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Heisenberg-limited measurements of displacement and rotations with cavity QED and ion traps

Luiz DAVIDOVICH
Pontificia Universidade Catolica de Rio de Janeiro
Departamento de Fisica
Rua Marques de Sao Vicente, 225
Caixa Postal 38071
RJ 22453-970 Rio de Janeiro
BRAZIL

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Heisenberg-limited measurement of displacements and rotations with cavity QED and ion traps

F. Toscano,[1] D.A.R. Dalvit,[2] L. Davidovich,[1] and W.H. Zurek[2]

[1] *Instituto de Física, Universidade Federal do Rio de Janeiro, Caixa Postal 68.528, 21.941-972, Rio de Janeiro, Brazil*

[2] *Theoretical Division, MS B213, Los Alamos National Laboratory, Los Alamos, NM 87545, USA*

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We show how sub-Planck phase-space structures can be used to achieve Heisenberg-limited sensitivity in weak force measurements. We explore the utility of nonclassical states of harmonic oscillators possessing such structures, such as superpositions of coherent states, for quantum metrology. We propose a way to measure weak forces that cause either translations or rotations in phase space of the state of the oscillator, by entangling the quantum oscillator with a two-level system. Implementations of this strategy in cavity QED and in ion traps settings are described.

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The two typical problems of small quantum parameter estimation are high precision phase measurements and the detection of weak forces [1]. Detection of a small relative phase between two superposed quantum states includes two equivalent techniques, *i.e.* Ramsey spectroscopy and Mach-Zehnder interferometry. They involve detection of a rotation of the quantum state in phase space around the origin. Thus, the problem of phase determination is ultimately associated with the estimation of a small rotation angle. Detection of weak forces can be traced back to the pioneering work on gravitational wave detectors that proposed to use a quantum-mechanical oscillator as an antenna [2, 3]. A weak force (exerted e.g. by the wave) induces a displacement of the quantum state in phase space in some direction. Thus, in this case the quantum parameter estimation can be reduced to the determination of a small linear displacement. The precision in quantum parameter estimation depends on the energy resources (e.g., number N of photons) involved in the measurement process. It is well known that using quasiclassical states the sensitivity is at the standard quantum limit (SQL). In particular, coherent states are associated with SQL: The phase space size of a coherent state is given by $\simeq \sqrt{\hbar}$ and its distance from the origin is $\simeq \sqrt{\hbar N}$. The smallest noticeable rotation that will lead to approximate orthogonality is equal to its angular size as "seen from the origin", $\sqrt{\hbar}/\sqrt{\hbar N} \simeq N^{-1/2}$, *i.e.*, the standard quantum limit. The same argument implies that the smallest detectable displacement is of the order of $\sqrt{\hbar}$, so SQL for weak force detection is independent of N, *i.e.*, it scales as N^0. The SQL limit can be surpassed by using entangled states, reaching the so-called Heisenberg limit (HL), in which the sensitivity is higher than SQL by $N^{-1/2}$ [4, 5, 6]. The HL has been recently achieved experimentally using entangled states of few photons [7, 8] and ions [9].

In this letter we show that, as is already anticipated by the brief discussion of SQL above, the sensitivity of the quantum state to displacements is related to the smallest phase space structures associated with its Wigner function W. This connection was conjectured by one of us [10] in the context of the discussion of the sub-Planck structures in W. The area of these structures can be as small as $a = \hbar^2/A$, where A is the action of the effective support of W. A is limited from above by the classical action of the state, but it can be much smaller than that. A is least for the minimum uncertainty wave packets. For example, for a coherent state $A \simeq \hbar$, which then leads to SQL. We shall however show that, for a fixed N, states with much larger values of $A \simeq \hbar N$ can be found, and we shall show that such states exhibit sensitivity set by $\sqrt{a} \simeq \sqrt{\hbar/N}$ to displacements, which then allows one to saturate the Heisenberg limit. In this way, we shall demonstrate that the sub-Planck scale $\hbar^2/A$ determines sensitivity of small parameter estimation. Using these ideas, we will propose a way for measuring small displacements and rotations with Heisenberg-limited sensitivity by using nonclassical states of a harmonic oscillator, suitably coupled to a two-level system (TLS). As concrete examples, we will discuss the implementation of state preparation, detection and measurement both in cavity QED and in ion trap experiments.

Let us consider superpositions of M coherent states, equidistantly placed on a circle $\mathcal{C}$ of radius $|\alpha| \gg 1$

$$|\text{cat}_M\rangle = \frac{1}{\sqrt{M}} \sum_{k=1}^{M} e^{i\gamma_k} \, |e^{i\phi_k}\alpha\rangle \;, \qquad (1)$$

where $\phi_k = 2\pi k/M$, and the γ_k's are arbitrary phases. States of the form (1) include the periodic case of the "generalized coherent states" introduced in [11] and first studied in [12], and the states that result from a quantum-nondemolition-measurement of the photon number in a cavity [13] or the vibrational number of a trapped ion [14]. Some properties of these states were studied in [15, 16]. Examples are the Schrödinger cat state $|\text{cat}_2\rangle = (|\alpha\rangle + |-\alpha\rangle)/\sqrt{2}$, and the compass state $|\text{compass}\rangle = |\text{cat}_4\rangle = (|\alpha\rangle + |-\alpha\rangle + |i\alpha\rangle + |-i\alpha\rangle)/2$ [10].

We show in the following that when a unitary perturbation $\hat{U}_x$ induces a small linear displacement of magnitude x, the overlap between the unperturbed state $|\text{cat}_M\rangle$ and the perturbed one $|\text{cat}_M(x)\rangle = \hat{U}_x|\text{cat}_M\rangle$ oscillates with a typical frequency $\sim |\alpha|$. Therefore, the least linear displacement $x = s$ needed to distinguish the two states is $s \sim 1/|\alpha|$. This scale defines the Heisenberg limit for displacement measurement. In the case of a rotation, $x = \theta$, quasi-orthogonality occurs when the rotation induces a linear displacement of the center of the circle $\mathcal{C}$ of the order of $s \sim \theta|\alpha|$, with $s \sim 1/|\alpha|$. Therefore, the detectable angle is $\theta \sim 1/|\alpha|^2$, defining in this case the Heisenberg limit for rotation measurements.

For a small linear displacement given by the unitary operator $\hat{D}(\beta) \equiv e^{\beta\hat{a}^\dagger - \beta^*\hat{a}}$ in an arbitrary direction $\beta = e^{i\varphi}\alpha s/|\alpha|$ with magnitude $|\beta| = s \ll 1$, the overlap function between the unperturbed state $|\text{cat}_M\rangle$ and the perturbed state $|\text{cat}_M(s)\rangle \equiv \hat{D}(\beta)|\text{cat}_M\rangle$ is

$$|\langle\text{cat}_M|\text{cat}_M(s)\rangle|^2 \approx \frac{1}{M^2}\left[M + \sum_{k=1}^{M}\sum_{l>k}^{M} 2\cos(2sa_{kl}|\alpha|)\right], \tag{2}$$

where $a_{kl} \equiv \sin(\varphi - \varphi_k) - \sin(\varphi - \varphi_l)$. Here we have neglected contributions $\langle e^{i\varphi_k}\alpha|\hat{D}(\beta)|e^{i\varphi_l}\alpha\rangle \approx \mathcal{O}(e^{-|\alpha|^2})$ (with $l \neq k$), and we have used the approximation $\hat{D}(\beta)|\alpha\rangle \approx e^{2i\text{Im}(\beta\alpha^*)}|\alpha\rangle$, valid for $|\beta| = s \ll 1$. The oscillatory behavior displayed in Eq. (2), with typical frequency $\propto |\alpha|$, comes from the overlap between the interference patterns in the Wigner functions of the unperturbed and perturbed states, and implies that the states $|\text{cat}_M\rangle$ are HL sensitive to displacements ($s \sim 1/|\alpha|$). Similar oscillations when the initial state is a Fock state were discussed in [17].

Small rotations in phase space, induced by the operator $\hat{R}(\theta) = e^{i\theta\hat{a}^\dagger\hat{a}}$, with $\theta \ll 1$, can be treated in a similar way, by considering the displaced state $|\overline{\text{cat}_M}\rangle \equiv \hat{D}(\eta)|\text{cat}_M\rangle$, with η chosen so that the circle $\mathcal{C}$ contains the origin. This can be achieved for instance with $\eta = \alpha$. The overlap between $|\overline{\text{cat}_M}\rangle$ and the rotated state $|\overline{\text{cat}_M}(\theta)\rangle \equiv \hat{R}(\theta)|\overline{\text{cat}_M}\rangle$ is

$$|\langle\overline{\text{cat}_M}|\overline{\text{cat}_M}(\theta)\rangle|^2 \approx |\langle\text{cat}_M|\hat{D}(\beta)|\text{cat}_M\rangle|^2, \tag{3}$$

where the linear displacement β is orthogonal to η, that is $\beta = i\eta s/|\eta|$, and has magnitude $s = |\alpha|\theta \ll 1$. The overlap function on the RHS is given by Eq. (2) with $s = \theta|\alpha|$ (see Fig(1)). This shows that the displaced states $|\overline{\text{cat}_M}\rangle$ are HL sensitive to rotations ($\theta \sim 1/|\alpha|^2$).

Let us consider the simplest case with $M = 2$, *i.e.* the cat state $|\text{cat}_2\rangle$. After a small displacement $\beta = i\alpha s/|\alpha|$, orthogonal to the direction of α, the overlap function according to Eq. (2) is

$$|\langle\text{cat}_2|\text{cat}_2(s)\rangle|^2 \approx [1 + \cos(4|\alpha|s)]/2 \ . \tag{4}$$

As we have seen, this is also the overlap function when we consider a small rotation, of angle $\theta = s/|\alpha|$, applied

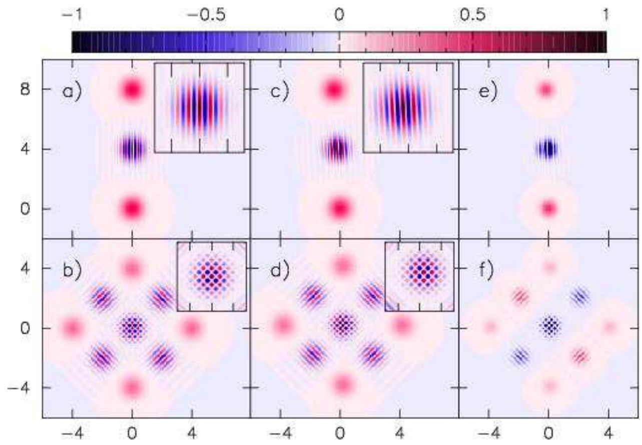

FIG. 1: The Wigner functions in the α-plane for: **a)** the displaced "cat state" $|\overline{\text{cat}_2}\rangle \equiv \hat{D}(\alpha)|\text{cat}_2\rangle$, and **b)** the "compass state" $|\text{compass}\rangle$ for $\alpha = 4i$. The displaced cat state is quasi-orthogonal to the rotated state $\hat{R}(\theta)|\overline{\text{cat}_2}\rangle$ ($\theta = \pi/4|\alpha|^2$) in **c)** at the Heisenberg limit scale $\theta \sim 1/|\alpha|^2$. The compass state is quasi-orthogonal to the translated compass state $\hat{D}(\beta)|\text{compass}\rangle$ ($\beta = e^{i\pi/4}\pi/2\sqrt{2}|\alpha|$) in **d)** at the Heisenberg limit scale $|\beta| \sim 1/|\alpha|$. The insets enlarge the central interference pattern of the displayed Wigner functions. In **e)** and **f)** we display the respective products of the unperturbed and perturbed Wigner functions. When performing the integration over the α-plane, the negative contributions (in blue) cancel the positive ones (in red), leading to quasi-orthogonality.

to the state $|\overline{\text{cat}_2}\rangle \equiv \hat{D}(\alpha)|\text{cat}_2\rangle = (1/\sqrt{2})(|2\alpha\rangle + |0\rangle)$. We see that if we could measure these overlap functions, we could determine the parameters s or $\theta = s/|\alpha|$ at the Heisenberg limit, *i.e.*, with a sensitivity proportional to $1/|\alpha|$ and $1/|\alpha|^2$ respectively. We show now that one can effectively realize this kind of measurement by entangling the system with a two level system (TLS).

The general method is the following: we initially prepare the oscillator in a large-amplitude coherent state $|\alpha\rangle$, and the TLS in one of its two states, say in the upper state $|e\rangle$. The composite system is then evolved during a certain time $t = T$ under a unitary evolution $\hat{U}$, which includes the interaction of the oscillator with the TLS as well as possible additional unitary operations acting only either on the states of the oscillator or on those of the TLS. The unitary perturbation $\hat{U}_x$ is then applied to the oscillator (assuming that it does not affect the state of the TLS), and finally the unitary evolution $\hat{U}$ is undone. The final entangled state of the composite system is

$$|\Psi_f\rangle = \hat{U}^\dagger(T)\hat{U}_x\hat{U}(T)|e,\alpha\rangle = \sqrt{P_e}|e,\Psi_S^e\rangle + \sqrt{P_g}|g,\Psi_S^g\rangle \ , \tag{5}$$

where P_e and $P_g = 1 - P_e$ are the probabilities of measuring the TLS in levels e and g, respectively. The unitary operator $\hat{U}$ must be such that the intermediate states $|\Psi\rangle = \hat{U}|e,\alpha\rangle$ and $|\Phi\rangle = \hat{U}_x\hat{U}|e,\alpha\rangle$ verify $|\langle\Psi|\Phi\rangle|^2 \approx |\langle\text{cat}_M|\text{cat}_M(x)\rangle|^2$. Given that $|\langle e,\alpha|\Psi_f\rangle|^2 = |\langle\Psi|\Phi\rangle|^2$, the information about the perturbation parameter x, contained in the overlap function $|\langle \text{cat}_M|\text{cat}_M(x)\rangle|^2$, is then translated into the probabilities, *i.e.*

$$P_e = 1 - P_g = \frac{|\langle e,\alpha|\Psi_f\rangle|^2}{|\langle\alpha|\Psi_S^e\rangle|^2} \approx \frac{|\langle \text{cat}_M|\text{cat}_M(x)\rangle|^2}{|\langle\alpha|\Psi_S^e\rangle|^2} \ . \quad (6)$$

Let us see how our method could be implemented when the interaction of the harmonic oscillator with the TLS is given by the Jaynes-Cummings (JC) model [18]. In a cavity QED scenario [13], the harmonic oscillator is a single mode of the quantized electromagnetic field in the cavity and the TLS is a Rydberg atom with a two-level electronic transition coupled to the field through the JC evolution. In an ion-trap scenario [19], the harmonic oscillator corresponds to the center-of-mass motion of the trapped ion, and it couples to the TLS (that corresponds to an internal atomic transition) when the ion is irradiated by a laser. In the following we adopt the cavity QED scenario, adding short remarks on issues that may be specific for trapped-ion implementations.

The coherent dynamics in the JC model is described by the Hamiltonian: $\hat{H}_{JC} = \hat{H}_A + \hat{H}_F + \hat{H}_{AF}$, where $\hat{H}_A = (\hbar\omega_0/2)\hat{\sigma}_z$ is the atomic TLS Hamiltonian, with $\hat{\sigma}_z \equiv |e\rangle\langle e| - |g\rangle\langle g|$, and ω_0 is the transition frequency between the lower $|g\rangle$ and the upper $|e\rangle$ states. The harmonic field mode is $\hat{H}_F \equiv \hbar\omega\hat{a}^\dagger\hat{a}$ and the interaction Hamiltonian is $\hat{H}_{AF} \equiv (\hbar\Omega_0/2)\left(\hat{\sigma}^\dagger\hat{a} + \hat{\sigma}\hat{a}^\dagger\right)$ where $\hat{\sigma} = |g\rangle\langle e|$ and Ω_0 is the vacuum Rabi frequency. It is more convenient to use the interaction picture with respect to the free evolution $\hat{H}_A + \hat{H}_F$, so the JC dynamics is described by

$$\hat{H}_{AF}^I = (\hbar\Omega_0/2)\left(e^{i\delta t}\hat{\sigma}^\dagger\hat{a} + e^{-i\delta t}\hat{\sigma}\hat{a}^\dagger\right) \ , \quad (7)$$

where $\delta \equiv \omega_0 - \omega$ is the detuning. Our method applies both to the dispersive and the resonant regime.

We assume first a dispersive interaction, with $\delta \gg \Omega_0\sqrt{\bar{n}}$, *i.e.*, the frequency of the field ω is far detuned from the transition frequency ω_0 of the TLS, and we assume that the atom has three relevant states $|g\rangle$, $|e\rangle$ and $|i\rangle$, so that the field in the high-Q cavity couples dispersively with the states $|g\rangle$ and $|i\rangle$, while transitions involving $|e\rangle$ can be neglected. We start with the atom in the state $|g\rangle$ and the field in the cavity in a large-amplitude coherent state $|\alpha\rangle$. Before the atom enters into the high-Q cavity it passes through a low-Q cavity and suffers a resonant $\pi/2$-pulse, so it evolves into $\hat{U}_{\pi/2}|g\rangle = (|e\rangle + |g\rangle)/\sqrt{2}$. The initial velocity of the atom is chosen in such a way that the transit time T up to the middle of the cavity satisfies $\Omega_0^2 T/4\delta = \pi$, where $\delta = \omega_{ie} - \omega$ is the detuning between the frequency of the field in the cavity, ω, and the frequency ω_{ie} of the transition $g \longleftrightarrow i$. Therefore, $\hat{U}_{JC}(T)|g,\alpha\rangle = |g,-\alpha\rangle$, while the state $|e,\alpha\rangle$ remains the same. The state of the system right before the application of the perturbation is $|\Psi\rangle = \hat{U}_{JC}(T)\hat{U}_{\pi/2}|g,\alpha\rangle$, and reads

$$|\Psi\rangle = [|e,\alpha\rangle + |g,-\alpha\rangle]\sqrt{2} \ . \quad (8)$$

Assume now a displacement perturbation, corresponding to the unitary operation $\hat{U}_s = \hat{D}(\beta)$, with $\beta = i\alpha s/|\alpha|$ and $|\beta| = s \ll 1$, is applied to this state. Displacements of the cavity field can be induced by injecting into the cavity coherent fields, produced for instance by a microwave generator, while in the ion-trap setting they can be generated by forces that displace the equilibrium position of the ion. For detecting a small rotation, we first apply the displacement operator $\hat{D}(\alpha)$, during a time $\Delta t \ll T$, which leads to the state

$$|\Psi\rangle = [|e,2\alpha\rangle + |g,0\rangle]/\sqrt{2} \ . \quad (9)$$

A small rotation of the cavity field can be implemented by a percussive dislocation of one of the mirrors of the cavity, thus changing the frequency of the mode by a small amount during a small time interval. Alternatively, one may send through the cavity a fast atom, which interacts dispersively with the field, and follows a trajectory that avoids the interaction with the first atom. In the ion-trap context, the same kind of perturbation can be implemented by slightly changing the frequency of the harmonic trapping potential. Note that for the state in Eq. (8) the overlap function $|\langle\Psi|\hat{D}(\beta)|\Psi\rangle|^2$ is equal to $|\langle \text{cat}_2|\text{cat}_2(s)\rangle|^2$ given by Eq. (4). In an analogous way, for the state in Eq. (9) we have $|\langle\Psi|\hat{R}(\theta)|\Psi\rangle|^2 = |\langle \text{cat}_2|\text{cat}_2(\theta)\rangle|^2$, also given by Eq. (4) with $s = \theta|\alpha|$. After the perturbation is applied, we undo the total unitary evolution $\hat{U}_{JC}(T)\hat{U}_{\pi/2}$ (or $\hat{D}(\alpha)\hat{U}_{JC}(T)\hat{U}_{\pi/2}$ for a rotation perturbation), by letting the atom interact with the cavity field again for a time T. Since T is half the period of the dispersive JC evolution, when the atom leaves the cavity at time $2T$ the JC dynamics is automatically undone. Up to a global phase, the final state is

$$|\Psi_f\rangle = \frac{1}{2}\left(1 - e^{i\,4\,|\alpha|\,s}\right)|e,\alpha\rangle + \frac{1}{2}\left(e^{i\,4\,|\alpha|\,s} + 1\right)|g,\alpha\rangle \ . \quad (10)$$

For a small rotation θ, we obtain the same final state with the displacement s replaced by $\theta|\alpha|$.

The probabilities that the atom exits the cavity in the upper and lower state depend on the small parameter s (equivalently $\theta = s/|\alpha|$),

$$P_e = 1 - P_g = [1 - \cos(4\,|\alpha|\,s)]/2 \ , \quad (11)$$

thus exhibiting the characteristic oscillation associated with the interference pattern of the Wigner function.

A good estimate of the unknown parameter s requires repeating the measurement several times. After R repetitions, the probability that the outcome $|e\rangle$ is obtained r times is given by a binomial distribution. In the large R limit, it is well approximated by a Gaussian distribution in the variable $\xi = r/R$, which can be regarded as effectively continuous [20]. In this limit the probability distribution for the estimator $\tilde{s} = \arccos(2r/R - 1)/4|\alpha|$ of the true displacement s is [21]

$$P(\tilde{s}) \approx \frac{1}{\sqrt{2\pi\Delta\tilde{s}^2}}\, e^{-\frac{(\tilde{s}-s)^2}{2\Delta\tilde{s}^2}}, \quad (12)$$

where the uncertainty of $\tilde{s}$ is $\Delta\tilde{s} = 1/8\sqrt{R\bar{n}}$, reaching the Heisenberg precision for displacement since $R\bar{n}$ is the total number of photons used in the measurement.

We discuss now a resonant-interaction implementation ($\delta = 0$), which has over the dispersive case the advantage of requiring much shorter transit times. The corresponding experimental setup leads to collapses and revivals of the atomic population [22]. We start with an initial product state of the TLS-oscillator composite system, $|e, \alpha\rangle$, with large photon number $\bar{n} = |\alpha|^2$. The joint evolution of the atom-field system inside the cavity is given by $\hat{U}_{JC} \equiv \exp\{-i\hat{H}^I_{AF} t/\hbar\}$, and it can be calculated following the approach developed in [23]. The velocity of the atom is set up so that the transit time T up to the middle of the cavity is half the revival time $T_R = 4\pi\sqrt{\bar{n}}/\Omega_0$. This transit time is much shorter than the one for the dispersive case. The evolved state $|\Psi\rangle = \hat{U}_{JC}(T)\,|e,\alpha\rangle$ turns out to be the product state [23],

$$|\Psi\rangle = \left[e^{-i\frac{\pi}{2}\bar{n}}|-i\alpha\rangle - e^{i\frac{\pi}{2}\bar{n}}|i\alpha\rangle\right]/\sqrt{2} \otimes |\phi\rangle_A \quad , \tag{13}$$

where $|\phi\rangle_A \equiv (1/\sqrt{2})\left(e^{-i\frac{\pi}{2}}|e\rangle + e^{-i\arg(\alpha)}|g\rangle\right)$. A small displacement is then applied to the field. At this point the JC dynamics must be inverted. This can be done by a procedure developed in [24]: one applies a percussive controlled phase kick corresponding to the unitary operation $\hat{U}_{\rm kick} = \hat{\sigma}_z$ that changes the sign of the relative phase between the atomic levels. This amounts to changing the sign of the interaction Hamiltonian ($\hat{\sigma} \to -\hat{\sigma}$), so the phase kick mimics the time-reversal operation. This idea was experimentally implemented in cavity QED [25] and can be similarly applied in the context of ion traps. The final state $|\Psi_f\rangle = \hat{U}^\dagger_{JC}(T)\hat{D}(\beta)\hat{U}_{JC}(T)|e,\alpha\rangle$, up to a global phase, is

$$|\Psi_f\rangle = \frac{1}{2}\left(e^{i\,4\,|\alpha|\,s} + 1\right)|e,\alpha\rangle + \frac{b}{2}\left(1 - e^{i\,4\,|\alpha|\,s}\right)|g,\alpha\rangle, \tag{14}$$

where $b \equiv e^{-i\arg(\alpha)}$. For small rotations, one proceeds as in the previous case, first displacing the field state in Eq. (13), then applying the rotation and subsequently inverting the displacement and the time evolution. With the replacement $s \to \theta|\alpha|$, one gets the same final state (14). Therefore, also in the case of resonant Jaynes-Cummings interaction one can measure weak forces at the Heisenberg limit, with interaction times much shorter than in the dispersive limit. One should note that, instead of applying the percussive time-inversion pulse, the same result would be obtained by letting the first atom go away of the cavity, after disentanglement, and then sending a second atom, prepared in the "time inverted" state, obtained from $|\phi\rangle_A$ by changing the sign of the relative phase between the states $|e\rangle$ and $|g\rangle$. Further shortening of the interaction time can be achieved by letting the atom interact with the field for a time $\Delta t < T_R/2$, so that in the intermediate state the atom is entangled with the two coherent states $|\alpha e^{\pm i\phi/2}\rangle$ ($\phi = \Omega_0\Delta t/2\sqrt{\bar{n}}$), and then inverting the dynamics. After an equal amount of time, one gets again a state like the one in Eq. (14), with s replaced by $s\sin(\phi/2)$, which implies reduced sensitivity, but does not change the scaling of the minimum detectable displacements and rotations.

In conclusion, we have shown that sub-Planck quantum phase space structures [10] have remarkable implications for quantum parameter estimation, as they are responsible for Heisenberg-limited sensitivity to perturbations. We have proposed a general method to measure perturbations with such high sensitivity, coupling a harmonic oscillator with a two level system. This method was applied to cavity QED and ion-trap settings, which should be within experimental reach.

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