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Workshop on
Noise and Instabilities in Quantum Mechanics

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Random circuits for efficient noise estimation

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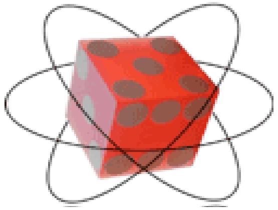
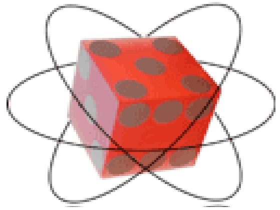


Random Circuits for Efficient Noise Estimation



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Noise and Instabilities in Quantum Mechanics

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Noise Determination for Quantum Information Processing

Knowledge of the **strength** and **type of noise** (e.g., unitary control errors, decoherence from the environment, etc) affecting a prototype quantum processor is crucial for:

- **Engineering of the device design**
  -To determine the physical mechanisms responsible for the noise
  -To benchmark various implementation schemes
  -Etc.

- **Optimizing algorithmic error-correction methods**
  - To determine the “block-size” of correlations
  - To identify the presence of decoherence free subspaces
  - Etc.

The Difficulty of Noise Determination

Full characterization of the noise can be achieved via Quantum Process Tomography (QPT) – this procedure is not *scalable*:

- QPT requires O(2^{4n}) experiments (n = number of qubits).
- Analysis of the tomographic data is complex and inefficient, requiring manipulation of matrices of dimension 2^{2n} £ 2^{2n}

> **Recent implementation of QPT for 3 qubits using NMR:**
> Y. Weinstein, J. Emerson, N. Boulant, M. Saraceno, D. Cory,
> *Quantum Process Tomography of the Quantum Fourier Transform*
> **J. of Chem. Phys. 121**, 6117 (2004).

Question: Can we measure a *benchmark* fidelity of the implementation without requiring an **exponentially large** number of experiments?

Answer: Yes, using *random unitary operators*!

Random Unitary Operators

Some basic concepts

• The "natural" measure that defines random unitary operators is the (unique) invariant group measure (ie., *the Haar measure*) for U(D):

$$\mu_H(dU) = \mu_H\,(W\ dU\ V)$$

where W and V are arbitrary unitary operators.

• An operator drawn from this measure is an element of the Circular Unitary Ensemble (CUE) - it is "*Haar*-random".

• Given some function *f* on U(D), we can define the Haar-average as,

$$\langle\, f(U)\, \rangle = \int_{U(D)} \mu_H(dU)\ f(U)$$

• Note that the Haar-average is often *analytically tractable*.

Random Unitary Operators

Some basic concepts

• The Haar-average is useful because of the "concentration of measure" effect:

For a *wide class* of test functions, (e..g, eigenvalue fluctuations), most unitary operators give values that are ***close to the Haar-average,*** **i.e.,**

$$\langle f(U)^2 \rangle - \langle f(U) \rangle^2 = O(1/D^a), \quad \text{with } a > 0 \quad (D=2^n).$$

• A similar "concentration of measure" effect holds for the Haar-induced measure on the space of pure quantum states:

$$\langle f(\psi) \rangle = \int \mu(d\psi)\, f(\psi)$$

$$\langle f(\psi)^2 \rangle - \langle f(\psi) \rangle^2 = O(1/D^a), \quad \text{with } a > 0.$$

Fidelity Loss under Imperfect Motion Reversal

The **fidelity loss** (instability) under perturbation was proposed initially by Peres (1984) as a means of **characterizing the presence of quantum chaos for a dynamical system.**

The **intuitive idea** was that the inner product between two states, one evolved under the system dynamical operator U, and one under the perturbed operator U_p , would decrease more rapidly with the number of iterations t when the system U is classically chaotic.

$$F_{\psi}(t) = \left| \langle \psi | (U_p)^{-t} (U)^t | \psi \rangle \right|^2$$

$$U_p = U \exp(-iV_p)$$

It turns out this wasn't exactly right – ***often regular systems can exhibit a faster rate of decay*** - **but it is still a very useful way to characterize stability**!

Universal Fidelity Loss for a Random Unitary

A fundamental conjecture of quantum chaos is that a classically ***chaotic quantum system*** may be modeled by ***a random unitary matrix.***

For a random unitary matrix, the fidelity loss is ***tightly concentrated*** about the Haar-average:

$$\left\langle F_\psi{}^2(t)\right\rangle - \left\langle F_\psi(t)\right\rangle^2 = O\left(D^{-1}\right) \equiv O\left(2^{-n}\right)$$

Due to this concentration of measure, the fidelity loss under imperfect motion reversal is *universal* for any *typical* or *generic* unitary operator and a generic initial state:

$$F_\psi(t) = \left\langle F_\psi(t)\right\rangle + O\left(D^{-1/2}\right)$$

FGR Fidelity Loss for a Chaotic System Modeled as Random Matrix

Under certain conditions, the fidelity loss for a classically chaotic unitary operator is given by the "Fermi Golden Rule" (P. Jacquod et al , PRE, 2001):

$$F_{\psi}(t) \cong \left\langle F_{\psi}(t) \right\rangle \cong \exp\left(-\Gamma\, t\right) \; ,$$

$$\Gamma = \frac{1}{D} \left\| V_p \right\|_2^2 = \frac{1}{D} \mathrm{Tr}\,(V^2)$$

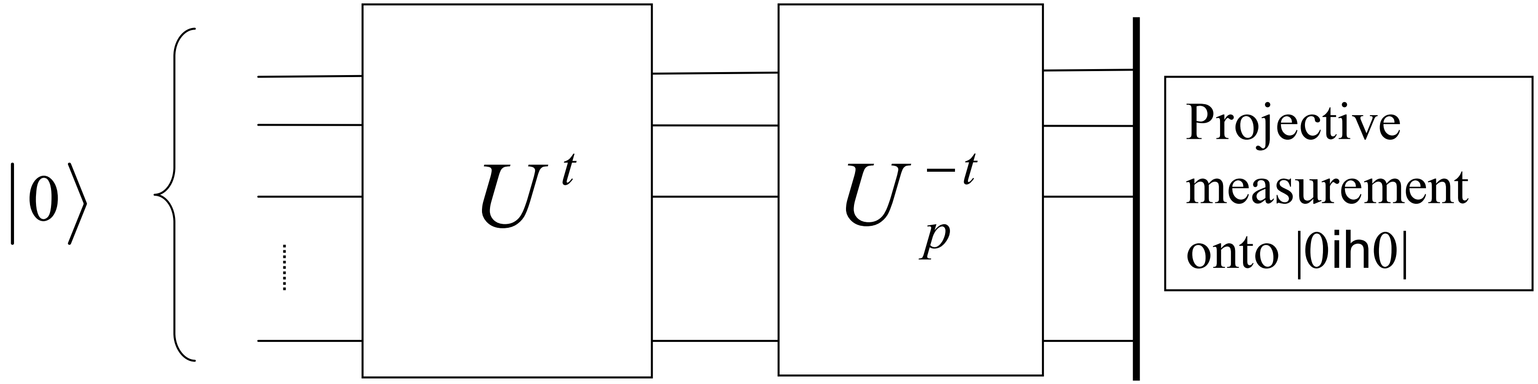
The decay rate **depends only** on the 2-norm of the perturbation operator $\boldsymbol{V_p}$.

Note: ***The random matrix assumption often fails!***

For example, ***if*** the perturbation operator has a semi-classical interpretation, ***then*** the FGR decay may not be observed, e.g., instead one sees a perturbation-independent Lyapunov decay (Jalabert and Pastawski, PRL, 2001).

Measurement of the Fidelity Loss under Imperfect Motion Reversal

The **fidelity loss** can be ***estimated efficiently***, e.g., via the following circuit:



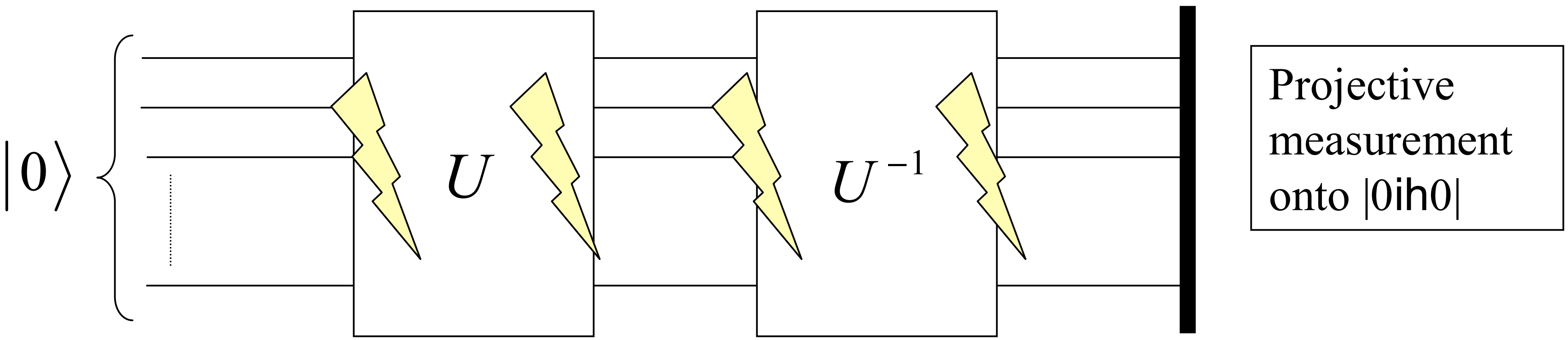
<u>Note:</u> This provides one efficient solution to the "read-out" problem, ie, measuring how some system under study responds to *a known perturbation.*

<u>Note:</u> ***Even a regular system can exhibit FGR decay when the perturbation is "complex"*** – so this is really a ***test of "relative randomness***".

J. Emerson, Y. Weinstein, S. Lloyd, D. Cory, **Phys. Rev. Lett**. **89**, 284102 (2002).

How to Measure Unknown Device Noise via a "Perfect" Motion Reversal Circuit

Now suppose that the unitary is chosen at "random" and that the only perturbation is due to the **actual (unknown) noise mechanisms** of the *device*:



$$F(U, \Lambda) = \sum_k \langle 0|U^\dagger A_k U|0\rangle\langle 0|U^\dagger A_k^\dagger U|0\rangle$$

where $\Lambda(\rho) = \sum_k A_k \, \rho \, A_k^\dagger$ is a ***CP map representing arbitrary (non-unitary) noise.***

Note that this CP map corresponds to the ***cumulative noise*** occurring ***throughout*** the implementation of the entire motion-reversal circuit.

Universal Fidelity Loss Under General Noise

<u>**Key Idea:**</u> Due to concentration of measure, the ***observed fidelity decay*** under this motion reversal experiment does ***not depend*** on the choice of random unitary (nor on the initial state), but ***depends only on the physical noise mechanisms affecting the implementation.***

$$\langle F(U,\Lambda)\rangle = p + \frac{(1\text{-}p)}{D}$$

$$p = \frac{\sum_k |Tr(A_k)|^2 - 1}{D^2 - 1} \leq 1$$

$$F(U,\Lambda) = \langle F(U,\Lambda)\rangle + O\left(D^{-1/2}\right)$$

- The fidelity loss under the experiment is uniquely characterized by a single invariant of the noise, the noise ***strength parameter p***.

J. Emerson, R. Alicki, and K. Zyczkowski, ***J. Opt. B: Quantum and Semiclassical Optics***, 7 (2005) S347-S352 (quant-ph/0503243).

Randomization in Quantum Information

➢There is a **growing body of work** making use of **Haar-randomization** across quantum information theory:

✓Random unitaries lead to **resource savings in remote state preparation** (Bennett et al., 2003)

✓ Random unitaries enable **superdense coding of quantum states** (Harrow et al., 2003)

✓ Random unitaries provide **approximate encryption and data-hiding of quantum states** (Hayden et al., 2003)

✓ Random unitaries can be used in quantum simulations on a quantum processor to study the **quantitative aspects of decoherence** under **engineered couplings** to a complex environment (Ryan et al, 2005).

➢Also, since random (and pseudo-random) numbers play a fundamental role in classical information theory, random (and pseudo-random) states and operators should play an equally fundamental role in quantum information theory.

Generating Haar-Random Unitary Operators on a Quantum Information Processor

➢The implementation of Haar-random unitary operators on a quantum processor is useful for randomization in quantum information theory.

Problem: any circuit generating *Haar*-random unitary operators requires **exponential** resources:

- $O(n^3 D^2)$ **quantum gates** (where $D = 2^n$)
- D^2 **random classical input parameters.**

From Haar-Random to “*Pseudo*-Random” Unitary Operators

Challenge: generate unitary operators that exhibit useful statistical features of uniformly random unitary operators but require only polylog(D) resources.
Call these “**pseudo-random**” unitary operators.

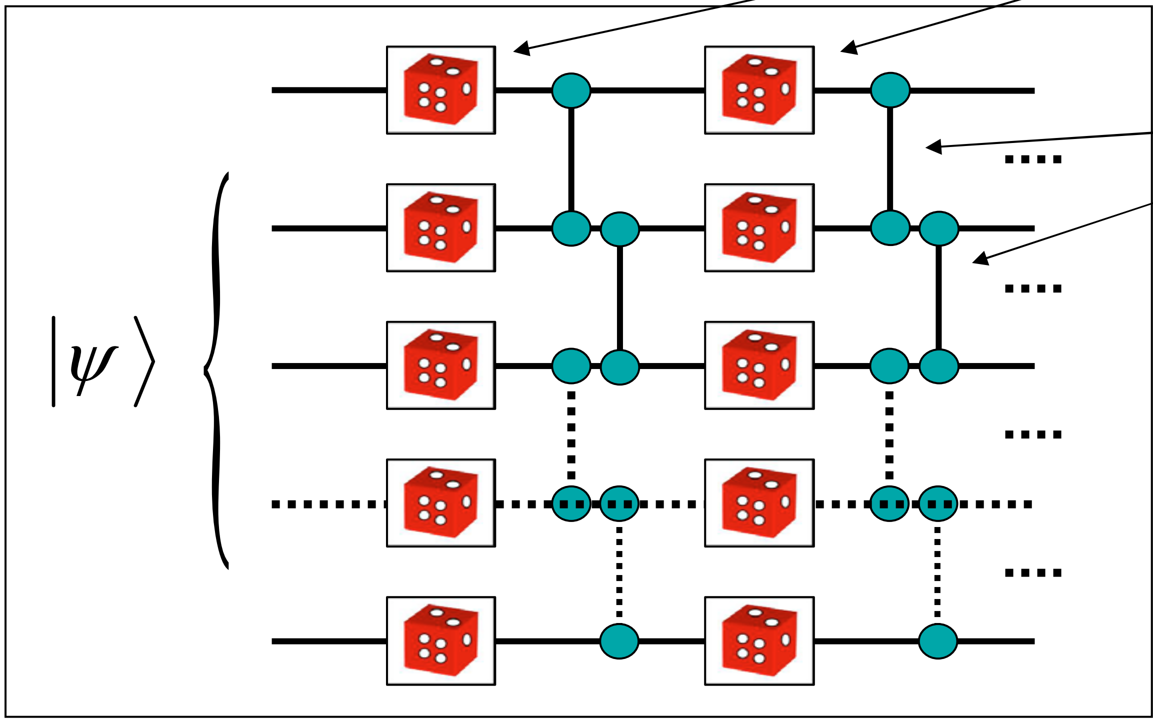
Question: Is it possible to efficiently generate pseudo-random operators that are ***sufficiently randomizing*** for practical applications?

Remark: We can expect this to be possible in light of ***efficient simulation algorithms for quantum chaos*** [R. Schack (1998), Georgeot and Shepelyansky (2001), Benenti et al (2001)].

Approach: generate ***maximally random circuits*** by drawing gates randomly from a **universal gate set**.

How to Generate Pseudo-Random Unitary Operators using Random Circuits

Schematic of *2 iterations* of a random circuit:



Apply random SU(2) rotations on each qubit

Couple the qubits with any "convenient" or "natural" coupling.

A Practical Circuit for Generating Pseudo-Random Unitary Operators in NMR

Basic Gate: independent *random* SU(2) rotations on each qubit followed by scalar couplings between nearest-neighbor pairs:

$$U = \left(U(2)^{\otimes n_q}\right) \prod_{j=1}^{n_q - 1} \exp(-i\, J\, \sigma_z^j \otimes \sigma_z^{j+1})$$

Note: This parameterized gate is universal for quantum computation.

Idea: Repeat this basic gate until the measure of random circuits can mimic (to within some ħ, any desired statistical feature of the Haar measure.

J. Emerson, Y. Weinstein, M. Saraceno, S. Lloyd, D. Cory, ***Science,*** vol. 302, 2098 (2003).

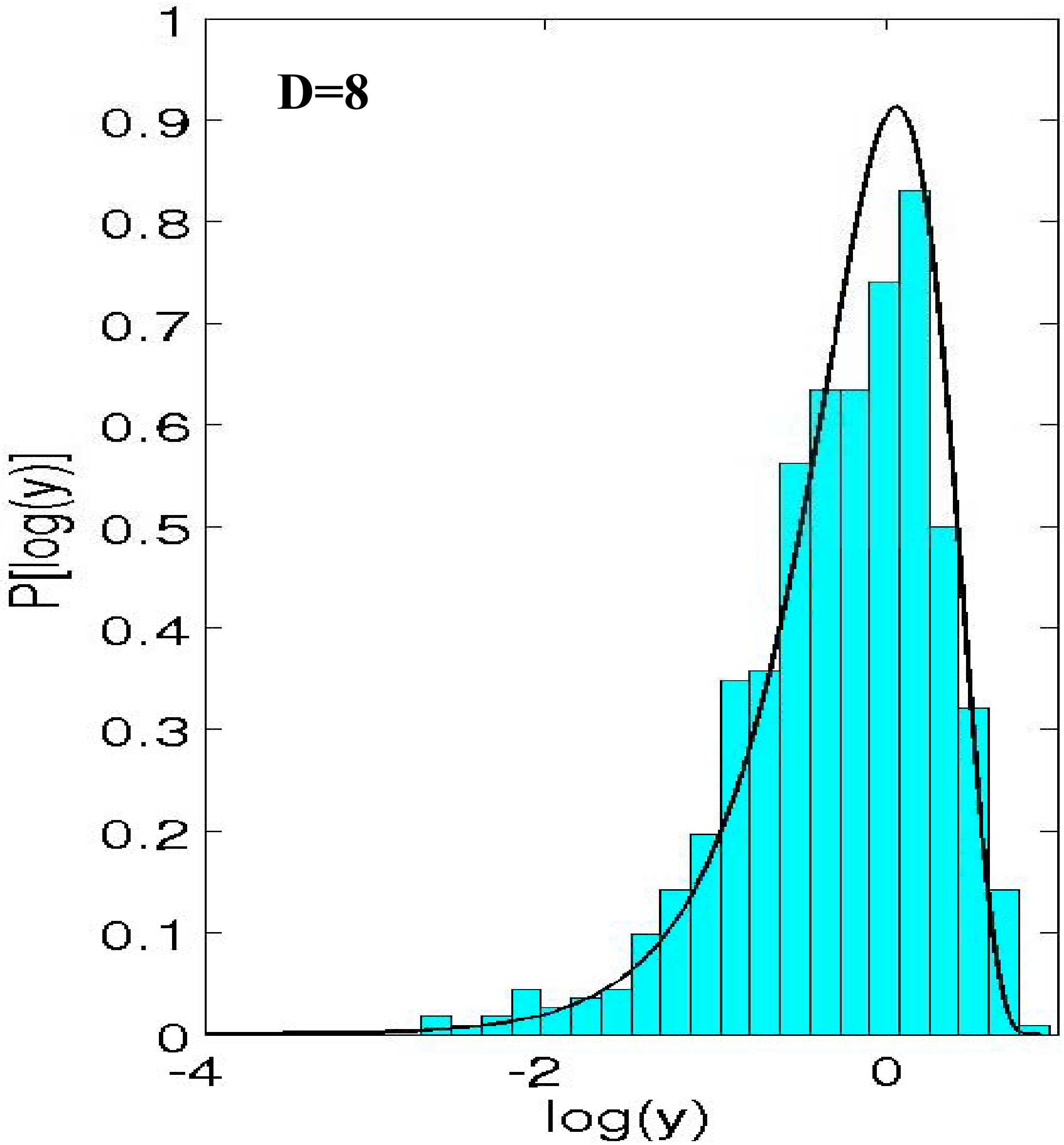
NMR Data for m=7 Iterations on 3 qubits

Histogram of the squared modulus of the matrix elements of the random circuit:

$y = D\, |U_{ij}|^2.$

Solid line: $P(y)$ from Haar-average.

The number of gates required for generating pseudo-random unitary operators is accessible with current levels of control in NMR.



Random Circuits Converge Exponentially to the Invariant (Haar) Measure

Given an initial probability measure ***p*** on ***U(D)***, the measure for the circuit composed of a product of ***m*** elements drawn from ***p*** is given by the convolution:

$$p^{*m} = p * p \cdots * p$$

If the probability measure ***p*** that has **support** over a subset generating **U(D)**, i.e, ***p*** contains a **universal gate set**, then ***p**m*** converges **exponentially** to the Haar measure.

<u>**Conjecture:**</u> **efficient random circuits** provide an adequate means of generating **pseudo-random distributions of unitary operators.**

Efficient means that ***m***, the number of iterations of the random circuit, does not increase exponentially with ***n***, the number of qubits.

Notes on the Proof of Convergence

- The Peter-Weyl theorem provides a discrete Fourier representation for functions on compact groups in terms of the irreps of the group.

- The Fourier transform of $\boldsymbol{p}^{*\boldsymbol{m}}$ is given by the ***m***'th power of the Fourier transform coefficient matrices (for each "irrep" *s*) of ***p*** ,

$$p^{*m} \Leftrightarrow [\widetilde{p}^{(s)}]^m$$

- It is possible to show that all eigenvalues of the Fourier coefficients are strictly less than 1.

- The ***largest spectral gap*** in the eigenvalues of the Fourier coefficients ***determines the rate of exponential convergence.***

J. Emerson, E. Livine, S. Lloyd, quant-ph/0503210.

Convergence to the Uniform Measure

For p continuous and p**2** L^2**(U(D)),** only a finite set of coefficients are needed in the Fourier representation to demonstrate ***uniform convergence***.

Warning: Exponential convergence (with respect to m) does not imply that random circuits are ***efficient*** because the exponent can depend on the dimension $D = 2^n$.

For example, ***for uniform convergence*** we know from simple counting arguments that the exponent must decrease at least as fast as $O(1/D^2)$ (since most unitary operators require an exponential number of gates).

This means that m, the number of iterations of the random circuit, must grow exponentially with the number of qubits n…
Of course, this isn't a worry since uniform convergence is unnecessarily strong!

Convergence Conditions for Practical Applications

For ***practical applications*** we only require convergence with respect to some class of ***test functions*** (ie, convergence with respect to the weak topology).

$$\mathbf{s}_{\mathbf{g2\ U(D)}}\ \mathbf{d\mu(g)}\ \boldsymbol{p^{*m}(g)\ f(g)}\ \mathbf{!}\ \mathbf{s}_{\mathbf{g2\ U(D)}}\ \mathbf{d\mu(g)}\ \boldsymbol{f(g)}$$

For ***polynomial test functions f,*** the degree of the polynomial implies a ***cut-off in the Fourier representation*** of ***p*** (the measure over random gates).

Hence the lower the degree of the polynomial, the smaller the number of Fourier coefficients that contribute to the rate of convergence.

For example, the ***fidelity under motion reversal*** is just a ***4th degree polynomial in the matrix elements of U***,

$$\mathbf{F(U,\Lambda) = \textstyle\sum_k h0|U^y A_k U|0ih0|U^y A_k{}^y U|0i}$$

$$\mathbf{h\ F(U,\Lambda)\ i\ /\ h\ U_{ij}\ U_{kl}\ U_{mn}{}^*\ U_{pq}{}^*\ i}$$

Numerical Test of Weak Convergence: Average Subsystem Purity

As a ***benchmark*** consider the ***average subsystem purity*** of the pure states generated by a ***set of random circuits*** acting on the $|0\rangle^{\otimes n}$ state.

The ***subsystem purity of these states is also a 4th degree polynomial*** in the matrix elements of the unitary operators generating them, so it will have the ***same convergence rate as the "fidelity under motion reversal".***

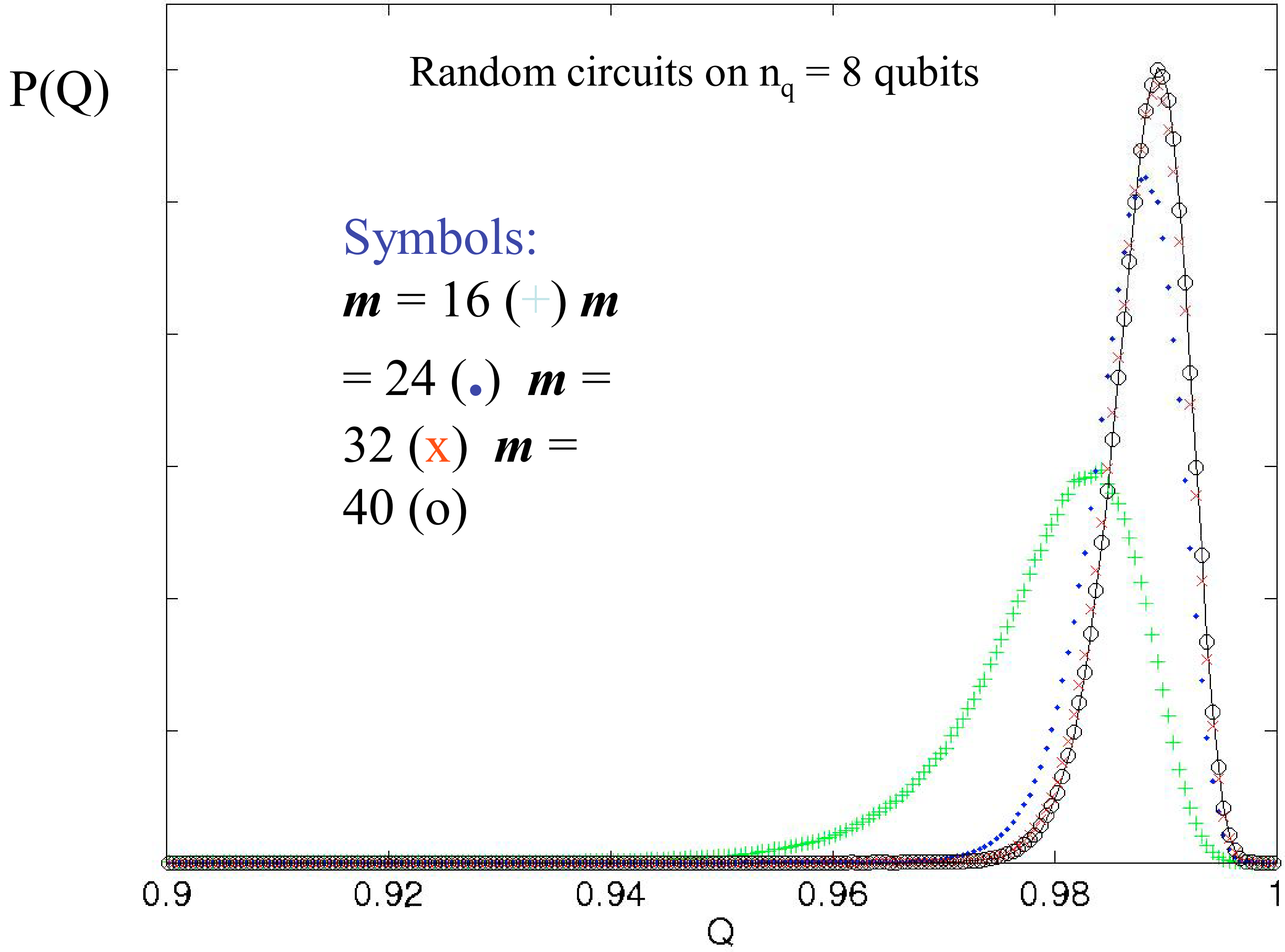
A practical indicator is the ***average qubit purity***:

$$Q = 2 - \frac{2}{n_q} \sum_{i=1}^{n_q} Tr\left[\rho_i^2\right]$$

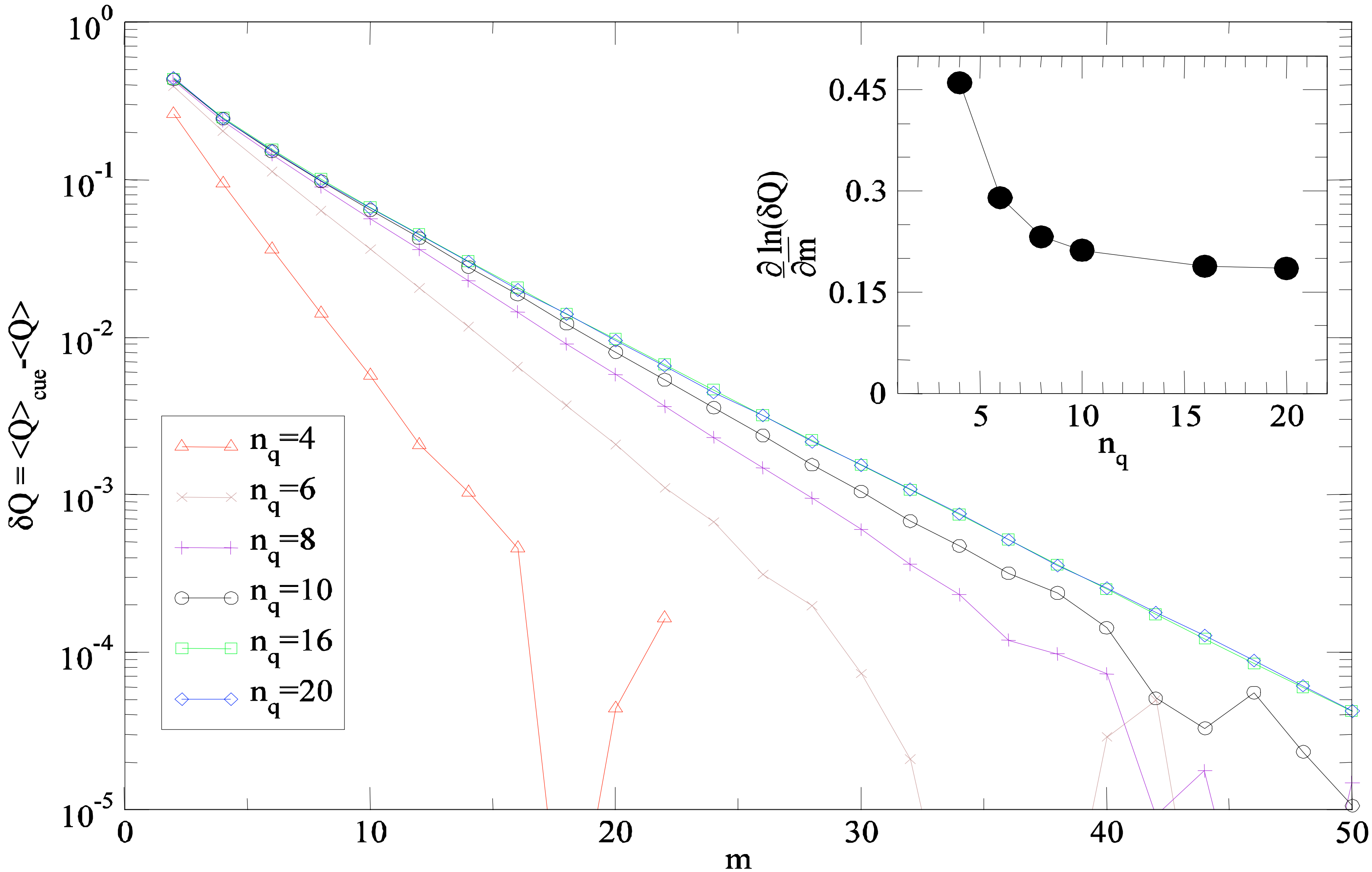
n_q = number of qubits

ρ_i = reduced density matrix for qubit i

Distribution of Average Subsystem Purity Generated by Random Circuits

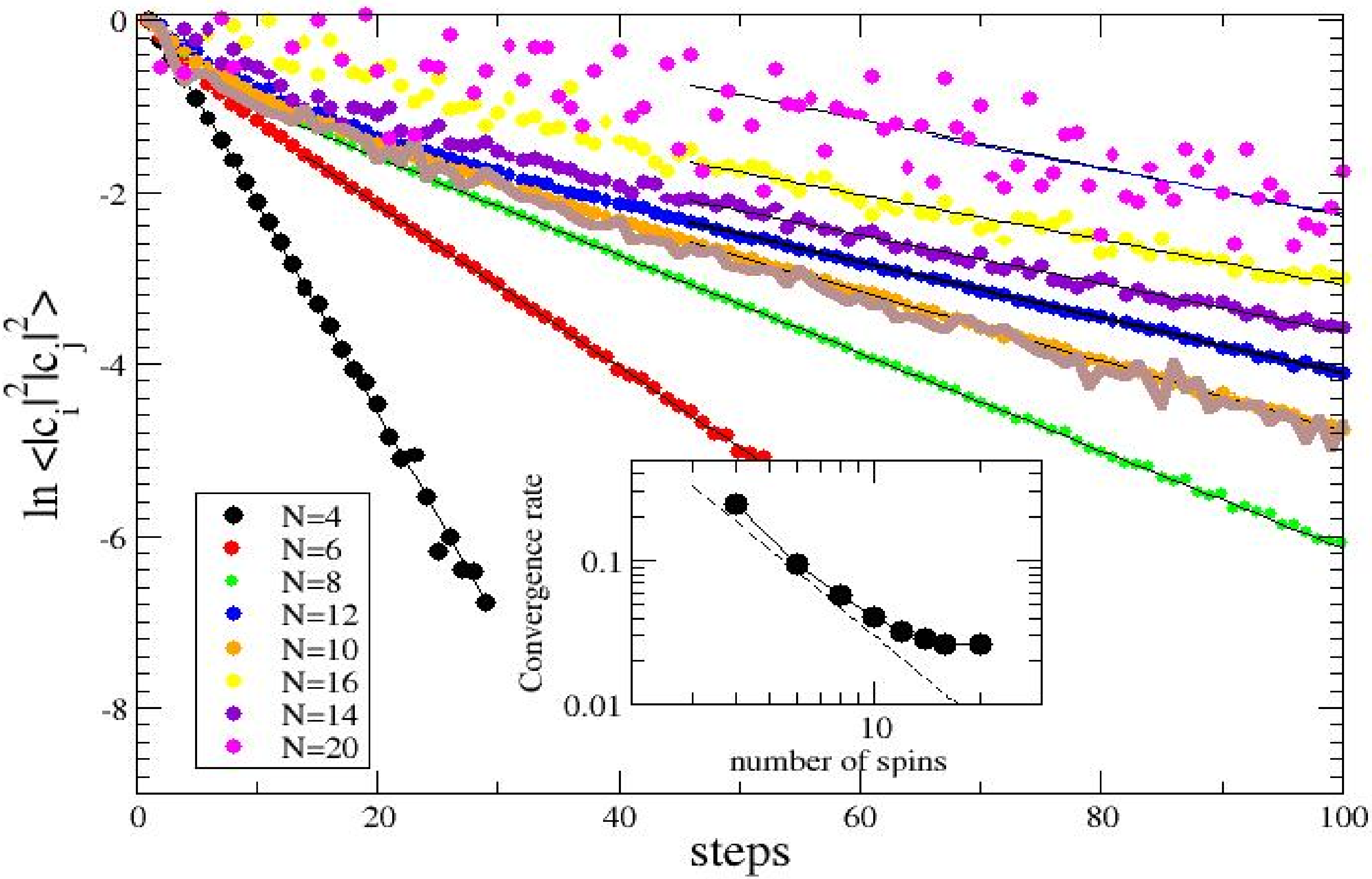


Saturation of exponential rate implies *efficiency*!



F. Cucchietti and J. Emerson, in preparation

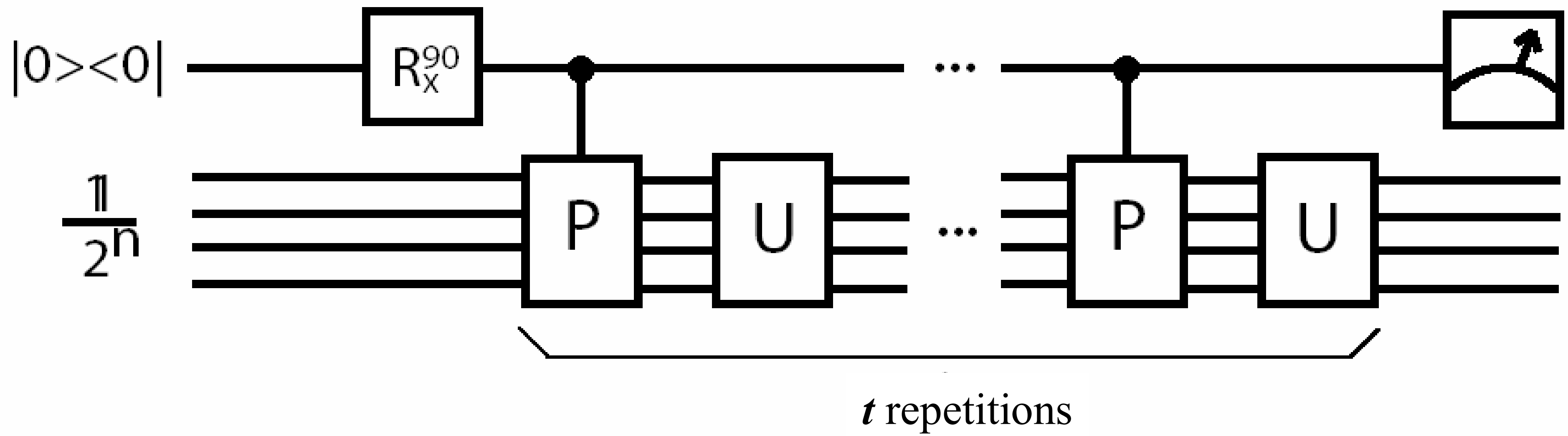
Saturation of exponential rate suggests efficiency



F. Cucchietti and J. Emerson, in preparation

Connection between fidelity decay and decoherence:

DQC1 circuit:



With DQC1 we get the exact ensemble average for free:

$$\rho_{12}(t) = \rho_{12}(0)\left\langle F_{\psi}(t)\right\rangle \cong \rho_{12}(0)\exp(-\Gamma t)$$

D. Poulin, R. Blume-Kahout, H. Ollivier, R. Laflamme, PRL (2003).

NMR Measurement of Decoherence Rates under for Arbitrary (Complex) Environment Dynamics

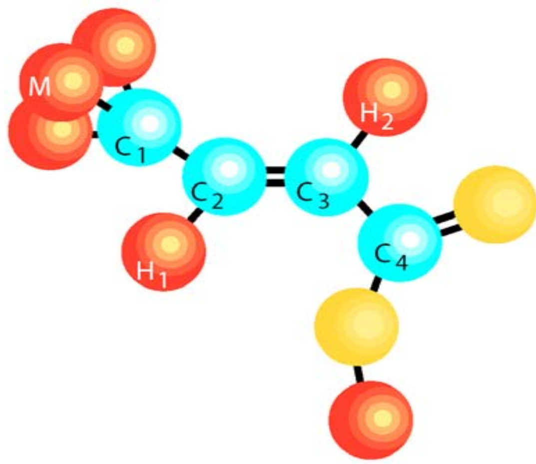
Implementation of DQC1 circuit with liquid state NMR:

4 Carbon spins: 4 qubits prepared in identity state

Hydrogens: decoupled (each prepared in σ_z state)

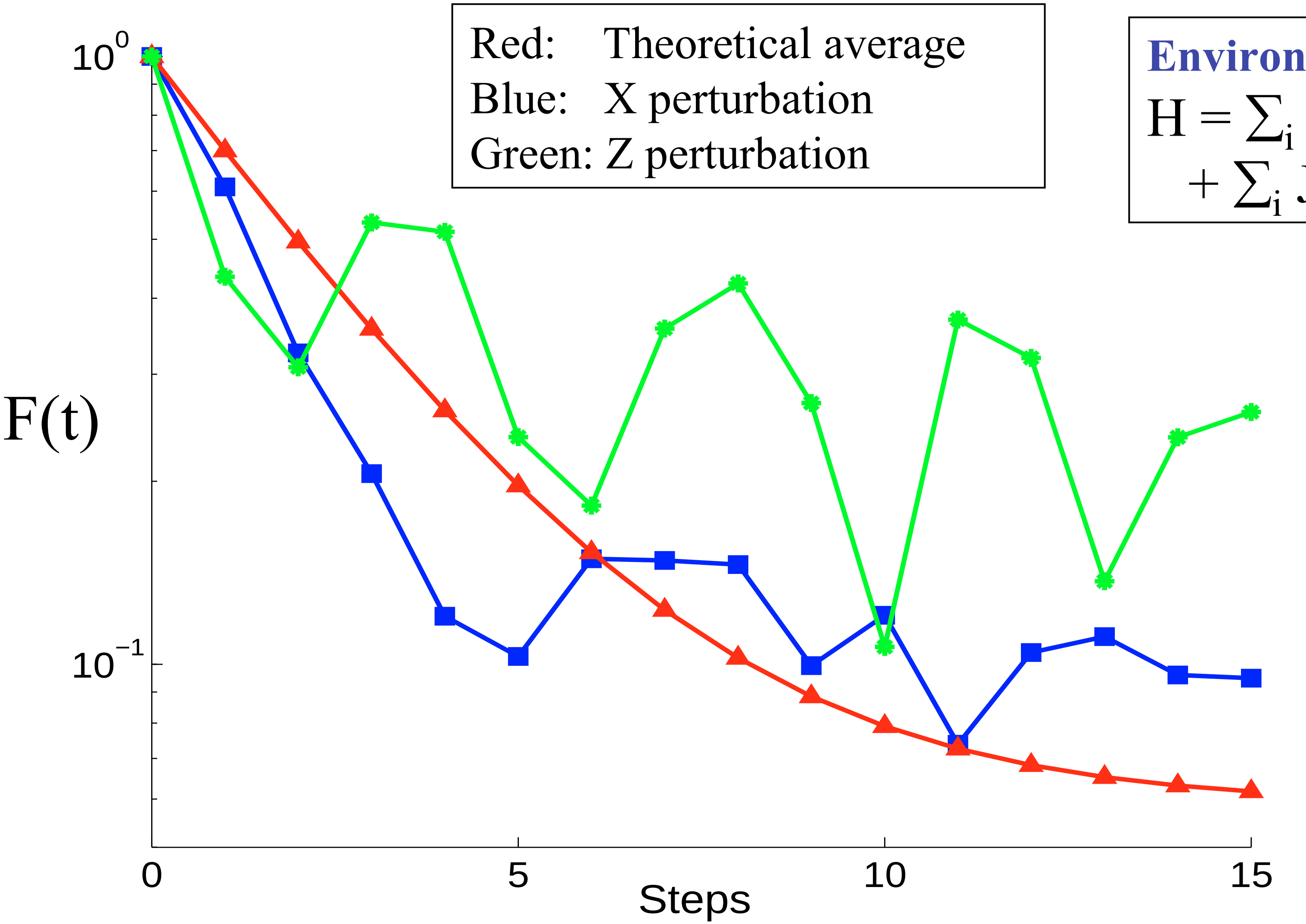
Methyl Group: pseudo-pure spin-1/2.

Crotonic Acid



C. Ryan, J. Emerson, C. Negreverne,
D. Poulin, R. Laflamme, in preparation.

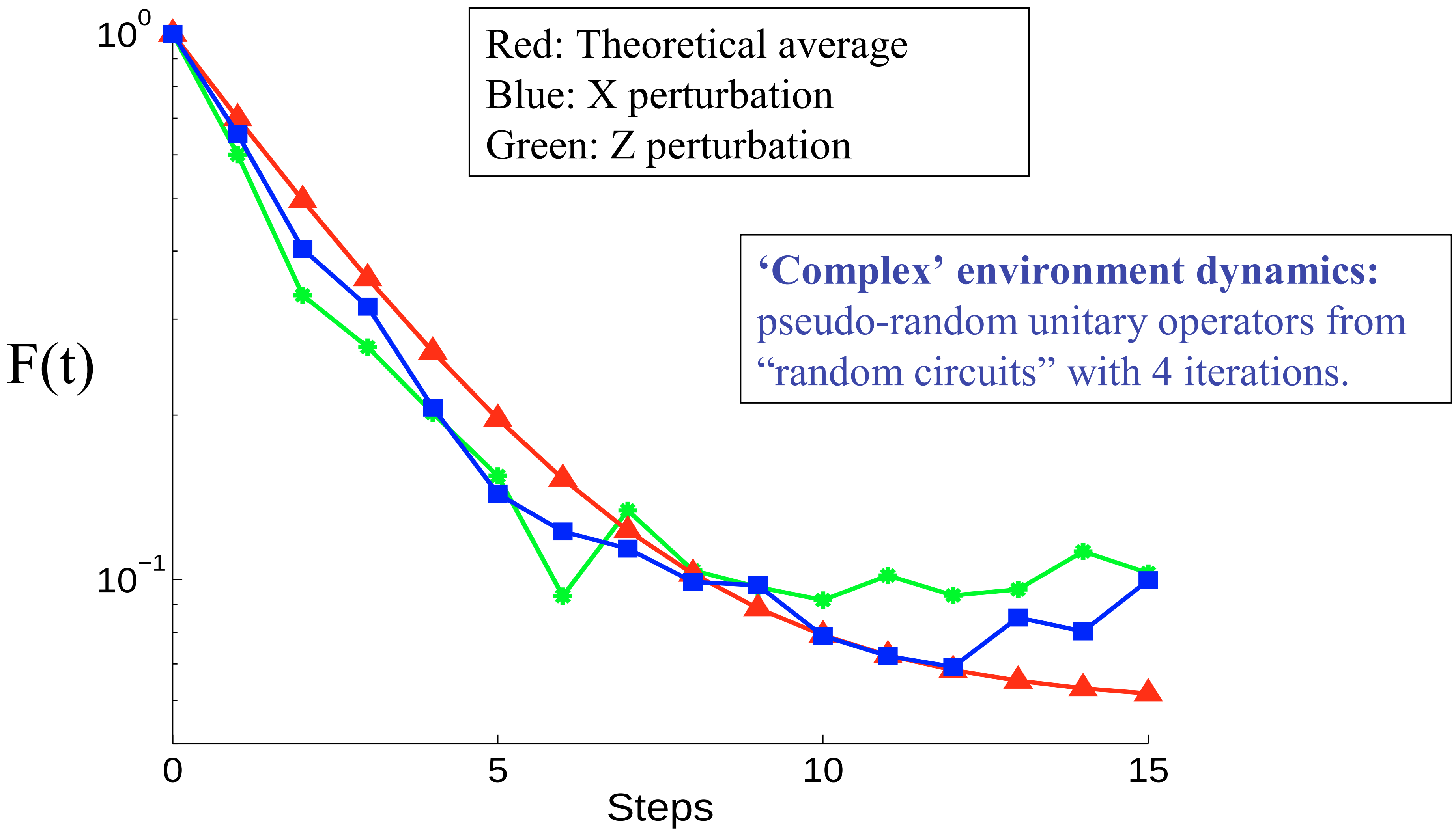
Non-Universal Fidelity Loss (Decoherence) for 'Regular' System (Environment) Dynamics



Environment dynamics:

$$H = \sum_i \omega_i \sigma_z^I + \sum_i J_i \sigma_z^i - \sigma_z^{i+1}$$

<u>Universal</u> Fidelity Loss (Decoherence) for "Complex" System (Environment) Dynamics



Some Important Open Problems

1. **Pseudo-Random Operators via Random Circuits:** analytically determine the ***actual convergence rates*** for the weaker randomization conditions defined by ***noise-estimation*** and ***other randomization applications***.

2. **Generalized Noise Estimation Methods:** develop ***generalized protocols*** for measuring additional parameters of the noise, e.g., **correlation length-scales and time-scales of the noise**.

Thank you for your attention!