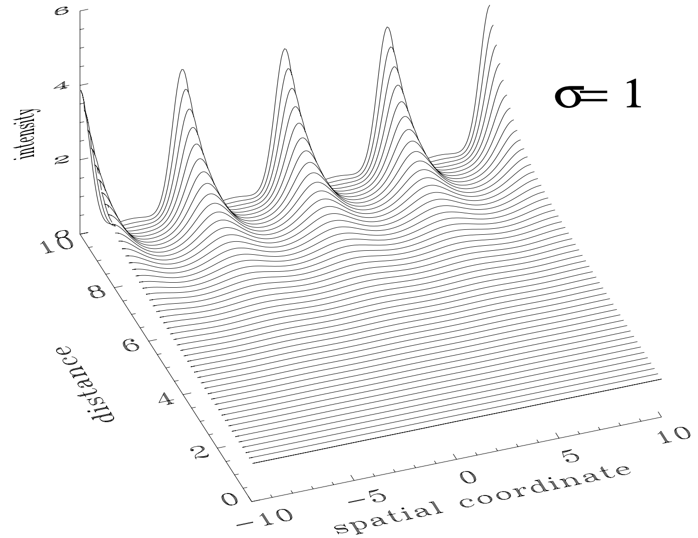
MI in focusing media

MI evolution $\rho_0 = 1,\ s = 1$
Gaussian response profiles:

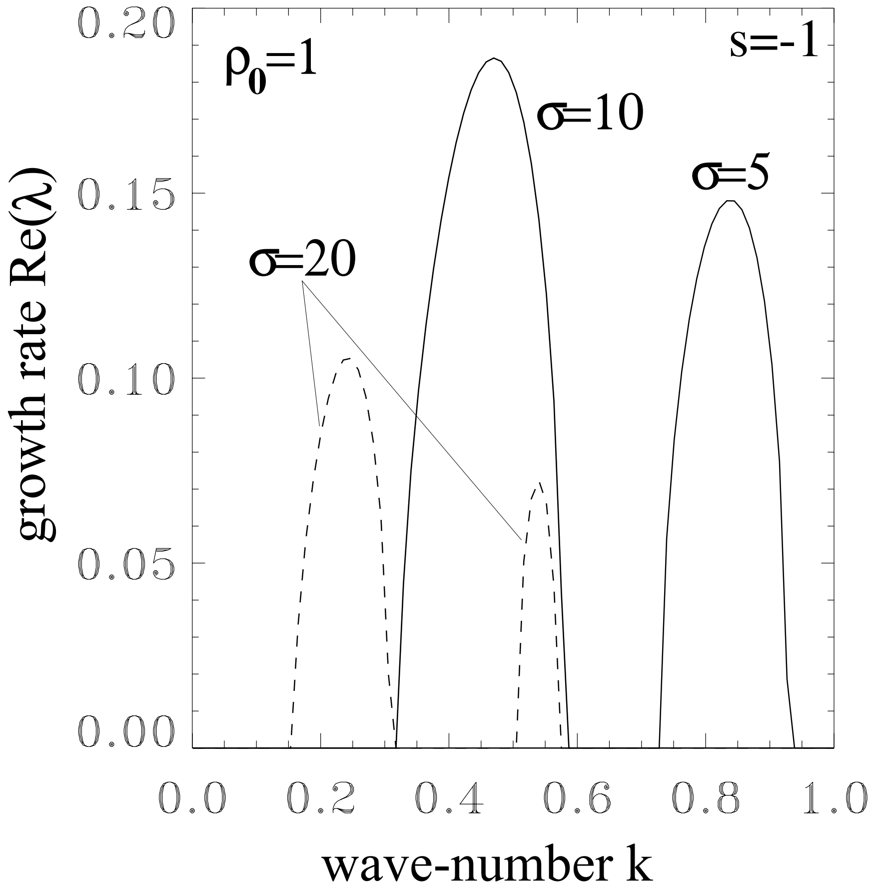
$$R(x) = \frac{1}{\sigma\sqrt{\pi}} \exp\left[-\frac{x^2}{\sigma^2}\right]$$

Experimental verification
Peccianti et al.
PRE **68** 025602 (2003)





MI in de-focusing media

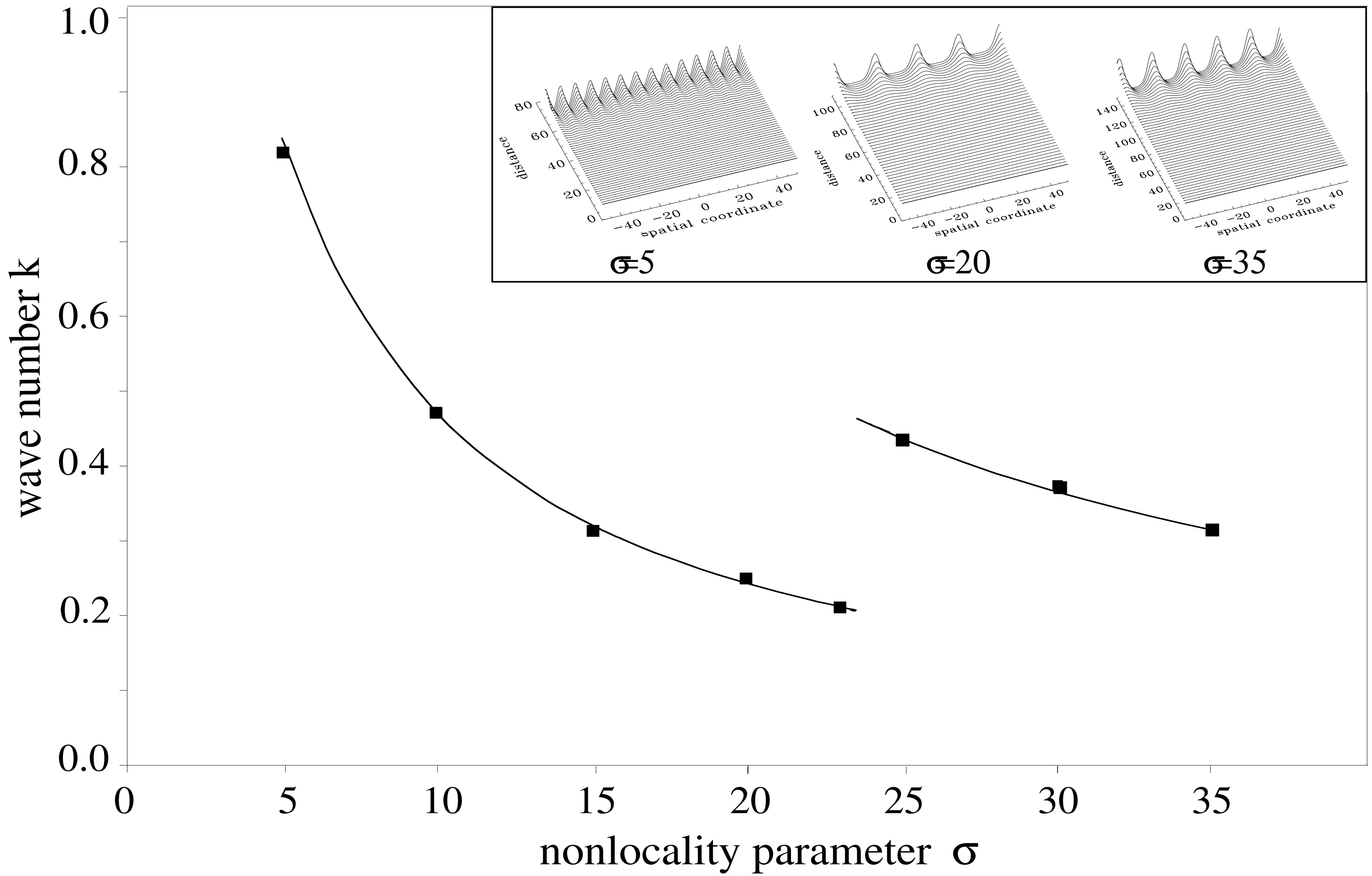


$$R(x) = \begin{cases} \frac{1}{2\sigma};\ |x| \le \sigma \\ 0;\ |x| > \sigma \end{cases}$$

$$\widehat{R}(k) = \frac{\sin(k\sigma)}{k\sigma}$$

MI in de-focusing nonlocal media with rectangular response function

MI in de-focusing media



Wave number for maximum gain versus σ. $s\rho_0 = -1$ and rectangular response function.

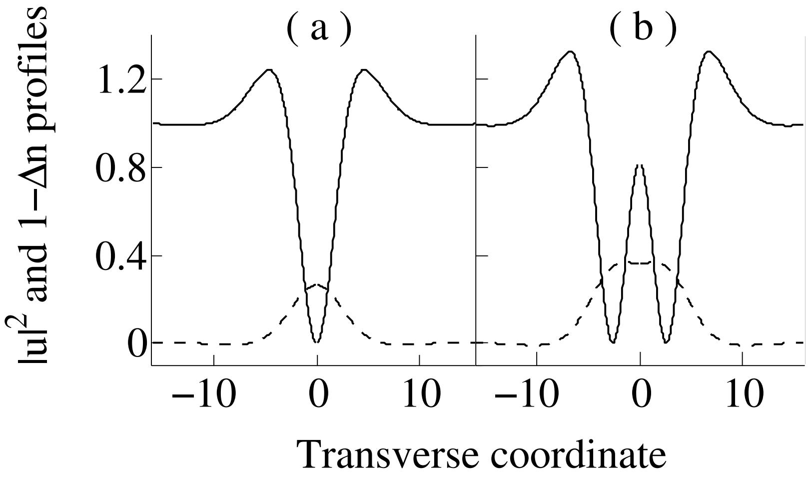
Dark solitons: interaction

$$i\partial_z u + \partial_x^2 u + \Delta n u = 0,$$

$$\Delta n(I) = -\int_{-\infty}^{\infty} R(x-\tau)I(\tau)d\tau,$$

$R(x) = (2\sigma)^{-1}\exp(-|x|/\sigma)$:

$$\Delta n - \sigma^2\partial_x^2\Delta n = -|u|^2,$$



Dark solitons:

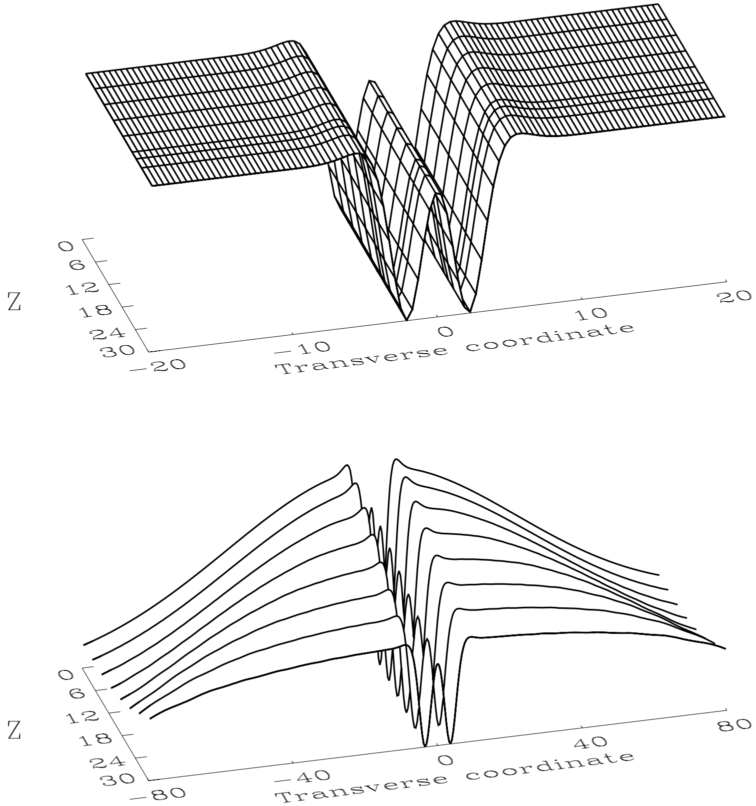
$u(x,z) = u(x)\exp(i\lambda z)$

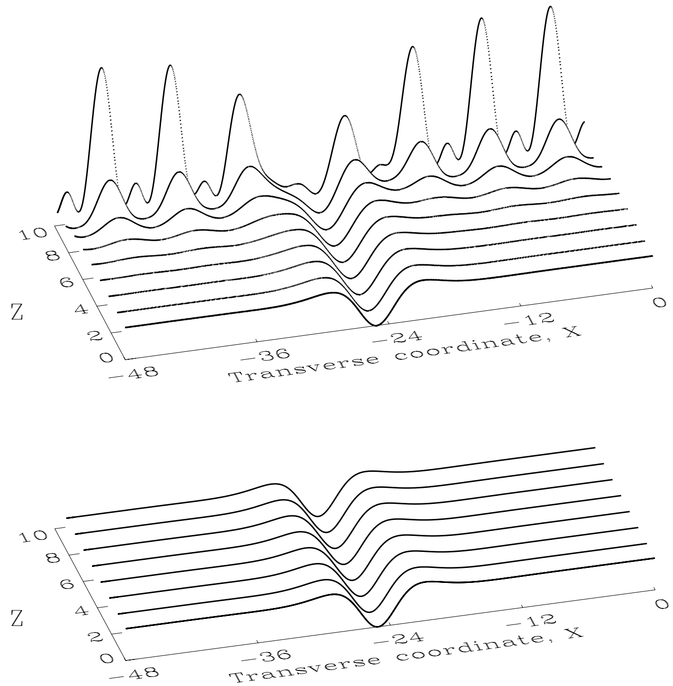
$u(0)$=0, π phase jump at $x = 0$

Equivalent $\chi^{(2)}$-solitons

Single dark soliton (a) and bound state (b). Nonlocal effects smear out Δn, creating an effective wavegui-de.

Dark solitons: interaction

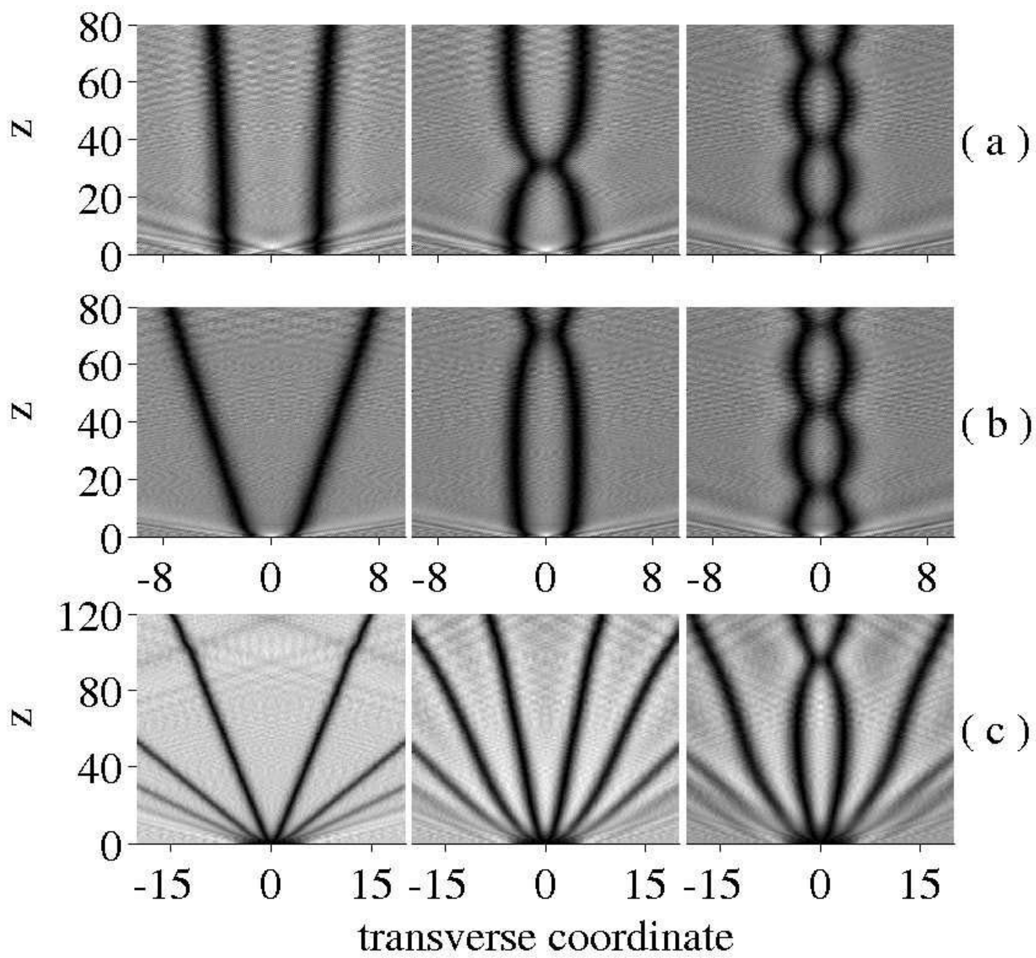




Evolution of bound state of two nonlocal solitons
Separation $\Delta x \approx 2\sigma$

Evolution of dark soliton in $\chi^{(2)}$-media and in nonlocal media

Dark solitons: interaction



Interaction of dark nonlocal solitons. a) Phase jump $= \pi$, $\sigma = 2$ and x_0 = 5.5, 4.0, 2.5 b) Phase jump $= 0.95\pi$, $\sigma =$ 0.1, 1.0, 2.0, and x_0 = 2.5. c) Intensity gap width 7.5 and $\sigma =$ 0.1, 3.0, 6.0. Critical separation $x_{0c} \approx 2\sigma$

Wave Collapse I

Local NLS: Collapse for $D \geq 2$

C. Sulem and P. Sulem, *The Nonlinear Schrödinger Equation*, Springer-Verlag, Berlin (1999)

$D = 2$: necessary and sufficient condition: Collapse (blow-up) for $P > P_{sol}$

$D = 3$: Sufficient condition $H < 0$
Kuznetsov *et al* Physica D **87**, 273 (1995)

Wave Collapse I

Nonlocal NLS: collapse is suppressed:

S.K. Turitsyn, Teor. Mat. Fiz. **64**, 226 (1985)

V.M. Pérez-García et al. Phys Rev E **62**, 4300 (2000)

Wave Collapse I

General case: symmetric non-singular response function:

Conservations of power, P, and Hamiltonian, H

$(N(I) = s \int R(\vec{\xi} - \vec{r}) I(\vec{\xi}, z) d\vec{\xi})$

$$P = \int I d\vec{r}, \quad H = ||\nabla \psi||_2^2 - \frac{1}{2} \int N I d\vec{r}.$$

Wave Collapse I

General case: symmetric non-singular response function:

$$P = \int I d\vec{r}, \quad H = ||\nabla\psi||_2^2 - \frac{1}{2}\int N I d\vec{r}.$$

Using: $|a-b| \geq ||a|-|b|| \geq |a|-|b|$ and $|\int f(\vec{x})d\vec{x}| \leq \int |f(\vec{x})|d\vec{x}$

$$|H| = |\ ||\nabla\psi||_2^2 - \frac{1}{2}\int N I d\vec{r}\ | \geq ||\nabla\psi||_2^2 - \frac{1}{2}|\int N I d\vec{r}|$$

Wave Collapse I

General case: symmetric non-singular response function:

$$P = \int I d\vec{r}, \quad H = ||\nabla\psi||_2^2 - \frac{1}{2}\int N I d\vec{r}.$$

$$|H| = |\; ||\nabla\psi||_2^2 - \frac{1}{2}\int N I d\vec{r}\; | \geq ||\nabla\psi||_2^2 - \frac{1}{2}|\int N I d\vec{r}|$$

Fourier space

$$\int N I d\vec{r} = \frac{1}{(2\pi)^D}\int \tilde{R}(\vec{k})|\tilde{I}(\vec{k})|^2 d\vec{k}$$

Wave Collapse II

$$\left| \int N I d\vec{r} \right| \leq \frac{1}{(2\pi)^D} \int |\tilde{R}(\vec{k})| |\tilde{I}(\vec{k})|^2 d\vec{k} \leq P^2 R_0$$

$$R_0 = R(0) \equiv \frac{1}{(2\pi)^D} \int |\tilde{R}(\vec{k})| d\vec{k}, \quad P \equiv \int I d\vec{r}$$

Demands $|\tilde{R}(\vec{k})|$ is integrable

$$||\nabla\psi||_2^2 \leq |H| + \frac{1}{2} R_0 P^2$$

Wave Collapse II

$$\left| \int N I d\vec{r} \right| \leq \frac{1}{(2\pi)^D} \int |\tilde{R}(\vec{k})| |\tilde{I}(\vec{k})|^2 d\vec{k} \leq P^2 R_0$$

$$R_0 = R(0) \equiv \frac{1}{(2\pi)^D} \int |\tilde{R}(\vec{k})| d\vec{k}, \quad P \equiv \int I d\vec{r}$$

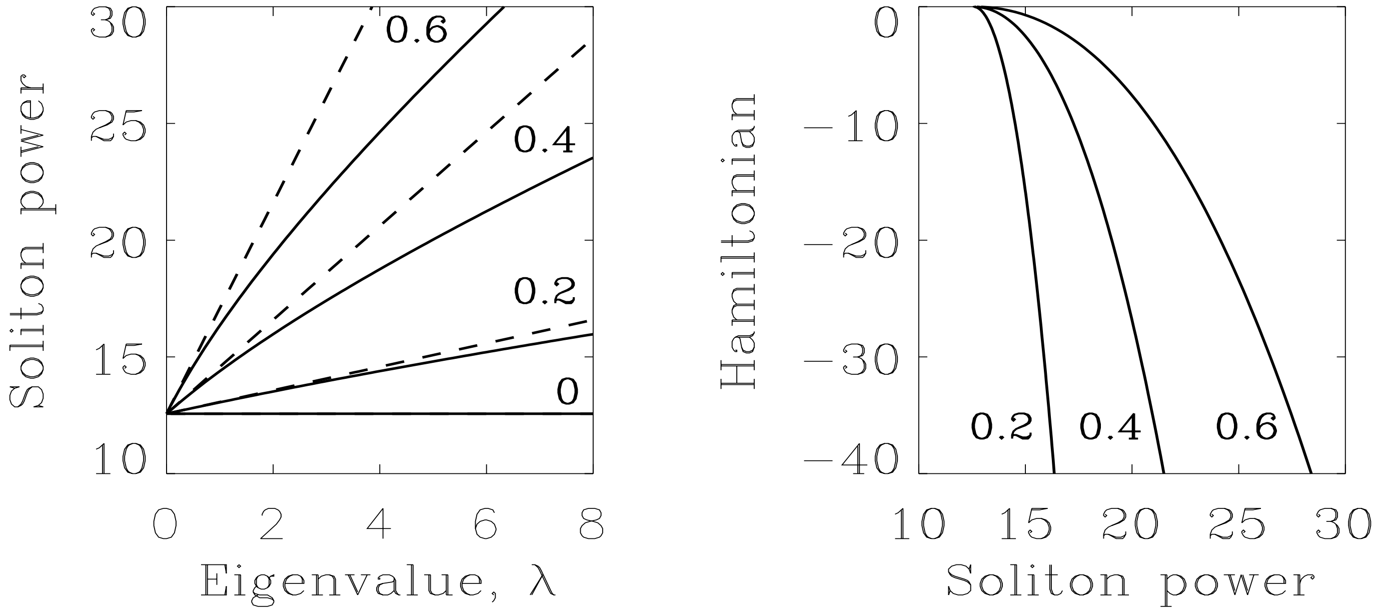
$$||\nabla \psi||_2^2 \leq |H| + \frac{1}{2} R_0 P^2$$

$||\nabla \psi||_2^2$ is bounded from above $\Rightarrow$ no blow-up!

Quasi collapse: smallest width $\approx$ width of $R(r)$.

Width $w \geq \frac{\sqrt{P}\sigma}{\sqrt{P/2\pi + H\sigma^2}}$

2D Beam Stability

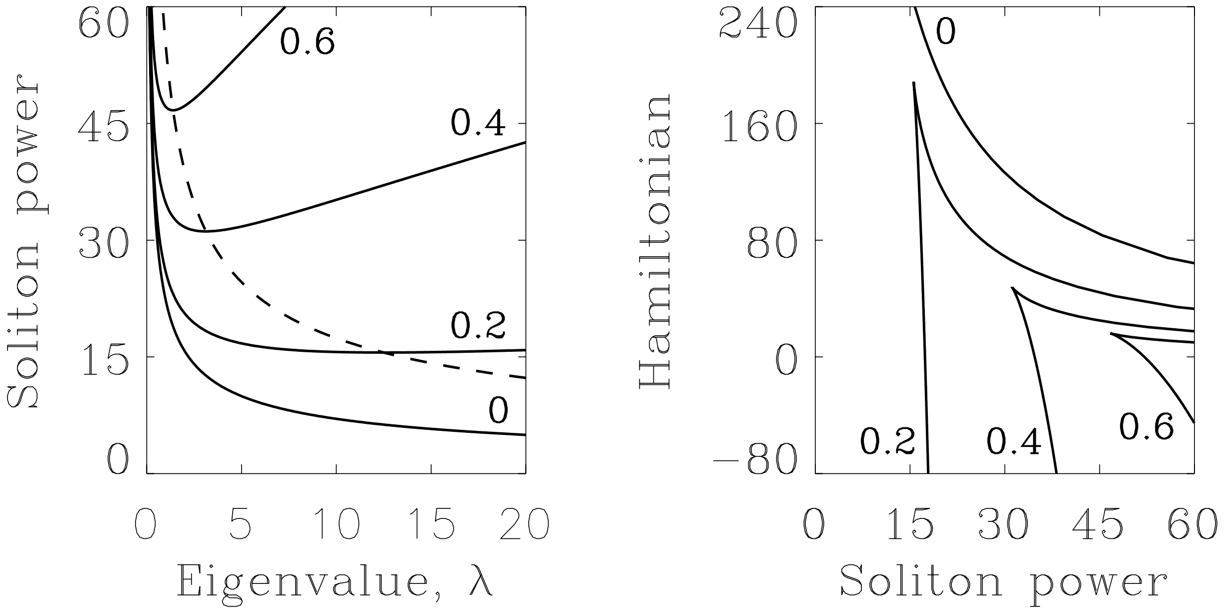


Soliton solution: $\psi(\vec{r}, z) = \phi(r) \exp(i\lambda z)$. Variational results for Gaussian response $R(\vec{r}) = \left(1/\pi\sigma^2\right)^{\frac{D}{2}} \exp\left(-|\vec{r}|^2/\sigma^2\right)$ and trial function $\phi(r) = \alpha \exp[-(r/\beta)^2]$

Different degrees of nonlocality, σ=0, 0.2, 0.4, and 0.6.

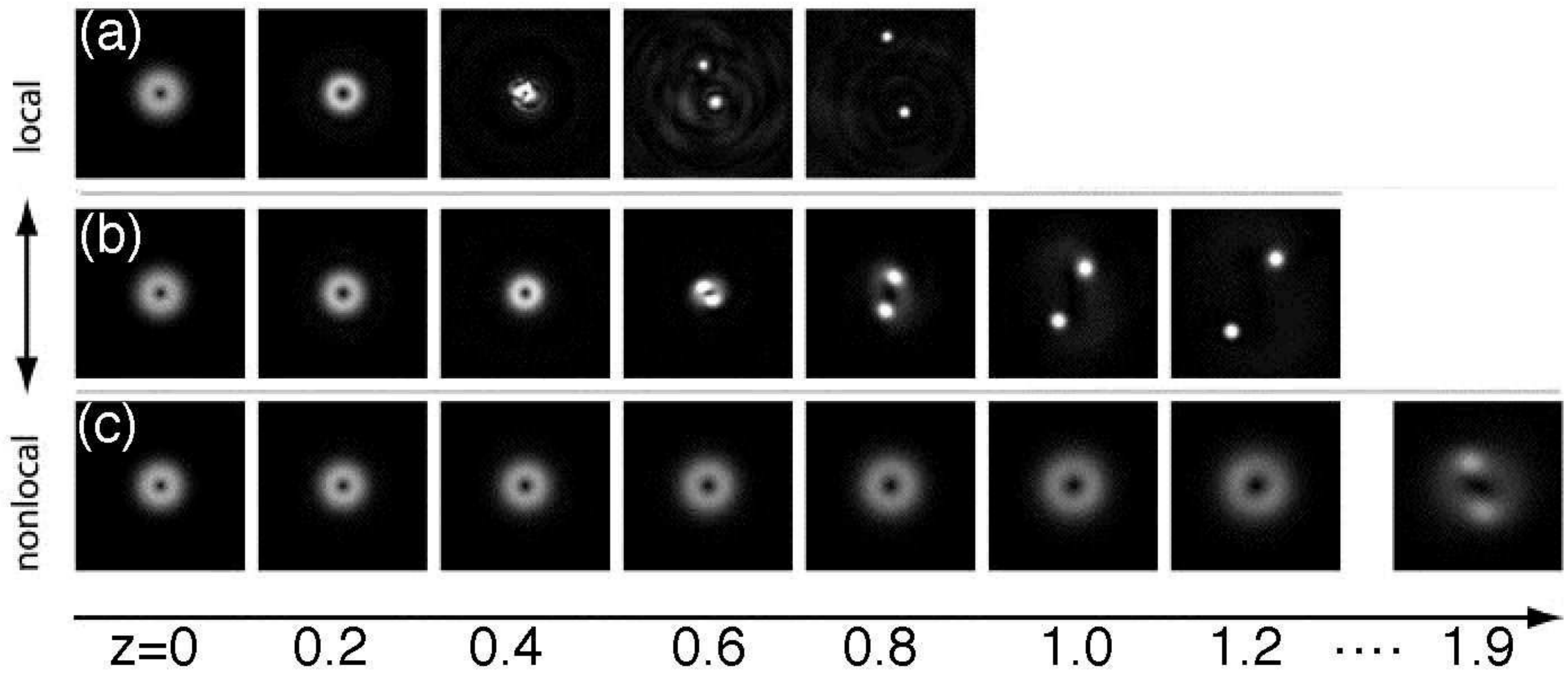
$\frac{dP}{d\lambda} > 0$ implies stability

3D Beam Stability



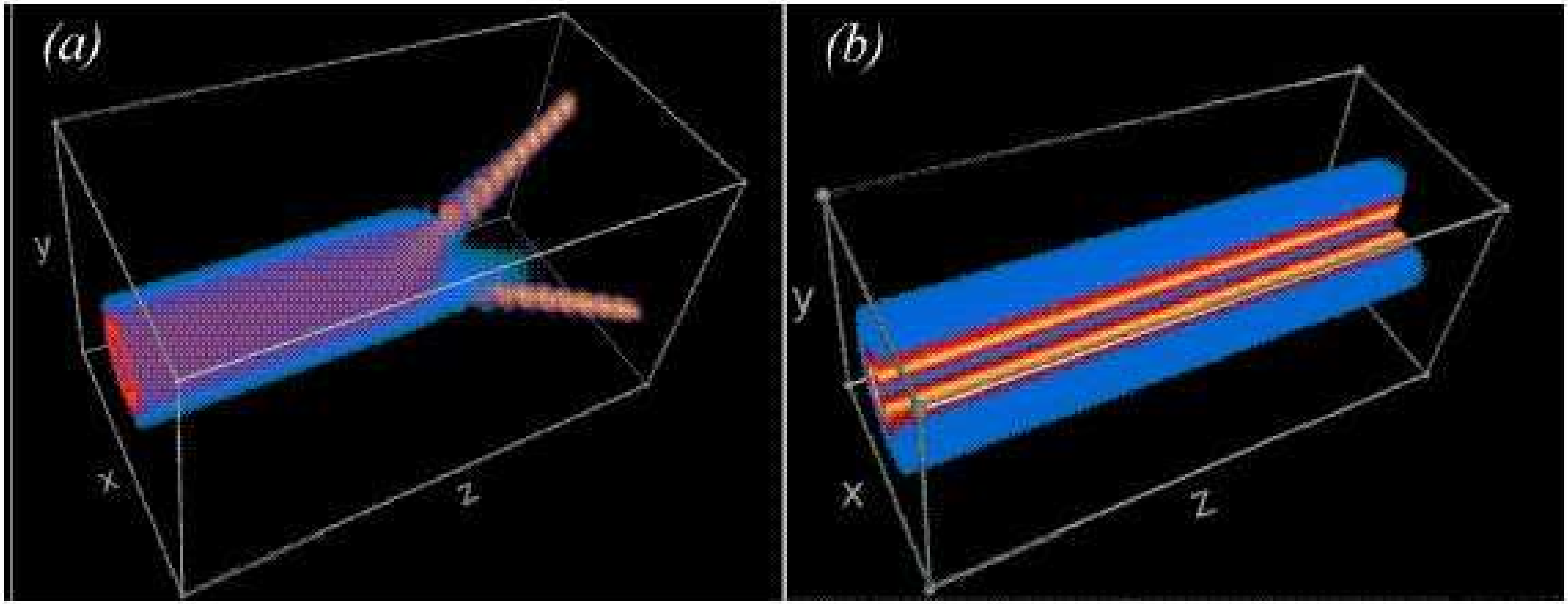
3D variational results with Gaussian response and trial function. σ=0, 0.2, 0.4, and 0.6. Dashed line threshold for instability

Stabilized vortex beam



Gaussian-Laguerre vortex beam $\psi(\vec{r}) = r\exp(-(r/r_0)^2\exp(i\theta)$ (charge 1, $r_0 = 1$) in a self-focusing nonlocal medium (Gaussian response function). (a) $\sigma = 0$, (b) $\sigma = 1$, (c) $\sigma = 10$.

Stabilized vortex beam



Propagation of a charge m = 1 vortex beam in a nonlocal medium.

Gaussian response function (a) $\sigma = 0.1$, (b) $\sigma = 10$.

Briedis *et al.* Optics Express **13** 435 (2005)

Conclusions

- Nonlocal effects have a strong influence on the nonlinear evolution
- The modulational instability of plane waves is suppressed for positive definite $R(k)$
- Higher order unstable bands appear for sign indefinite $R(k)$ and unstable bands may appear in de-focusing media.
- Nonlocality influences soliton interactions and may introduce 2(n)-soliton bound states.
- Nonlocality prevents blow-up, stabilize 2 and 3-D solitons; quasi collapse is possible.