The Abdus Salam
International Centre for Theoretical Physics

SMR.1675 - 15

**Workshop on
Noise and Instabilities in Quantum Mechanics**

3 - 7 October 2005

Multipartite entanglement and decoherence

**Andreas BUCHLEITNER
Max-Planck-Institute for the Physics of Complex Systems
Noethnitzer Str. 38
D-01187 Dresden
GERMANY**

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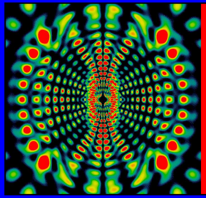
Andreas Buchleitner

Max-Planck-Institut für Physik komplexer Systeme, Dresden

Nonlinear Dynamics in Quantum Systems

Noise and Instabilities in Quantum Mechanics, ICTP Trieste, 6 October 2005

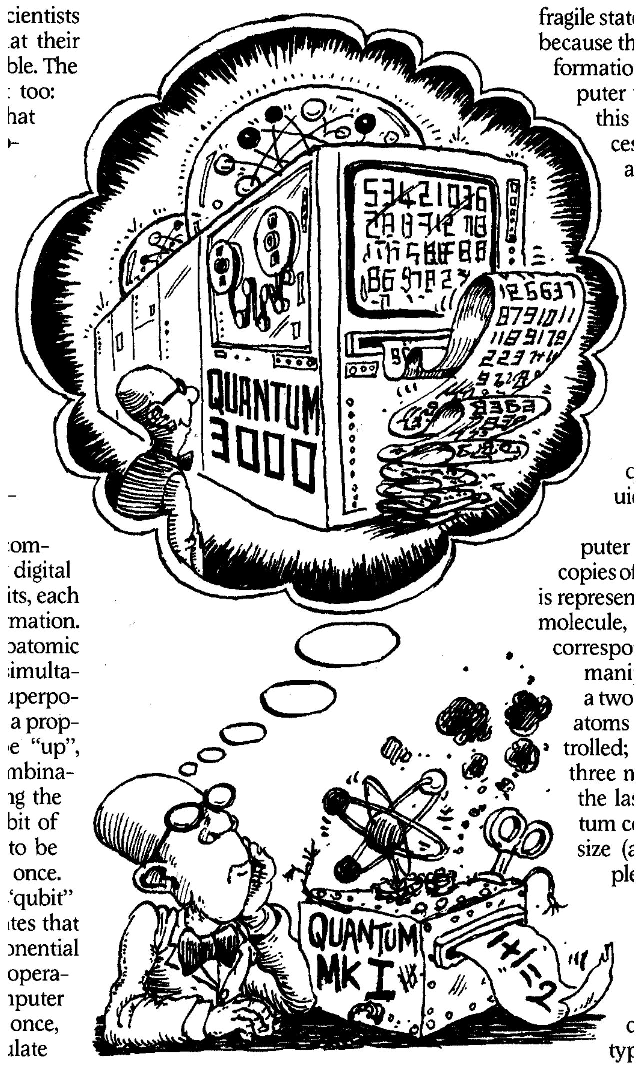


Nonlinear Dynamics in Quantum Systems

Entanglement, a central resource of quantum information processing

Key challenges

- Coherent evolution over sufficiently long time scales
- Controlled entanglement of *a large number* of qubits/subsystems
- How long does this special fuel "entanglement" last, under realistic conditions?
- Scalability – how do size and coherence requirements compete?

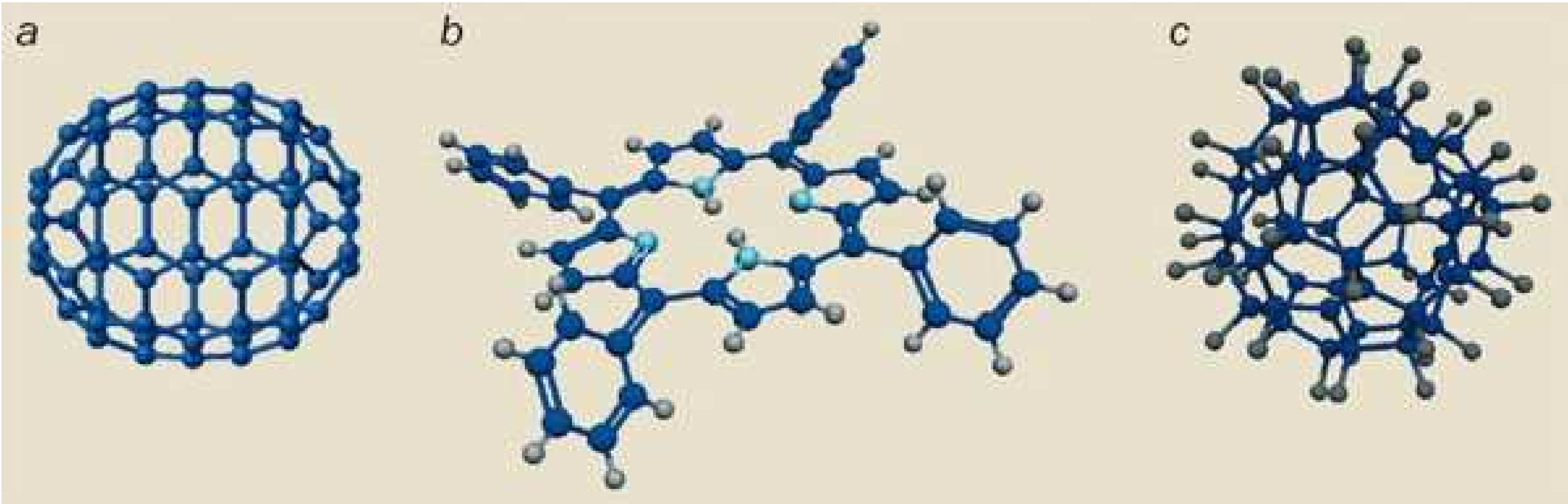


[The Economist, *Quantum Dreams*, 10/3/2001]

Open, multipartite systems

Quantum information processing with
• photons • atoms, molecules • trapped atoms or ions
faces residual coupling to uncontrolled degrees of freedom
• electromagnetic vacuum • atomic continua • nonresonantly coupled states,
i.e., **decoherence**

We know little about decoherence in many degrees of freedom systems,
let alone about the entanglement between these different degrees of freedom in the presence of decoherence

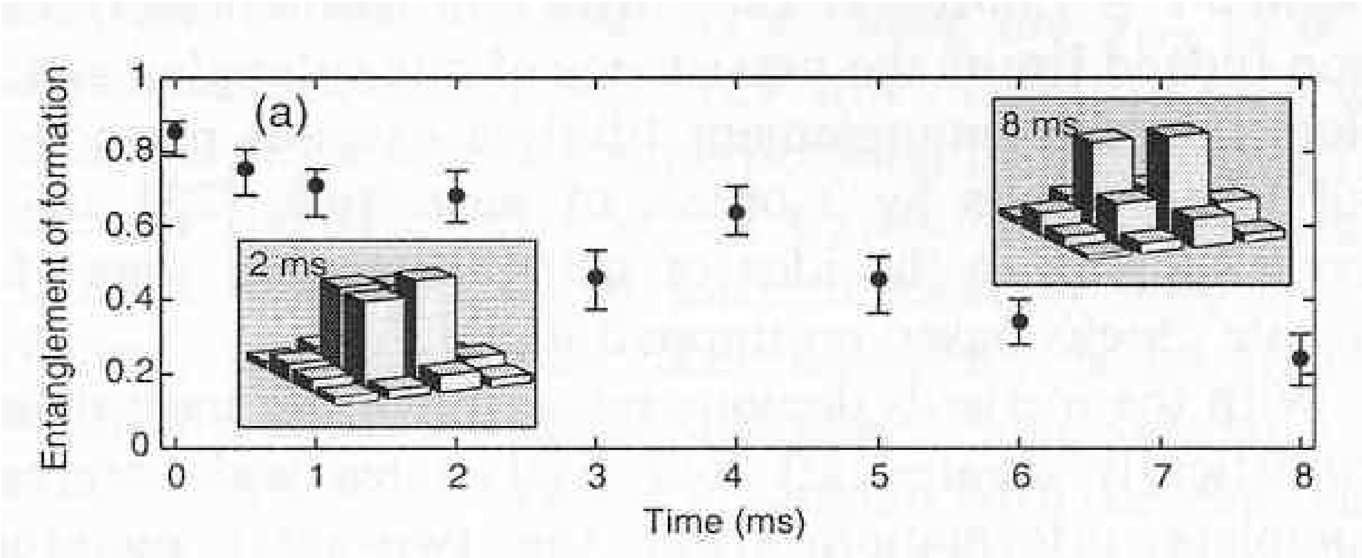


[Arndt, Hornberger & Zeilinger, Physics World 2005]

What we want, and what we need

Wanted! *"Life time" of entanglement for large quantum registers* – need

- . . . efficient entanglement detector
- . . . efficient entanglement quantifier
- . . . description of **entanglement dynamics**



experiments: life times of atomic Bell states [Roos et al., PRL 2004]

Entanglement measures – pure states

Any pure state $|\Psi\rangle$ which *cannot* be written as a product $|\phi\rangle \otimes |\chi\rangle$, i.e.,

$$|\Psi\rangle \neq |\phi\rangle \otimes |\chi\rangle$$

is **nonseparable** or **entangled**.

A useful measure of entanglement is provided by the **concurrence**

[Rungta, Buzek, Caves, Hillery, Milburn, PRA 2001]

$$c(\Psi) = \sqrt{2(1 - \mathrm{tr}\rho_r^2)} \ ,$$

which vanishes precisely for separable states, for which

$$\mathrm{tr}\rho_r^2 = 1 \ .$$

Further quantifiers are, e.g., Schmidt coefficients, von Neumann entropy, etc . . .

Entanglement measures – mixed states

Given an arbitrary (generally mixed) quantum state ρ, with a pure state decomposition

$$\rho = \sum_j p_j |\Psi_j\rangle\langle\Psi_j| \ , \tag{1}$$

one *might* be tempted to generalize pure state concurrence,

$$c(\rho) \rightarrow \sum_j p_j c(\Psi_j) \ .$$

However, this is **WRONG**, since the decomposition (1) is **NOT UNIQUE**!
Need to *optimize over all possible decompositions of* ρ:

$$c(\rho) = \inf_{\{p_j, \Psi_j\}} \sum_j p_j c(\Psi_j) \ .$$

Unfavourable scaling of optimization space! Explicit solution only available for pairs of qubits [Wootters, PRL 1998].

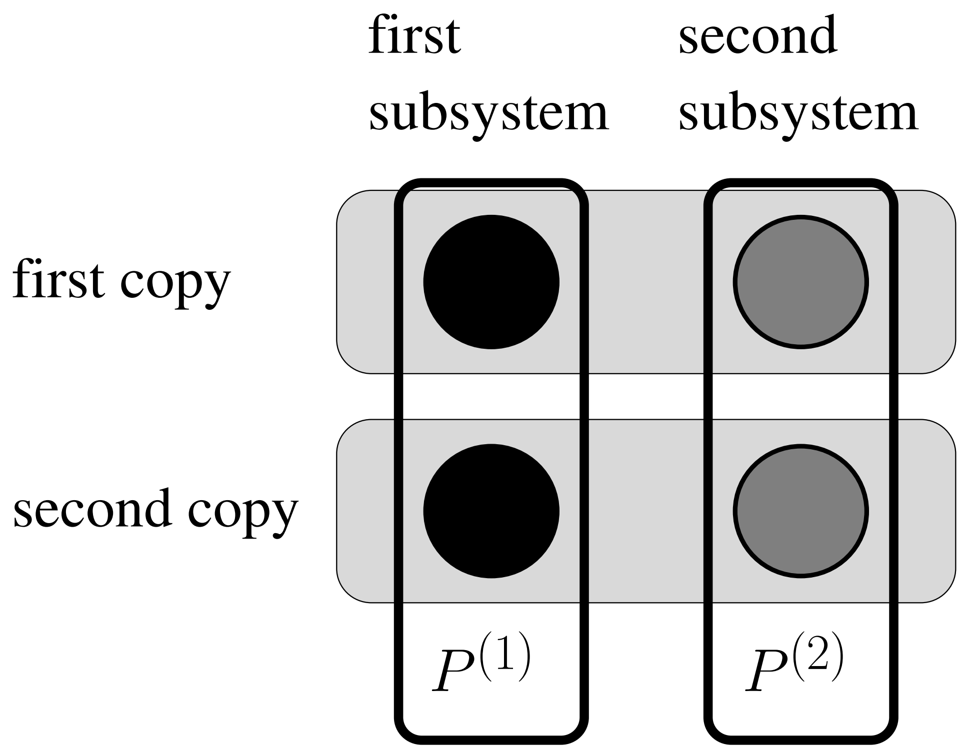
Reformulating concurrence

An efficient quantifier for entanglement is derived by rewriting pure state concurrence as

$$c(\Psi) = \sqrt{\langle\Psi| \otimes \langle\Psi|A|\Psi\rangle \otimes |\Psi\rangle}\ ,$$

where A acts on *two copies* of the given state $|\Psi\rangle$.

$A \sim P^{(1)} \otimes P^{(2)}$ *projects on the antisymmetric subspaces of the underlying factor spaces. Thus,* c *vanishes for states which are invariant under exchanges of the individual copies.*
Possible interpretation in terms of suitable measurement(s) on two copies.



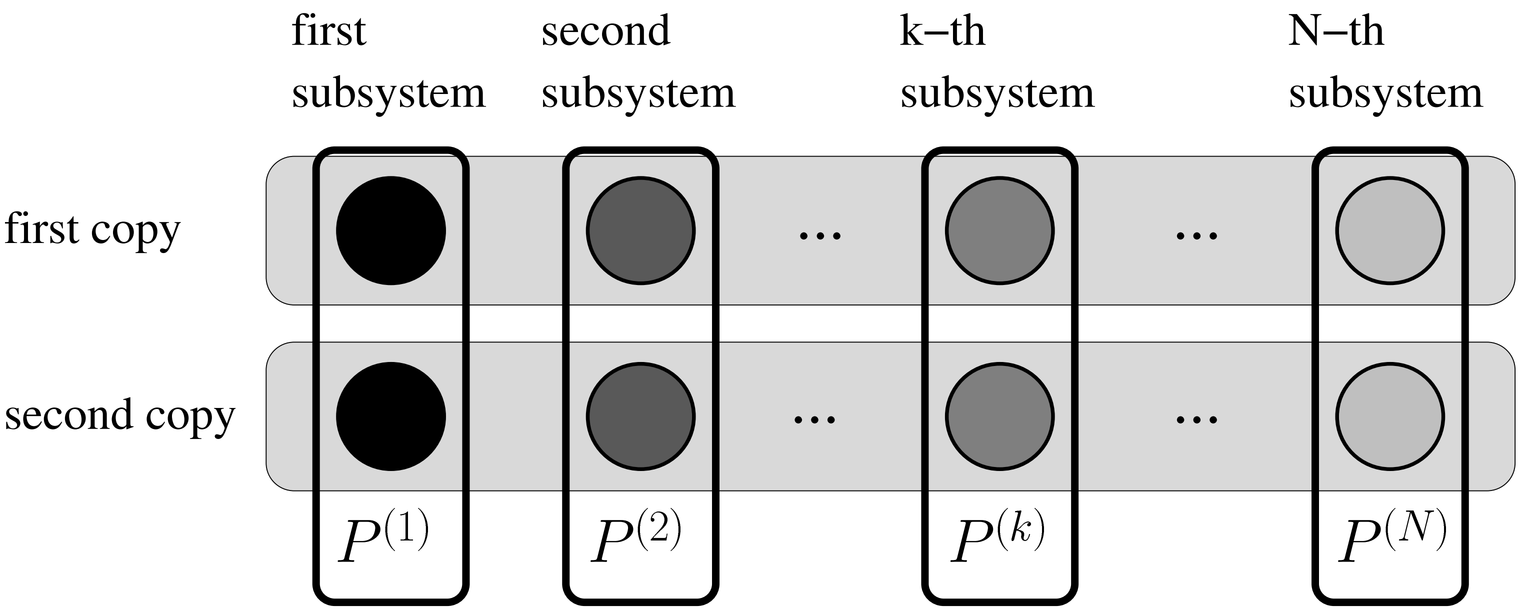
Mixed state concurrence in higher dimensions

For mixed states of bi- or multipartite states of *arbitrary finite dimension* we obtain

$$c(\rho) = \inf_{\{p_j,\Psi_j\}} \sum_j \sqrt{\langle\Psi_j| \otimes \langle\Psi_j|A|\Psi_j\rangle \otimes |\Psi_j\rangle}\ ,$$

which can be immediately generalized to multipartite systems, by generalizing the representation of A.

[Mintert, Kuś & A.B., PRL, in press]



This provides the desired tool for our assessment of the crucial scaling properties!

Explicit evaluations

The (numerical) evaluation of the infimum

$$c(\rho) = \inf_{\{p_j, \Psi_j\}} \sum_j \sqrt{\langle \Psi_j | \otimes \langle \Psi_j | A | \Psi_j \rangle \otimes | \Psi_j \rangle} \tag{2}$$

provides an *upper* bound of $c(\rho)$ . . . we need *lower bounds*!

The algebraic structure of (2) leads to a hierarchy of approximations from below

1. **optimized lower bound** (implies numerical optimization over lower dimensional optimization space) [Mintert, Kuś & A.B., PRL 2004]

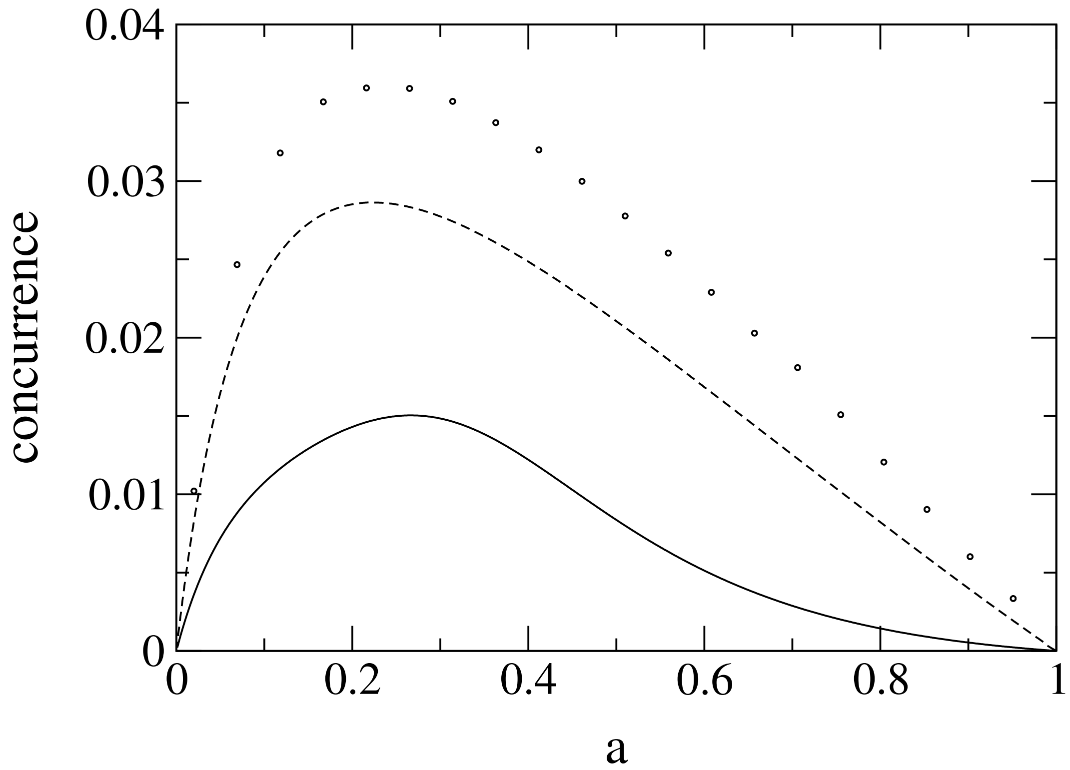
2. **algebraic lower bound** (implies diagonalization of a matrix of dimension equal to the maximal rank of A)

3. **quasi pure approximation (qpa)** [Mintert & A.B., PRA 2005]

Quasi pure approximation (qpa)

- implies diagonalization of a matrix only of the dimension of ρ! (computationally "cheap"!)

- *inspired by state of the art experiments: ρ is initially quasi pure*, i.e., it posesses one largely dominant eigenvalue $\lambda_0 \gg \lambda_j$, for all $j \neq 0$.



Nontrivial test case:
nonseparable state with *positive* partial transpose, parametrized by a
[Horodecki, Phys. Lett. A 1997,

Mintert & A.B., PRA 2005]

qpa turns out to be a lower bound of the concurrence of ρ

Dynamics under nonvanishing environment coupling

Three types of dynamics

1. random system-environment time evolution with subsequent trace over the environment

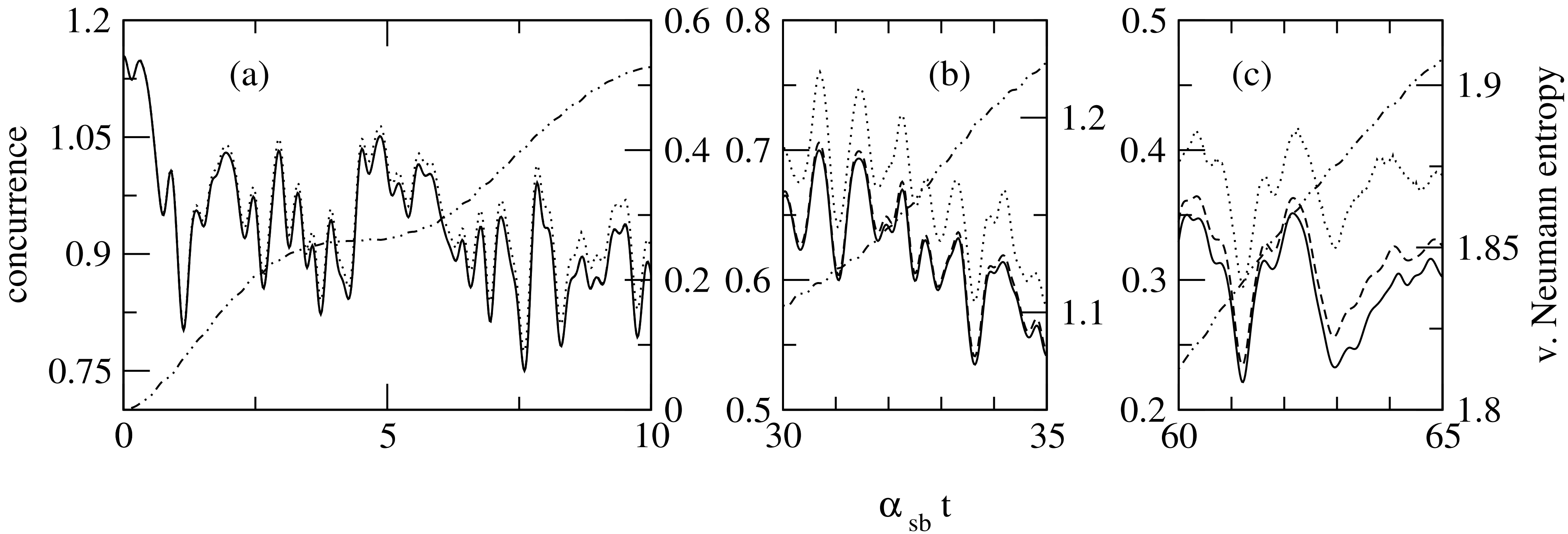
$$H = H_{\mathrm{sys-env}} + H_{\mathrm{sys}} \otimes \mathbb{1}_{\mathrm{env}} + \mathbb{1}_{\mathrm{sys}} \otimes H_{\mathrm{env}}$$

2. entanglement decay due to coupling of subsystems to "private" baths

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_{\mathrm{sys}}, \rho] + \mathcal{L}\rho = -\frac{i}{\hbar}[H_{\mathrm{sys}}, \rho] + \sum_j \frac{\Gamma_j}{2}\left(2\, d_j\, \rho\, d_j^\dagger - d_j^\dagger\, d_j\, \rho - \rho\, d_j^\dagger\, d_j\right)$$

3. entanglement generation vs. decoherence

Random time evolution



solid line: qpa – **dashed line**: optimized lower bound – **dotted line**: upper bound

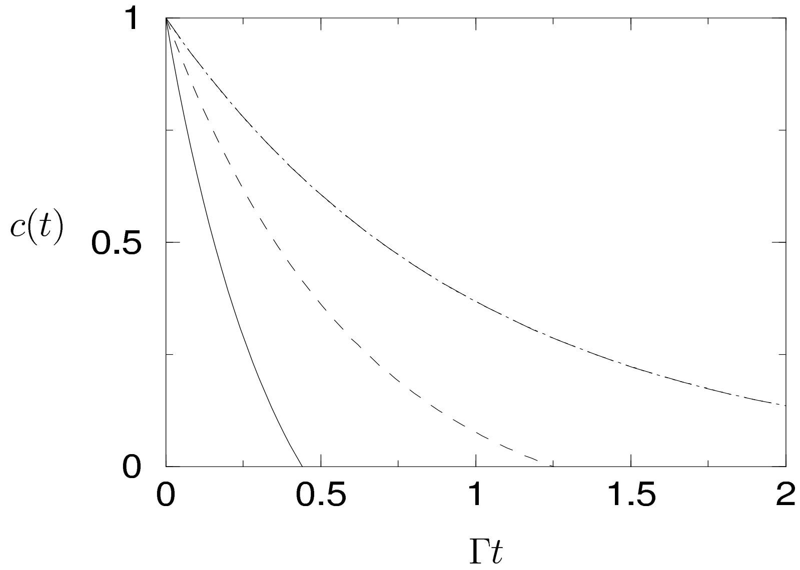
- Concurrence for an initially pure, maximally entangled 3×5 bipartite state $|\Psi_0\rangle = \sum_{j=1}^{3} |jj\rangle/\sqrt{3}$ under random, non-unitary time evolution; $\alpha_{\rm sb}$ – system-environment coupling strength

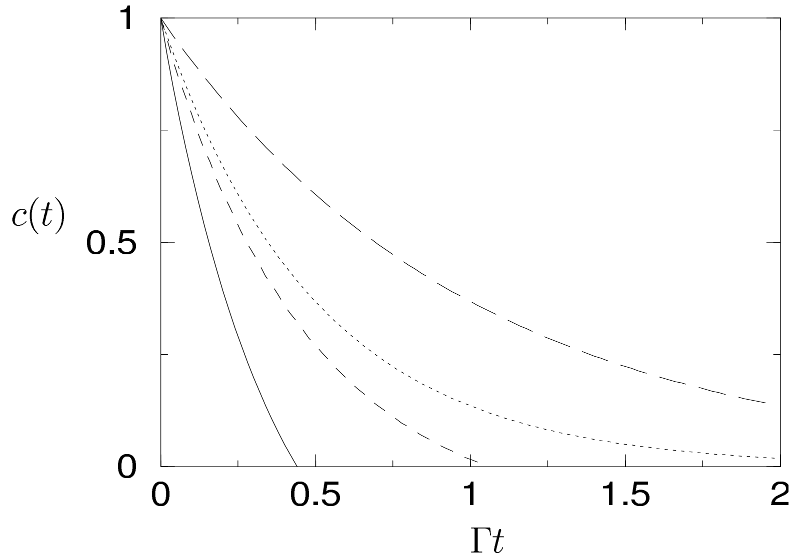
- dash-double dotted line: von Neumann entropy $S = -{\rm tr}\rho_{\rm sys} \ln \rho_{\rm sys}$ (measures mixing)

[Mintert & A.B., PRA 2005]

Entanglement decay of bipartite two-level systems

Initial states $|\Psi_s\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$ (left) and $|\Psi_s\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ (right)





- coupling to **thermal bath** with
  - **zero temperature** (only spontaneous emission; dotted line)
  - **finite temperature** ($\bar{n} = 0.1$ thermal photon in the environment; dashed)
  - **infinite temperature** (noisy environment; solid)

- or to **dephasing reservoir** (only coherence loss; long dashed)

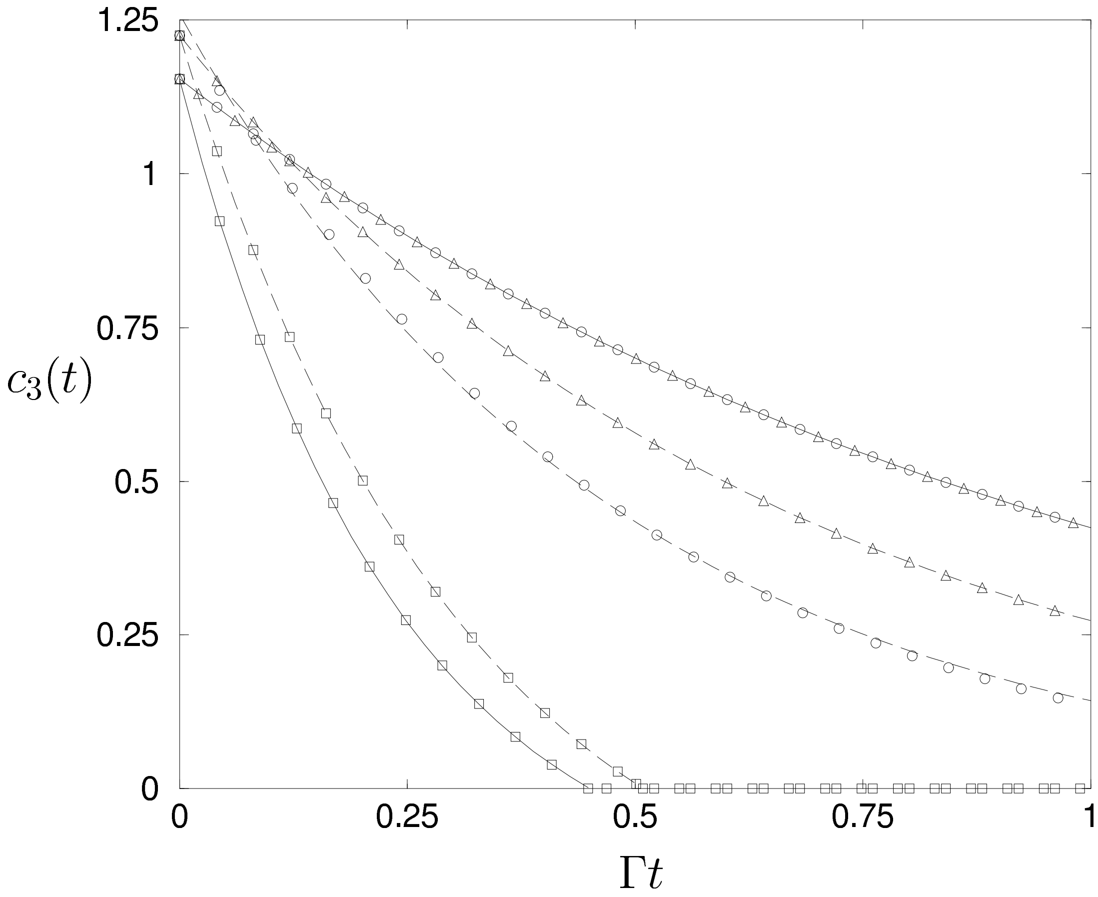
(multi-) exponential decay with finite or infinite separability times

[Mintert, Carvalho, Kuś & A.B., Phys. Rep. 2005]

N-partite entanglement

We generalize bipartite concurrence $c_2(\Psi) = \sqrt{2(\langle\Psi|\Psi\rangle^2 - \mathrm{tr}\rho_r^2)}$ for N-partite systems (with j counting all possible partitions):

$$c_N(\Psi) = 2^{1-\frac{N}{2}} \sqrt{(2^N - 2)\langle\Psi|\Psi\rangle^2 - \sum_j \mathrm{tr}\rho_j^2}$$



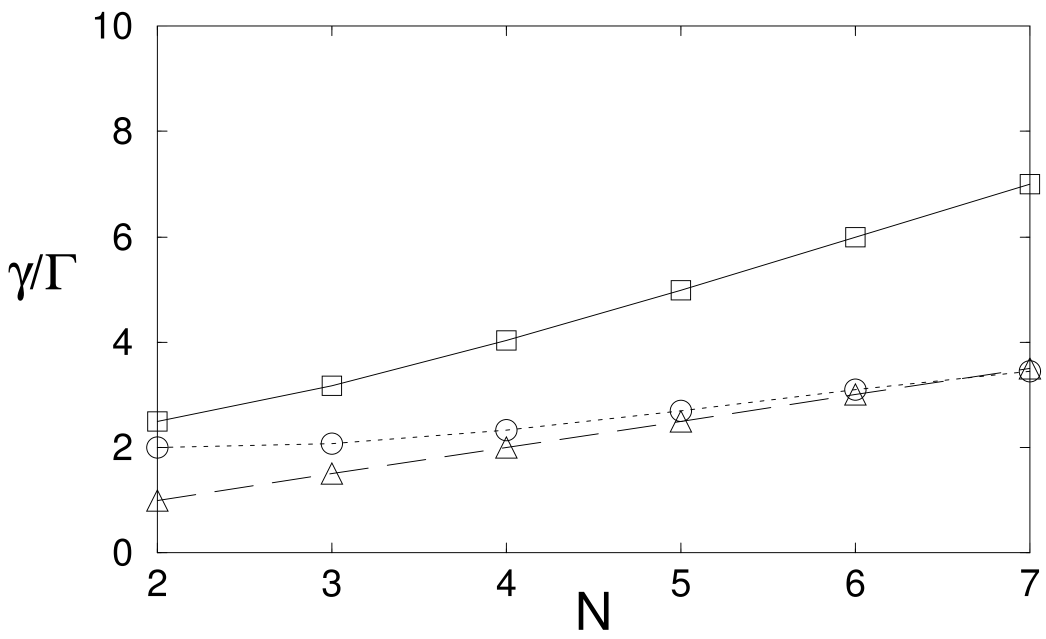
$c_N(t)$ for $N = 3$

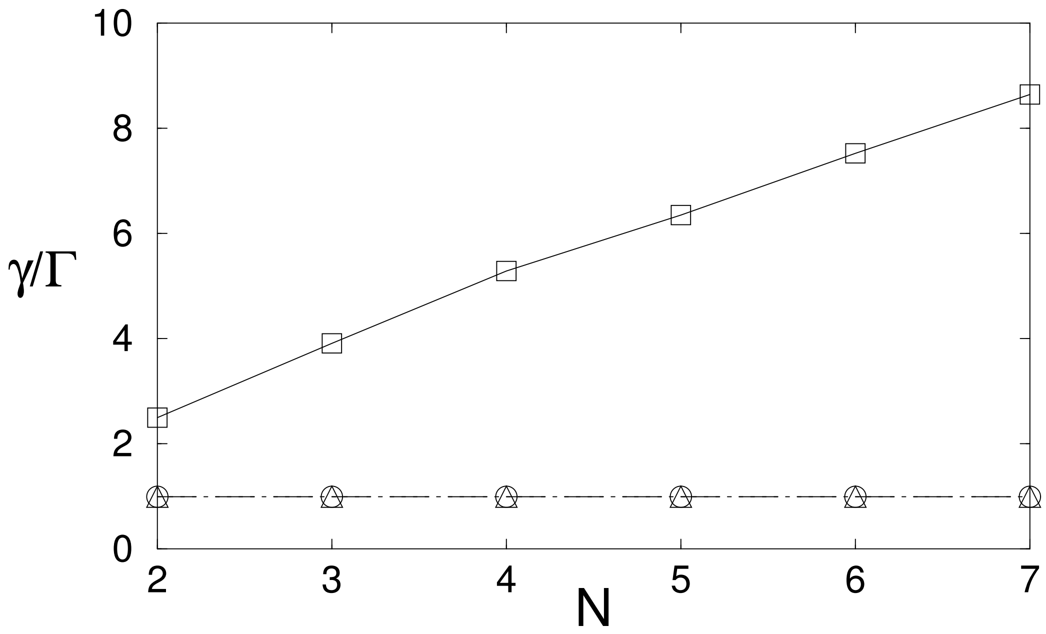
W-states (solid lines)
$|\mathrm{W}_N\rangle = (|00\ldots01\rangle + |00\ldots10\rangle + \ldots + |10\ldots00\rangle)/\sqrt{N}$

GHZ-states (dashed lines)
$|\mathrm{GHZ}_N\rangle = (|00\ldots0\rangle + |11\ldots1\rangle)/\sqrt{2}$

zero temperature (circles), **infinite temperature** (squares), and **dephasing** (triangles) environments.

Scaling of the decay rates γ





top: GHZ – **bottom**: W

circles: zero temperature

squares: infinite temperature

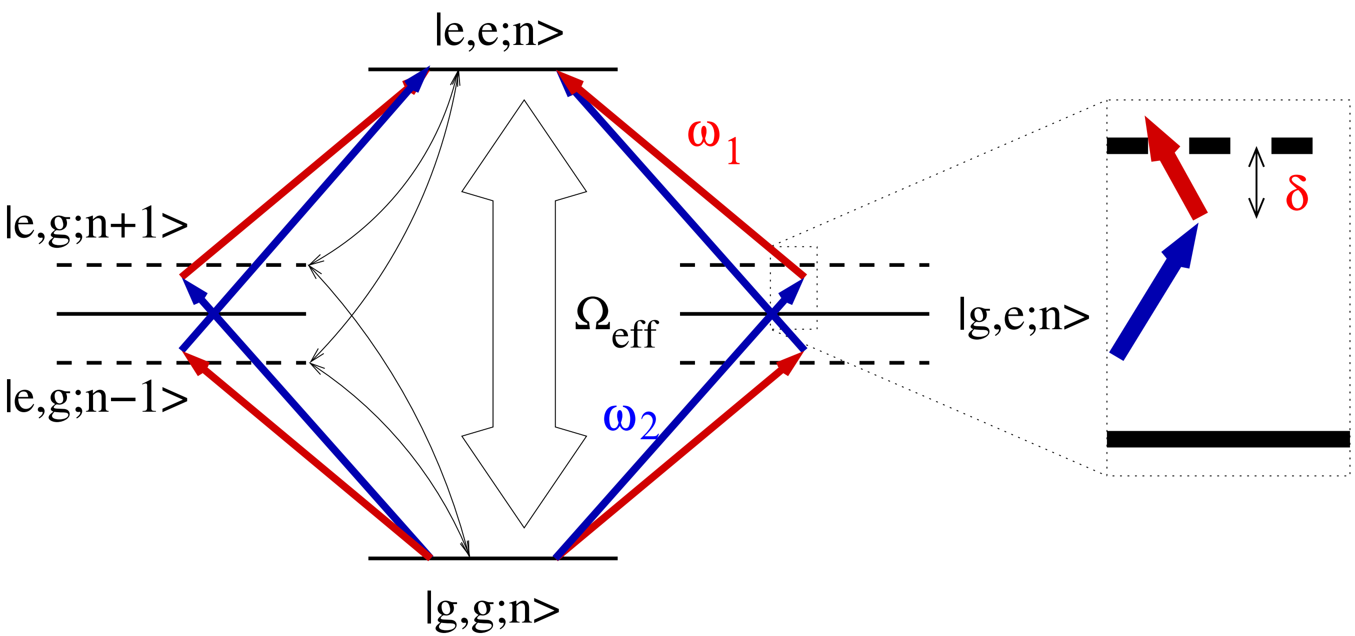
triangles: dephasing

W-states' decay rates *independent* of N for zero temperature and dephasing!

[Carvalho, Mintert & A.B., PRL 2004]

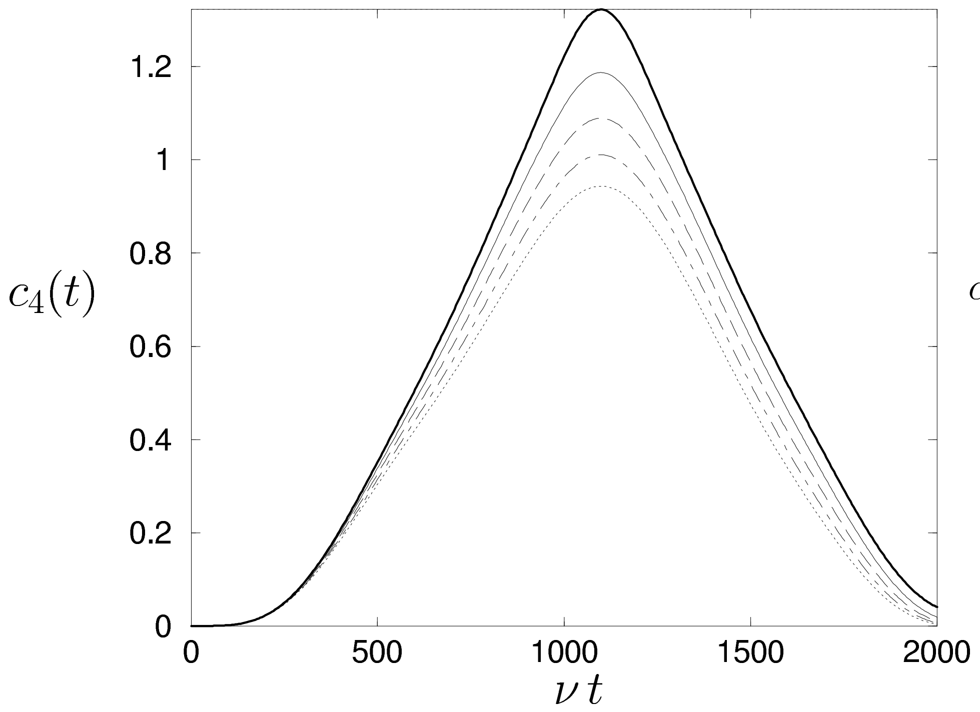
Entanglement generation vs. decoherence

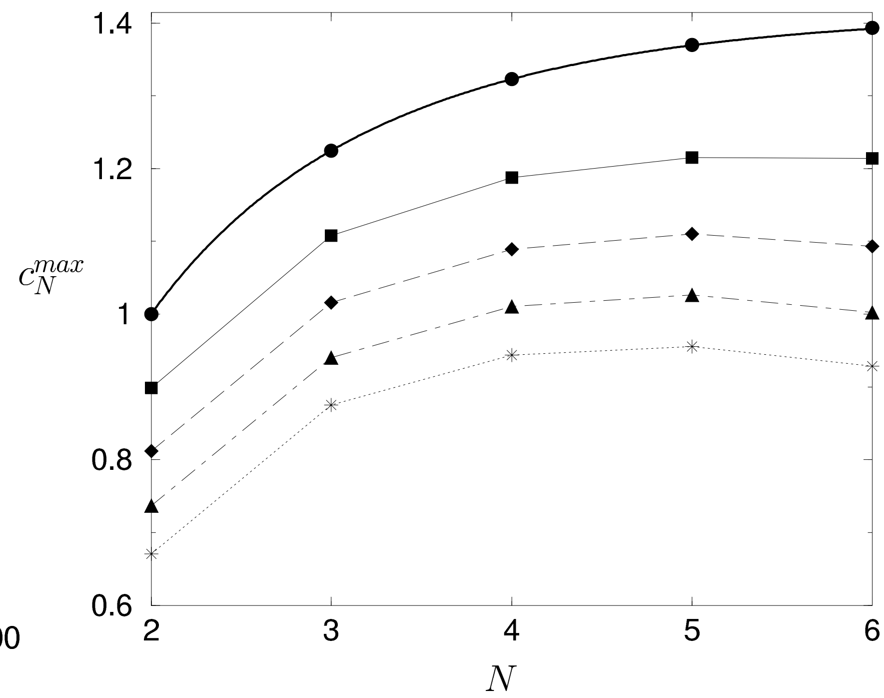
Mølmer-Sørensen scheme to prepare GHZ-like states
$|\Psi_N\rangle = (|00\ldots0\rangle + e^{i\phi_N}|11\ldots1\rangle)/\sqrt{2}$ [Mølmer & Sørensen, PRL 1999]



- motional decoherence limits the preparation efficiency
- only *after* the preparation process are internal and vibrational degrees of freedom uncorrelated

Preparation efficiency vs. t and N





- various coupling strengths to vibrational noise $\Gamma/\nu = 0$ (thick line), 1×10^{-4} (full line), 2×10^{-4} (dashed), 3×10^{-4} (dash-dotted), 4×10^{-4} (dotted)

  [Turchette et al., PRA 2000]

- **left:** concurrence c_4 of four ions vs. preparation time

- **right:** maximum concurrence vs. ion number – nonmonotonous behaviour indicates that environment effects become more detrimental with increasing N!

Dynamics revisited

So far

1. Solve master equation

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_{\rm sys}, \rho] + \sum_k \frac{1}{2}\left(2\, J_k\, \rho\, J_k^\dagger - J_k^\dagger\, J_k\, \rho - \rho\, J_k^\dagger\, J_k\right),$$

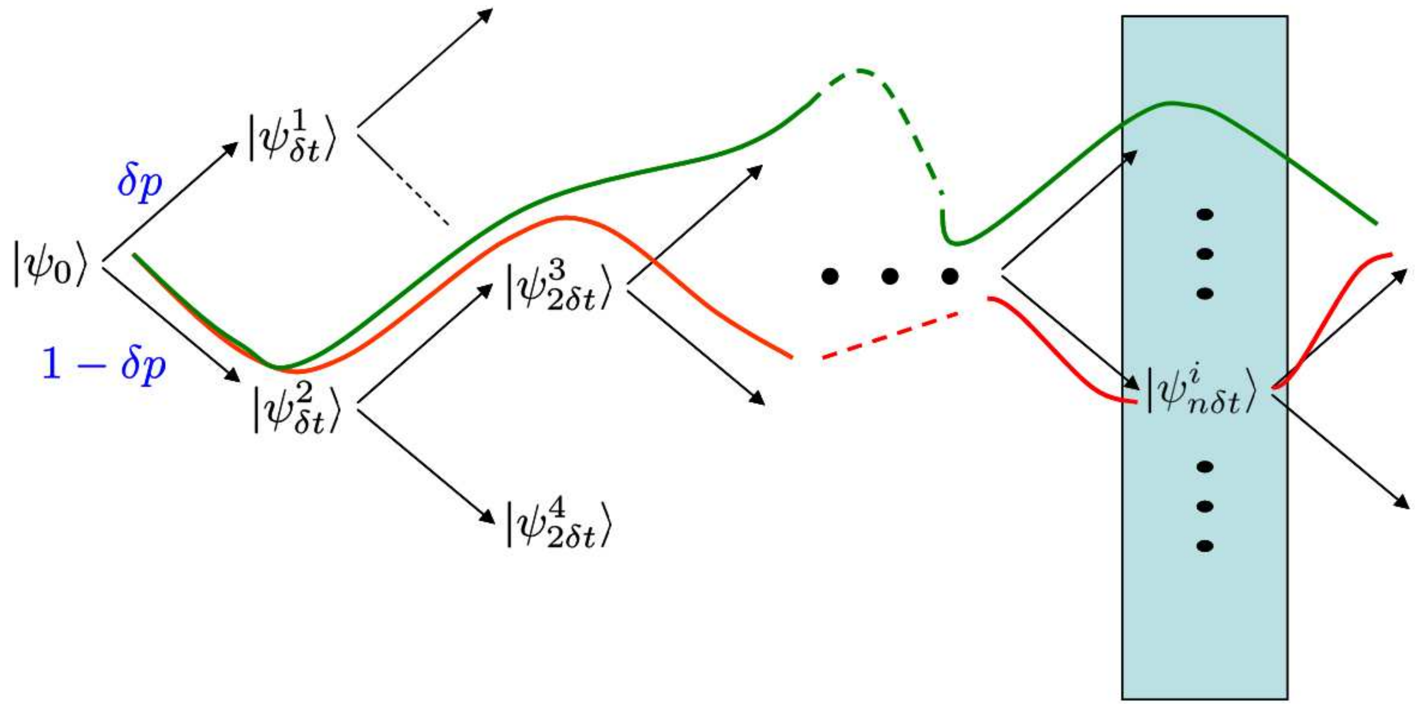
2. and evaluate entanglement measure $M(\rho(t))$ for all t

Now: Unravel Entanglement! [Carvalho, Busse, Brodier, Viviescas, A.B., quant-ph/0510006]

start with pure state $|\Psi_0\rangle$ *and generate* **stochastic pure state evolution** *under action of*

- jumps mediated by jump operators J_k, with probability δp_k,
- alternating with nonhermitian free evolution, generated by $H_{\rm eff} = H_{\rm sys} - i\hbar \sum_k J_k^\dagger J_k/2$

Quantum trajectories



. . . naturally generate a pure state decomposition of $\rho(t)$!

$$\rho(t) \simeq \sum p_i |\Psi^i(t)\rangle\langle\Psi^i(t)|\,.$$

. . . but is it optimal (remember the inf above!) ???

Optimal unravelling

- Jump operators which generate the same master equation are not uniquely defined – most general form:

$$L_{k,\pm} = \frac{\mu \, \mathrm{Id} \pm \sum_i U_{ki} J_i}{\sqrt{2}},$$

U unitary, μ complex.

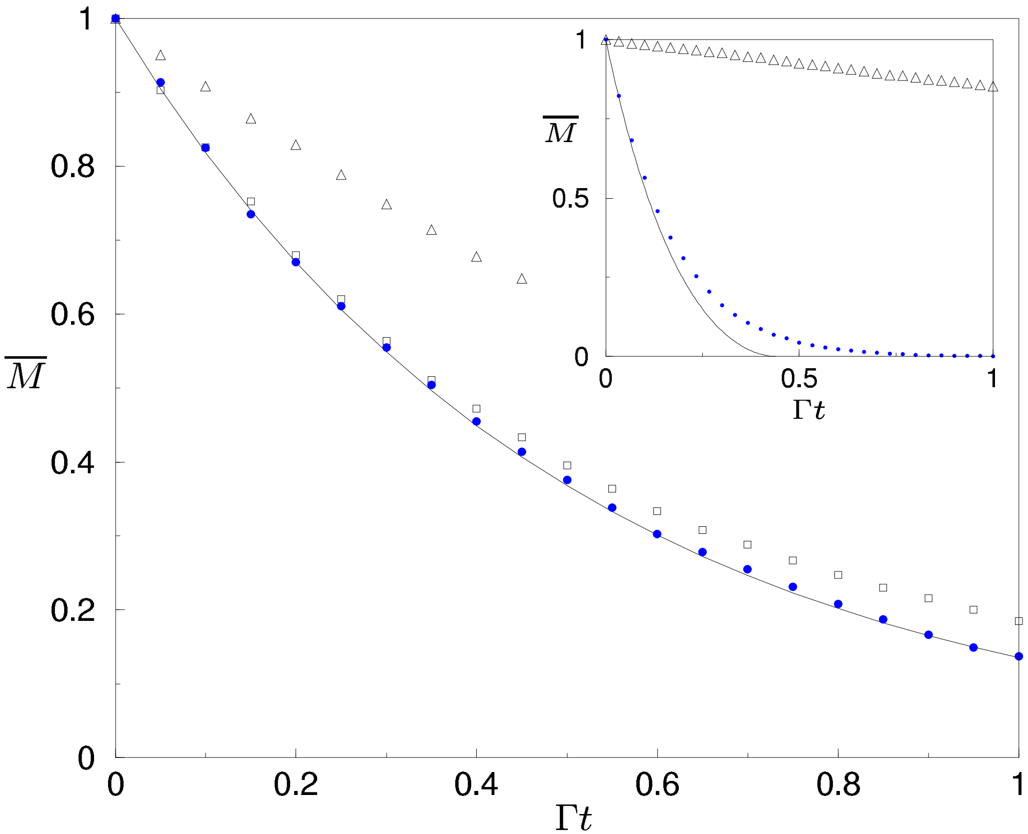
- different jump operators correspond to different measurement prescriptions, i.e. different ways of monitoring the system

- minimize average entanglement $\overline{M}(\delta t)$ after first time step δt

$$\overline{M}(\delta t) = (1 - \sum_{k=1}^{N} \delta p_k) M(\Psi_{\delta t}^{N+1}) + \sum_{k=1}^{N} \delta p_k M(\Psi_{\delta t}^{k})$$

Two qubits – zero and infinite temperature environment

$\overline{M} \equiv \sum_i p_i M(\Psi^i(t))$ measures entanglement!



$|\Psi_0\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$

zero temperature (spontaneous emission)

$$J_1 = \sqrt{\Gamma}\sigma_-^{(1)} \; , \; J_2 = \sqrt{\Gamma}\sigma_-^{(2)}$$

infinite temperature (inset)

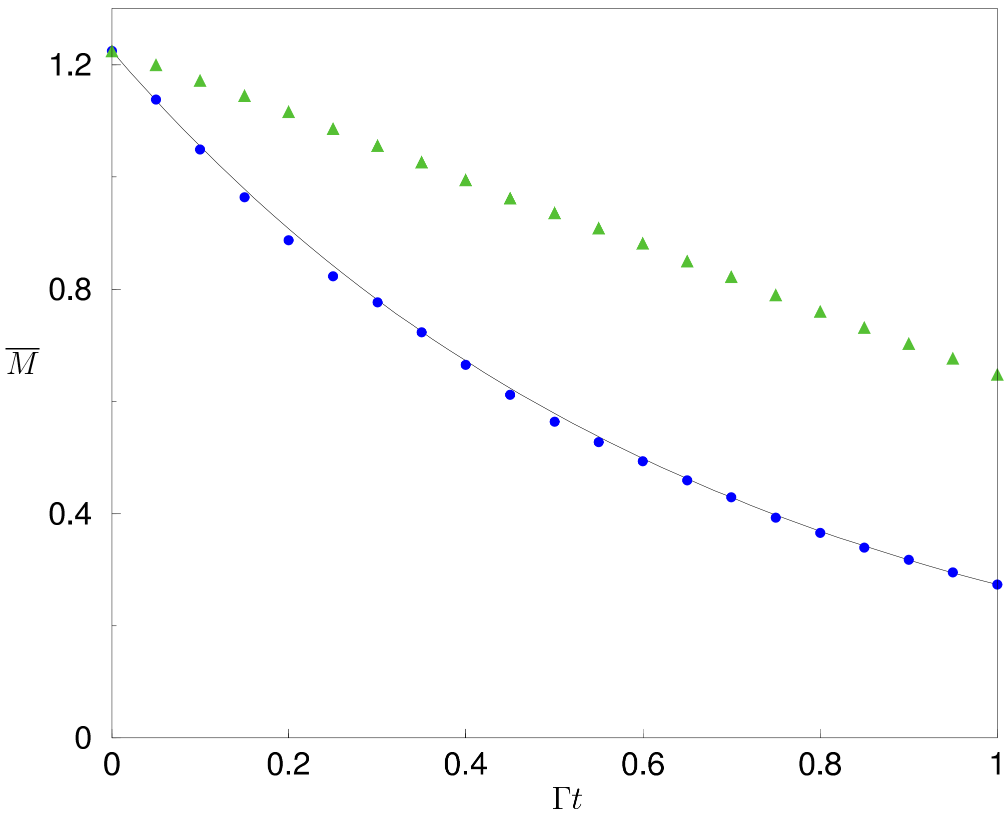
$$J_{1,3} = \sqrt{\Gamma}\sigma_\pm^{(1)} \; , \; J_{2,4} = \sqrt{\Gamma}\sigma_\pm^{(2)}$$

optimize

$$L_{k,\pm} = \frac{\mu \,\mathrm{Id} \pm \sum_i U_{ki} J_i}{\sqrt{2}}$$

Three qubits – dephasing environment

tripartite concurrence $C_3(t)$



$$|\Psi_0\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$$

jump operators for ions $k = 1, 2, 3$

$$J_k = \sigma_+^{(k)} \sigma_-^{(k)}$$

(here already optimal with coupling operators from master equation)

. . . works equally well for W states under zero temperature environment coupling

What to take away . . .

!!Scalability of entanglement dynamics!!

! With a suitable reformulation of concurrence, we obtained an efficient quantifier of mixed state entanglement
• in arbitrary finite dimensions • for bi- as well as N-partite systems

! W-states outperform GHZ-states in terms of their robustness against decoherence, with increasing number of entangled qubits

! Our tool box also provides a means to scrutinize the competition between coherent entanglement generation schemes and decoherence

! unravelling entanglement by optimal measurements

! no ambiguity of mixed state entanglement with respect to different unravellings (in contrast to Nha & Carmichael, PRL 2004)

What to work on . . .

? Which observable(s) to measure to obtain a reliable estimate of the entanglement properties? Scaling properties?

? How robust are these entanglement measures against errors in the determination of the density matrix' elements?

? "Universal" time evolution of entanglement alike the Lindblad form? General classification of arbitrary mixed states? Will unravelling always work?

? Decoherence and entanglement in *large/heavy* objects with many degrees of freedom?

Literature (selection):

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Mintert, Carvalho, Kuś & A.B., Phys. Rep. 415, 207 (2005)