On some questions in the theory of flowing plasmas – non-Hermitian dynamics of waves in shear flows

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I. WHAT DO WE KNOW ABOUT "HERMITIAN" SYSTEMS?

- Standard form of a Hermitian system:

$$i\partial_t u = Hu \qquad (H^* = H). \tag{1}$$

- We may assume

$$u(x,t) = f(t)\varphi(x) = e^{-i\omega t}\varphi(x)$$

to convert (1) into a "dispersion relation"

$$(\omega I - H)\varphi = 0. \tag{2}$$

- The eigenvalue problem (2) gives particular solutions of (1):

$$u_j(x,t) = e^{i\omega_j t}\varphi_j(x) \quad (j = 1, 2, \cdots). \tag{3}$$

Here we assume that u_j has a finite energy ($\|\varphi_j\|^2 < \infty$).

- **von Neumann's theorem**: The totality of the particular solutions (3), together with singular (infinite energy) solutions with "continuous spectra", is "complete" to represent every solution to an "initial value problem ($u(x,0) = u_0(x)$):

$$u(x,t) = \sum_j e^{i\omega_j t}\langle\varphi_j, u_0\rangle\varphi_j \tag{4}$$

To include continuous spectra, we have to generalize (4) as

$$u(x,t) = \int e^{i\omega t}\langle \varphi_\mu, u_0\rangle \varphi_\mu d\omega(\mu), \tag{5}$$

where φ_u may be a singular eigenfunction, $\omega(\mu)$ is a monotone (non-decreasing) function of $\mu \in \mathbf{R}$ ($d\omega(\mu) = \sum_j \delta(\omega_j - \mu)$ yields (4)). More appropriate representation is

$$u(x,t) = \int e^{i\mu t} dE(\mu) u_0 \tag{6}$$

where $E(\mu)$ is the orthogonal projector onto a subspace parameterized by μ such that $E(\mu)E(\mu') = E(\min(\mu,\mu'))$.

- The solution operator (propagator)

$$e^{-itH} = \int e^{i\mu t} dE(\mu) \quad (t \in \mathbf{R})$$

constitutes a "one parameter group" of unitary transforms:

$$\|e^{-itH} u_0\| = \|u_0\| \quad (t \in \mathbf{R}),$$

which implies the conservation of the wave quanta.

- The energy is also a constant of motion:

$$\langle u(t), Hu(t)\rangle = \text{constant} \quad (u(t) = e^{-itH}u_0,\ t \in \mathbf{R}).$$

II. HOW IS NON-HERMITIAN SYSTEM INTERESTING (DIFFICULT)?

- There may exist (non-exponential) instabilities even if all ω are real:

$$i\frac{d}{dt}\boldsymbol{u} = \begin{pmatrix} \omega & 1 \\ 0 & \omega \end{pmatrix} \boldsymbol{u} \quad \Rightarrow \quad te^{-i\omega t}$$

Jordan block represents "resonance". Modes with overlapping frequencies may interact resulting in secular amplification.

- Energy principle is weakened. It is generally difficult to find the necessary and sufficient conditions for (exponential) stability/instability.

  Consider

$$\partial_t u = i\Omega u + \Gamma u.$$

  Let λ_Γ denote an eigenvalue of the operator Γ.

$$\exists\lambda_\Gamma > 0 \Rightarrow \text{necessary for } \textit{instability}$$
$$\forall\lambda_\Gamma \leq 0 \Rightarrow \text{sufficient for } \textit{stability}$$

  (proof) Let $u = e^{it\Omega}\varphi$. Then,

$$\partial_t \varphi = \gamma(t)\varphi \qquad \gamma(t) = e^{-it\Omega}\Gamma e^{it\Omega}.$$

  If $\forall\lambda_\Gamma \leq 0$,

$$\frac{d}{dt}\|u\|^2 = \frac{d}{dt}\|\varphi\|^2 = 2(\varphi, \gamma(t)\varphi) = 2(\varphi, \Gamma\varphi) \leq 0.$$

  But, $\exists\lambda_\Gamma > 0$ is not sufficient for instability, because the unstable and stable phases mix. The energy principle (being unaware of the Ω term) is, thus, weakened.

III. A CLASS OF NON-HERMITIAN SYSTEMS

- Let us consider a second-order dynamical system

$$\partial_t^2 q = -\Omega^2 q \qquad (\Omega^2 : \text{ Hermitian}), \tag{7}$$

  which is formally equivalent to

$$\partial_t \phi = \pm i\Omega\phi. \tag{8}$$

  A rigorous interpretation of (8) is given by defining $u = (q, p)^t$, and rewriting (7) as

$$\partial_t u = \partial_t \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \Omega^2 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}. \tag{9}$$

Denoting $\phi_\pm = p \pm i\Omega q$, (9) reads

$$\partial_t \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} = \begin{pmatrix} i\Omega & 0 \\ 0 & -i\Omega \end{pmatrix} \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix}. \tag{10}$$

The evolution equation (10) is a Hamiltonian system: Defining

$$\mathcal{H}(u) = \frac{1}{2}\left[\langle q, \Omega^2 q\rangle + \langle p, p\rangle\right], \tag{11}$$

we may rewrite (10) in a canonical form

$$\partial_t u = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \partial_u \mathcal{H} \quad \Leftrightarrow \partial_t \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \partial_p \mathcal{H} \\ -\partial_q \mathcal{H} \end{pmatrix} \tag{12}$$

- Adding a first-order term (representing the flow effect) to (7), we obtain a non-Hermitian system

$$\partial_t^2 q + L\partial_t q = -\Omega^2 q \qquad (iL: \text{ Hermitian}). \tag{13}$$

Then, (12) modifies as

$$\partial_t u = \begin{pmatrix} 0 & I \\ -I & L \end{pmatrix} \partial_u \mathcal{H}. \tag{14}$$

- Hitherto, we consider a "non-canonical" Hamiltonian system such as

$$i\partial_t u = AHu, \tag{15}$$

where A is a Hermitian operator (iA is an anti-symmetric operator), $Hu = \partial_u \mathcal{H}$ (H is a Hermitian operator).

Note that AH is not self-adjoint when A and H do not commute.

- Conservation of the energy is the direct consequence of the anti-symmetry of iA:

$$\frac{1}{2}\langle u, Hu\rangle \ (= \mathcal{H}) = \text{constant}.$$

- The anti-symmetric operator iA is a generalization of i in the Schrödinger equation (1). The propagator e^{-itAH} (if it may be generated) is no longer a unitary operator, and hence, the wave quanta may not be conserved.

- The range $\mathcal{R}(A)$ of A may be smaller than the total Hilbert space. The orthogonal compliment of $\mathcal{R}(A)$ is spanned by constants of motion that are called "Casimirs".

- For a Hermitian operator

$$H = \int_{-\infty}^{+\infty} \mu dE(\mu) \tag{16}$$

let us define

$$H^{1/2} = H_+^{1/2} + iH_-^{1/2} \tag{17}$$

$$H_+^{1/2} = \int_0^{+\infty} \sqrt{\mu} dE(\mu) \quad H_-^{1/2} = \int_{-\infty}^{0} \sqrt{-\mu} dE(\mu). \tag{18}$$

Then, we observe

$$H^{-1/2} = H_+^{-1/2} - iH_-^{-1/2} = \int_0^{+\infty} \frac{1}{\sqrt{\mu}} dE(\mu) - i\int_{-\infty}^{0} \frac{1}{\sqrt{-\mu}} dE(\mu).$$

Denoting $H^{1/2}u = \psi$, we may rewrite (15) as

$$i\partial_t \psi = H^{1/2} A H^{1/2} \psi. \tag{19}$$

The generator $H^{1/2}AH^{1/2}$ may be decomposed into the real and imaginary parts:

$$\begin{aligned} H^{1/2}AH^{1/2} &= (H_+^{1/2}AH_+^{1/2} - H_-^{1/2}AH_-^{1/2}) + i(H_+^{1/2}AH_-^{1/2} + H_-^{1/2}AH_+^{1/2}) \\ &= L_1 + iL_2. \end{aligned} \tag{20}$$

Here, both L_1 and L_2 are self-adjoint.

- Note that $L_2 = 0$ ($H^{1/2}AH^{1/2}$ is Hermitian) if H is positive $H_-^{1/2} = 0$ (or negative $H_+^{1/2} = 0$). An instability is due to the couping of positive and negative energy modes.

- Generating e^{-itL_1}, we define

$$\psi(t) = e^{-itL_1}\varphi(t).$$

Then, (19) reads as

$$\partial_t\varphi = \mathcal{L}_2(t)\varphi \quad (\mathcal{L}_2(t) = e^{itL_1}L_2e^{-itL_2}). \tag{21}$$

Since e^{-itL} is a unitary transform, the spectra of $\mathcal{L}_2(t)$ is the same as that of L_2 for every t. However,

$$e^{\int^t \mathcal{L}_2(s)ds}$$

may not glow even when some spectra are positive.

IV. LYAPUNOV STABILITY

- Equilibria are the extremers of constants of motion, i.e., isolated points of the levelsets of the constants of motion.

- The minimum of the "Hamiltonian" is often a trivial equilibrium (for example, the minimum of $(p^2 + \omega^2 q^2)/2$ (harmonic oscillator) is the stationary point).

- Combination of some other constants of motion (than the energy), which is often called Casimirs, yields a diversity of nontrivial equilibria. Then, a "Lyapunov function" (= combination of constants of motion) may give the necessary condition for the stability [12].

- In an infinite-dimension Hilbert space, we have to check the "coercivity" of the Lyapunov function.

- Example: Beltrami state

  - Ideal MHD system:

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \nabla p = 0, \tag{22}$$

$$\partial_t \boldsymbol{B} - \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = 0. \tag{23}$$

  We assume boundary conditions

$$\boldsymbol{n} \cdot \boldsymbol{v} = 0, \quad \boldsymbol{n} \cdot \boldsymbol{B} = 0 \quad \text{on } \Gamma \tag{24}$$

  and flux conditions

$$\int_{\Sigma_\ell} \boldsymbol{n} \cdot \boldsymbol{B} \; ds = K_\ell \quad (\ell = 1, \cdots, m), \tag{25}$$

  where the fluxes through the cuts are given constants.

  - The dynamics allows three important constants of motion:

$$H_0 = \|\boldsymbol{v}\|^2 + \|\boldsymbol{B}\|^2 \quad \text{(energy)}, \tag{26}$$

$$H_1 = (\mathcal{P}\boldsymbol{A}, \boldsymbol{B}) \quad \text{(magnetic helicity)}, \tag{27}$$

$$H_2 = 2(\boldsymbol{v}, \boldsymbol{B}) \quad \text{(cross helicity)}. \tag{28}$$

  - The variational principle

$$\delta(H_0 - \mu_1 H_1 - \mu_2 H_2) = 0 \tag{29}$$

  gives Beltrami fields defined by

$$(1 - \mu_2^2)\nabla \times \boldsymbol{B} = \mu_1 \boldsymbol{B}, \tag{30}$$

$$\boldsymbol{v} = \mu_2 \boldsymbol{B}. \tag{31}$$

– We find that the integral

$$G(\tilde{\boldsymbol{B}}, \tilde{\boldsymbol{v}}) = \|\tilde{\boldsymbol{v}}\|^2 + \|\tilde{\boldsymbol{B}}\|^2 - \mu_1(\mathcal{P}\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{B}}) - 2\mu_2(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{B}}) \tag{32}$$

is a constant of motion for the perturbations $\tilde{\boldsymbol{B}}$ and $\tilde{\boldsymbol{v}}$ satisfying the nonlinear equation (22)-(23), or their linearized equations. The flux condition (25) demands $\tilde{\boldsymbol{B}} \in L^2_\Sigma(\Omega)$.

– We now prove the inequality

$$(\mathcal{P}\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{B}}) \le |\lambda|^{-1}\|\tilde{\boldsymbol{B}}\|^2, \tag{33}$$

where $|\lambda| = \min_j |\lambda_j|$ [λ_j $(j = 1, 2, \cdots)$ are the eigenvalues of the self-adjoint curl operator]. Invoking the spectral resolution theorem due to Yoshida-Giga [11], we expand $\boldsymbol{u} = \sum(\boldsymbol{u}, \boldsymbol{\psi}_j)\boldsymbol{\psi}_j$ $(\forall \boldsymbol{u} \in L^2_\Sigma(\Omega))$, where $\boldsymbol{\psi}_j$ is the eigenfunction of the self-adjoint curl operator belonging to an eigenvalue λ_j, and write

$$\tilde{\boldsymbol{B}} = \sum(\tilde{\boldsymbol{B}}, \boldsymbol{\psi}_j)\boldsymbol{\psi}_j,$$

and

$$\mathcal{P}\tilde{\boldsymbol{A}} = \sum(\tilde{\boldsymbol{B}}, \boldsymbol{\psi}_j)\boldsymbol{\psi}_j/\lambda_j,$$

leading to the promised inequality

$$\begin{aligned}
(\mathcal{P}\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{B}}) &\le \|\mathcal{P}\tilde{\boldsymbol{A}}\| \cdot \|\tilde{\boldsymbol{B}}\| \\
&= \left[\sum(\tilde{\boldsymbol{B}}, \boldsymbol{\psi}_j)^2/\lambda_j^2\right]^{-1/2} \left[\sum(\tilde{\boldsymbol{B}}, \boldsymbol{\psi}_j)^2\right]^{-1/2} \\
&\le |\lambda|^{-1} \sum(\tilde{\boldsymbol{B}}, \boldsymbol{\psi}_j)^2 \\
&= |\lambda|^{-1}\|\tilde{\boldsymbol{B}}\|^2.
\end{aligned}$$

– Using

$$2(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{B}}) \le \alpha\|\tilde{\boldsymbol{v}}\|^2 + \alpha^{-1}\|\tilde{\boldsymbol{B}}\|^2 \quad (\forall \alpha > 0),$$

we observe

$$G(\tilde{\boldsymbol{B}}, \tilde{\boldsymbol{v}}) \geq (1-\alpha|\mu_2|)\,\|\tilde{\boldsymbol{v}}\|^2 + \left(1 - \frac{|\mu_2|}{\alpha} - \frac{|\mu_1|}{|\lambda|}\right)\|\tilde{\boldsymbol{B}}\|^2. \tag{34}$$

The choices $\alpha = 1/|\mu_2|$, and $\alpha = |\mu_2|/(1-|\mu_1|/|\lambda|)$ convert (34) to

$$G(\tilde{\boldsymbol{B}}, \tilde{\boldsymbol{v}}) \geq \left(1 - \mu_2^2 - \frac{|\mu_1|}{|\lambda|}\right)\|\tilde{\boldsymbol{B}}\|^2, \tag{35}$$

and

$$G(\tilde{\boldsymbol{B}}, \tilde{\boldsymbol{v}}) \geq \left(1 - \frac{\mu_2^2}{1-|\mu_1|/|\lambda|}\right)\|\tilde{\boldsymbol{v}}\|^2, \tag{36}$$

respectively. If $1 - \mu_2^2 - |\mu_1|/|\lambda| > 0$, then (35) and (36) give bounds for the energy associated with the magnetic ($\tilde{\boldsymbol{B}}$) as well as the velocity ($\tilde{\boldsymbol{v}}$) fluctuations.

– The "sufficient condition" for the stability, therefore, consists of the simultaneous inequalities

$$\mu_2^2 < 1. \tag{37}$$

$$\sigma \equiv \frac{|\mu_1|}{1-\mu_2^2} < |\lambda|, \tag{38}$$

where σ stands for the eigenvalue of the Beltrami equation (30) for $\mu_1 > 0$. The first stability condition requires that the flow velocity must not exceed the local Alfvén speed [see (31)], while the second condition demands that σ must not exceed the minimum of $|\lambda_j|$ (λ_j is the eigenvalue of the self-adjoint curl operator).

- General theorem:

  – Let $f(a, b)$ be a bilinear map. We define $\mathcal{F}(u) = f(u, u)$, and consider an abstract nonlinear evolution equation

$$\partial_t u = \mathcal{F}(u). \tag{39}$$

We further suppose that there are symmetric bilinear forms $h_j(a, b)$ ($j = 1, \cdots, \nu$) such that

$$h_j(u, \mathcal{F}(u)) = 0 \quad (j = 1, \cdots, \nu,\ \forall u). \tag{40}$$

– It is easy to show that $H_j(u) = h_j(u,u)$ (u is a solution of (39)) is a constant of motion for the evolution equation (39);

$$\begin{aligned} \frac{d}{dt} H_j(u) &= 2h_j(u, \partial_t u) \\ &= 2h_j(u, \mathcal{F}(u)) = 0. \end{aligned} \tag{41}$$

Let u_0 be a stationary point (equilibrium) of (39), i.e., $\mathcal{F}(u_0) = 0$. We assume that u_0 solves

$$\delta \left[\sum_{j=1}^{\nu} \mu_j H_j(u) \right] = 0 \tag{42}$$

with some fixed real numbers μ_j ($j = 1, \cdots, \nu$). We call such a u_0 as a "Beltrami field".

Remark 2. If (42) has a unique (or isolated) solution u_0, then this u_0 is an equilibrium of (39). Indeed, any departure from u_0 will change the value of $G(u) \equiv \sum_{j=1}^{\nu} \mu_j H_j(u)$, while $G(u)$ is a constant of motion.

Theorem 1. *Suppose that $u = u_0 + \tilde{u}$ (u_0 is a Beltrami field) satisfies either (39) or its "linearized" equation*

$$\partial_t \tilde{u} = f(u_0, \tilde{u}) + f(\tilde{u}, u_0). \tag{43}$$

Then,

$$G(\tilde{u}) = \sum_{j=1}^{\nu} \mu_j H_j(\tilde{u}) \tag{44}$$

is a constant of motion.

(proof) Using (40), we observe

$$\begin{aligned} 0 &= \sum \mu_j h_j(u, \mathcal{F}(u)) \\ &= \sum \mu_j h_j(u_0 + \tilde{u}, \mathcal{F}(u_0 + \tilde{u})) \\ &= \sum \mu_j h_j(u_0, \mathcal{F}(u_0 + \tilde{u})) \\ &\qquad + \sum \mu_j h_j(\tilde{u}, \mathcal{F}(u_0 + \tilde{u})). \end{aligned} \tag{45}$$

Since (42) implies $\sum \mu_j h_j(u_0, \delta) = 0$ $(\forall \delta)$, the first sum in (45) vanishes. Hence, if u solves (39), we obtain

$$\begin{aligned} \frac{d}{dt} G(\tilde{u}) &= 2 \sum \mu_j h_j(\tilde{u}, \partial_t \tilde{u}) \\ &= 2 \sum \mu_j h_j(\tilde{u}, \mathcal{F}(u_0 + \tilde{u})) = 0. \end{aligned} \tag{46}$$

We can rewrite (45) as

$$\begin{aligned} 0 = \sum \mu_j h_j(\tilde{u}, f(u_0, \tilde{u}) + f(\tilde{u}, u_0)) \\ + \sum \mu_j h_j(\tilde{u}, \mathcal{F}(\tilde{u})). \end{aligned} \tag{47}$$

By (40), the second term of (47) vanishes. If $\tilde{u}$ is a solution of (43), we obtain

$$\frac{d}{dt} G(\tilde{u}) = 2 \sum \mu_j h_j(\tilde{u}, f(u_0, \tilde{u}) + f(\tilde{u}, u_0)) = 0. \tag{48}$$

□

– Although each functional H_j occurring in the sum that defines G is a constant of motion for the total field u, it is only the special linear combination (44) that is conserved for the perturbation, $\tilde{u}$. The coefficients μ_j included in G are the structure (Beltrami) parameters characterizing the equilibrium.

– If a continuous quadratic form $F(v)$ satisfies (on a Hilbert space V)

$$F(v) \geq c\|v\|^2 \quad (\forall v \in V) \tag{49}$$

with some positive constant c ($\|v\|$ is the norm of v in V), $F(v)$ is said to be "coercive".

Proposition 1. *If $G(v) = \sum_{j=1}^{\nu} \mu_j H_j(v)$ with given μ_j is a coercive form, then*

1. *$G(u)$ has a unique "minimizer" that is given by the variational principle (42),*
2. *the minimizer u_0 of $G(u)$ is a stationary point (equilibrium) of (39),*
3. *the minimizer u_0 is "stable"; the norm of every perturbation $\tilde{u}$ is bounded by a constant that depends upon $G(\tilde{u}|_{t=0})$.*

V. KELVIN'S METHOD –GENERALIZED EIGENFUNCTION FOR FLOWING SYSTEMS

- Because of the non-Hermitian nature, conventional modal approach (spectral resolution of the generator) does not apply for shear-flow systems. Here, we invoke the idea of Lord Kelvin [6] to derive particular solutions that describe deformation of "modes" in a shear flow. We use these solutions as "flowing eigenfunctions" in expanding fluctuations.

- We explain the method using an abstract evolution equation. Let $\mathcal{A}$ be the generator of a no-flow ($\boldsymbol{v} = 0$) system. Adding the convection term $\mathcal{F}\boldsymbol{u} = (\boldsymbol{v} \cdot \nabla)\boldsymbol{u}$, we consider an evolution equation governing a fluctuation $\boldsymbol{u}$:

$$\partial_t \boldsymbol{u} + \mathcal{F}\boldsymbol{u} = \mathcal{A}\boldsymbol{u}. \tag{50}$$

The generator $\mathcal{A} - \mathcal{F}$ is generally a non-Hermitian operator.

Solving

$$(\partial_t + \mathcal{F})\boldsymbol{\varphi}(\boldsymbol{x}, t; \boldsymbol{\mu}) = 0, \tag{51}$$

we determine the deformation of a function $\boldsymbol{\varphi}$ in the flow $\boldsymbol{v}$, where $\boldsymbol{\mu}$ is a certain parameter (quantum number). If this $\boldsymbol{\varphi}$ satisfies, for each t,

$$\mathcal{A}\boldsymbol{\varphi}(\boldsymbol{x}, t; \boldsymbol{\mu}) = \lambda(t; \boldsymbol{\mu})\boldsymbol{\varphi}(\boldsymbol{x}, t; \boldsymbol{\mu}), \tag{52}$$

we call $\boldsymbol{\varphi}$ a "flowing eigenfunction".

- If the set

$$\{\boldsymbol{\varphi}(\boldsymbol{x}, t; \boldsymbol{\mu});\ \ \boldsymbol{\mu} \in \sigma\}$$

is an orthogonal complete system (for each t), we may expand

$$\boldsymbol{u}(\boldsymbol{x},t) = \int_\sigma q(t;\boldsymbol{\mu})\boldsymbol{\varphi}(\boldsymbol{x},t;\boldsymbol{\mu})\ d\mu. \tag{53}$$

Then, the evolution equation (50) decomposes into independent ordinary differential equations (ODE):

$$q'(t;\boldsymbol{\mu}) = \lambda(t;\boldsymbol{\mu})q(t;\boldsymbol{\mu}) \quad (\forall\boldsymbol{\mu}\in\sigma). \tag{54}$$

- While the differential equation (54) is "integrable", the amplitude $q(t;\boldsymbol{\mu})$ may exhibit rather complex behavior, because the eigenvalue λ is a function of t.

VI. LAGRANGIAN OF NON-CONSERVATIVE SYSTEM

- We assume that the system is (approximately) Fourier analyzable in x and y. The wave numbers $\boldsymbol{\mu} = (k_x, k_y)$ are "good quantum numbers". We consider an ambient flow with a constant shear such that $\boldsymbol{v} = sx\boldsymbol{e}_y$ (s is a real constant number). For a Fourier mode $\varphi = e^{i(k_x x + k_y y)}$, (51) yields a flowing eigenfunction

$$\varphi(x,y,t;k_x,k_y) = e^{i[(k_x - sk_y t)x + k_y y]}. \tag{55}$$

- In MHD models, the evolution equation (50), governing a vector-valued variable $\boldsymbol{u}$, can be often cast into a second-order differential equation, and the corresponding "dispersion relation", for the case of $\boldsymbol{v}=0$, yields $\omega = \Omega(k_x,k_y)$ such that ω^2 is real. Using the flowing eigenfunction (55), we obtain a second-order ODE such as [10,9]

$$q'' + a(t)q' + \omega^2(t)q = 0. \tag{56}$$

The coefficients $a(t)$ and $\omega(t)$ depend on the quantum number $\boldsymbol{\mu} = (k_x, k_y)$ that is fixed in (56). For example, a model of interchange modes yields [9]

$$a(t) = -\frac{2sk_y K_x(t)}{K_x(t)^2 + k_y^2}, \tag{57}$$

$$\omega^2(t) = 1 - G\frac{k_y^2}{K_x(t)^2 + k_y^2}, \tag{58}$$

where

$$K_x(t) = k_x - sk_yt$$

and G is a positive parameter measuring the driving force of interchange instabilities. By (58), we observe that the stretching effect of the shear flow ($\lim_{t\to\infty}|K_x(t)| = \infty$) finally removes the instability. The second term on the left-hand side of (56) is analogous to a "friction", which represents the phase mixing effect of the shear flow.

- While the system (56) is non-conservative, it has a Lagrangian

$$L = \frac{\rho(t)q'^2}{2} - V(q,t), \tag{59}$$

where we define

$$\rho(t) = \exp\int_0^t a(t)dt, \tag{60}$$

$$V(q,t) = \frac{\rho(t)\omega^2(t)q^2}{2}. \tag{61}$$

The "canonical momentum" is given by

$$p = \frac{\partial L}{\partial q'} = \rho(t)q'. \tag{62}$$

- In the Lagrangian formalism (59), the phase-mixing effect [the friction term of (56)] is represented by a time-dependent "effective mass" $\rho(t)$. When we observe the fluctuations in the space (q,q'), the coefficient $\rho(t)^{-1}$ yields volume reduction. Using (57), we estimate $a(t) \propto t^{-1}$ for large t [10,9], and hence, we obtain

$$\rho(t) \propto t \quad (t \to \infty). \tag{63}$$

We note that the volume reduction (non-conservative property) is not exponential, but is algebraic [15].

VII. KINETIC THEORY OF NON-HERMITIAN WAVE SYSTEM

- In the phase space (q,p), the system (56) is a standard Hamiltonian system; The Hamiltonian is

$$H = \frac{p^2}{2\rho(t)} + \frac{\rho(t)\omega^2(t)q^2}{2}. \tag{64}$$

When $\omega^2 > 0$ and $|a(t)| \ll |\omega|$, we may invoke the adiabatic invariance of the action

$$I = \frac{1}{2\pi}\oint_{H=E} p \ dq,$$

where the path integral is taken through an approximate closed orbit characterized by $H(q,p,t) = E$ [temporal change of H is assumed to be small during the one cycle of the orbit $(2\pi/\omega)$]. The well-known relation $I = E/\omega$ allows us to interpret I as the number of wave quanta [8,3].

- Integrating over the quantum number $\boldsymbol{\mu} = (k_x, k_y)$, we obtain the wave field

$$\boldsymbol{u}(\boldsymbol{x},t) = \int \boldsymbol{\psi}(t;k_x,k_y)e^{i[(k_x - sk_y t)x + k_y y]} \ dk_x dk_y, \tag{65}$$

where the "mode amplitude"

$$\boldsymbol{\psi}(t;k_x,k_y) = \begin{pmatrix} q(t;k_x,k_y) \\ p(t;k_x,k_y) \end{pmatrix} \tag{66}$$

$$\approx \begin{pmatrix} q_0(k_x,k_y) \\ p_0(k_x,k_y) \end{pmatrix} e^{-i\int \omega(t;k_x,k_y) \ dt} \tag{67}$$

is determined by the Hamiltonian (64). One may include a slow variation of the frequency ω as a function of (x,y), and then the energy density becomes inhomogeneous in space. The energy density of the wave field is given by

$$\mathcal{E}(x,y,t;k_x,k_y) = \rho(t)\omega^2(x,y,t;k_x,k_y)q_0(k_x,k_y)^2.$$

The action (number density of wave quanta) is

$$\mathcal{I}(x,y,t;k_x,k_y) = \frac{\mathcal{E}(x,y,t;k_x,k_y)}{\omega(x,y,t;k_x,k_y)}.$$

This $\mathcal{I}(x,y,t;k_x,k_y)$ is an adiabatic invariant along the "eikonal" where the variation of the phase

$$\begin{aligned} &S(x,y,t;k_x,k_y) \\ &\quad = \int -\omega(x,y,t;k_x,k_y)\ dt + (k_x - sk_y t)x + k_y y \end{aligned}$$

is minimized. The variational principle

$$\delta S(x,y,t;k_x,k_y) = 0$$

yields the eikonal equation that defines the Cauchy characteristics of the wave kinetic equation describing the adiabatic conservation of the action:

$$\frac{\partial}{\partial t}\mathcal{I} + \{\Omega, \mathcal{I}\} = 0, \tag{68}$$

where

$$\begin{aligned} \Omega(x,y,t;k_x,k_y) &= -\frac{\partial S}{\partial t} \\ &= \omega(x,y,t;k_x,k_y) + sk_y x \end{aligned}$$

and $\{\ ,\ \}$ is the standard Poisson bracket.

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