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Workshop on
Noise and Instabilities in Quantum Mechanics

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Noise models for open quantum maps and algorithms

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Noise Models for Open Quantum Maps and Algorithms

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- Quantum Maps
- Open Quantum Maps
- Models of Noise

  a) Diagonal noise in a phase space basis

  b) Diagonal noise in a qubit basis
- Spectral Properties
- The quantum-classical correspondence
  (a spectral perspective)
- Asymptotic decays

Phys.Rev.Lett.91,064101(2003); Phys.Rev.A70,062301(2004); Phys.Rev.E69,056211(2004); quant-ph/0504211(2005)

with L. Aolita, I. Garcia-Mata, G. Carlo

Models of Unitary Dynamics - Quantum Maps

•Area preserving maps model hamiltonian dynamics without the need to integrate differential equations.

•Quantization leads to finite unitary matrices of dimension $N = (2\pi\hbar)^{-1}$

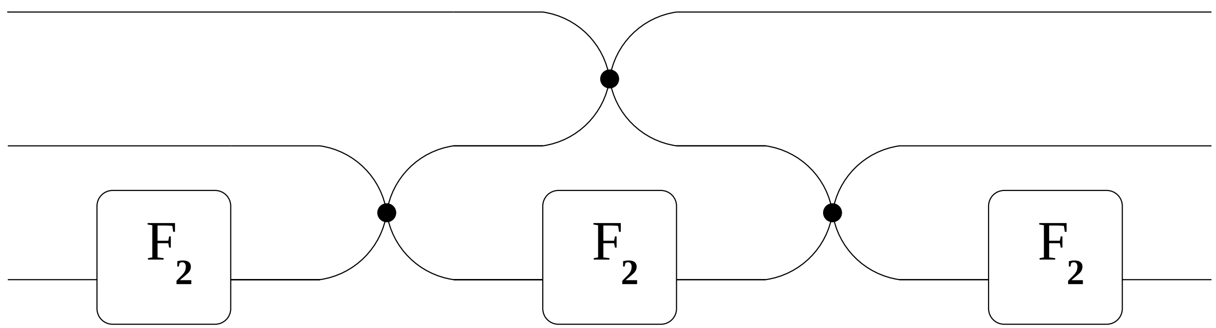
•Maps are the simplest classical systems where chaos can occur.

•Many maps are available :

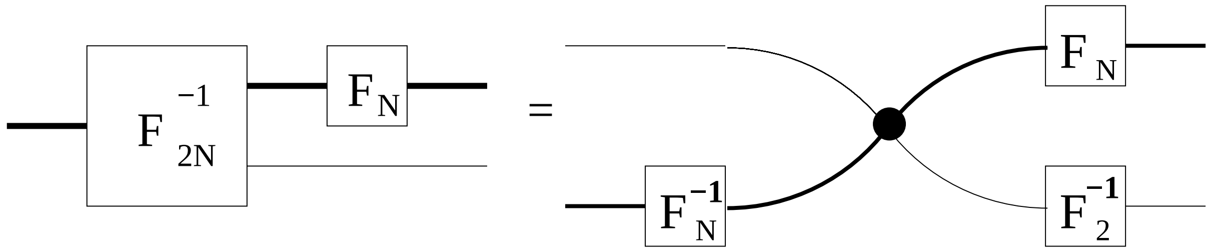
hyperbolic: baker, cats, sawtooth,....
elliptic: cats, FFT,
parabolic: shears, translations
mixed: kicked systems

•Many of these maps have efficient qubit circuit decompositions

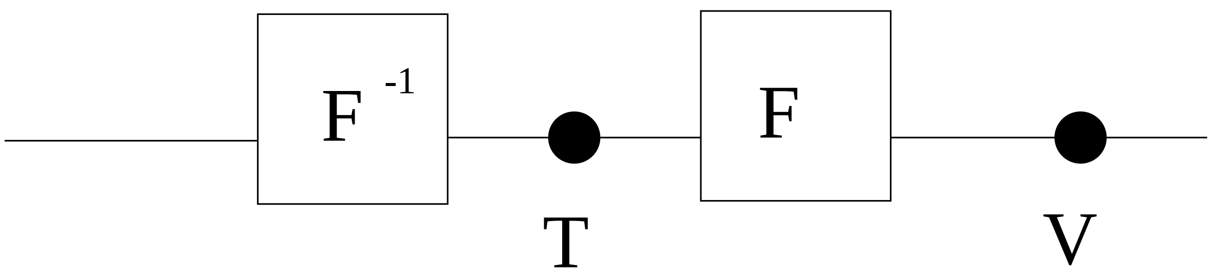
- The quantum Fourier Transform



- The baker's map



- Kicked maps



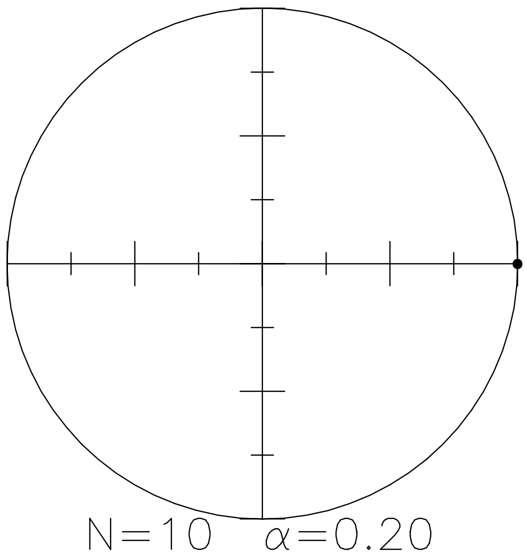
Unitary Spectrum

•Floquet spectrum of the quantum map

$$\hat{U}|\,\phi_k\,\rangle = \mathrm{e}^{\mathrm{i}\xi_k}|\,\phi_k\,\rangle$$

•Propagation of the density matrix is $\rho' = \hat{U}\rho\hat{U}^\dagger = \mathbf{U}(\rho)$. The spectrum of $\mathbf{U}$ is unitary and given by

$$\mathbf{U}(|\,\phi_k\,\rangle\langle\,\phi_j\,|) = \mathrm{e}^{\mathrm{i}(\xi_k - \xi_j)}\,|\,\phi_k\,\rangle\langle\,\phi_j\,|$$



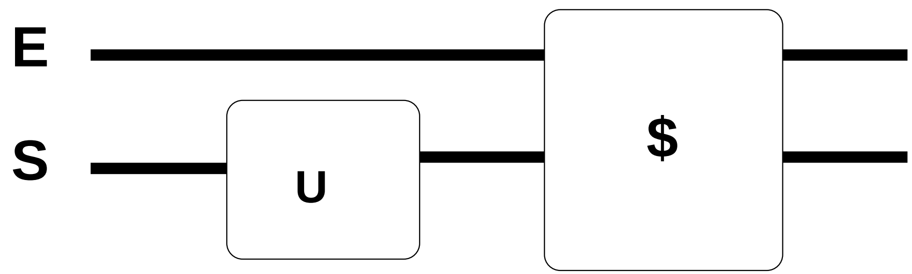
Open Quantum Maps

In the Markovian Approximation open quantum systems are described by the Lindblad equation

$$\mathrm{i}\hbar\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = [\hat{H},\hat{\rho}] + \mathrm{i}\sum_j (\hat{L}_j\hat{\rho}\,\hat{L}_j^\dagger - \frac{1}{2}\hat{L}_j^\dagger\hat{L}_j\hat{\rho} - \frac{1}{2}\hat{\rho}\,\hat{L}_j^\dagger\hat{L}_j)$$

whose solution for a finite time step has the Kraus form $\mathbf{L}(\rho) = \sum_i \hat{M}_i\rho\hat{M}_i^\dagger$ with $\sum_i \hat{M}_i^\dagger\hat{M}_i = \mathbb{1}$

We consider the evolution split into a unitary step followed by a noisy step $\mathbf{L} = \mathbf{\$} \circ \mathbf{U}$



Models of Noise

We choose an orthonormal basis Λ_α (the noise basis) for the space $\mathcal{H}_{N^2}$. In this basis the most general action on density matrices is

$$\begin{aligned}
\$(\rho) &= \sum_{\alpha,\beta} c_{\alpha,\beta} \Lambda_\alpha \rho \Lambda_\beta^\dagger \\
c_{\alpha,\beta} &\geq 0 \quad \text{for complete positivity} \\
\sum_{\alpha,\beta} c_{\alpha,\beta} \Lambda_\alpha^\dagger \Lambda_\beta &= \mathbb{1} \qquad \text{for trace preservation}
\end{aligned}$$

If $c_{\alpha,\beta}$ is diagonalized with non-negative eigenvalues this form reduces to the Kraus representation. Different models are obtained by choosing a basis Λ_α and a matrix of coefficients $c_{\alpha,\beta}$.

Quantum Mechanics on the Torus

•Classical phase space with periodic boundary conditions (2-torus)
•Hilbert space of finite dimension N ($2\pi\hbar = \frac{1}{N}$)
•Discrete values for position and momentum

$$
\begin{aligned}
|\, q_i \,\rangle &= |\, i/N \,\rangle \quad |\, p_j \,\rangle = |\, j/N \,\rangle \\
\langle\, p_j |\, q_i \,\rangle &= \frac{1}{\sqrt{N}} \mathrm{e}^{-\mathrm{i}\frac{2\pi}{N} ij} \quad (DFT)
\end{aligned}
$$

•Unitary translation operators $\hat{T}_{(q,p)}$ $(q, p = 0, N-1)$

$\hat{T}_{(q,p)}$ is a set of N^2 orthogonal operators that can be used to expand an arbitrary operator as

$$
\hat{A} = \sum_{q,p} a(q,p) \hat{T}_{(q,p)} \qquad a(q,p) = \mathrm{tr}[\hat{A}\hat{T}^{\dagger}_{(q,p)}]
$$

(the chord representation).

Diagonal noise in the unitary translation basis

$$\mathbf{\$} = \sum_{q,p} c(q,p)\; \hat{T}_{(q,p)} \otimes \hat{T}^{\dagger}_{(q,p)}$$

with $\sum c(q,p) = 1$ to preserve the trace.

The spectrum of $\mathbf{\$}$ is easily obtained as

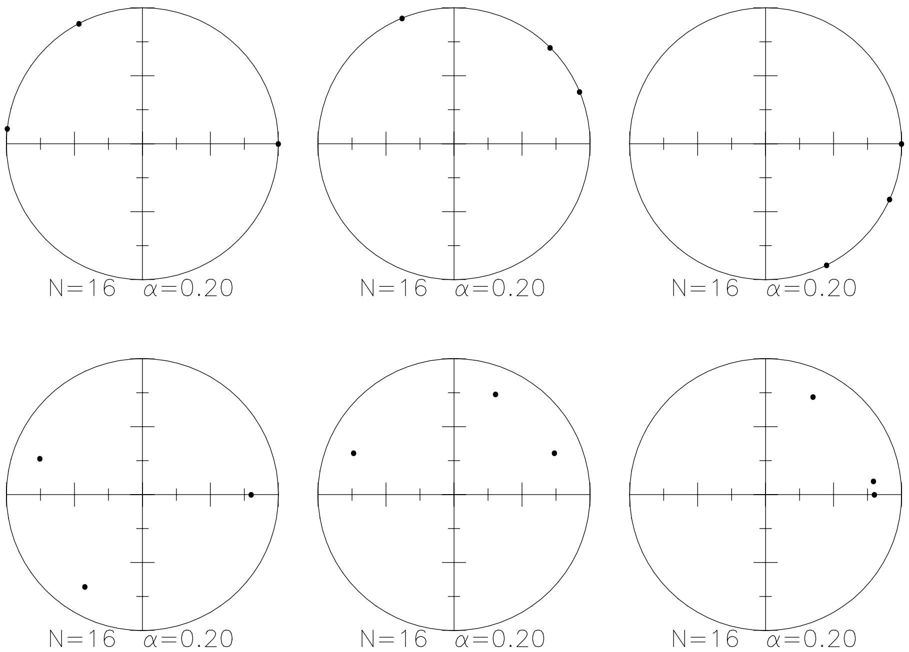
$$\begin{aligned}
\mathbf{\$}\left(\hat{T}_{(\mu,\nu)}\right) &= \sum_{q,p} c(q,p)\hat{T}_{(q,p)}\hat{T}_{(\mu,\nu)}\hat{T}^{\dagger}_{(q,p)} \\
&= \sum_{q,p} c(q,p) \exp\left[\mathrm{i}\frac{2\pi}{N}(\nu q - \mu p)\right] \hat{T}_{(\mu,\nu)} \\
&= \widetilde{c}(\mu,\nu)\; \hat{T}_{(\mu,\nu)}
\end{aligned}$$

Different models can be specified either by $c(q,p)$ or by $\widetilde{c}(\mu,\nu)$.
They all share the property that $\hat{T}_{(q,p)}$ are eigenoperators of $\mathbf{\$}$.

Depolarizing Noise

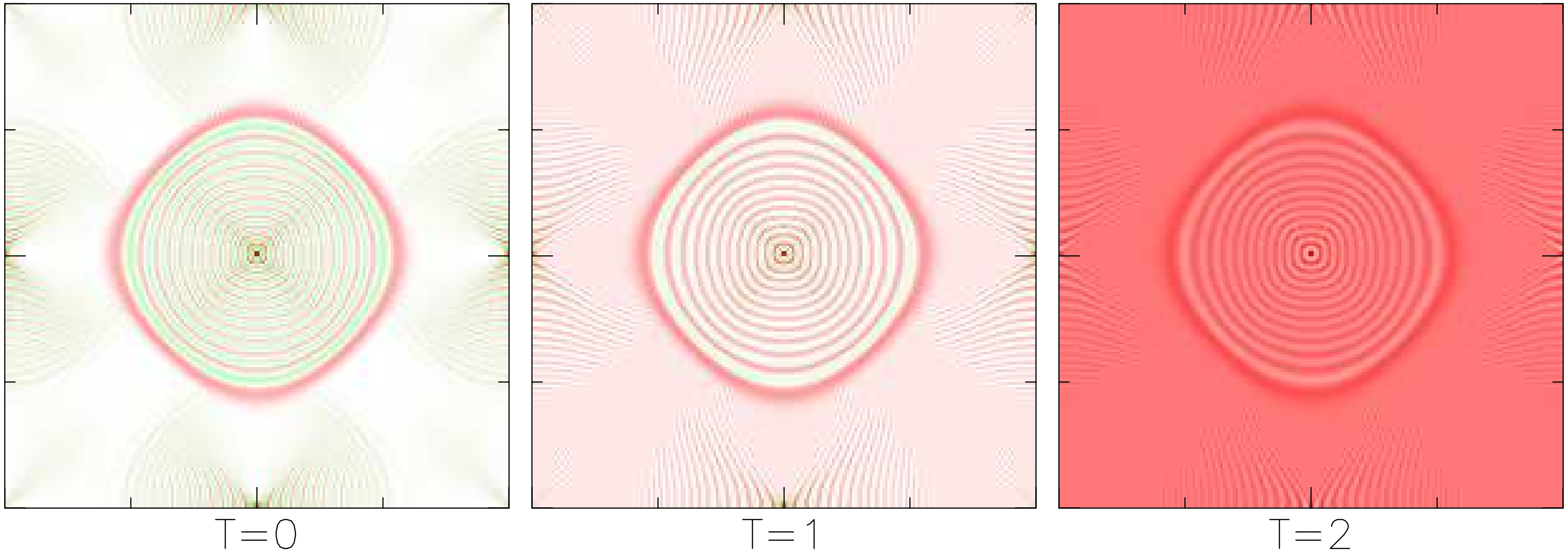
$$\mathbf{\$}_{dep} = (1-\alpha)\hat{T}_{(0,0)} \otimes \hat{T}^{\dagger}_{(0,0)} + \frac{\alpha}{N^2}\sum_{q,p} \hat{T}_{(q,p)} \otimes \hat{T}^{\dagger}_{(q,p)}$$

The eigenvalues are $\widetilde{c}(0,0) = 1$ and $\widetilde{c}_{\epsilon}(\mu,\nu) = 1-\alpha$ otherwise.



Hyperbolic Elliptic Mixed

Depolarizing Noise

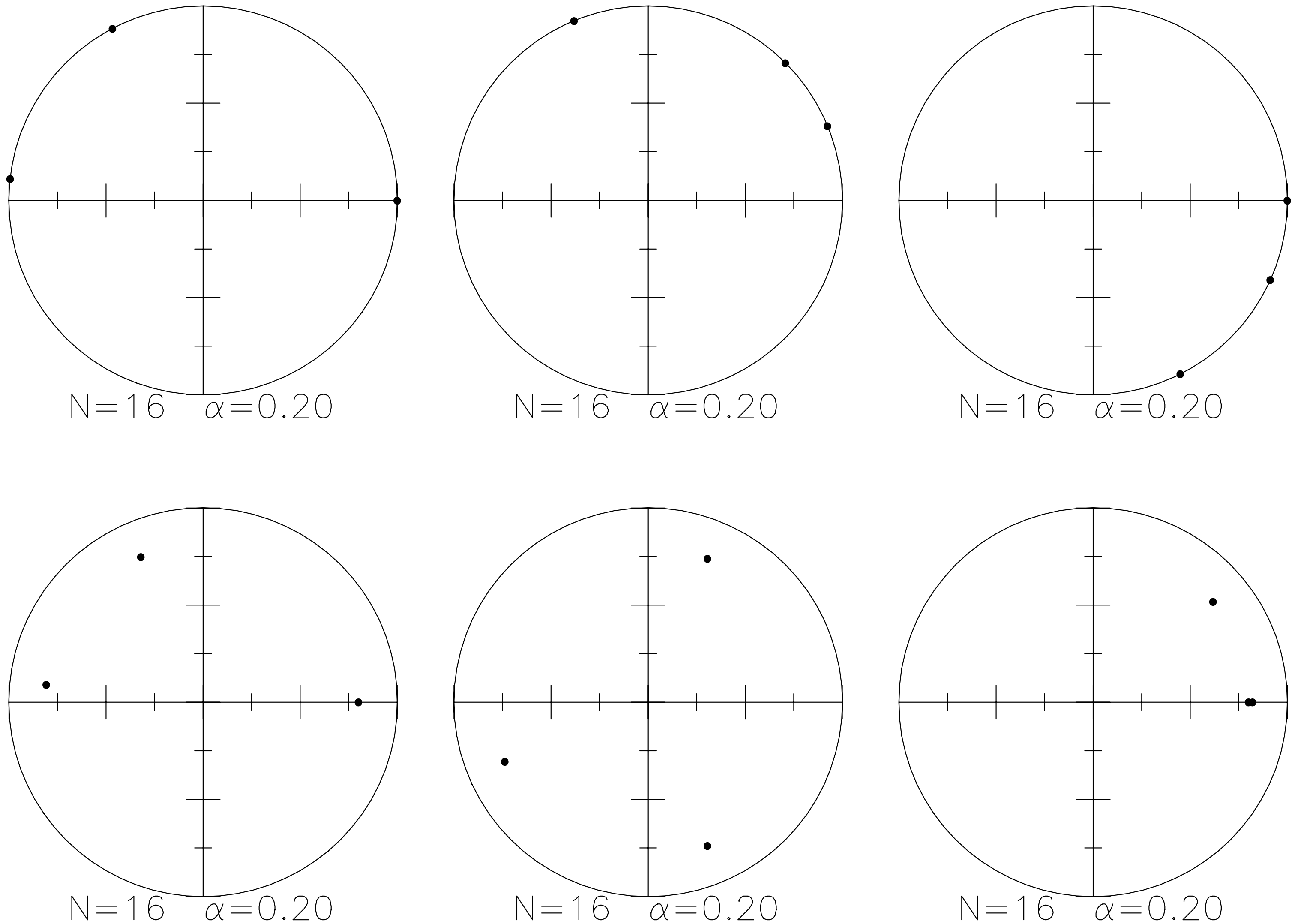


Phase Damping Noise

$$\$_{phd} = (1-\alpha)\hat{T}_{(0,0)} \otimes \hat{T}^{\dagger}_{(0,0)} + \frac{\alpha}{N}\sum_{p} \hat{T}_{(0,p)} \otimes \hat{T}^{\dagger}_{(0,p)}$$

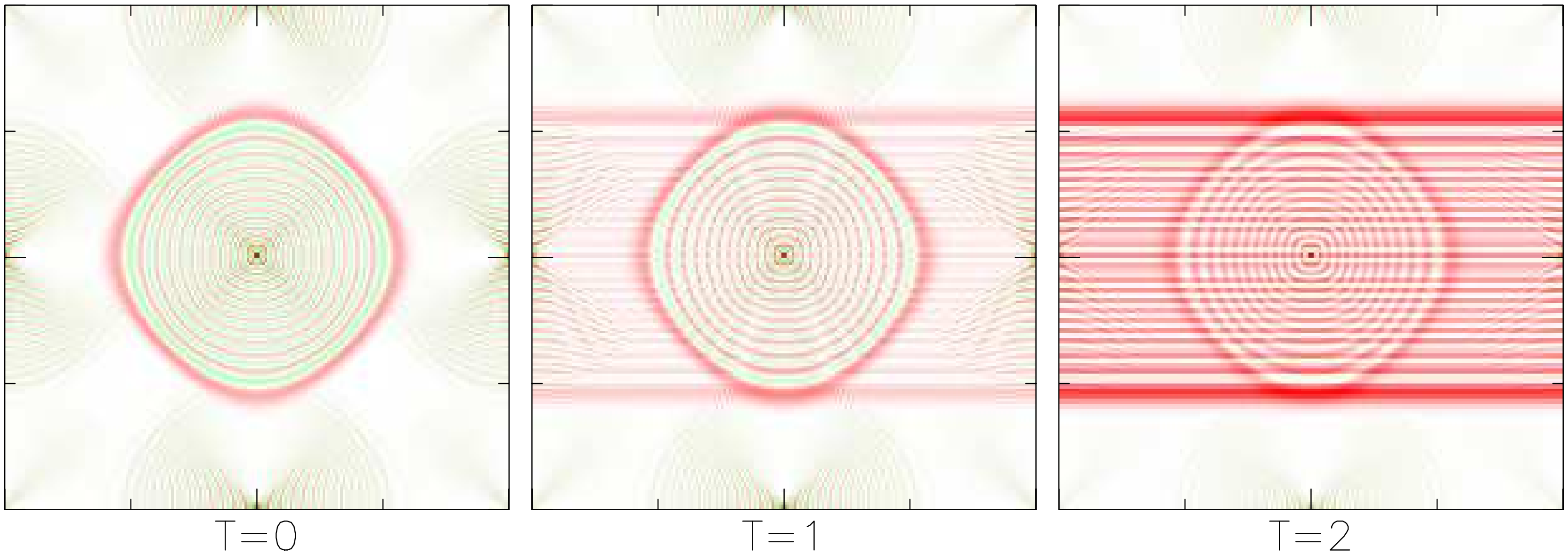
The eigenvalues are $\widetilde{c}(0,p) = 1$ for $p = 0, ...N-1$ and $\widetilde{c_{\epsilon}}(\mu,\nu) = 1-\alpha$ otherwise.

This noise leaves invariant the diagonal elements of the density matrix in the momentum basis and reduces by a factor $1-\alpha$ the off diagonal ones. Thus it leads to a classical distribution with the momentum as a pointer basis.



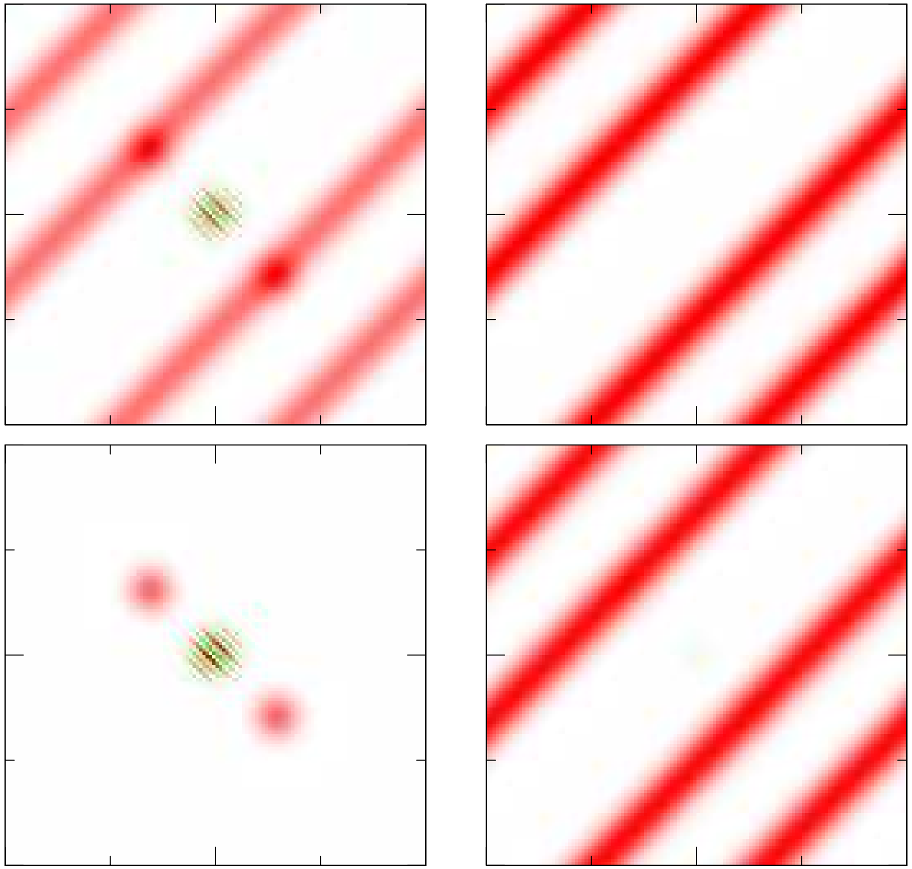
Hyperbolic Elliptic Mixed

Phase Damping Noise



Phase damping on a selected pointer basis

Using the symplectic invariance we can change at will the pointer basis and produce the decoherence in a basis related to any line in phase space.



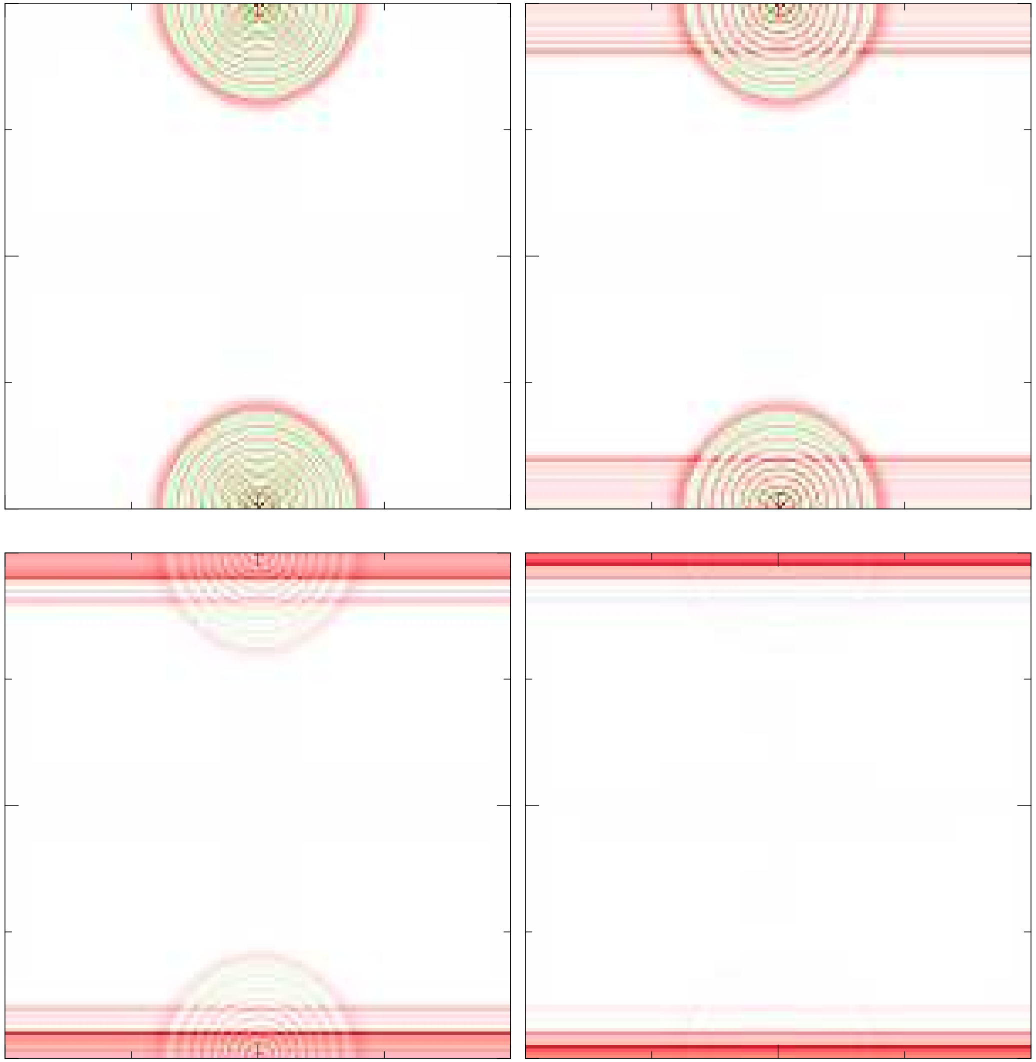
Dissipative noise

Dissipative processes can be modelled by **non-unital** channels. They are characterized by Kraus operators that are not normal. The operator

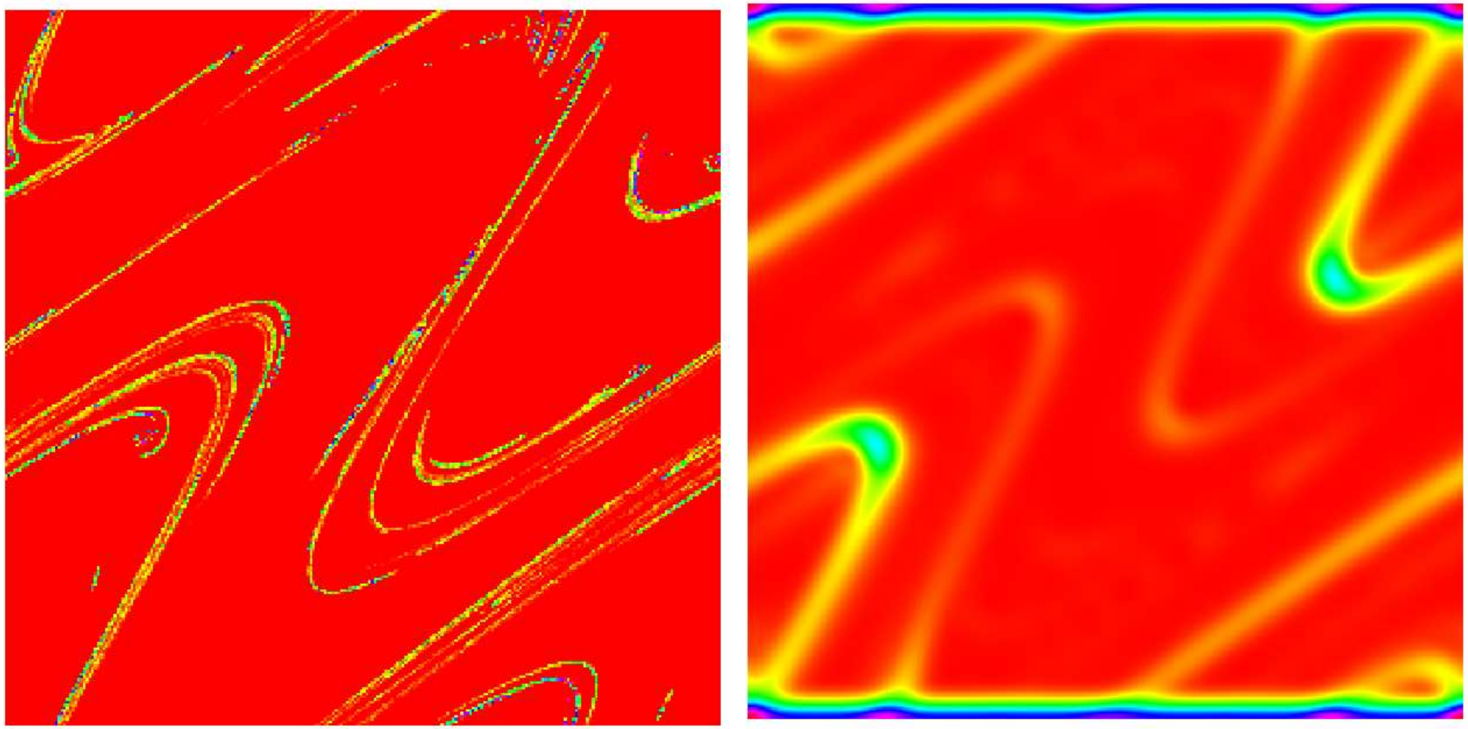
$$\hat{\Gamma} = \mathbf{\$}(I/N) - I/N = \sum_{\mu} [\hat{M}_{\mu}, \hat{M}_{\mu}^{\dagger}]$$

is related in the classical limit to the local rate of phase space contraction.

Processes of this type, in conjunction with unitary maps lead to a complex invariant state of a fractal nature.



p



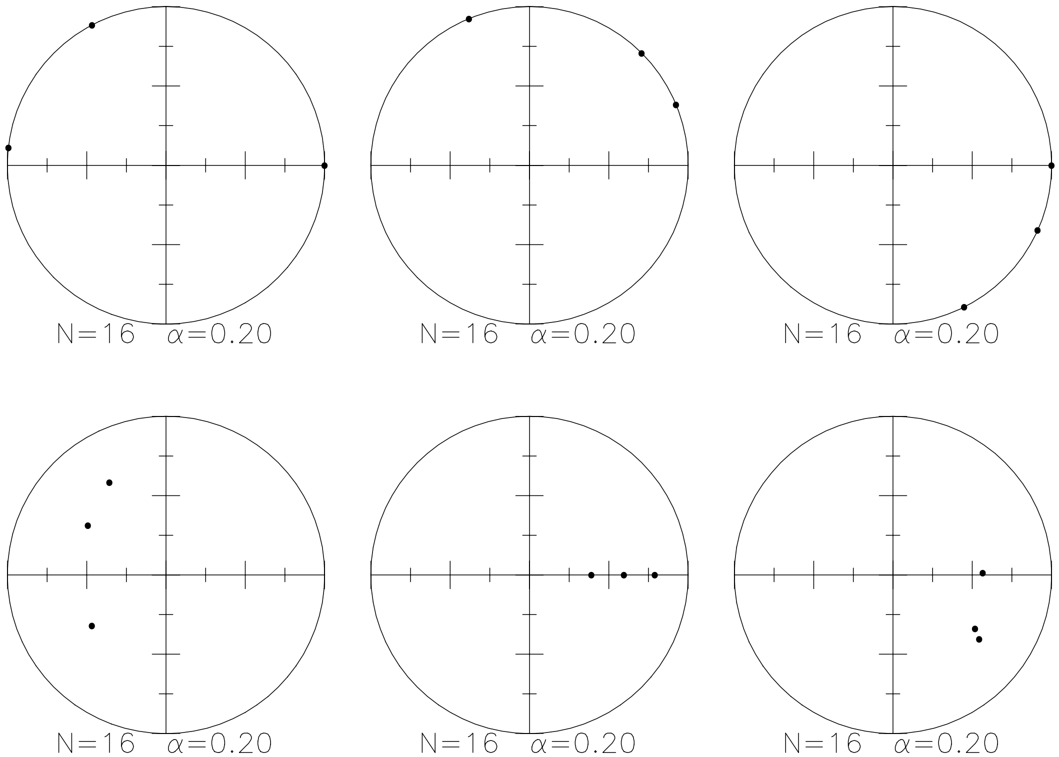
q q

classical *quantum*

Diffusive noise - random gaussian

$$\mathsf{D}_\epsilon = \sum_{q,p} c_\epsilon(q,p)\, \hat{T}_{(q,p)} \otimes \hat{T}^\dagger_{(q,p)}$$

$$\widetilde{c}_\epsilon(\mu,\nu) = \mathrm{e}^{-\frac{1}{2}\left(\frac{\epsilon N}{\pi}\right)^2 \left(\sin^2[\pi\mu/N]+\sin^2[\pi\nu/N]\right)}$$

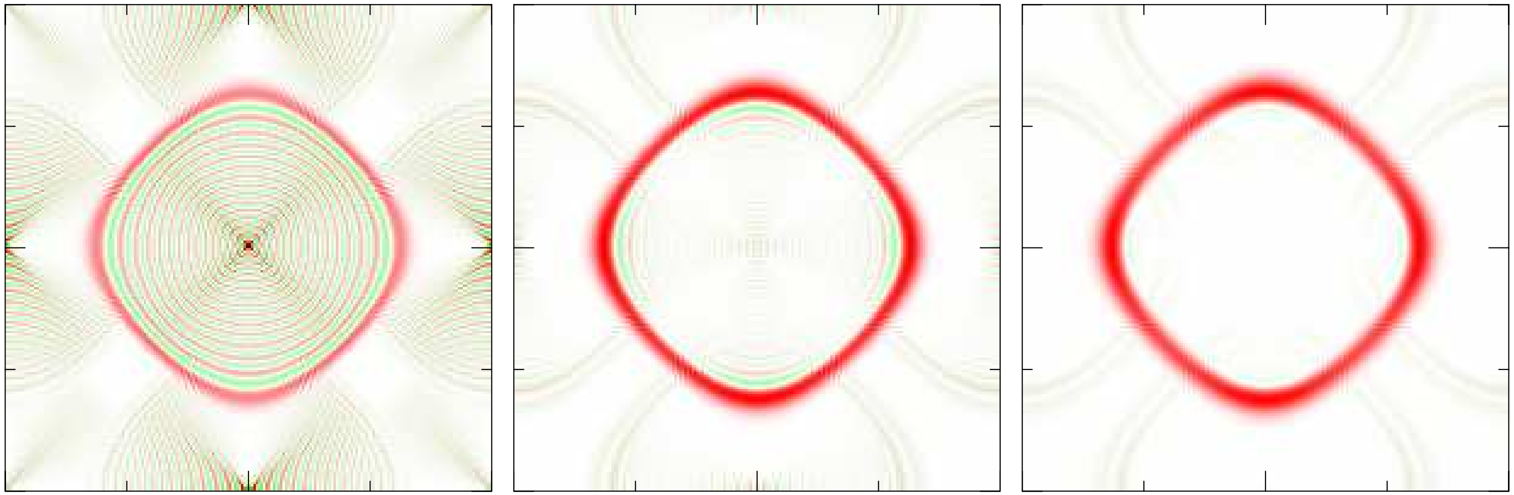


Hyperbolic Elliptic Mixed

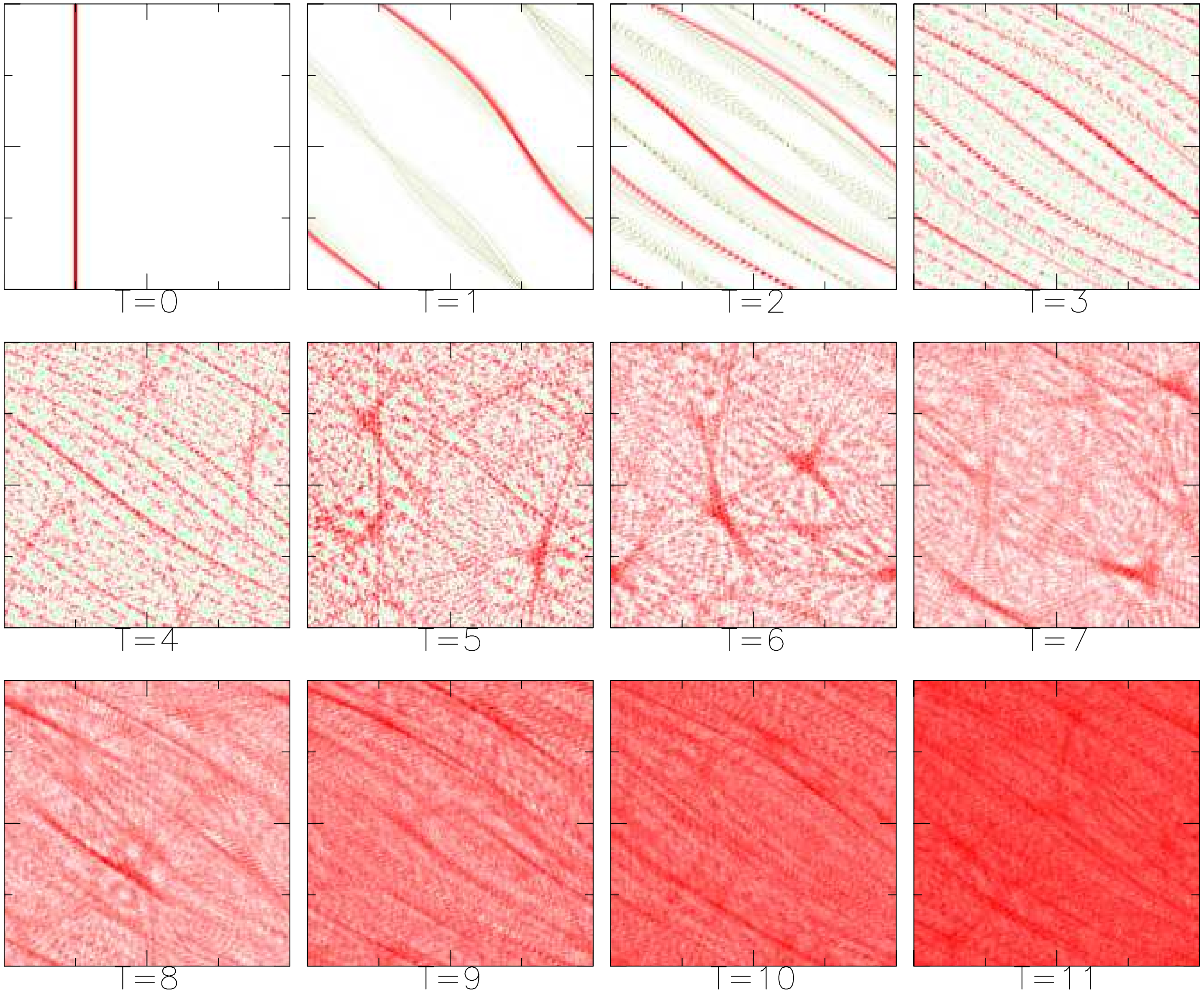
Action of diffusive noise

The action of the noise $\mathbf{D}_\epsilon$ is best seen in the chord representation.

$$\begin{aligned} \hat{\rho} &= \sum_{\mu,\nu} \rho(\mu,\nu)\hat{T}_{\mu,\nu} \\ \mathbf{D}_\epsilon\hat{\rho} &= \sum_{\mu,\nu} \widetilde{c}_\epsilon(\mu,\nu)\rho(\mu,\nu)\hat{T}_{\mu,\nu} \end{aligned}$$



Long chords, corresponding to high frequency modes, are suppressed by $\widetilde{c}_\epsilon(\mu,\nu)$.


T=0
T=1
T=2
T=3
T=4
T=5
T=6
T=7
T=8
T=9
T=10
T=11

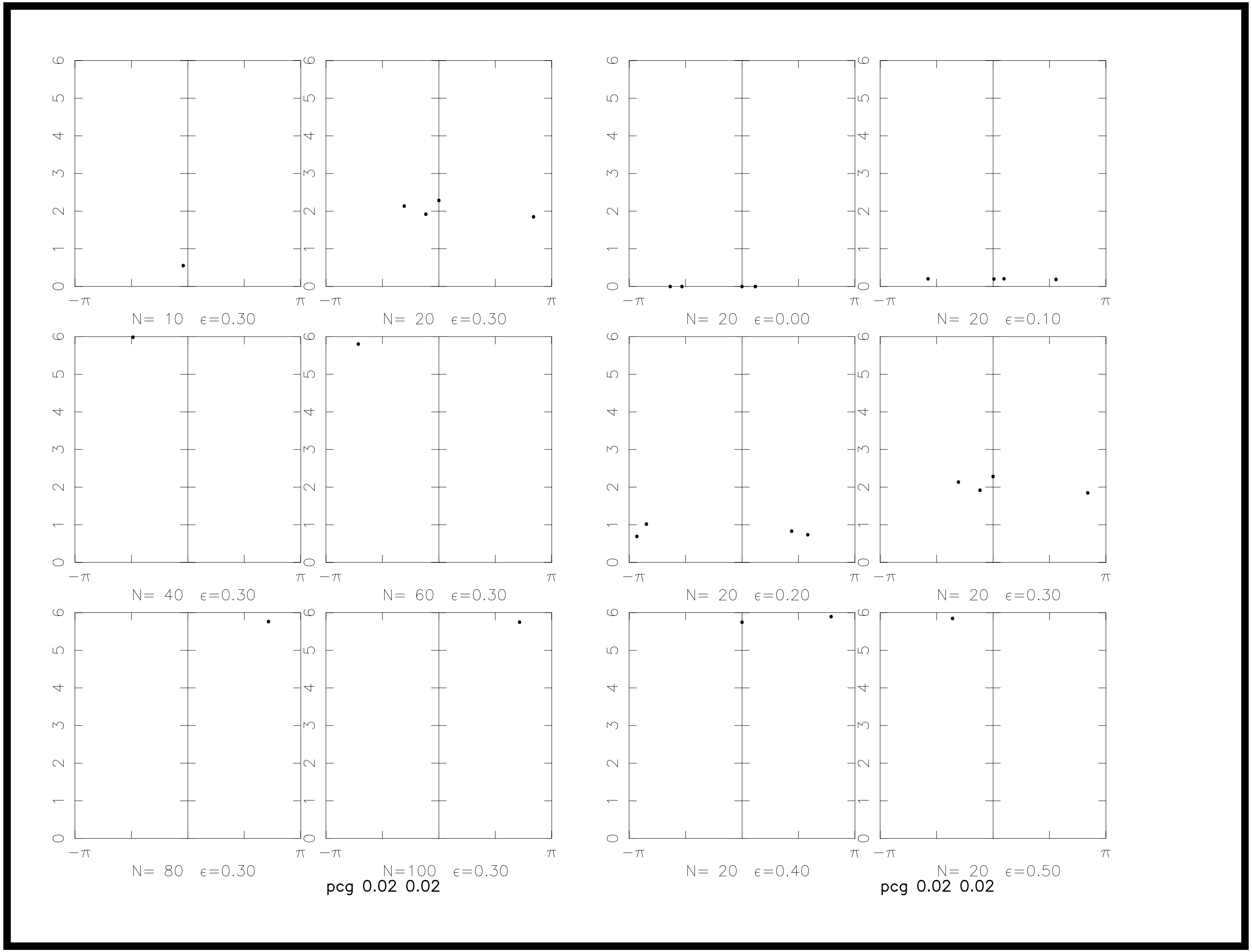
Spectrum of the Noisy Propagator

$$\mathbf{L}_\epsilon(\rho) = \mathbf{D}_\epsilon \circ \mathbf{U}(\rho)$$

•$\mathbf{L}_\epsilon(\rho_\infty) = \rho_\infty \qquad \rho_\infty = 1/N$
•Spectrum is symmetric *w.r.* to the real axis
•$\mathbf{L}_\epsilon$ is not a normal linear operator, and has distinct left and right eigenvectors.
•The $N^2 \times N^2$ matrix is best diagonalized in the chord representation

$$(\mathbf{L}_\epsilon)_{\mu,\nu;\mu',\nu'} = \widetilde{c}_\epsilon(\mu,\nu)\ \mathrm{tr}\left[\hat{T}^\dagger_{(\mu,\nu)}\mathbf{U}\hat{T}_{(\mu',\nu')}\right]$$

The unitary matrix $\mathbf{U}$ is effectively truncated by $\widetilde{c}_\epsilon(\mu,\nu)$ and the resulting *contracting* submatrix yields the leading spectrum.
•To extract classical features of the map $\mathbf{U}$ we need to explore the limits of this procedure when $N \to \infty$ and $\epsilon \to 0$. These limits should be taken toghether and they do not commute.


N= 10 ϵ=0.30
N= 20 ϵ=0.30
N= 20 ϵ=0.00
N= 20 ϵ=0.10
N= 40 ϵ=0.30
N= 60 ϵ=0.30
N= 20 ϵ=0.20
N= 20 ϵ=0.30
N= 80 ϵ=0.30
N=100 ϵ=0.30
N= 20 ϵ=0.40
N= 20 ϵ=0.50
pcg 0.02 0.02
pcg 0.02 0.02
−π
π
0
1
2
3
4
5
6

Relationship with the classical propagator

Theorem:

(S.Nonnenmacher, Nonlinearity **16**,1685 (2003).)

For smooth maps on the torus the spectrum away from zero of $\mathbf{L}_\epsilon$ converges to the classical Perron-Frobenius $\mathcal{L}_\epsilon$ as $N \to \infty$.

Consequences:

•From a practical point of view the classical spectrum can be obtained from quantum mechanics with finite matrices.

•These are operator statements that are independent of the particular representation (Weyl, Chord, Husimi, Kirkwood...)

•The behaviour and the distribution of eigenvalues of the quantum operator is determined by classical properties of the map.

•For chaotic maps the classical spectrum consists of isolated resonances with a finite spectral gap.

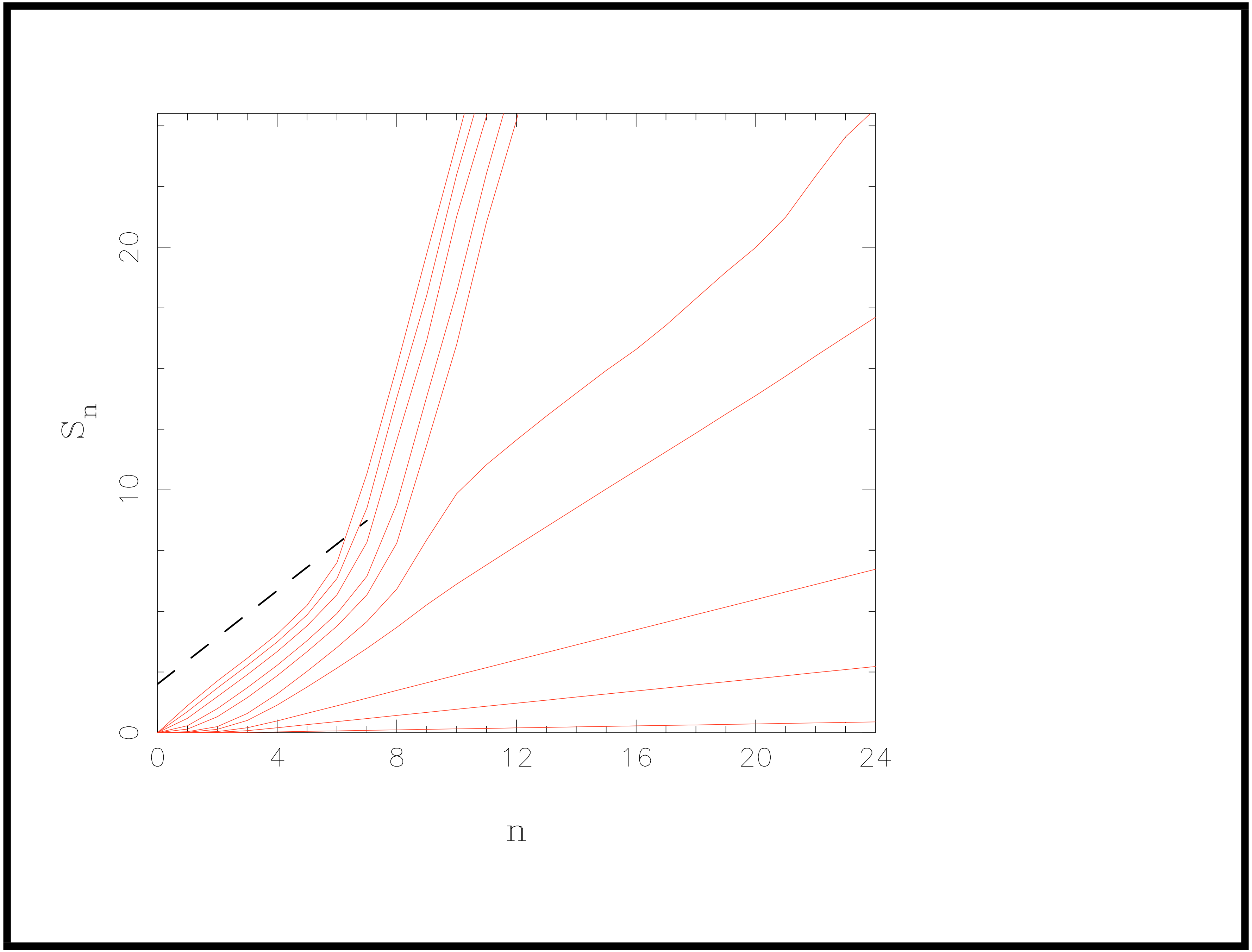
•For regular maps the spectrum accumulates on lines that touch the unit circle.

Asymptotic decays

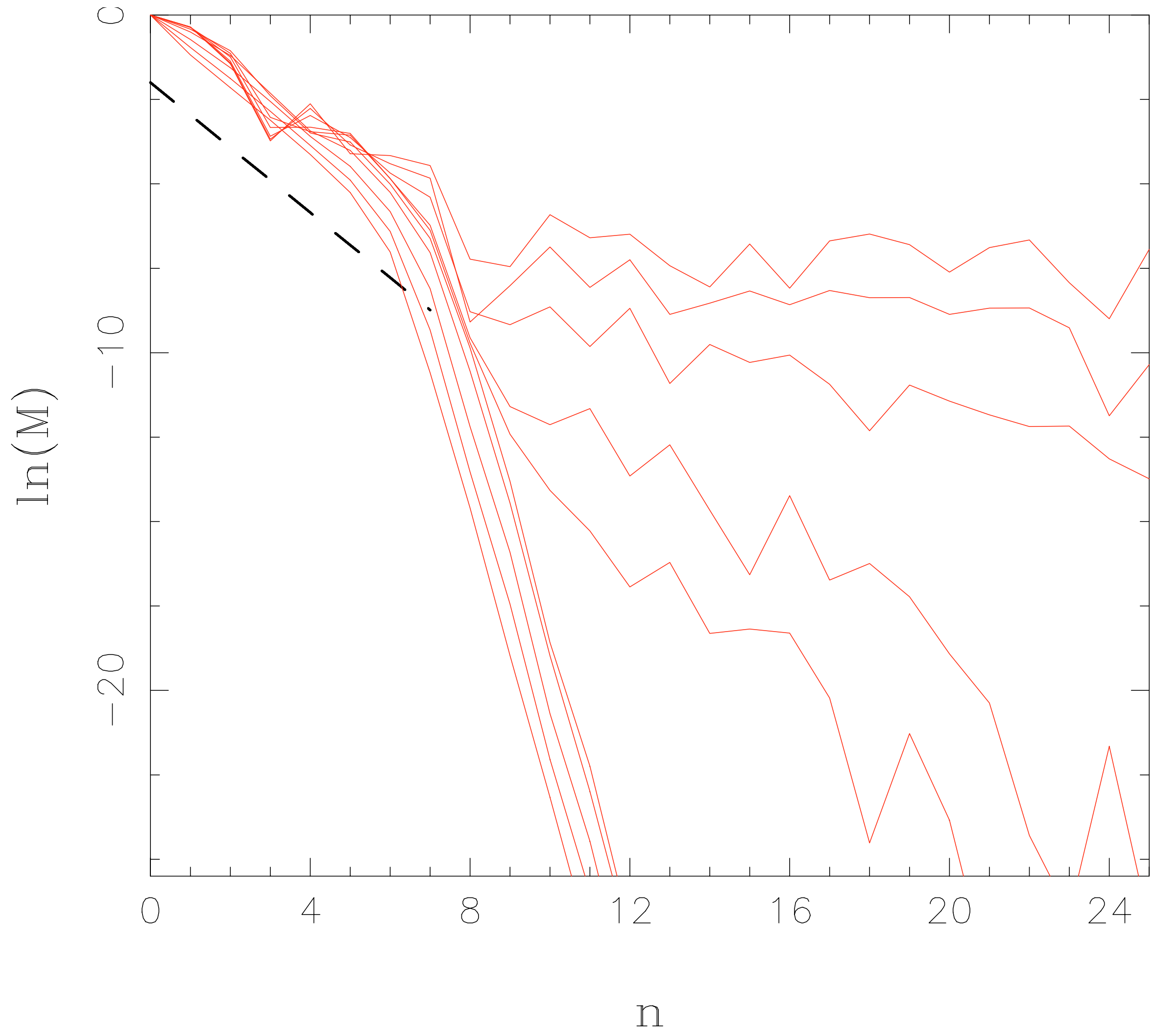
The resonances are emergent classical properties of the quantum map. They become manifest in the decay of correlations, the Loschmidt echo and the growth of entropy and they result in rates that are independent of ϵ and N

$$S_n = -\ln \mathbf{tr}(\rho_n - \rho_\infty)^2 \qquad (Entropy)$$

$$M_n = \mathbf{tr}[(\rho_n - \rho_\infty)(\rho'_n - \rho_\infty)] \quad (Echo)$$



0
10
20
S_n
0
4
8
12
16
20
24
n



Both quantities show two distinct regimes independent of the noise.

Diagonal noise in a qubit tensor product basis

A very similar structure occurs in the tensor product basis appropriate for a system of n qubits. One can define "translations" as

$$\hat{T}(\mathbf{a},\mathbf{b}) = X_{\mathbf{a}} Z_{\mathbf{b}} \mathrm{e}^{\mathrm{i}\frac{\pi}{2}\mathbf{a}\cdot\mathbf{b}} \equiv$$
$$\equiv X^{a_0} Z^{b_0} \mathrm{e}^{\mathrm{i}\frac{\pi}{2}a_0\cdot b_0} \otimes \ldots \otimes X^{a_{n-1}} Z^{b_{n-1}} \mathrm{e}^{\mathrm{i}\frac{\pi}{2}a_{n-1}\cdot b_{n-1}}$$

with X, Z the usual Pauli matrices for each particle. The set of these $2^n \times 2^n$ unitary and hermitian operators is orthonormal and complete. Its elements are labelled by a pair of binary strings

$$(\mathbf{a},\mathbf{b}) = (a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1})$$

•Composition: $\hat{T}(\mathbf{a},\mathbf{b})\hat{T}(\mathbf{q},\mathbf{p}) = phase \hat{T}(\mathbf{a}\oplus\mathbf{q},\mathbf{b}\oplus\mathbf{p})$

•Conjugation $\hat{T}(\mathbf{a},\mathbf{b})\hat{T}(\mathbf{q},\mathbf{p})\hat{T}(\mathbf{a},\mathbf{b}) = \hat{T}(\mathbf{q},\mathbf{p})\mathrm{e}^{\mathrm{i}\pi\mathbf{a}\cdot\mathbf{p}}\mathrm{e}^{\mathrm{i}\pi\mathbf{q}\cdot\mathbf{b}}$

Diagonal noise in this basis is characterized by the superoperator

$$\mathbf{\$} = \sum_{\mathbf{a},\mathbf{b}} c(\mathbf{a},\mathbf{b})\hat{T}(\mathbf{a},\mathbf{b}) \otimes \hat{T}(\mathbf{a},\mathbf{b})$$

The physical meaning is that all possible qubit noise (or errors) occur with probability $c(\mathbf{a},\mathbf{b})$. The spectral properties are again very simple and explicit

$$\mathbf{\$}(\hat{T}(\mathbf{a},\mathbf{b})) = \tilde{c}(\mathbf{a},\mathbf{b})\hat{T}(\mathbf{a},\mathbf{b})$$

where now the eigenvalues $\tilde{c}(\mathbf{a},\mathbf{b})$ are obtained as a Hadamard transform of $c(\mathbf{a},\mathbf{b})$.

The depolarizing channel corresponds to $c(\mathbf{a},\mathbf{b}) = (1/2)^{2n}$ and has the same effect.

Phase damping channels are characterized by a set S of 2^n *commuting* translations.

$$\$ = (1-\alpha)\hat{T}_{(0,0)} \otimes \hat{T}_{(0,0)} + \alpha \sum_{\mathbf{a},\mathbf{b}\in S} \hat{T}(\mathbf{a},\mathbf{b}) \otimes \hat{T}(\mathbf{a},\mathbf{b})$$

The set S defines a **stabilizer** basis. States in this basis are not affected by this noise while coherences are damped. This basis is the pointer basis for this channel.

Conclusions

•We have studied some noise superoperators quite independently from the specific physical processes that produce them with the aim of extracting some general properties and how they modify the unitary dynamics.

•Unital processes model diffusive behaviour, whose invariant state is the uniform density matrix. Non-unital processes model dissipative behaviour, possibly leading to complex invariant states when acting on unitary maps.

•Spectral analysis of the superoperators is numerically accessible in many cases and yields important information on the asymptotic time evolution.

•It is possible to explore the classical limit of open quantum maps by spectral techniques. For diffusive noise the spectrum reflects the chaotic or regular features of the classical map and yields a different perspective on the classical-quantum correspondence.

•In chaotic systems, the Lyapounov regime lasts up to the Ehrenfest time. A different regime, characteristic of the decay towards the uniform density, sets in at later times and is dominated by the spectrum of Ruelle-Pollicott resonances.

•The same techniques can be used to study the response to other kinds of noise or for quantum maps representing algorithms of interest to quantum information.

•Other phase space geometries (sphere, cylinder) can be explored.

(nlin.CD/0312062; nlin-CD0406001; quant-ph/0406236)

Thank you!