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Methods in accelerator-driven systm dynamics

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Abstract. The present paper illustrates some models and methods for the kinetic evaluation of neutron multiplying systems. After introducing the general physical features of time-dependent neutronics, the simple point kinetics model is analysed, giving details on the physical meaning and mathematical structure of the equations. The spatial characterization of the neutronic transient is then investigated, with special regards to source-driven systems. It is evidenced that in a source-dominated structures, the space transients are less important than in systems departing from near criticality. The factorization technique used for quasi-static analysis is then presented and extended to source-driven systems and some considerations on the problem concerning the choice of the weghting function are made. The paper is concluded by a discussion of the physics of fluid-fuel systems, a consistent model is described and the factorization procedure is applied.

1. Introduction

The study of the neutron evolution in nuclear systems is a challenging mathematical physical problem and it constitutes a fundamental topic in the design of critical and subcritical reactors. The need to study subcritical multiplying structures driven by an external neutron source which are being proposed for long lived products transmutation and for safe and acceptable energy production is introducing new and interesting features which need to be properly accounted for. These lectures are specially devoted to source driven systems. However, a general description of reactor kinetics is obviously needed to settle the basis for such a specific application. Therefore, the body of the lectures is divided in the following chapters:

* General aspects of neutron time dependent problems

* Point kinetics and its features

* Spatial characterization of neutronic transients

* Factorization methods and quasi statics

* Extension of models to source driven systems

* Developments: the multipoint scheme

* A harder problem: the modeling of fluid fuel systems

The numerical results presented are intended to enlighten some specific physical aspects of the material presented.

2. General aspects of neutron time-dependent problems

The evolution of the neutron population within a multiplying system is strongly characterized by broadly different time scales. Specifically, the different physical phenomena determine the appearance of the following typical scales:

* prompt neutron scale, connected to the lifetime in the multiplication process of prompt neutrons; it is a very fast scale (10^{-4} 10^{-6} s);

* delayed emission scale, connected to the evolution of delayed neutron precursors (10^{-1} 10^{1} s);

* thermal hydraulic scale associated to feedback, connected to the evolution of temperatures and hy draulic parameters (10^{-1} 10^{2} s);

* control scale, connected to the movement of masses in the system (control rods, poisons); it can range very broadly from seconds to hours;

* nuclide transmutation scale, connected to neutron transmutation phenomena (from $>$ to $\gg 10^{2}$ s).

The simultaneous existence of so many different time characteristics makes the physico mathematical problem stiff. This fact implies that numerical methods have to be specifically tailored to the problem needs. Often one is not interested in obtaining information on all the aspects of the evolution. In the following presentation, the interest is focused mainly on the neutron evolution determined by relatively fast phenomena. Therefore the long term effects due to the nuclide transmutation are not accounted for.

The neutron evolution in a nuclear reactor is strongly affected by the delayed emissions from the ra dioactive decay of some fission products. Although the fractions are usually rather small, the effect is of slowing down the evolution of the neutron population, thus making the system controllable [1]. Several different precursors have been identified. They are grouped according to similar decay characteristics, usually in six families [2]. Recently, a IAEA endeavor has been undertaken to review existing data for delayed neutrons and evaluate and compile new and more up to date data coming from recent experi mental activities [3]. It has been advised a regrouping of the delayed precursors in eight families.

Each family is characterized by:

* the fraction of fission neutrons β_i appearing in the i th family ($\beta = \sum_{i=1}^{6} \beta_i$);

* the decay constant λ_i [s^{-1}];

* the emission spectrum χ_i, which is much softer that the prompt spectrum, thus significantly changing the contribution of delayed neutrons to the chain reaction process.

The delayed fractions depend dramatically on the fissile nuclide being considered.

The neutron evolution is governed by the time dependent Boltzmann transport equation including de layed neutrons [4]:

$$\begin{cases} \dfrac{\partial n(\mathbf{r},E,\Omega,t)}{\partial t} = \hat{\mathrm{B}}(t)n(\mathbf{r},E,\Omega,t) + \displaystyle\sum_{i=1}^{6} \lambda_i \frac{\chi_i(E)}{4\pi} C_i(\mathbf{r},t) + S(\mathbf{r},E,\Omega,t), \\ \dfrac{\partial(\chi_i(E)C_i(\mathbf{r},t)/4\pi)}{\partial t} = \hat{\mathrm{M}}_i(t)n(\mathbf{r},E,\Omega,t) - \lambda_i \dfrac{\chi_i(E)}{4\pi} C_i(\mathbf{r},t), \end{cases} \tag{1}$$

which is a linear version of the general non linear equation written for gases by the great scientist Ludwig

Boltzmann *(1844-1906)* [5].

The operators appearing in Eq. (1) can all be time dependent, since the properties of the neutron diffusing material can in general change in time. The change can be a linear effect, when it is not driven by the neutron population itself but is driven by external (known) causes, or non linear, when it is determined by a functional of the neutron distribution. They are named and defined as follows:

* balance operator:

$$\hat{\mathrm{B}}(t) = \hat{\mathrm{L}}(t) + \hat{\mathrm{M}}_p(t); \tag{2}$$

* leakage operator:

$$\hat{\mathrm{L}}(t) = -\Omega \cdot \nabla v(E) - \Sigma(\mathbf{r}, E, t)v(E) + \int dE' \oint d\Omega' v(E')\Sigma_s(\mathbf{r}, E' \to E, \Omega' \cdot \Omega, t); \tag{3}$$

* prompt multiplication operator:

$$\hat{\mathrm{M}}_p(t) = \sum_j \frac{\chi_p^j(E)}{4\pi} \int dE' \oint d\Omega' v(E')(1-\beta^j)\nu^j(E')\Sigma_f^j(\mathbf{r}, E', t), \tag{4}$$

where the sum is carried out over all fissile nuclides;

* delayed multiplication operator:

$$\hat{\mathrm{M}}_i(t) = \sum_j \frac{\chi_i^j(E)}{4\pi} \int dE' \oint d\Omega' v(E')\beta_i^j\nu^j(E')\Sigma_f^j(r, E', t); \tag{5}$$

* total multiplication operator:

$$\hat{\mathrm{M}}(t) = \hat{\mathrm{M}}_p(t) + \sum_{i=1}^{6} \hat{\mathrm{M}}_i(t); \tag{6}$$

* effective emission spectrum for each fissile nuclide:

$$\chi^j(E) = (1-\beta^j)\chi_p^j(E) + \sum_{i=1}^{6} \beta_i^j \chi_i^j(E). \tag{7}$$

Some considerations are in order. The solution of the Boltzmann equation (1), associated to initial con ditions for the neutron flux and delayed neutron concentrations and to boundary conditions for the flux (usually zero incoming neutron flux for a system exposed to vacuum) is a formidable task. Moreover, it may yield too much physical detail, since in real systems only integral quantities can be observed. Consequently, it is suitable to construct simplified models (multigroup, diffusion, ...) based on some adequate physical assumptions and then is it required to develop numerical algorithms (discretizations, expansions). These necessary steps introduce some approximations which need to be assessed in order to establish their adequateness for the problem considered.

3. Point kinetics and its features

Certainly the simplest model is point kinetics. Its formulation dates back to the early times of reactor physics [6], [7], [8]. It has proved to be a powerful tool for many reactor evaluations, both in operation and in safety transient situations. The name itself indicates that the neutron distribution is supposed to evolve as a point, in the sense that each point is representative of the whole system. For a system that is source free this is perfectly equivalent to saying that only the fundamental eigenfunction of the kinetic operator appears in the neutron distribution at all instants. This statement is not true when the system is injected by an external source, in which case the distribution is dominated by the source injection and

may involve a superposition of many eigenfunctions. In this situation the point like behavior amounts to saying in a quite general fashion that the neutron distribution can be factorized in the product of an amplitude function (time dependent only) and a shape function (time independent, but dependent on all phase space variables).

How is it possible to derive point equations consistently starting from the balance transport model (1) or one of its approximations? This question is differed a while to the section 5.1, while it is useful and instructive to consider some applications to deepen the meaning of a point behavior.

Let us consider a simpler and easier problem, i.e. space one group diffusion in a homogeneous slab geometry with one delayed family and time constant properties. The basic equations are the following:

$$\begin{cases} \dfrac{1}{v}\dfrac{\partial\Phi(x,t)}{\partial t} = D\dfrac{\partial^2\Phi(x,t)}{\partial x^2} - \Sigma_a\Phi(x,t) + (1-\beta)\nu\Sigma_f\Phi(x,t) + \lambda C(x,t) + S(x,t), \\ \dfrac{\partial C(x,t)}{\partial t} = -\lambda C(x,t) + \beta\nu\Sigma_f\Phi(x,t), \end{cases} \tag{8}$$

with boundary and initial conditions, as:

$$\begin{aligned} &\Phi(0,t>0) = \Phi(H,t>0) = 0, \\ &\Phi(x,t=0) = \Phi_0(x), \\ &C(x,t=0) = C_0(x). \end{aligned} \tag{9}$$

This physical problem can be given an exact solution by an eigenfunction expansion [9], [10]. The Helmholtz eigenfunctions constitute a complete and orthogonal system, the most suitable base for rep resenting the solution of the diffusion problem

$$\begin{aligned} &\frac{d^2\varphi_n(x)}{dx^2} = -B_n^2\varphi_n(x), \\ &\varphi_n(0) = \varphi_n(H) = 0. \end{aligned} \tag{10}$$

Hence the space functions appearing in Eq. (8) can be expanded as:

$$\begin{aligned} \Phi(x,t) &= \sum_{n=0}^{\infty} a_n(t)\varphi_n(x), \\ C(x,t) &= \sum_{n=0}^{\infty} c_n(t)\varphi_n(x), \\ S(x,t) &= \sum_{n=0}^{\infty} s_n(t)\varphi_n(x), \end{aligned} \tag{11}$$

where the source components are known time dependent functions given by:

$$s_n(t) = \int_0^H dx S(x,t)\varphi_n(x) \equiv (\varphi_n, S(t))\,. \tag{12}$$

Also the initial conditions can be expanded:

$$\begin{aligned} \Phi_0(x) &= \sum_{n=0}^{\infty} a_{n0}\varphi_n(x) \equiv \sum_{n=0}^{\infty} (\varphi_n, \Phi_0)\,\varphi_n(x), \\ C_0(x) &= \sum_{n=0}^{\infty} c_{n0}\varphi_n(x) \equiv \sum_{n=0}^{\infty} (\varphi_n, C_0)\,\varphi_n(x). \end{aligned} \tag{13}$$

After introduction of the above expansion into problem (8), making use of the orthogonality of the Helmholtz eigenfunctions, the system of equations for the components of the unknown solutions can be given a compact matrix form introducing the vectors:

$$|x_n(t)\rangle = \left| \begin{array}{c} a_n(t) \\ c_n(t) \end{array} \right\rangle, \qquad |s_n(t)\rangle = \left| \begin{array}{c} s_n(t) \\ 0 \end{array} \right\rangle, \tag{14}$$

as

$$\frac{d}{dt}|x_n(t)\rangle = \hat{M}_n |x_n(t)\rangle + |s_n(t)\rangle, \tag{15}$$

where

$$\hat{M}_n = \begin{pmatrix} v\left[(1-\beta)\nu\Sigma_f - DB_n^2\right] & v\lambda \\ \beta\nu\Sigma_f & -\lambda \end{pmatrix}. \tag{16}$$

The solution is expressed in terms of the eigenvectors of the characteristic matrix $\hat{M}_n$. To this purpose the following direct and adjoint eigenproblems are solved:

$$\begin{aligned} \hat{M}_n |u_n\rangle &= \omega_n |u_n\rangle, \\ \langle u_n| \hat{M}_n &= \omega_n \langle u_n|, \end{aligned} \tag{17}$$

where the eigenvalues ω_n are the solutions of the algebraic generalized inhour equation associated to each spatial eigenfunction:

$$\det\left(\hat{M}_n - \omega_n \hat{\Im}\right) = 0. \tag{18}$$

In this case there are two (seven, when six families are considered) distinct real eigenvalues $\omega_n^{(j)}$, $j = 1, 2$. The un normalized eigenvectors turn out to take the form:

$$\begin{aligned} |u_n\rangle &= \left| \begin{array}{c} 1 \\ \dfrac{\beta\nu\Sigma_f}{\omega_n + \lambda} \end{array} \right\rangle, \\ \langle u_n| &= \left\langle \begin{array}{cc} 1 & \dfrac{v\lambda}{\omega_n + \lambda} \end{array} \right|. \end{aligned} \tag{19}$$

In conclusion, the analytical full closed form solution is:

$$\begin{aligned} |x_n(t)\rangle = \sum_{j=1}^{2} \frac{1}{\left\langle u_n^{(j)} \middle| u_n^{(j)} \right\rangle} \left[\left\langle u_n^{(j)} |x_n(0)\rangle e^{\omega_n^{(j)} t} + \int_0^t dt' \left\langle u_n^{(j)} | s_n(t') \right\rangle e^{\omega_n^{(j)}(t-t')} \right] \left| u_n^{(j)} \right\rangle \equiv \\ \sum_{j=1}^{2} \left[b_{n0}^{(j)} e^{\omega_n^{(j)} t} + \int_0^t dt' \sigma_n^{(j)}(t') e^{\omega_n^{(j)}(t-t')} \right] \left| u_n^{(j)} \right\rangle. \end{aligned} \tag{20}$$

The explicit form for the neutron flux is:

$$\begin{aligned} \Phi(x,t) = \sum_{n=0}^{\infty} \left\{ \sum_{j=1}^{2} \left(1 + \frac{\beta v \Sigma_f v \lambda}{(\omega_n^{(j)} + \lambda)^2} \right)^{-1} \times \right. \\ \left. \left[(\varphi_n, \Phi_0)\, e^{\omega_n^{(j)} t} + \frac{v\lambda}{\omega_n^{(j)} + \lambda} (\varphi_n, C_0)\, e^{\omega_n^{(j)} t} + \int_0^t dt' \, (\varphi_n, S(t'))\, e^{\omega_n^{(j)}(t-t')} \right] \right\} \varphi_n(x). \end{aligned} \tag{21}$$

A point reactor is here defined as a system evolving according to the fundamental eigenfunction φ_0 only. In this situation any of the following statements is true:

* no space distortion appear during the transient;

* the evolution is space time separable;

* any point is representative of the whole system.

Obviously, this can be realized only if the source is distributed according to the fundamental eigenfunc tion.

When six delayed families are considered, it is straightforward to generalize the point solution as:

$$\Phi(x,t) = \sum_{j=1}^{7}\left(1+\frac{\beta\upsilon\Sigma_f\upsilon\lambda}{(\omega^{(j)}+\lambda)^2}\right)^{-1}\left[(\varphi_0,\Phi_0)\,e^{\omega^{(j)}t}+\frac{\upsilon\lambda}{\omega^{(j)}+\lambda}\,(\varphi_0,C_0)\,e^{\omega^{(j)}t}+\right. \tag{22}$$

$$\left.\int_0^t dt'\,\left(\varphi_0,S(t')\right)\,e^{\omega^{(j)}(t-t')}\right]\varphi_0(x), \tag{23}$$

$$C(x,t) = \sum_{j=1}^{7}\left(1+\frac{\beta\upsilon\Sigma_f\upsilon\lambda}{(\omega^{(j)}+\lambda)^2}\right)^{-1}\left[(\varphi_0,\Phi_0)\,e^{\omega^{(j)}t}+\frac{\upsilon\lambda}{\omega^{(j)}+\lambda}\,(\varphi_0,C_0)\,e^{\omega^{(j)}t}+\right. \tag{24}$$

$$\left.\int_0^t dt'\,\left(\varphi_0,S(t')\right)\,e^{\omega^{(j)}(t-t')}\right]\frac{\beta\nu\Sigma_f}{\omega^{(j)}+\lambda}\varphi_0(x), \tag{25}$$

where $\omega^{(j)}$ are solutions of the inhour equation:

$$\frac{\omega\Lambda}{1+\omega\Lambda}+\frac{\omega}{1+\omega\Lambda}\sum_{i=1}^{6}\frac{\beta_i}{\omega+\lambda_i}-\rho=0, \tag{26}$$

The following integral kinetic parameters can be defined:

$$\rho=\frac{k_{eff}-1}{k_{eff}} \qquad \text{reactivity} \tag{27}$$

$$k_{eff}=\frac{\upsilon\Sigma_f/\Sigma_a}{1+L^2B_0^2} \qquad \text{multiplication constant} \tag{28}$$

$$\Lambda=\frac{1}{\upsilon\Sigma_a\left(1+L^2B_0^2\right)} \qquad \text{prompt neutron lifetime} \tag{29}$$

It is worth while to remark on some features of the real roots $\omega^{(j)}$ of the inhour equation (26):

$\star$ six roots are close to and approach each $-\lambda_i$ as subcriticality increases;

$\star$ the seventh one, $\omega^{(7)}$, is much larger in absolute value and negative and it determines the prompt response of the neutron population connected to the inverse of the prompt lifetime;

$\star$ with a time constant source, asymptotically the solution is driven by the exponential associated to the dominant root:

$$\Phi_{as}(x,t) = \left(1+\frac{\beta\upsilon\Sigma_f\upsilon\lambda}{(\omega^{(0)}+\lambda)^2}\right)^{-1}\left[(\varphi_0,\Phi_0)\,e^{\omega^{(0)}t}+\frac{\upsilon\lambda}{\omega^{(0)}+\lambda}\,(\varphi_0,C_0)\,e^{\omega^{(0)}t}+\right. \tag{30}$$

$$\left.\frac{(\varphi_0,S)}{\omega^{(0)}}\left(1-e^{\omega^{(0)}t}\right)\right]\varphi_0(x), \tag{31}$$

$$C_{as}(x,t) = \left(1+\frac{\beta\upsilon\Sigma_f\upsilon\lambda}{(\omega^{(0)}+\lambda)^2}\right)^{-1}\left[(\varphi_0,\Phi_0)\,e^{\omega^{(0)}t}+\frac{\upsilon\lambda}{\omega^{(0)}+\lambda}\,(\varphi_0,C_0)\,e^{\omega^{(0)}t}+\right. \tag{32}$$

$$\left.\frac{(\varphi_0,S)}{\omega^{(0)}}\left(1-e^{\omega^{(0)}t}\right)\right]\frac{\beta\nu\Sigma_f}{\omega^{(0)}+\lambda}\varphi_0(x); \tag{33}$$

$\star$ the ratio of the precursor density to the neutron density is:

$$\frac{C}{n}=\frac{vC}{\Phi}=\frac{\upsilon\beta\nu\Sigma_f}{\omega^{(0)}+\lambda}=\beta\frac{k_{eff}}{\Lambda}\frac{1}{\omega^{(0)}+\lambda}, \tag{34}$$

which may assume values of the order of 10^3 - 10^4! This is physically due to the fact that delayed precursors can largely accumulate with respect to prompt neutrons owing to their extremely large value of their mean time before being released by radioactive decay as compared to the prompt lifetime;

$\star -\lambda_1$ has a special role, since it limits the negative value of the fundamental time constant of the system; consequently, the "averaging" of the properties of delayed precursors to produce the delayed families is a rather delicate task.

4. Spatial characterization of neutronic transients

The object of this section is to derive a mathematical framework to characterize space and energy effects in neutron transients. To attain such scope, an analytical approach is always used analyzing simplified configurations, referring to the diffusion theory model with one group for neutrons and one delayed family for precursors. This study can help to understand the physics of the neutron evolution in multiplying systems and to establish limits of simplified models such as point kinetics [9], [11]. Some special considerations are devoted to the physics of subcritical systems, with reference to the dominance of the source.

The following parameters are introduced to characterize a spatial transient:

• asymptotic ratio, as relative difference with respect to the asymptotic value of the solution Φ_{asy}:

$$\mathcal{R}_{asy} = \left| \frac{\Phi - \Phi_{asy}}{\Phi_{asy}} \right| ; \tag{35}$$

• dominance ratio, as relative difference with respect to the dominant portion of the solution $\Phi_{\mathcal{D}}$, which is particularly useful to characterize source-driven systems:

$$\mathcal{R}_{\mathcal{D}} = \left| \frac{\Phi - \Phi_{\mathcal{D}}}{\Phi_{\mathcal{D}}} \right| ; \tag{36}$$

• spatiality parameter, as norm of the difference between the full solution and a reference steady-state flux distribution:

$$\xi \equiv \frac{1}{\langle \Phi \rangle} \sqrt{\left\langle \left(\Phi - \frac{\langle \Phi \rangle}{\langle \Psi \rangle} \Psi \right)^2 \right\rangle}. \tag{37}$$

Two cases are now discussed.

Case 1. The critical reactor.

An initially critical reactor in absence of delayed emissions is firstly considered. The analytical solution is as follows [10]:

$$\omega_n = v \left[\nu \Sigma_f - \left(D B_n^2 + \Sigma_a \right) \right] ; \tag{38}$$

$$\Phi(x,t) = \sum_{n=0}^{\infty} a_{n0} e^{\omega_n t} \varphi_n(x); \tag{39}$$

$$\Phi_{asy}(x,t) = a_{00} e^{\omega_0 t} \varphi_0(x), \tag{40}$$

hence:

$$\mathcal{R}_{asy} \to e^{(\omega_1 - \omega_0)t}, \tag{41}$$

which assigns a special role to the time-eigenvalue separation of the system.

The initially critical reactor is now considered with delayed emissions. In this case it is useful to separate the contribution of the fundamental eigenfunction:

$$\Phi_{fund}(x,t) = \left(b_{\mathbf{00}}^{(1)} e^{\omega_{\mathbf{0}}^{(1)} t} + b_{\mathbf{00}}^{(2)} e^{\omega_{\mathbf{0}}^{(2)} t}\right) \varphi_{\mathbf{0}}(x). \tag{42}$$

Noting that:

$$\left|\omega_n^{(1)}\right| \ll \left|\omega_n^{(2)}\right|, \tag{43}$$

and $\omega_n^{(2)} < 0$, it is possible to write:

$$\Phi_{asy}(x,t) = b_{\mathbf{00}}^{(1)} e^{\omega_{\mathbf{0}}^{(1)} t} \varphi_{\mathbf{0}}(x), \tag{44}$$

hence:

$$\mathcal{R}_{asy} \to e^{(\omega_1^{(1)} - \omega_{\mathbf{0}}^{(1)})t}. \tag{45}$$

Case 2. The source driven reactor.

For the subcritical system different considerations need to be made. The presence of the source intro duces a convolution term in the solution:

$$\int_{\mathbf{0}}^{t} dt' \sigma_n^{(j)}(t') e^{\omega_n^{(j)}(t-t')} = -\frac{\sigma_n^{(j)}}{\omega_n^{(j)}} \left(1 - e^{\omega_n^{(j)} t}\right). \tag{46}$$

For the no delayed neutron and constant source situation the asymptotic portion can be given the form:

$$\Phi_{asy}(x,t) = -\sum_{n=\mathbf{0}}^{\infty} \frac{\sigma_n}{\omega_n} \varphi_n(x). \tag{47}$$

It is then clear that it includes contribution from **all** eigenfunctions.

In the evolution towards the asymptotic behavior, also the contribution of the dominant transient portion needs to be taken into account. The dominant flux includes the contribution from the source induced asymptotic portion and from the evolution of the fundamental eigenfunction:

$$\Phi_{\mathcal{D}}(x,t) = \left(a_{\mathbf{0}} + \frac{\sigma_{\mathbf{0}}}{\omega_{\mathbf{0}}}\right) e^{\omega_0 t} \varphi_{\mathbf{0}}(x) + \Phi_{asy}(x,t), \tag{48}$$

hence:

$$\mathcal{R}_{\mathcal{D}} \to \left| \frac{\left(a_1 + \frac{\sigma_1}{\omega_1}\right) e^{\omega_1 t} \varphi_1(x)}{\Phi_{\mathcal{D}}} \right|. \tag{49}$$

If the delayed neutrons are taken into account:

$$\Phi_{asy}(x,t) = \sum_{n=\mathbf{0}}^{\infty} \left(\frac{\sigma_n^{(1)}}{\omega_n^{(1)}} + \frac{\sigma_n^{(2)}}{\omega_n^{(2)}}\right) \varphi_n(x), \tag{50}$$

and consequently:

$$\Phi_{\mathcal{D}}(x,t) = \sum_{j=1}^{2} \left(b_{\mathbf{00}}^{(j)} + \frac{\sigma_{\mathbf{0}}^{(j)}}{\omega_{\mathbf{0}}^{(j)}}\right) e^{\omega_{\mathbf{0}}^{(j)} t} \varphi_{\mathbf{0}}(x) + \Phi_{asy}(x,t). \tag{51}$$

Therefore the dominance ratio takes the form:

$$\mathcal{R}_{\mathcal{D}} \to \left| \frac{\left(b_{\mathbf{10}}^{(1)} + \frac{\sigma_1^{(1)}}{\omega_1^{(1)}}\right) e^{\omega_1^{(1)} t} \varphi_1(x)}{\Phi_{\mathcal{D}}} \right|, \tag{52}$$

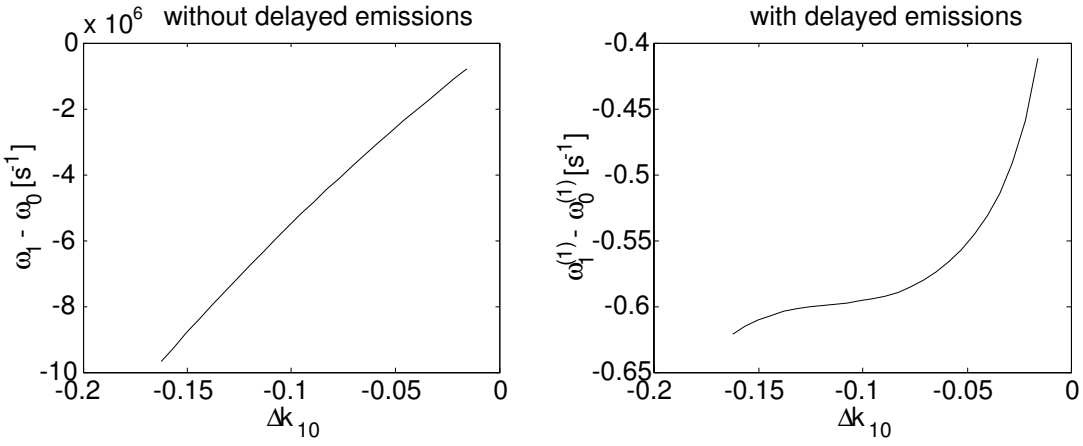

FIG. 1. Relation between the separation of the first static multiplication eigenvalues and the separation of the first time eigenvalues.

which shows the particular role of the first eigenvalue in the characterization of spatial transients in source driven systems, differently from the situation of critical reactors.

Some numerical results are presented in Figs. 1 to 6.

The conclusions on the observation of the graphs can be synthesized in the following considerations:

1. The response to a perturbation in a critical reactor is spatially more significant in a large system, i.e. R_{asy} is larger and takes longer to reduce to zero, and thus the contribution of higher order harmonics is more persistent;

2. The evolutions of both R_{asy} and $R_{\mathcal{D}}$ for subcritical systems show that the importance of higher order harmonics increases with increasing subcriticality, as the systems are more source dominated;

3. The comparison of initially critical and subcritical systems shows that the spatial feature of the transients is larger in systems departing from criticality; therefore, one can expect that the point model may have obvious limitations of applicability in these situations, while it may be more efficient in source driven systems.

The conclusion in 3. has been proved in some recent independent papers [12]. The practical consequence may be that quasi statics can prove to be highly efficient for ADS.

5. Factorization methods and quasi-statics

Multidimensional evaluations always require large computational effort for a direct numerical solution. Quasi statics is an attracting method, because it can yield accurate results with a limited amount of computational effort. The literature on the subject is very large. For instructional purposes, the classical references to the work by Alan F. Henry [7] and by Jacques Devooght [13] are particularly useful. It is appropriate to summarize here the steps that lead to the classic quasi static method as a background to its extension to subcritical systems.

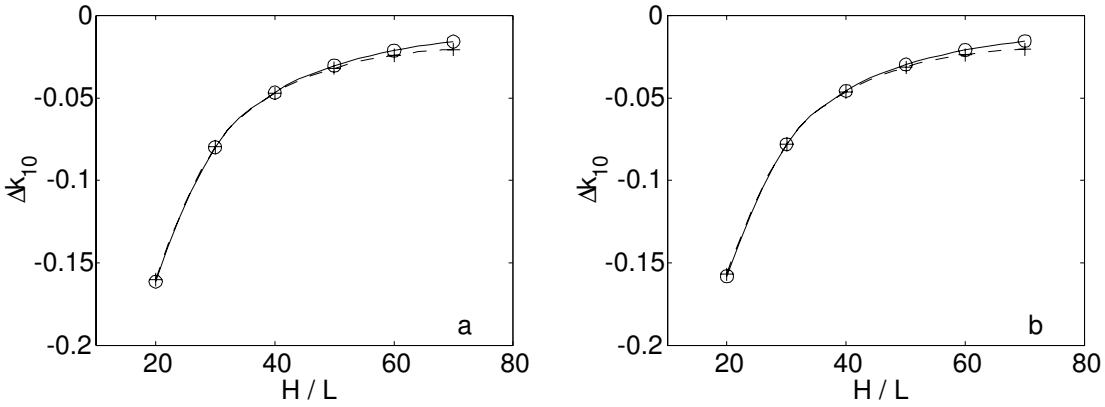

FIG. 2. Eigenvalue separation for critical and subcritical systems. The circles indicate results for the homogeneous system and crosses for a system with an absorber that introduces a change in k of -500 pcm. Graph (a) on the left refers to the case in which the homogeneous system is critical, while graph (b) refers to the case in which the homogeneous system is subcritical ($k = 0.98$).

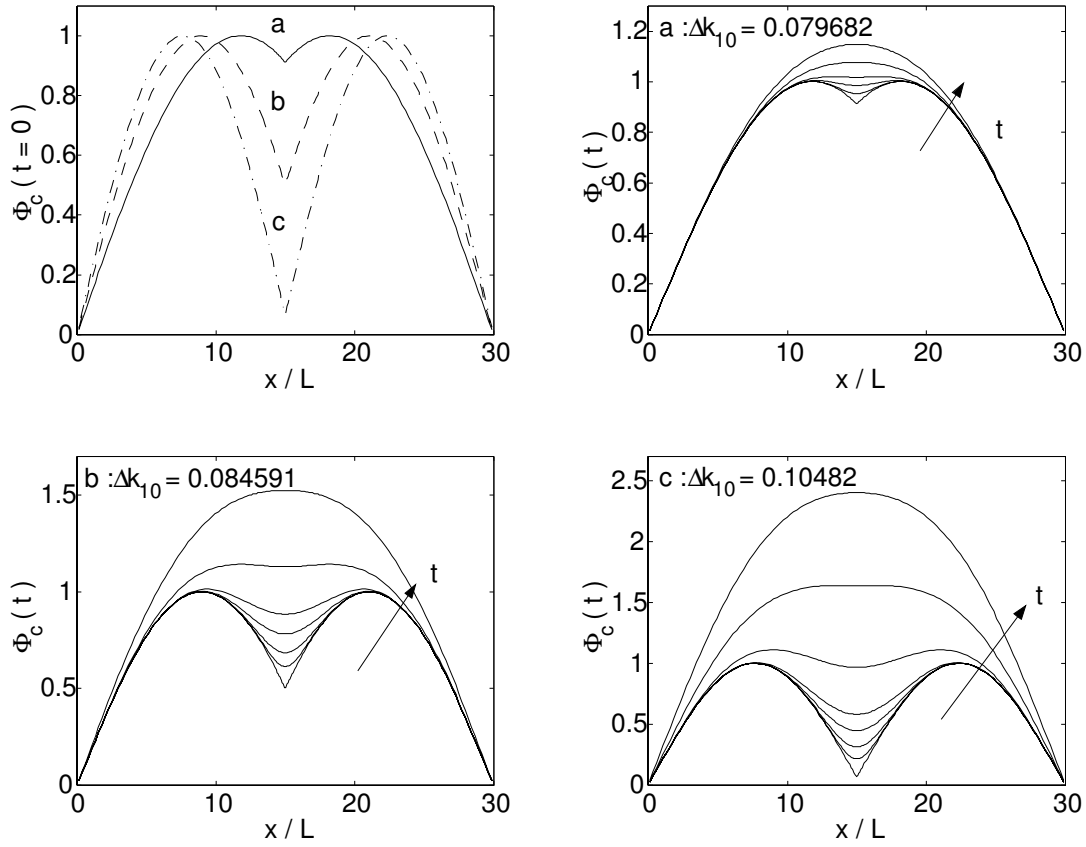

FIG. 3. Spatial evolution in initially critical systems with different eigenvalue separations. The top left graph reports the initial flux distributions for three values of the greyness of the control device.

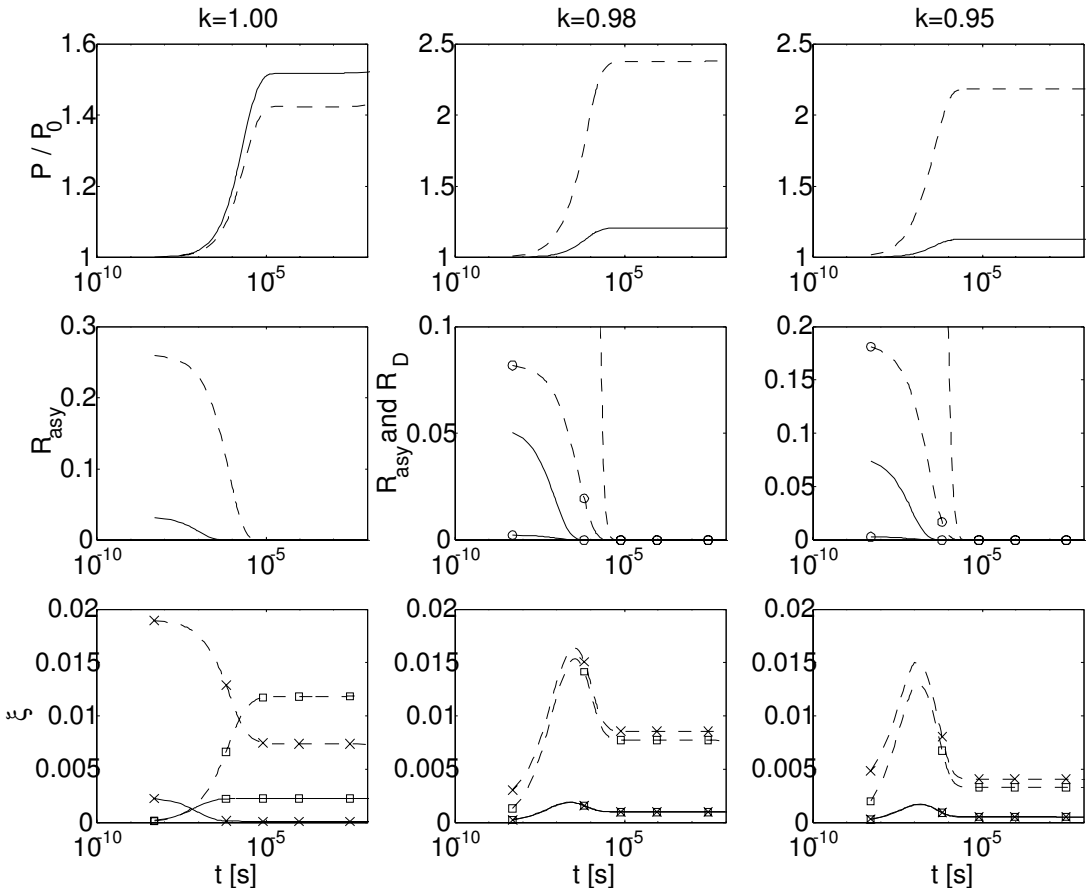

FIG. 4. Spatiality of transients in critical and subcritical systems, for a fixed reactivity insertion of 500 pcm. The power evolution for the initially critical case is diverging and such behavior appears at longer times than shown in the graph. Solid line refers to a small system ($H = 20L$) and broken line to a large system ($H = 70L$). Crosses ($\times$) results for ξ are produced using as reference Ψ the final flux distribution, while squares ($\Box$) using the initial state. For subcritical systems circles ($\circ$) indicate the ratio $\mathcal{R}_{asy}$.

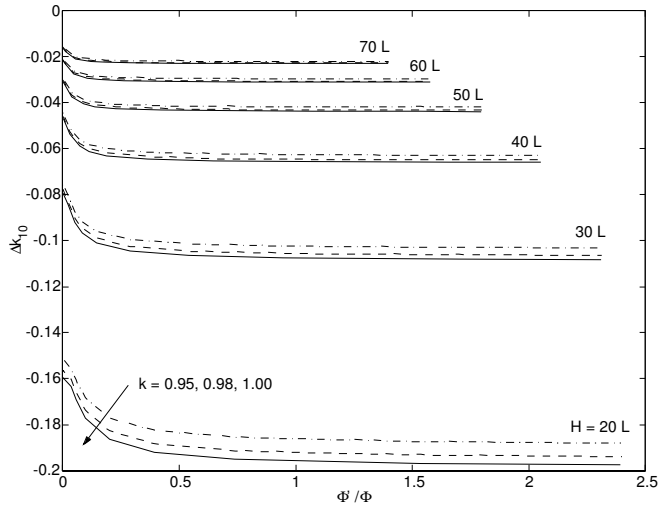

FIG. 5. Eigenvalue separation for different values of the greyness of the control device and of the physical dimensions of the system.

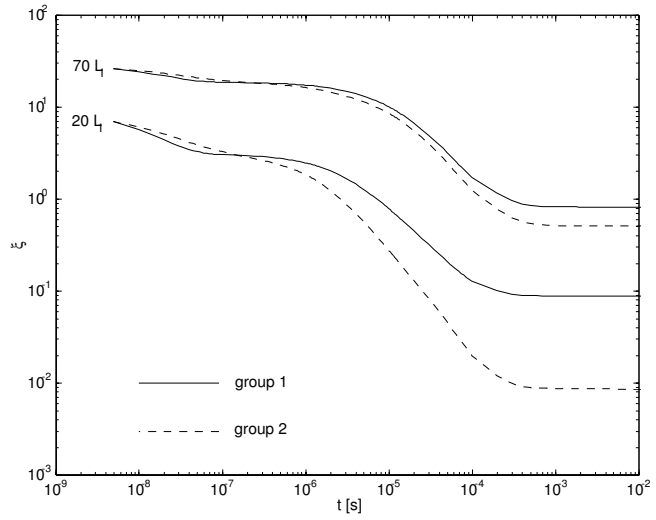

FIG. 6. Evolution of the spatiality parameter ξ for two subcritical systems having $\Delta k/k = -1.4114\beta$. The reference Ψ is the final flux distribution.

5.1. Derivation of the factorized equations

It is supposed the initial system is a critical source free reactor in steady state condition. This system is denoted as the reference reactor, and its neutron distribution is the solution of the following homoge neous problem, for which a non vanishing solution is then assumed to exist:

$$(\hat{\mathrm{L}}_0 + \hat{\mathrm{M}}_0)N_0(\mathbf{r}, E, \Omega) = 0, \qquad N_0(\mathbf{r}_S, E, \Omega_{in}) = 0, \tag{53}$$

where the steady state (global, prompt + delayed) multiplication operator is introduced:

$$\hat{\mathrm{M}}_0 = \sum_j \frac{\chi^j(E)}{4\pi} \int dE' \oint d\Omega' v(E')\nu^j(E')\Sigma_f^j(\mathbf{r}, E', 0); \tag{54}$$

both the multiplication and the leakage operators $\hat{\mathrm{M}}_0$ and $\hat{\mathrm{L}}_0$ are evaluated with the initial cross section data. The global fission spectrum is defined by:

$$\chi^j(E) = (1-\beta)\,\chi_p^j(E) + \sum_{i=1}^{R} \beta_i \chi_i^j(E). \tag{55}$$

For this system it is possible also to determine the solution of the adjoint equation:

$$(\hat{\mathrm{L}}_0^\dagger + \hat{\mathrm{M}}_0^\dagger)N_0^\dagger(\mathbf{r}, E, \Omega) = 0, \qquad N_0^\dagger(\mathbf{r}_S, E, \Omega_{out}) = 0, \tag{56}$$

which is well known to assume the meaning of neutron importance, in the classical sense [6], [14].

The neutron density, whose knowledge is the object of all our efforts, is factorized as:

$$n(\mathbf{r}, E, \Omega, t) = A(t)\varphi(\mathbf{r}, E, \Omega; t). \tag{57}$$

$A(t)$ is the amplitude function, while $\varphi(\mathbf{r}, E, \Omega; t)$ is referred to as the shape function. It is obvious that as such the factorization is not unique and further constraints need to be introduced. The idea underlying the above factorization is that the evolution follows two scales in time: the scale for the amplitude may be much faster than the one for the shape. In such a way one can deal with the stiffness of the problem. Furthermore, if the scale for the shape is slower, it can be recalculated fewer times along a transient, with a reduction of the computational effort, since it is apparent that it is much more difficult to calculate the shape that contains information on the neutron behavior in phase space, while the amplitude is connected to the evolution of the bulk of the neutron population and it is thus dependent only on time.

The factorized density is now introduced into the balance equations:

$$\begin{cases} A\dfrac{\partial\varphi}{\partial t} + \varphi\dfrac{dA}{dt} = A\hat{\mathrm{B}}\varphi + \displaystyle\sum_{i=1}^{6} \lambda_i \left(\frac{\chi_i}{4\pi}C_i\right) + S, \\ \dfrac{\partial(\chi_i C_i/4\pi)}{\partial t} = A\hat{\mathrm{M}}_i\varphi - \lambda_i \left(\dfrac{\chi_i}{4\pi}C_i\right). \end{cases} \tag{58}$$

These are to be regarded as equations for the shape function, were the amplitude behavior known. It is now possible to solve for the delayed neutron precursor concentrations:

$$\frac{\chi_i(E)}{4\pi}C_i(\mathbf{r}, t) = \frac{\chi_i(E)}{4\pi}C_i(\mathbf{r}, t = t_0)e^{-\lambda_i(t-t_0)} + \int_{t_0}^{t} A(t')\hat{\mathrm{M}}_i\varphi(\mathbf{r}, E, \Omega; t')e^{-\lambda_i(t-t')}dt'. \tag{59}$$

If this expression were cast into the balance equation for the neutron density, the problem would be given an integro differential form in time. This expression will be used in the following.

Equation (58) is now projected on the solution to the adjoint problem:

$$\begin{cases} A\dfrac{d}{dt}\left\langle N_0^\dagger \middle| \varphi \right\rangle + \dfrac{dA}{dt}\left\langle N_0^\dagger \middle| \varphi \right\rangle = A\left\langle N_0^\dagger \middle| \hat{\mathrm{B}}\varphi \right\rangle + \displaystyle\sum_{i=1}^{6}\lambda_i \left\langle N_0^\dagger \middle| \frac{\chi_i}{4\pi}C_i \right\rangle + \left\langle N_0^\dagger \middle| S \right\rangle, \\ \dfrac{d}{dt}\left\langle N_0^\dagger \middle| \dfrac{\chi_i}{4\pi}C_i \right\rangle = A\left\langle N_0^\dagger \middle| \hat{\mathrm{M}}_i\varphi \right\rangle - \lambda_i \left\langle N_0^\dagger \middle| \dfrac{\chi_i}{4\pi}C_i \right\rangle. \end{cases} \tag{60}$$

The projection operation could be performed using a weighting function other than the importance function. However, on observing the factorization (57), one can conclude that the use of such weighting function implies that the amplitude plays the role of component of the total neutron density along the fundamental eigenfunction of the reference system.

It is now possible to introduce a further constraint to make the factorization unique. A normalization condition is imposed for the shape function:

$$\frac{d}{dt}\left\langle N_0^\dagger \middle| \varphi \right\rangle = 0; \tag{61}$$

consequently, the first term in Eq. (60) will drop out.

Some definitions are now listed in the following chain of equalities:

$$\begin{aligned} &\hat{\mathrm{B}}(t) = \hat{\mathrm{L}}(t) + \hat{\mathrm{M}}_p(t) = (\hat{\mathrm{L}}_0 + \hat{\mathrm{M}}_0) + \delta\hat{\mathrm{B}}(t) - \sum_{i=1}^{6}\hat{\mathrm{M}}_i(0) - \sum_{i=1}^{6}\delta\hat{\mathrm{M}}_i(t) + \sum_{i=1}^{6}\delta\hat{\mathrm{M}}_i(t) = \\ &= (\hat{\mathrm{L}}_0 + \hat{\mathrm{M}}_0) + \delta\left[\hat{\mathrm{B}}(t) + \sum_{i=1}^{6}\hat{\mathrm{M}}_i(t)\right] - \sum_{i=1}^{6}\hat{\mathrm{M}}_i(t), \end{aligned} \tag{62}$$

with the introduction of the perturbation operator:

$$\delta\hat{\mathrm{K}} = \delta\hat{\mathrm{B}}(t) + \sum_{i=1}^{6}\delta\hat{\mathrm{M}}_i(t). \tag{63}$$

At last the equations for the amplitude are obtained (point like equations):

$$\begin{cases} \dfrac{dA(t)}{dt} = \dfrac{\rho(t)-\tilde{\beta}}{\Lambda}A(t) + \displaystyle\sum_{i=1}^{6}\lambda_i\tilde{C}_i(t) + \tilde{S}(t), \\ \dfrac{d\tilde{C}_i(t)}{dt} = \dfrac{\tilde{\beta}_i}{\Lambda}A(t) - \lambda_i\tilde{C}_i(t). \end{cases} \tag{64}$$

It is immediate to note that, if the shape were constant (and known), the standard point model would be obtained, with the same structure as each of the equations (15). If that is not the case, the model is not consistent, since the coefficients of (64) contain the shape φ which can be determined from (58) only once the amplitude A is available.

Before explaining how a consistent numerical technique can be developed, it is worth to write explicitly all the kinetic parameters appearing in Eq. (64):

$\Rightarrow$ system reactivity

$$\rho(t) = \frac{\left\langle N_0^\dagger \middle| \delta\hat{\mathrm{K}}\varphi \right\rangle}{\left\langle N_0^\dagger \middle| \hat{\mathrm{M}}\varphi \right\rangle}; \tag{65}$$

$\Rightarrow$ effective mean prompt neutron generation time

$$\Lambda = \frac{\left\langle N_0^\dagger \middle| \varphi \right\rangle}{\left\langle N_0^\dagger \middle| \hat{\mathrm{M}}\varphi \right\rangle}; \tag{66}$$

$\Rightarrow$ effective delayed neutron fractions

$$\tilde{\beta}_i = \frac{\left\langle N_0^\dagger \middle| \hat{\mathrm{M}}_i \varphi \right\rangle}{\left\langle N_0^\dagger \middle| \hat{\mathrm{M}} \varphi \right\rangle}. \tag{67}$$

New unknowns are also introduced as effective delayed neutron precursor concentrations:

$$\tilde{C}_i(t) = \frac{\left\langle N_0^\dagger \middle| \frac{\chi_i}{4\pi} C_i \right\rangle}{\left\langle N_0^\dagger \middle| \varphi \right\rangle} = \frac{1}{\Lambda} \frac{\left\langle N_0^\dagger \middle| \frac{\chi_i}{4\pi} C_i \right\rangle}{\left\langle N_0^\dagger \middle| \hat{\mathrm{M}} \varphi \right\rangle}, \tag{68}$$

and the effective external source function:

$$\tilde{S}(t) = \frac{\left\langle N_0^\dagger \middle| S \right\rangle}{\left\langle N_0^\dagger \middle| \varphi \right\rangle} = \frac{1}{\Lambda} \frac{\left\langle N_0^\dagger \middle| S \right\rangle}{\left\langle N_0^\dagger \middle| \hat{\mathrm{M}} \varphi \right\rangle}. \tag{69}$$

5.2. The numerical quasi-static scheme

For the numerical solution of the problem two time intervals are introduced:

* the shape interval (slow phenomena) Δt_φ;

* the amplitude interval (fast phenomena) Δt_A.

The integro differential equation for the shape is discretized across Δt_φ:

$$T = t_0 + \Delta t_\varphi,$$

$$\begin{aligned} &A(T)\frac{\varphi(T) - \varphi(t_0)}{\Delta t_\varphi} + \varphi(T)\dot{A}(T) = A(T)\hat{\mathrm{B}}\varphi(T) + \\ &\sum_{i=1}^{6} \lambda_i \left[\frac{\chi_i(E)}{4\pi} C_i(t_0) e^{-\lambda_i \Delta t_\varphi} + \int_{t_0}^{T} A(t') \hat{\mathrm{M}}_i \varphi(t_0) e^{-\lambda_i (T-t')} dt' \right] + S(T), \end{aligned} \tag{70}$$

with the objective of determining $\varphi(T)$ through the solution of a stationary like problem of the same type and difficulty as Eq. (53).

In general, $\varphi(T)$ obtained by the solution of Eq. (70) will not satisfy the required normalization condi tion, Eq. (61), namely:

$$\gamma(T) = \left\langle N_0^\dagger \middle| \varphi(T) \right\rangle \neq \gamma(t_0) = \left\langle N_0^\dagger \middle| \varphi(t_0) \right\rangle. \tag{71}$$

To obtain a shape satisfying, as required, condition (61), an iterative process is necessary. Different requirements can be applied to run the iteration. It is suggested here to preserve the continuity of the amplitude function, thus allowing a discontinuity in its time derivative. Introducing an iteration numbering index n, the steps of the scheme follow:

$$\begin{aligned} &A(T)\frac{\varphi^{(n)}(T) - \varphi(t_0)}{\Delta t_\varphi} + \varphi^{(n)}(T)\dot{A}^{(n)}(T) = \\ &A(T)\hat{\mathrm{B}}\varphi^{(n)}(T) + \sum_{i=1}^{6} \lambda_i \left[\frac{\chi_i(E)}{4\pi} C_i(t_0) e^{-\lambda_i \Delta t_\varphi} + \int_{t_0}^{T} A(t') \hat{\mathrm{M}}_i \varphi(t_0) e^{-\lambda_i (T-t')} dt' \right] + S(T), \end{aligned} \tag{72}$$

$$\gamma^{(n)}(T) = \left\langle N_0^\dagger \middle| \varphi^{(n)}(T) \right\rangle, \tag{73}$$

$$\varphi^{(n+1/2)}(T) = \frac{\gamma(t_0)}{\gamma^{(n)}(T)} \varphi^{(n)}(T). \tag{74}$$

The derivative of the amplitude function is updated according to Eq. (64):

$$\dot{A}^{(n+1)}(T) = A(T)\frac{\left\langle N_0^\dagger \middle|\, \hat{\mathrm{B}}\varphi^{(n+1/2)} \right\rangle}{\left\langle N_0^\dagger \middle|\, \varphi^{(n+1/2)}(T) \right\rangle} + \sum_{i=1}^{6}\lambda_i \frac{\left\langle N_0^\dagger \middle|\, \frac{\chi_i}{4\pi} C_i \right\rangle}{\left\langle N_0^\dagger \middle|\, \varphi^{(n+1/2)}(T) \right\rangle} + \frac{\left\langle N_0^\dagger \middle|\, S(T) \right\rangle}{\left\langle N_0^\dagger \middle|\, \varphi^{(n+1/2)}(T) \right\rangle}, \quad (75)$$

up to convergence.

6. Extension of models to source-driven systems

It is foreseen that the analysis of ADS may require spatial and spectral neutron kinetics. It is advisable therefore to try to adapt quasi statics also to subcritical systems [15], [16]. This is the scope of the present section.

6.1. Application of the factorization procedure to source-driven systems

The reference reactor is driven by an external source, hence it is obvious that the initial shape should be assumed as solution of the steady state equation:

$$(\hat{\mathrm{L}}_0 + \hat{\mathrm{M}}_0)N_0(\mathbf{r}, E, \Omega) + S_0 = 0, \qquad N_0(\mathbf{r}_S, E, \Omega_{in}) = 0. \quad (76)$$

In the projection procedure the problem that immediately arises is the choice of the weighting function. Although the method can be constructed irrespective of such a choice, the accuracy may depend signifi cantly on it. One possibility is to construct a fictitious homogeneous adjoint equation by the introduction of a multiplication eigenvalue:

$$(\hat{\mathrm{L}}_0^\dagger + \frac{1}{k_0}\hat{\mathrm{M}}_0^\dagger)N_{0,cr}^\dagger(\mathbf{r}, E, \Omega) = 0, \qquad N_{0,cr}^\dagger(\mathbf{r}_S, E, \Omega_{out}) = 0. \quad (77)$$

Alternatively, a non homogeneous adjoint equation may be considered, which requires an adjoint source, as

$$(\hat{\mathrm{L}}_0^\dagger + \hat{\mathrm{M}}_0^\dagger)N_{0,s}^\dagger(\mathbf{r}, E, \Omega) + S^\dagger = 0, \qquad N_{0,s}^\dagger(\mathbf{r}_S, E, \Omega_{out}) = 0. \quad (78)$$

Now the problem of the definition of the adjoint source arises. This is connected to the physical meaning one may want to assign to the importance function. The solution to the problem is not unique. If the importance is defined as the number of fission neutrons to be produced per neutron injected at a point in phase space, the adjoint source stems as:

$$S^\dagger = \nu\Sigma_f. \quad (79)$$

The choice of the weighting function has a consequence on the reactivity of the system; if the reactivity is split as:

$$\rho = \rho_0 + \rho_p, \quad (80)$$

where ρ_p is the perturbation reactivity and ρ_0 (always negative) is the initial subcriticality level, one has, alternatively:

$$\rho_0 = \frac{\left\langle N_{0,cr}^\dagger \middle|\, \hat{\mathrm{M}}_0\varphi \right\rangle}{\left\langle N_{0,cr}^\dagger \middle|\, \hat{\mathrm{M}}\varphi \right\rangle}\left(\frac{k-1}{k}\right), \quad (81)$$

$$\rho_0 = -\frac{\left\langle S^\dagger|\varphi \right\rangle}{\left\langle N_{0,s}^\dagger \middle|\, \hat{\mathrm{M}}\varphi \right\rangle}. \quad (82)$$

6.2. Discussion on the choice of the weighting function in separation schemes

The most suitable choice of the weighting function for subcritical systems is an open problem for dis cussion. No univocal answer can be given. The *best* option may be strongly dependent on the physical

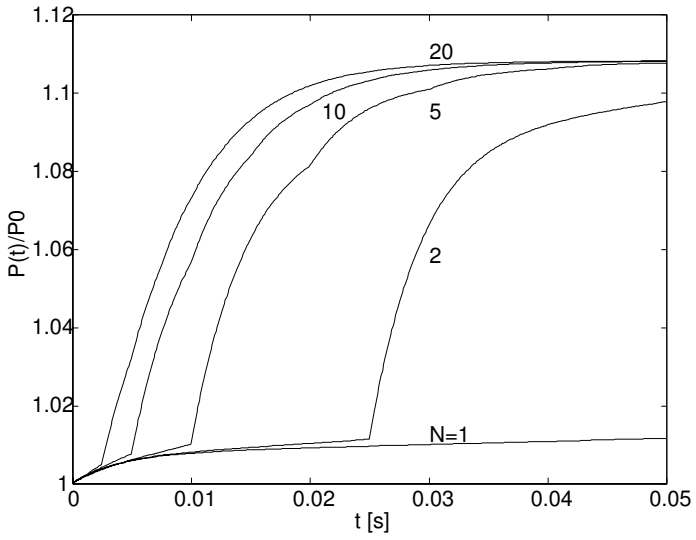

FIG. 7. A typical transient in a source-driven system computed by the quasi-static method. The weight function is a source adjoint problem solution [18]. *N indicates the number of shape recalculations.*

problem considered. This consideration is proved by the following Figs. 8 to 10. Here the problem is evidenced and no general solution is proposed.

6.3. A computational tool

A code is available for the evaluation of transients in accelerator driven systems. The code is the result of the coupling of a neutronic module with a thermal model, in order to compute also feedback effects. The neutronic module solves the multigroup diffusion equations in cylindrical geometry with the presence of delayed emissions. The thermal calculation is performed by the channel code TIESTE developed at ENEA (Italy) for accelerator driven systems cooled by lead bismuth [17]. The code determines the axial distribution of temperatures for the fuel, the cladding and the coolant, for different channels. The instants for the thermal calculation are chosen according to the power change of the system. The average values of the power density are computed for pre defined zones of the multiplying structure and input into the thermal module. At the end of the thermal calculation, the cross sections are updated according to a linear interpolation process between tables of data generated at different fuel and coolant temperatures.

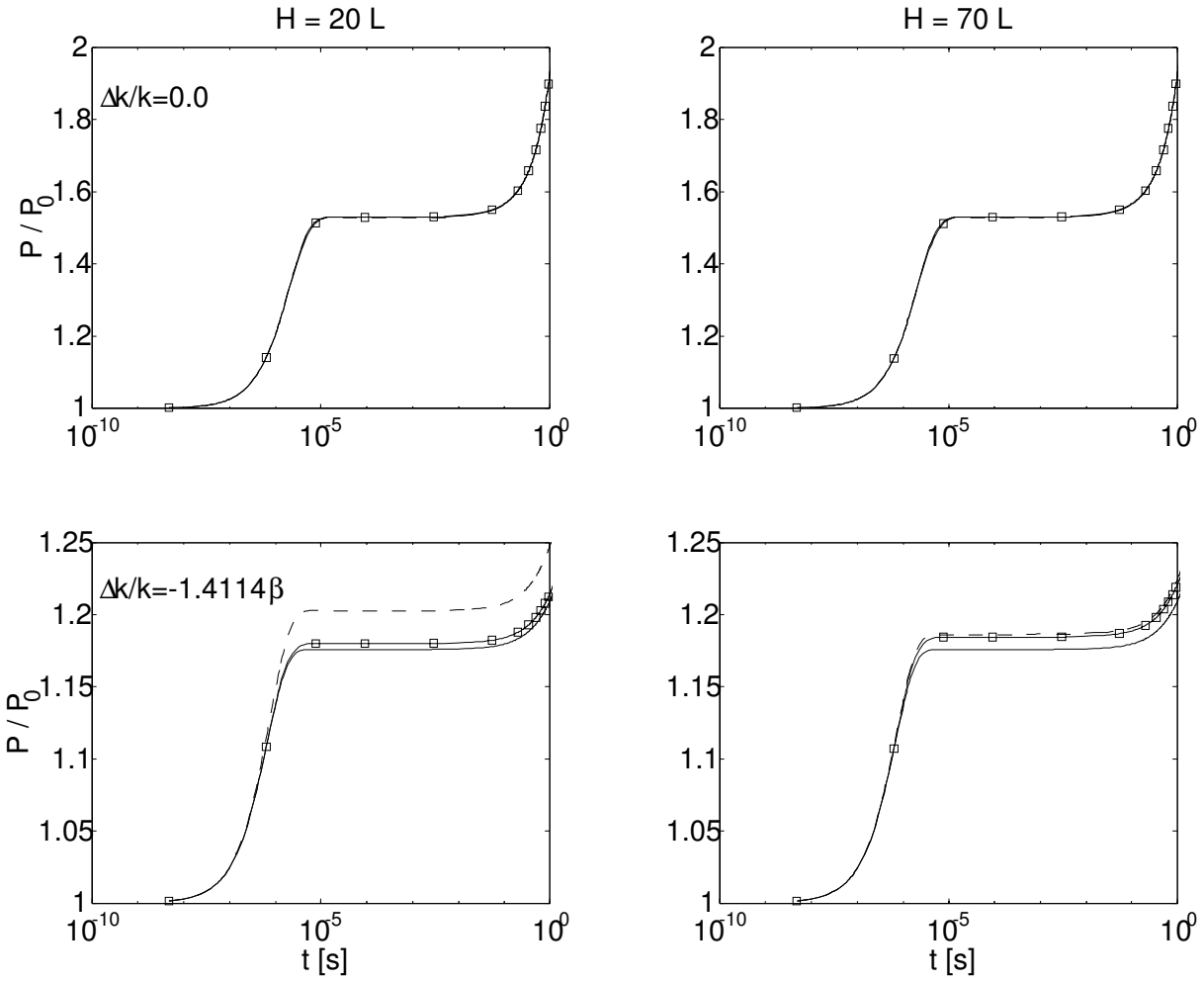

FIG. 8. Homogeneous perturbation, systems with different k. Top: initial critical system; bottom: initial subcritical system, $k = 0.98$. Squares: reference; broken line: constant adjoint; solid line: critical adjoint; dot point line: source driven adjoint.

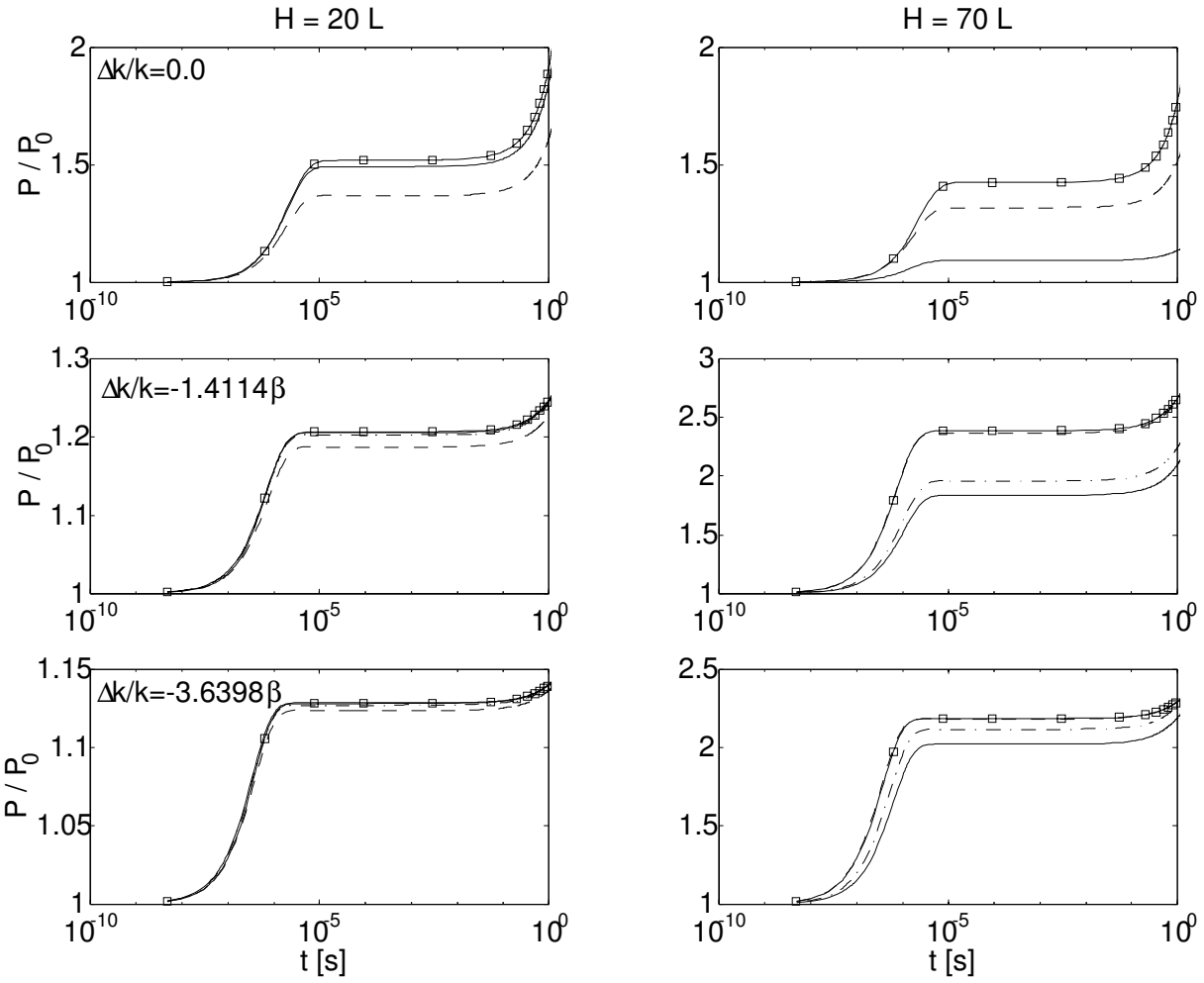

FIG. 9. Comparison of the power evolutions calculated using different weighting functions in response to a localized perturbation. Squares: reference results; broken line: constant adjoint; solid line: critical adjoint; dot point line: source driven adjoint.

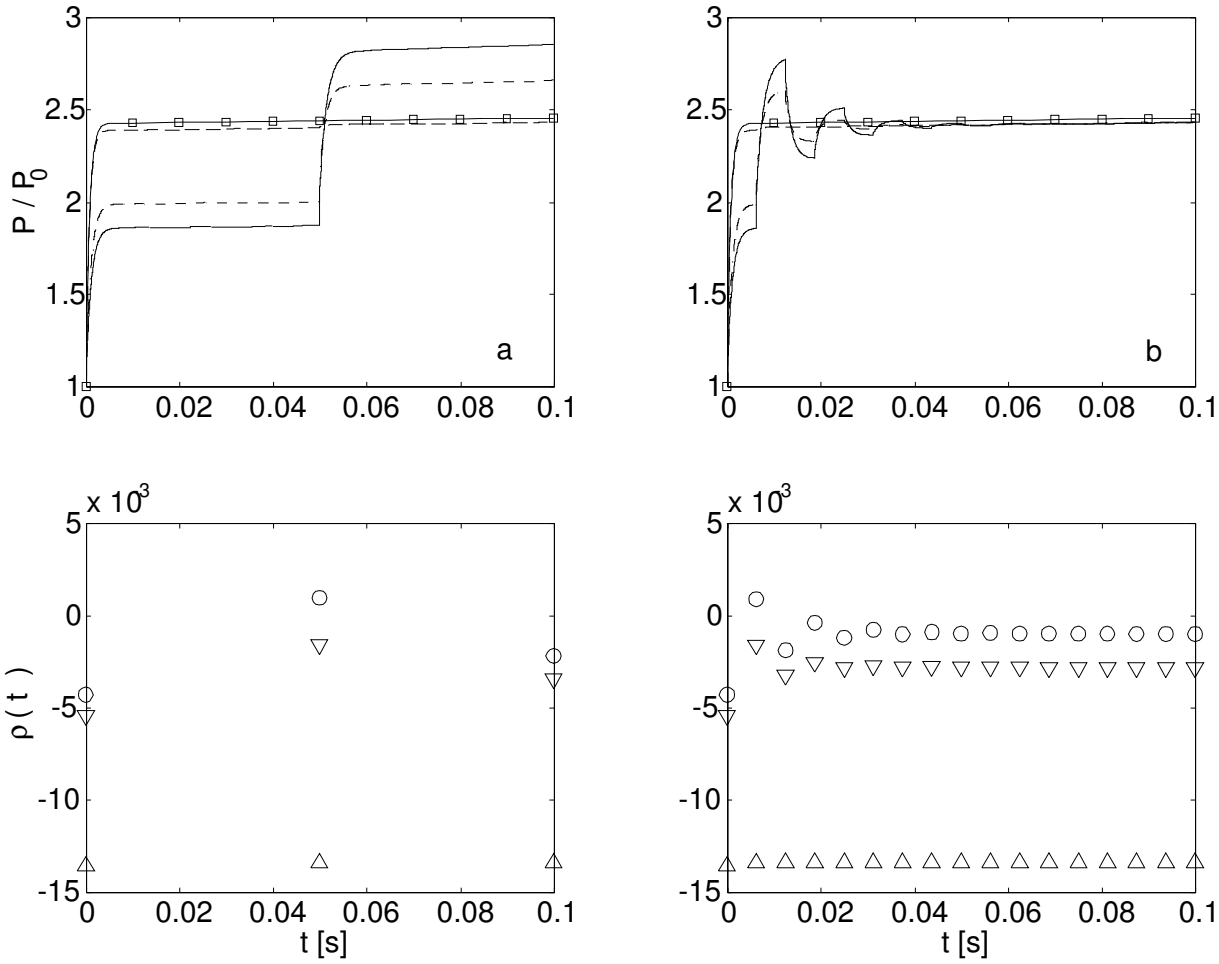

FIG. 10. Quasi static calculations with $H = 70L$ and $\Delta k/k = -1.4114\beta$, compared to the full spatial results. One shape update for graphs a), 15 recalculations for case b). Reactivity at the bottom; ∇: source adjoint; $\triangle$: constant; $\circ$: critical adjoint.

7. A development: the multipoint scheme

The quasi static method proves to be a powerful and computationally efficient tool for the analysis of source driven systems [12], [18], [19]. Accurate predictions of the transient behavior in non linear conditions dominated by thermal feed back can also be attained by a coupling of the neutronic module with a thermal hydraulic tool. However, significant improvements can be reached with little increase in computational effort passing to a multipoint scheme. This is a generalization of the method originally proposed by Avery [20] and more recently revived by Kobayashi [21]. It is seen that it can easily be included in quasi static framework. In the following the method will be developed in a factorization projection formalism, extending the Henry technique.

It is simple to start from the balance equations in discretized form:

$$\begin{cases} \dfrac{1}{v_m}\dfrac{d\phi_{nm}}{dt} = \displaystyle\sum_{n'}\sum_{m'} k_{nm,n'm'}\phi_{n'm'} + \sum_{i=1}^{6}\lambda_i\chi_{i,m}C_{i,n} + S_{nm}, \\ \dfrac{dC_{i,n}}{dt} = \beta_i \displaystyle\sum_{m'} f_{nm'}\phi_{nm'} - \lambda_i C_{i,n} \qquad\qquad i = 1,2,...,6, \end{cases} \tag{83}$$

where:

$$\phi_{nm}(t) = \phi(\mathbf{r}_n, V_m, t), \qquad C_{i,n}(t) = C_i(\mathbf{r}_n, t). \tag{84}$$

One can notice that system (83) shows a *multipoint* structure, in the sense that it describes the evolution of the neutron population at each point nm as the results of the phenomena taking place at this point as well as of the transfer from other points in phase space. However, the discretization is carried out in order to capture the physical features of the neutron migration, hence it has to be tuned to the characteristics of the problem. This involves then *too many* points. The method presented intends to reduce drastically the number of points, to *very few*. Point kinetics is the limiting model.

The phase space of the problem is subdivided into macroregions, Γ_{NM}, where indexes N and M are used to denote space and velocity, respectively. A factorization is now applied to each macroregion.

$$\phi_{nm}(t) = A_{NM}(t)\varphi_{nm}(t) \qquad \mathbf{r}_n, V_m \in \Gamma_{NM}. \tag{85}$$

A definition of a regionwise inner product needs to be preliminarily given:

$$\langle w|\ g\rangle = \left[\sum_n\sum_m\right]_{NM} w_{nm}g_{nm}. \tag{86}$$

The factorization is introduced into the balance equations:

$$\begin{cases} \dfrac{1}{v_m}\varphi_{nm}\dfrac{dA_{NM}}{dt} + \dfrac{1}{v_m}A_{NM}\dfrac{d\varphi_{nm}}{dt} = \\ \displaystyle\sum_{N'}\sum_{M'}\left[\sum_{n'}\sum_{m'}\right]_{N'M'} k_{nm,n'm'}\varphi_{n'm'}A_{N'M'} + \sum_{i=1}^{6}\lambda_i\chi_{i,m}C_{i,n} + S_{nm}, \\ \dfrac{dC_{i,n}}{dt} = \beta_i \displaystyle\sum_{M'}\left[\sum_{m'}\right]_{M'} f_{nm'}\varphi_{nm'}A_{NM'} - \lambda_i C_{i,n}, \\ \qquad\qquad i = 1,2,...,6, \qquad \mathbf{r}_n, V_m \in \Gamma_{NM}, \end{cases} \tag{87}$$

and the equations are multiplied by w_{nm} and summed on NM. Following what is done is standard factorization projection procedures, a normalization condition is imposed:

$$\frac{d}{dt}\left[\sum_n\sum_m\right]_{NM} w_{nm}\frac{1}{v_m}\varphi_{nm}(t) = \frac{d}{dt}\gamma_{NM} = 0. \tag{88}$$

The point to point transfer term can be easily manipulated as:

$$\frac{1}{\gamma_{NM}}\left[\sum_n\sum_m\right]_{NM} w_{nm}\sum_{N'}\sum_{M'}\left[\sum_{n'}\sum_{m'}\right]_{N'M'} k_{nm,n'm'}\varphi_{n'm'}A_{N'M'} =$$

$$\sum_{N'}\sum_{M'}\frac{1}{\gamma_{NM}}\left[\sum_n\sum_m\right]_{NM}\left(w_{nm}\left[\sum_{n'}\sum_{m'}\right]_{N'M'} k_{nm,n'm'}\varphi_{n'm'}\right)A_{N'M'} = \sum_{N'}\sum_{M'}K_{NM,N'M'}A_{N'M'}. \tag{89}$$

At last, the multipoint equations can be cast into the following form:

$$\begin{cases} \dfrac{dA_{NM}}{dt} = \displaystyle\sum_{N'}\sum_{M'}K_{NM,N'M'}A_{N'M'} + \sum_{i=1}^{6}\lambda_i C_{i,NM} + S_{NM}, \\ \dfrac{dC_{i,NM}}{dt} = \beta_i\displaystyle\sum_{M'}F_{i,NM,M'}A_{NM'} - \lambda_i C_{i,NM} \qquad i = 1,2,...,6, \end{cases} \tag{90}$$

which is a system of coupled point models. The following definitions hold for the effective multipoint source (known):

$$S_{NM} = \frac{1}{\gamma_{NM}}\left[\sum_n\sum_m\right]_{NM} w_{nm}S_{nm}, \tag{91}$$

and the effective multipoint delayed precursor concentrations (unknown):

$$C_{i,NM} = \frac{1}{\gamma_{NM}}\left[\sum_n\sum_m\right]_{NM} w_{nm}\chi_{i,m}C_{i,n}. \tag{92}$$

The delayed neutron production coefficients appearing in Eq. (90) are defined according to the following expression:

$$F_{i,NM,M'} = \frac{1}{\gamma_{NM}}\left[\sum_n\sum_m\right]_{NM} w_{nm}\chi_{i,m}\beta_i\left[\sum_{m'}\right]_{M'} f_{nm'}\varphi_{nm'}. \tag{93}$$

It is rather easy to include the multipoint can be included into a quasi static scheme, following the standard steps [1]:

* Solution of the slow shape equation at time t_0 (either the equation for the steady state reference system at $t_0 = 0$, or Eq. (87) at successive steps);

* Determination of the multipoint equation parameters (point to point transfer terms and delayed neu tron production coefficients) together with the effective multipoint source;

* Solution of the multipoint system (90) in the interval $[t_0, t_0 + \Delta t_\varphi]$. Defining $T = t_0 + \Delta t_\varphi$, after time discretization the shape equation is written as:

$$\begin{cases} \dfrac{1}{v_m}\dot{A}_{NM}(T)\varphi_{nm}(T) + \dfrac{1}{v_m}A_{NM}(T)\dfrac{\varphi_{nm}(T) - \varphi_{nm}(t_0)}{\Delta t_\varphi} = \\ \displaystyle\sum_{N'}\sum_{M'}\left[\sum_{n'}\sum_{m'}\right]_{N'M'} k_{nm,n'm'}(T)\varphi_{n'm'}(T)A_{N'M'}(T) + \sum_{i=1}^{6}\lambda_i\chi_{i,m}C_{i,n}(T) + S_{nm}(T), \\ C_{i,n}(T) = C_{i,n}(t_0)e^{-\lambda_i\Delta t_\varphi} + \displaystyle\int_{t_0}^{T}\beta_i\sum_{M'}\left[\sum_{m'}\right]_{M'} f_{nm'}(T)\varphi_{nm'}(t_0)A_{N'M'}(t')e^{-\lambda_i(T-t')}dt'. \end{cases} \tag{94}$$

To fulfill the normalization condition, as for the standard quasi static scheme, an iteration sequence is required, according to the following steps:

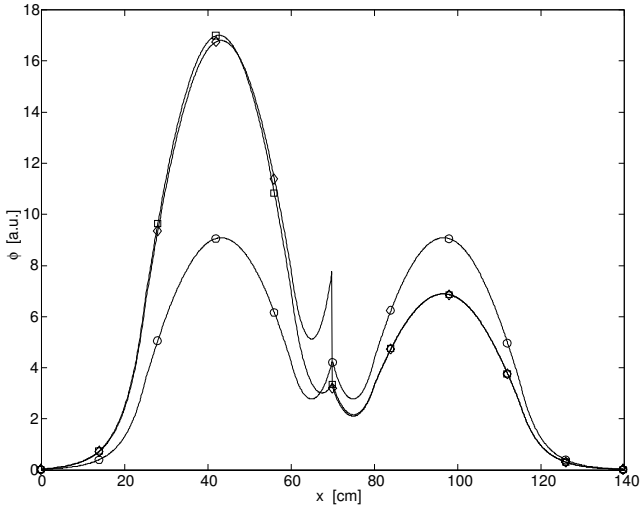

FIG. 11. Flux at end of transient: ∘ *: point kinetics;* ◇ *: 2 point kinetics;* □ *: exact solution.*

$$\frac{1}{v_m}\dot{A}^{(l)}_{NM}(T)\varphi^{(l)}_{nm}(T)+\frac{1}{v_m}A_{NM}(T)\frac{\varphi^{(l)}_{nm}(T)-\varphi_{nm}(t_0)}{\Delta t_\varphi}= \sum_{N'}\sum_{M'}\left[\sum_{n'}\sum_{m'}\right]_{N'M'} k_{nm,n'm'}(T)\varphi^{(l)}_{n'm'}(T)A_{N'M'}(T)+\sum_{i=1}^{6}\lambda_i\chi_{i,m}C_{i,n}(T)+S_{nm}(T), \tag{95}$$

$$\gamma^{(l)}_{NM}(T)=\left[\sum_{n}\sum_{m}\right]_{NM} w_{nm}\frac{1}{v_m}\varphi^{(l)}_{nm}(T), \tag{96}$$

$$\varphi^{(l+1/2)}_{nm}(T)=\frac{\varphi^{(l)}_{nm}(T)}{\gamma^{(l)}_{NM}(T)}\gamma_{NM}(t_0), \tag{97}$$

$$\dot{A}^{(l+1)}_{NM}(T)=\gamma^{(l+1/2)}_{NM}(T)\sum_{N'}\sum_{M'}\left[\sum_{n}\sum_{m}\right]_{NM}\left(w_{nm}\left[\sum_{n'}\sum_{m'}\right]_{N'M'}k_{nm,n'm'}(T)\varphi^{(l+1/2)}_{n'm'}(T)\right)A_{N'M'}+ \frac{\left[\sum_{n}\sum_{m}\right]_{NM}w_{nm}\chi_{i,m}C_{i,n}(T)}{\gamma^{(l+1/2)}_{NM}(T)}+\frac{\left[\sum_{n}\sum_{m}\right]_{NM}w_{nm}S_{nm}(T)}{\gamma^{(l+1/2)}_{NM}(T)}. \tag{98}$$

The calculations performed on the multipoint scheme allow to make the following conclusions [9]:

* multipoint is effective in many reactor kinetics problems, specially for source driven decoupled con figurations;

* the method can easily be included within quasi statics, greatly enhancing its performance.

A typical transient is reported in Fig. 11. A 1D plane source driven system (spatially localized and sym metrically located source) is materially perturbed, homogeneously increasing the absorption on the half right of the slab and simultaneously decreasing it on the left, thus maintaining the same multiplication eigenvalue. The curves report the flux distribution at the end of the transient. Multipoint kinetics is con

structed with two point, for each two halves of the system. It can be seen that point kinetics is highly on the side of unsafety and it is unable to follow the space transient, since no distortion is allowed. The two point model is much closer to the real solution (computed numerically) and it yields almost the exact value of the power (integral of the flux). A spatial discontinuity can be seen, owing to the fact that the two amplitudes are allowed to evolve separately (although connected through the multipoint system). This shortcoming can be easily overcome in a quasi static framework.

It seems also particularly interesting a development to apply the multipoint procedure to angular schemes in transport calculations. However, some information on the nature of the transient to be studied is needed, to perform the most suitable subdivision of the phase space of the problem in order to enhance the capability to capture the physical features of the evolution [22].

8. A harder problem: the modeling of fluid-fuel systems

Fluid fuel systems are today proposed within the Generation IV project as viable means for energy production and radioactive product transmutation. Fuel should be a mixture of fissile and fertile molten salts. The fuel itself is to be circulated and transport the fission generated energy into a heat exchanger. There are obvious advantages in the fuel cycle and in the management of fissile materials and there is a possibility of on line removal of fission products, thus reducing the radioactivity inventory in the core with meaningful consequences on the safety of the system. Unfortunately, many problems still exist concerning material compatibility and corrosion. However, experimental systems of this type were built and operated in the sixties, e.g. the Molten Salt Reactor Experiment, MSRE, carried out at ORNL, and a lot of data is actually available. It seems possible to run this systems both in the critical and subcritical state.

The study of the physics of these systems involves very interesting mathematical problems. In the fol lowing a model is presented and discussed and some numerical techniques for its solutions are outlined.

The Fig. 12 illustrates the schematics of a molten salt reactor.

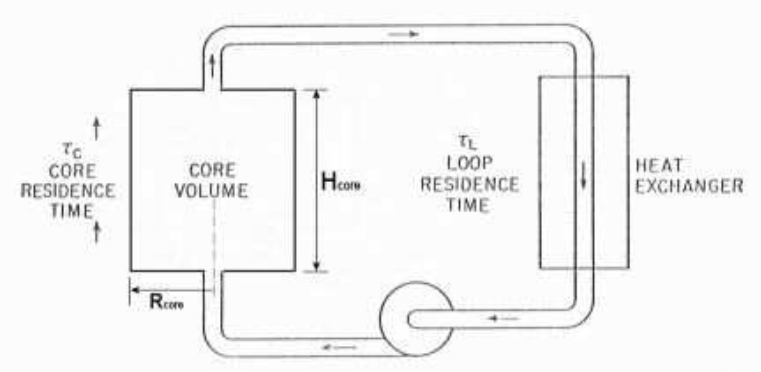

FIG. 12. Layout of a circulating fuel reactor.

8.1. General model

An essential feature of the neutron kinetics of a recirculated fluid fuel system is the motion through and outside the core of the delayed neutron precursors produced by fissions. As a consequence, while the structure of the balance equation for neutrons is essentially unchanged with respect to solid fuel systems,

a streaming term appears in the balance for delayed neutron precursors:

$$\begin{cases} \dfrac{\partial n(\mathbf{r},E,\Omega,t)}{\partial t} = \left[\hat{\mathrm{L}}(t)+\hat{\mathrm{M}}_p(t)\right] n(\mathbf{r},E,\Omega,t) + \displaystyle\sum_{i=1}^{R} \mathcal{E}_i(\mathbf{r},E,t) + S(\mathbf{r},E,\Omega,t), \\ \dfrac{1}{\lambda_i}\dfrac{\partial \mathcal{E}_i(\mathbf{r},E,t)}{\partial t} + \dfrac{1}{\lambda_i}\boldsymbol{\nabla}\cdot(\mathbf{u}\mathcal{E}_i(\mathbf{r},E,t)) = \hat{\mathrm{M}}_i(t) n(\mathbf{r},E,\Omega,t) - \mathcal{E}_i(\mathbf{r},E,t), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i=1,2,...,R, \end{cases} \tag{99}$$

where the isotropic delayed neutron emissivity $\mathcal{E}_i$ is introduced according to the following definition:

$$\mathcal{E}_i(\mathbf{r},E,t) = \lambda_i C_i(\mathbf{r},t)\frac{\chi_i(E)}{4\pi}. \tag{100}$$

Since the equation for delayed precursors is first order differential in space, an appropriate boundary condition must then be introduced [23]. The most general, though rather complicated, form of this condition is as follows:

$$\mathcal{E}_i(\mathbf{r},E,t)\mathbf{u}(\mathbf{r})\cdot(-\mathbf{n}) = \int\limits_{\substack{\mathcal{A}_{out} \\ \mathbf{r}\in\mathcal{A}_{in},}} \mathcal{E}_i(\mathbf{r}',E,t-\tau(\mathbf{r}'\to\mathbf{r}))e^{-\lambda_i\tau(\mathbf{r}'\to\mathbf{r})}\mathbf{u}(\mathbf{r}')\cdot\mathbf{n}'\mathfrak{F}(\mathbf{r}'\to\mathbf{r})\,d\mathcal{A}', \tag{101}$$

where the function $\mathfrak{F}(\mathbf{r}'\to\mathbf{r})$ describes the transfer form the exiting point to the re entering point of the system through the external circuit, and $\tau(\mathbf{r}'\to\mathbf{r})$ is the corresponding recirculation time. Of course, the velocity field $\mathbf{u}$ has to be determined by the simultaneous solution of the fluid equations. The physical model is completely defined once initial conditions are associated to Eqs. (99).

It is useful for the physical comprehension to write a simplified version of problem (99), assuming the multigroup diffusion model in cylindrical geometry for the neutron and a one dimensional (axial) slug flow, imposed by the externally driven devices. Therefore, $\mathbf{u}(\mathbf{r}) = u\mathbf{e}_z$, and it follows:

$$\begin{cases} \dfrac{1}{v_g}\dfrac{\partial \Phi_g}{\partial t} = \bigtriangledown\cdot D_g \bigtriangledown \Phi_g - \Sigma_g\Phi_g + \displaystyle\sum_{g'}\left[\chi_{p,g}\nu\Sigma_{fg'}(1-\beta)+\Sigma_{g'\to g}\right]\Phi_{g'} + S_g + \sum_{i=1}^{R}\lambda_i\chi_{i,g}C_i, \\ \dfrac{\partial C_i}{\partial t} = -\lambda_i C_i + \beta_i\displaystyle\sum_g \nu\Sigma_{fg}\Phi_g - \frac{\partial}{\partial z}(uC_i), \qquad\qquad i=1,2,...,R, \end{cases} \tag{102}$$

with boundary conditions:

$$u(0)C_i(z=0,r,t) = u(H)\frac{e^{-\lambda_i\tau_L}}{\mathcal{A}_{core}}\int\limits_{\mathcal{A}_{core}} C_i(z=H,r,t-\tau_L)d\mathcal{A}. \tag{103}$$

8.2. Discussion of the dynamic effects of fuel motion

The main physical features connected to the motion of the fissile material are now briefly discussed. First of all, it is important to notice that the delayed precursor equations cannot be eliminated in the steady state configuration. It is straightforward for a solid fuel problem in absence of the streaming term to express the emissions from precursors as functions of the fission term as:

$$\mathcal{E}_i(\mathbf{r},E) = \hat{\mathrm{M}}_i n(\mathbf{r},E,\Omega), \tag{104}$$

and back substitute them into the neutron balance equation. In the case of circulating fuel the equations for precursors are still differential for the space variable and their concentrations can not be made explicit and substituted:

$$\boldsymbol{\nabla}\cdot(\mathbf{u}\mathcal{E}_i(\mathbf{r},E)) = \lambda_i\hat{\mathrm{M}}_i n(\mathbf{r},E,\Omega) - \lambda_i\mathcal{E}_i(\mathbf{r},E). \tag{105}$$

A direct consequence of this fact is that the multiplication eigenvalue depends on delayed neutron and

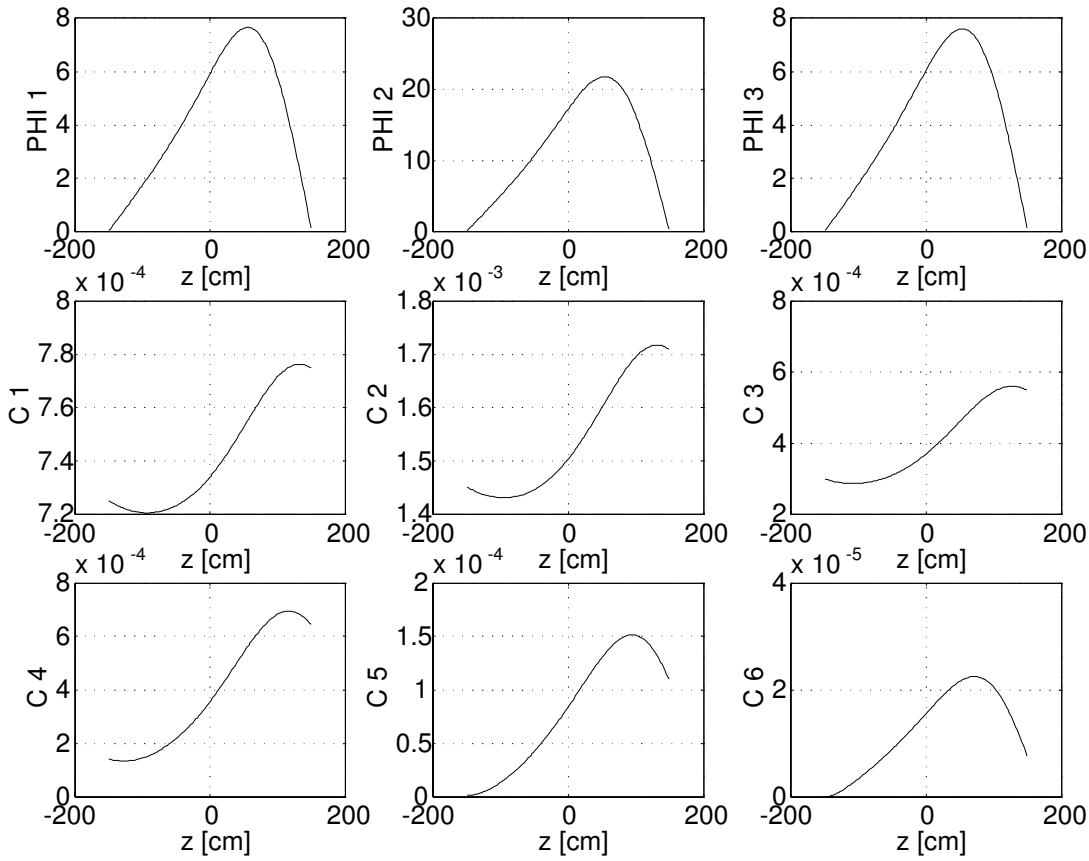

FIG. 13. Steady state distributions of fluxes and precursors.

flow characteristics.

The space distribution of delayed precursors is completely different from what is expected in the case of a solid fuel system. Figure 13, well illustrates this fact, reporting the typical neutron flux (in a three group model) and the delayed neutron concentrations [24].

The role of delayed emissions is significantly reduced with respect to standard systems. This is due to two causes: i) the space redistribution of the delayed precursors, as delayed precursors can emit neutrons at positions having different importances with respect to the position where the fission event took place and ii) the external recirculation, as delayed emission can take place outside the core region and thus the neutrons cannot take part to the chain reaction process. These effects cause a reduction of the effective delayed neutron fraction. In the following some results will quantify the extent of such effect.

Owing to the peculiar space distributions of the delayed precursors, the factorization schemes for de riving the neutron kinetics equations should unavoidably be applied to both neutrons and precursors. Therefore, the point kinetic model needs a specific formulation, which is presented in the following sec tion.

8.3. Derivation of consistent point kinetics

To derive a consistent point kinetic model, the Henry factorization procedure is now applied to the balance equations (99). For this purpose, a reference configuration is introduced as described by the

equations:

$$\begin{cases} \left[\hat{\mathrm{L}}_0 + \hat{\mathrm{M}}_{p,0}\right] N_0(\mathbf{r},E,\Omega) + \sum_{i=1}^{R} \mathcal{E}_{i,0}(\mathbf{r},E) + S_0(\mathbf{r},E,\Omega) = 0, \\ \frac{1}{\lambda_i}\boldsymbol{\nabla}\cdot(\mathbf{u_0}\mathcal{E}_{i,0}(\mathbf{r},E)) = \hat{\mathrm{M}}_{i,0} N_0(\mathbf{r},E,\Omega) - \mathcal{E}_{i,0}(\mathbf{r},E), \qquad i=1,2,...,R, \end{cases} \tag{106}$$

with appropriate boundary conditions. For the projection procedure it is necessary to define the inner product between elements $\mathbf{w} = (n, \mathcal{E}_1, ..., \mathcal{E}_R)^t$ of the direct and $\mathbf{w}^\dagger = (n^\dagger, \mathcal{E}_1^\dagger, ..., \mathcal{E}_R^\dagger)$ of the adjoint space as:

$$(\mathbf{w}^\dagger, \mathbf{w}) = \sum_{n=1}^{R+1} \langle w_n^\dagger \mid w_n \rangle = \sum_{n=1}^{R+1} \int_V dV \int_E dE \int_{4\pi} d\Omega \cdot w_n^\dagger w_n. \tag{107}$$

It is physically meaningful to use for the weighting function a proper mathematical adjoint. The im portance is defined consistently as the number of fission neutrons that would be produced within the multiplying system by the injection of a neutron at a given point in phase space. Thence, the system of equations for the importance takes the form:

$$\begin{cases} \left[\hat{\mathrm{L}}_0^\dagger + \hat{\mathrm{M}}_{p,0}^\dagger\right] N_0^\dagger(\mathbf{r},E,\Omega) + \sum_{i=1}^{R} \hat{\mathrm{M}}_{i,0}^\dagger \mathcal{E}_{i,0}^\dagger(\mathbf{r},E) + S_0^\dagger(\mathbf{r},E,\Omega) = 0, \\ \frac{1}{4\pi}\oint d\Omega N_0^\dagger(\mathbf{r},E,\Omega) + \frac{1}{\lambda_i}\mathbf{u_0}\cdot\boldsymbol{\nabla}\left(\mathcal{E}_{i,0}^\dagger(\mathbf{r},E)\right) - \mathcal{E}_{i,0}^\dagger(\mathbf{r},E) = 0, \qquad i=1,2,...,R, \end{cases} \tag{108}$$

with boundary conditions:

$$\mathcal{E}_i^\dagger(\mathbf{r},E) = \int_{\mathcal{A}_{in}} \mathcal{E}_i^\dagger(\mathbf{r}',E) e^{-\lambda_i \tau(\mathbf{r}\to\mathbf{r}')} \mathfrak{F}\left(\mathbf{r}\to\mathbf{r}'\right) d\mathcal{A}', \qquad \mathbf{r} \in \mathcal{A}_{out}. \tag{109}$$

These boundary conditions are symmetric with respect to Eq. (101), and can be physically interpreted consequently.

Both flux and delayed emission distributions are factorized with an amplitude shape formula:

$$\begin{aligned} n(\mathbf{r},E,\Omega,t) &= A(t)\varphi(\mathbf{r},E,\Omega;t), \\ \mathcal{E}_i(\mathbf{r},E,t) &= G_i(t) e_i(\mathbf{r},E;t) \qquad i=1,2,...,R. \end{aligned} \tag{110}$$

Afterwards, the factorized formula is introduced into the balance equations (99), which are then pro jected on the adjoint solution. Normalization conditions are imposed on the shape functions, for both neutrons and precursors, to make the factorization unique:

$$\begin{aligned} &\frac{d}{dt}\left\langle N_0^\dagger \right| \varphi\rangle = 0, \\ &\frac{d}{dt}\left\langle \mathcal{E}_{i,0}^\dagger \right| e_i\rangle = 0, \qquad i=1,2,...,R. \end{aligned} \tag{111}$$

The amplitude functions are found to obey a generalized point like model [24]:

$$\begin{cases} \Lambda_A \frac{dA}{dt} = \left(\rho_S - \tilde{\beta}\right) A + \sum_{i=1}^{R} \lambda_i \Gamma_i + \tilde{S}, \\ \Lambda_i \frac{d\Gamma_i}{dt} = \left(\tilde{\beta}_i + \rho_i\right) A - (\lambda_i + \mu_{u,i} + \mu_{\xi,i})\Gamma_i + \sigma_i, \qquad i=1,...,R. \end{cases} \tag{112}$$

The structure of system (112) is equivalent to the point kinetic model for solid fuel systems. However, the kinetic parameters and the effective delayed neutron functions Γ_i have different definitions, and unconventional coefficients appear, such as ρ_i, connected to the perturbation of the delayed neutron precursor production, $\mu_{u,i}$, due to the perturbation of the fluid velocity, and $\mu_{\xi,i}$, due to the perturbation of the recirculation time. The definitions of the parameters and of the effective source and delayed neutron precursor functions follow:

- normalization factor:

$$\mathcal{F} = \sum_{i=1}^{R}\sum_{n=1}^{G}\left\langle \Phi_n^{\dagger}\middle|\, \chi_{i,n}\lambda_i C_{i,\mathbf{0}}\right\rangle + (1-\beta)\sum_{n=1}^{G}\sum_{g=1}^{G}\left\langle \Phi_n^{\dagger}\middle|\, \chi_n\left(\nu\Sigma_f\right)_{g,\mathbf{0}}\Phi_{g,\mathbf{0}}\right\rangle; \tag{113}$$

- effective delayed neutron precursor functions:

$$\Gamma_i = \frac{1}{\mathcal{F}}\sum_{n=1}^{G}\left\langle \Phi_n^{\dagger}\middle|\, \chi_{i,n}C_{i,\mathbf{0}}\right\rangle G_i(t); \tag{114}$$

- effective external neutron source:

$$\tilde{S} = \frac{1}{\mathcal{F}}\sum_{n=1}^{G}\left\langle \Phi_n^{\dagger}\middle|\, S_n\right\rangle; \tag{115}$$

- effective delayed neutron fractions

$$\tilde{\beta}_i = \frac{1}{\mathcal{F}}\sum_{n=1}^{G}\left\langle \Phi_n^{\dagger}\middle|\, \chi_{i,n}\lambda_i C_{i,\mathbf{0}}\right\rangle, \qquad \tilde{\beta} = \sum_{i=1}^{R}\tilde{\beta}_i; \tag{116}$$

- reactivity $\rho_S = \rho_{\mathbf{0}} + \rho_p$, where:

$$\rho_{\mathbf{0}} = -\frac{1}{\mathcal{F}}\sum_{n=1}^{G}\left\langle S_n^{\dagger}\middle|\, \Phi_{n,\mathbf{0}}\right\rangle; \tag{117}$$

in diffusion theory (Eqs. (102)), the perturbation reactivity term takes the form:

$$\begin{aligned}\rho_p = \frac{1}{\mathcal{F}}\Bigg\{ &-\sum_{n=1}^{G}\left\langle \boldsymbol{\nabla}\Phi_n^{\dagger}\cdot\middle|\, \delta D_n\boldsymbol{\nabla}\Phi_{n,\mathbf{0}}\right\rangle + \sum_{n=1}^{G}\sum_{g=1}^{G}\left\langle \Phi_n^{\dagger}\middle|\, (1-\beta)\,\chi_n\delta\left(\nu\Sigma_f\right)_g\Phi_{g,\mathbf{0}}\right\rangle - \\ &\sum_{n=1}^{G}\left\langle \Phi_n^{\dagger}\middle|\, \delta\Sigma_{R,n}\Phi_{n,\mathbf{0}}\right\rangle + \sum_{n=1}^{G}\sum_{g=1}^{G}\left\langle \Phi_n^{\dagger}\middle|\, \delta\Sigma_{g\to n}\Phi_{g,\mathbf{0}}\right\rangle\Bigg\};\end{aligned} \tag{118}$$

- effective prompt neutron lifetime:

$$\Lambda_A = \frac{1}{\mathcal{F}}\sum_{n=1}^{G}\left\langle \Phi_n^{\dagger}\middle|\, \frac{1}{v_n}\Phi_{n,\mathbf{0}}\right\rangle; \tag{119}$$

- effective generalized precursor lifetime:

$$\Lambda_i = \frac{\left\langle C_i^{\dagger}\middle|\, C_{i,\mathbf{0}}\right\rangle}{\sum_{n=1}^{G}\left\langle \Phi_n^{\dagger}\middle|\, \chi_{i,n}C_{i,\mathbf{0}}\right\rangle}; \tag{120}$$

- unconventional coefficients:

$$\rho_i = \frac{1}{\mathcal{F}} \sum_{g=1}^{G} \left\langle C_i^\dagger \middle| \beta_i \delta(\nu\Sigma_f)_g \Phi_{g,\mathbf{0}} \right\rangle ;$$

$$\mu_{u,i} = \frac{\left\langle C_i^\dagger \middle| \dfrac{\partial}{\partial z} \left(\delta u C_{i,\mathbf{0}}\right) \right\rangle}{\displaystyle\sum_{n=1}^{G} \left\langle \Phi_n^\dagger \middle| \chi_{i,n} C_{i,\mathbf{0}} \right\rangle}; \tag{121}$$

$$\mu_{\xi,i} = \frac{u_{\mathbf{0}}(H) \displaystyle\int_{\mathcal{A}_{core}} C_i^\dagger(H) C_{i,\mathbf{0}}(r,H) d\mathcal{A}}{\displaystyle\sum_{n=1}^{G} \left\langle \Phi_n^\dagger \middle| \chi_{i,n} C_{i,\mathbf{0}} \right\rangle};$$

- apparent precursor source:

$$\sigma_i = \begin{cases} \dfrac{\Xi_i}{\displaystyle\sum_{n=1}^{G} \left\langle \Phi_n^\dagger \middle| \chi_{i,n} C_{i,\mathbf{0}} \right\rangle} \Gamma_{i,\mathbf{0}}, & \text{if } t \le \tau, \\ \dfrac{\Xi_i}{\displaystyle\sum_{n=1}^{G} \left\langle \Phi_n^\dagger \middle| \chi_{i,n} C_{i,\mathbf{0}} \right\rangle} \Gamma_i(t-\tau) & \text{if } t > \tau, \end{cases} \tag{122}$$

where:

$$\Xi_i = \frac{1}{\mathcal{A}_{core}} u_{\mathbf{0}}(H) \left(\xi_{i,\mathbf{0}} + \delta\xi_i\right) \int_{\mathcal{A}_{core}} C_i^\dagger(r,0) d\mathcal{A} \int_{\mathcal{A}_{core}} C_{i,\,\mathbf{0}}(r,H). \tag{123}$$

It is worth to present some values of the effective delayed neutron fractions, Table I, which are computed for a reference system in a 1D model in a Uranium fueled reactor [25]. The column for $\tau = 0$ is ideal, but it gives an indication of the extent of the effect due to the spatial redistribution only, since no external decay is accounted for. It is seen that this effect alone is about 16 17%. Globally the reduction of the role of delayed neutrons can be rather significant, up to almost 70%. For source driven systems, this is to be added to the reduction that has to be associated to subcriticality [26].

If the shape function in the factorization (111) is kept constant and coincident with the solution of the reference problem (106), the point model is obtained. However, a quasi static procedure can easily be implemented [23].

Table I. Ratio $\tilde{\beta}/\beta$ as a function of τ and k for different fluid velocities.

τ [s] $\rightarrow$	0	5	10	15	u [cm/s] $\downarrow$
$k = 0.95009$	0.842	0.542	0.470	0.443	60
	0.835	0.422	0.363	0.332	100
$k = 0.97048$	0.843	0.540	0.469	0.441	60
	0.834	0.420	0.353	0.330	100
$k = 0.99028$	0.842	0.539	0.467	0.440	60
	0.833	0.419	0.352	0.329	100
$k = 1.00001$	0.843	0.539	0.468	0.440	60
	0.834	0.420	0.352	0.329	100

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