The Abdus Salam
International Centre for Theoretical Physics

United Nations
Educational, Scientific
and Cultural Organization

International Atomic
Energy Agency

SMR1735/10

Advanced School and Conference on Representation Theory and Related Topics

(9 - 27 January 2006)

Cellular algebras

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Abstract

This is a series of four lectures on cellular algebras, which are given on 9-27 January 2006, at ICTP, Italy. In the lectures we introduce the theory of cellular algebras which were defined first by Graham and Lehrer in 1996 to deal systematically with parameterizing irreducible representations of algebras and related problems. If a finite-dimensional algebra is given by quiver with relations, the irreducible representations over such an algebra are very easy to describe. But in the real world, algebras, especially those with the interesting applications in mathematics and physics, appear very often not in this form. It turns out that the determination of the irreducible representations of these algebras is a quite hard problem. The difficulty encountered is well-known, for instance, in the representation theory of finite groups. The theory of cellular algebras enables one to solve this problem and study the irreducible representations axiomatically in terms of linear algebra. Moreover, some homological properties of cellular algebras can be simply detected via linear algebra method.

Cellular structures of algebras seem to be very common; in the last a few years, a large variety of algebras appearing in both mathematics and physics have been proved to have a cellular structure. For example, the Hecke algebras and the q-Schur algebras of type A, Brauer algebras; inverse semigroup algebras; Temperley-Lieb algebras, partition algebras, Birman-Wenzl algebras, and many other diagram algebras. It is worthy to notice that the cellular algebra method can also be used to study algebras of infinite dimension.

The contents of the lectures will include the definitions (original definition of Graham and Lehrer in terms of a basis, and the one which is basis-free), some examples and basic properties of cellular algebras; representation theory of cellular algebras; homological aspects of cellular algebras, and relationship with quasi-hereditary algebras; applications to algebras in mathematics and physics.

This is a very primary version. In the notes many topics that are of interest could not be included, for example, the consideration of structures of cellular algebras, and the references are surely not complete. I hope I could write a satisfied version in future. So, any critical comments and suggestions are welcome.

AMS Classification: 16G10, 16P10, 16S20; 18G20.

Key words: Cellular algebras, cell ideals, quasi-hereditary algebras, Cartan matrices, global dimensions

1 Definitions, Examples and basic properties

Cellular algebras were introduced by Graham and Lehrer in 1996 in the paper [12], and were defined by the existence of a basis with certain nice multiplicative properties. This gives an axiomatic method to treat the hard problem of parameterizing simple modules in the representation of algebraic groups and related topics. In fact, the method reduces systematically many hard problems to standard ones in linear algebra.

1.1 Definition of cellular algebras of Graham and Lehrer

Let us first introduce the basis definition, which reflexes many combinatorial features of cellular algebras.

Definition 1.1 *(Graham and Lehrer [12]) Let R be a commutative Noetherian integral domain. An associative R–algebra A is called a* **cellular algebra** *with cell datum (Λ, M, C, i) if the following conditions are satisfied:*

$(C1)$ The finite set Λ is partially ordered. Associated with each $\lambda \in \Lambda$ there is a finite set $M(\lambda)$. The algebra A has an R–basis $C^{\lambda}_{S,T}$ where (S,T) runs through all elements of $M(\lambda)\times M(\lambda)$ for all $\lambda \in \Lambda$.

$(C2)$ The map i is an R–linear anti–automorphism of A with $i^2 = id$ which sends $C^{\lambda}_{S,T}$ to $C^{\lambda}_{T,S}$.

$(C3)$ For each $\lambda \in \Lambda$ and $S, T \in M(\lambda)$ and each $a \in A$ the product $aC^{\lambda}_{S,T}$ can be written as $(\sum_{U\in M(\lambda)} r_a(U,S)C^{\lambda}_{U,T}) + r'$ where r' is a linear combination of basis elements with upper index μ strictly smaller than λ, and where the coefficients $r_a(U,S) \in R$ do not depend on T.

Remarks. (1) There is a similar version of (C3) for right multiplication. This follows from (C3) and (C2) automatically.

(2) The axioms of cellular algebras are motivated from the properties of the Kazhdan-Lusztig canonical basis for Hecke algebras of type A [19]. It can be also considered as a deformation of semisimple algebras in some sense (see Proposition 1.5 below).

In the following we shall call a k–linear anti–automorphism i of A with $i^2 = id$ an involution of A. In case A is a cellular algebra, then the basis in Definition 1.1 will be called a cellular basis.

1.2 A basis-free definition of cellular algebras

The following definition reflexes structural properties of cellular algebras.

Definition 1.2 *(see [21]) Let A be an R–algebra where R is a commutative noetherian integral domain. Assume there is an involution i on A. A two–sided ideal J in A is called a* **cell ideal** *if and only if $i(J) = J$ and there exists a left ideal $\Delta \subset J$ such that Δ is finitely generated and free over R and that there is an isomorphism of A–bimodules $\alpha : J \simeq \Delta \otimes_R i(\Delta)$ (where $i(\Delta) \subset J$ is the i–image of Δ) making the following diagram commutative:*

$$\begin{array}{ccc} J & \xrightarrow{\alpha} & \Delta \otimes_R i(\Delta) \\ i \downarrow & & \downarrow x \otimes y \mapsto i(y) \otimes i(x) \\ J & \xrightarrow{\alpha} & \Delta \otimes_R i(\Delta) \end{array}$$

The algebra A (with the involution i) is called **cellular** *if and only if there is an R–module decomposition $A = J'_1 \oplus J'_2 \oplus \cdots \oplus J'_n$ (for some n) with $i(J'_j) = J'_j$ for each j and such that setting $J_j = \oplus_{l=1}^{j} J'_l$ gives a chain of two–sided ideals of A: $0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$ (each of them fixed by i) and for each j $(j = 1, \ldots, n)$ the quotient $J'_j = J_j/J_{j-1}$ is a cell ideal (with respect to the involution induced by i on the quotient) of A/J_{j-1}.*

The $\Delta's$ obtained from each section J_j/J_{j-1} are called **standard modules** of the cellular algebra A, and the above chain of ideals in A is called a cell chain of A. Note that all simple A–modules can be obtained from standard modules [12]. (Standard modules are called cell modules in [12]).

Proposition 1.3 *The two definitions of cellular algebras are equivalent.*

Proof. Assume that A is cellular in the sense of Graham and Lehrer. Fix a minimal index λ. Define $J(\lambda)$ to be the R–span of the basis elements $C^{\lambda}_{S,T}$. By (C3), this is a two–sided ideal. By (C2) it is fixed by the involution i. For any fixed index $T \in M(\lambda)$, we define Δ as the R–span of $C^{\lambda}_{S,T}$ (where S varies). Defining α by sending $C^{\lambda}_{U,V}$ to $C^{\lambda}_{U,T} \otimes i(C^{\lambda}_{V,T})$ gives the required isomorphism. Thus $J(\lambda)$ is a cell ideal. Continuing by induction, it follows that A is cellular in the sense of the new definition.

Conversely, if a cell ideal, say J, in the sense of the new definition is given, we choose any R–basis, say $\{C_S\}$, of Δ, and denote by $C_{S,T}$ the inverse image under α of $C_S \otimes i(C_T)$. Since Δ is a left module, (C3) is satisfied. (C2) follows from the required commutative diagram. This finishes the proof for those basis elements occurring in a cell ideal. Induction (on the length of the chain of ideals J_j) provides us with a cellular basis of the quotient algebra A/J. Choosing any pre-images in A of these basis elements together with a basis of J as above we produce a cellular basis of A. $\blacksquare$

Remark. (1) The cellularity of an algebra depends on the involution and the basis. We display an example to show that an algebra may be cellular with respect to one involution, while it might not be cellular with respect to another involution.

Let A be the group algebra of the cyclic group of order 3 over a field of characteristic 3. If we take the involution to be the identity, then the algebra is cellular with respect to this involution. Now if we take another involution which is induces by sending the group element to its inverse.

(2) (A, i) is cellular if and only if (A^{op}, i) is cellular.

(3) A cellular algebra may have many different cellular bases.

Finally, we recall the definition of quasi-hereditary algebras. which is another class of algebras of interest in representation theory of Lie algebras and algebraic groups.

Definition 1.4 *(Cline, Parshall and Scott [7]) Let A be a k-algebra. An ideal J in A is called a* **heredity ideal** *if J is idempotent, $J(rad(A))J = 0$ and J is a projective left (or, right) A-module. The algebra A is called* **quasi-hereditary** *provided there is a finite chain $0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$ of ideals in A such that J_j/J_{j-1} is a heredity ideal in A/J_{j-1} for all j. Such a chain is then called a* ***heredity chain*** *of the quasi-hereditary algebra A.*

It is well-known that quasi-hereditary algebras have finite global dimension. For more information on quasi-hereditary algebras with the representation theory of algebraic groups and Lie algebras we refer to [32].

1.3 Some elementary properties of cellular algebras

To check an ideal or algebra over a field to be cellular, we often use the following property.

Proposition 1.5 *Let A be a k–algebra with an involution i. Suppose A is generated as algebra by $a_1, ..., a_m$. Let J be a subspace of A with a basis*

$$\begin{array}{cccc} C_{11}, & C_{12}, & \cdots, & C_{1n}, \\ C_{21}, & C_{22}, & \cdots, & C_{2n}, \\ \cdots & \cdots & \cdots & \cdots \\ C_{n1}, & C_{n2}, & \cdots, & C_{nn} \end{array}$$

such that $i(C_{lk}) = C_{kl}$ for k, l. Define $c_j := (C_{1j}, C_{2j}, ..., C_{nj})$ for $1 \leq j \leq n$ and $a_l c_j := (a_l C_{1j}, ..., a_l C_{nj})$. If $a_l c_j \in \sum_{t=1}^{n} k c_t$ for all l, j, then J is a cell ideal in A.

Proof. It is clear that J is an ideal in A with $i(J) = J$. Fix an index j and define $\Delta = kC_{j1} + kC_{j2} + \cdots + kC_{jn}$. Then Δ is a left ideal contained in J. We define

$$\alpha : \Delta \otimes_k i(\Delta) \longrightarrow J$$

by sending $C_{jl} \otimes C_{kj}$ to C_{kl}. Obviously, α is a k–linear bijection. We shall show that it is also a $A-A$–bimodule homomorphism. Take $a_s \in A$, consider the image of $a_s C_{jl} \otimes C_{kj}$ under the map α. Suppose $a_s C_{jl} = \sum_t \lambda_t C_{jt}$. This implies that $a_s c_l = \sum_t \lambda_t c_t$ and $a_s C_{kl} = \sum_t \lambda_t C_{kt}$. Thus we deduce that α is a bimodule homomorphism. It is clear that the diagram

$$\begin{array}{ccl} J & \stackrel{\alpha}{\longrightarrow} & \Delta \otimes_k i(\Delta) \\ i \downarrow & & \downarrow x \otimes y \mapsto i(y) \otimes i(x) \\ J & \stackrel{\alpha}{\longrightarrow} & \Delta \otimes_k i(\Delta) \end{array}$$

is commutative. Hence J is a cell ideal in A. ■

Cell ideals are of two different kinds, one of them being familiar from the theory of quasi–hereditary algebras [7].

Proposition 1.6 *Let A be an R–algebra ($R = k$ any field) with an involution i and J a cell ideal. Then J satisfies one of the following (mutually exclusive) conditions:*
(A) J has square zero.
(B) There exists a primitive idempotent e in A such that J is generated by e as a two–sided ideal. In particular, $J^2 = J$. Moreover, eAe equals $Re \simeq R$, and multiplication in A provides an isomorphism of A–bimodules $Ae \otimes_R eA \simeq J$. In other words, J is a heredity ideal in A.

Proof. By assumption, J has an R–basis $C_{S,T}$ whose products satisfy the rule (C3). If all the products $C_{S,T}C_{U,V}$ are zero, then we are in situation (A). Thus we may assume that there is one such product which is not zero. Since the coefficients do not depend on S or V, the product $C_{U,T}C_{U,T}$ also is not zero. But by [12], 1.7 (or a direct comparison of the two ways writing this product as a linear combination of basis elements, using (C3) and its dual), this product is a scalar multiple of $C_{U,T}$. Hence there is an idempotent in J, which thus cannot be nilpotent.

So, J contains a primitive idempotent, say e, and Ae is a left ideal which is contained in J. The cell ideal J as a left A–module is a direct sum of copies of a standard module Δ. But Ae is a submodule, hence a direct summand of the left ideal $J = Ae \oplus J(1-e)$. It follows that Ae is a direct summand of Δ which we can decompose into $Ae \oplus M$ for some A–module M. Because of $J \simeq \Delta \otimes_R i(\Delta)$ we can decompose J as left module into $Ae^m \oplus M^m$ where m is the R–dimension of Δ (which equals the R–dimension of $i(\Delta)$). Of course, Ae^m is contained in the trace X of Ae inside J (that is, the sum of all images of homomorphisms $Ae \to J$). This trace X is contained in the trace AeA of Ae in A. But the dimension of AeA is less than or equal to the product of the dimension of Ae with the dimension, say n, of eA since there is a canonical bimodule surjective map $Ae \otimes_R eA \longrightarrow AeA$. Thus $m \cdot \dim_R(Ae) \leq \dim_R(AeA) \leq \dim_R(Ae)\dim_R(eA) = n \cdot \dim_R(Ae)$, and $m \leq n$. The number n equals the dimension of $Ai(e)$, which (by the same argument) also is a direct summand of Δ. Hence $n = \dim_R Ai(e) \leq \dim_R(\Delta) = m$. So we have $m = n$. This implies that $\Delta \simeq Ai(e)$ is indecomposable since $i(e)$ is primitive idempotent element in A, and that $\Delta = Ae$ (that is, $M = 0$) and $\dim_R AeA = m^2$. In particular, J equals AeA and also $Ae \otimes_R eA \simeq AeA$ as bimodules. If we multiply e on both sides, then we have $eAe \otimes_R eAe \simeq eAe$. This implies that $\dim_R eAe = 1$. It follows that eAe must be equal to $Re \simeq R$. ■

Proposition 1.7 *Suppose R is a field. Let A be an R-algebra with a cell ideal J which is equal to A. Then A is isomorphic to a full matrix ring over the ground field R.*

Proof. The assumption says that A has an involution i and can be written as $\Delta \otimes_R i(\Delta)$ for some left ideal Δ. Hence there is an R–isomorphism $A \simeq \mathrm{Hom}_A(A, A) \simeq \mathrm{Hom}_A(\Delta \otimes_R i(\Delta), A) \simeq \mathrm{Hom}_R(i(\Delta), \mathrm{Hom}_A(\Delta, A))$. Denote the R–dimension of the Δ by m. Then A has R–dimension m^2, and as left module, A is isomorphic to m copies of Δ. Hence, $\mathrm{Hom}_A(\Delta, A)$ (which is a submodule of the R–space $\mathrm{Hom}_R(\Delta, A)$, hence has R–dimension at least m. But by the above isomorphism it cannot have larger dimension. Thus, the A–endomorphism ring E of Δ has dimension one and contains R, hence E is equal to R. Thus $\mathrm{End}({}_AA) \simeq \mathrm{Hom}_A(\oplus\Delta^m, \oplus\Delta^m) \simeq M_m(E) = M_m(R)$, where $M_n(R)$ stands for the full n by n matrix algebra over R. ■

Proposition 1.8 *Each cell chain of a cellular algebra is maximal, that is, the chain cannot be refined to another cell chain with longer length.*

Proof. The Proposition follows by induction on the minimal number of ideals in a cell chain from the definition of cellular algebras and the following Claims:

Claim 1. (a) If J is an n^2-dimensional cell ideal in A with an involution i, then the k–dimension of $Fix_i(J) := \{x \in J \mid i(x) = x\}$ satisfies

$$dim_k(Fix_i(J)) = n(n+1)/2.$$

(b) If $0 \subset J_1 \subset J_2 \subset \cdots \subset J_m = A$ is a cell chain with the corresponding cell ideals having k–dimensions $n_1^2, n_2^2, \ldots, n_m^2$, then the k–dimension of $Fix_i(A)$ satisfies

$$dim_k(Fix_i(A)) = n_1(n_1+1)/2 + n_2(n_2+1)/2 + \cdots + n_m(n_m+1)/2.$$

The proof of (a) is straightforward from linear algebra since the symmetric $n \times n$–matrices form a vector space of dimension $n(n+1)/2$.

In order to also prove (b) we use the following observation: If a vector space V can be decomposed as $U \oplus W$ in such a way that a given involution i acting on V sends both U and W into itself, then the k–dimension of the space of fixed points is additive, since in fact the spaces of fixed points add up: $Fix_i(V) = Fix_i(U) \oplus Fix_i(W)$. Thus (b) follows by noting that A has a cell basis $C^\lambda_{S,T}$, hence can be written as a direct sum of spaces $V^\lambda_{S,T}$, each of them generated by the one or two basis elements $C^\lambda_{S,T}$ and $C^\lambda_{T,S}$ and each $V^\lambda_{S,T}$ being fixed under i.

Claim 2. Let J be a cell ideal in a cellular algebra A with respect to an involution i. Suppose J_1 is another cell ideal such that there is an inclusion $0 \subseteq J_1 \subseteq J$ and that J/J_1 is filtered by a chain of ideals with subquotients being cell ideals. Then $J_1 = J$.

Proof. Denote the k–dimension of J by n^2. Denote the chain of ideals filtering J by $J_1 \subseteq J_2 \cdots \subseteq J_m = J$ for some $m \geq 1$ and the corresponding k–dimensions of cell ideals by $n_1^2, n_2^2, \ldots, n_m^2$. We have (by the argument which proved Claim 1(b)) the following equalities of dimensions:

$$dim_k(J) = n^2 = n_1^2 + n_2^2 + \cdots + n_m^2,$$
$$dim_k(Fix_i(J)) = n(n+1)/2 = n_1(n_1+1)/2 + n_2(n_2+1)/2 + \cdots + n_m(n_m+1)/2.$$

This implies another equation

$$n = n_1 + n_2 + \cdots + n_m,$$

which together with the first equation implies the desired equality $m = 1$. This finishes the proof. ■

Proposition 1.9 *If for a cellular algebra A there is a cell chain $0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$ such that the cell ideal J_j/J_{j-1} is not nilpotent, then A ia a quasi-hereditary algebra.*

Proof. This follows from 1.6 immediately.

This proposition says that if a cell chain of a cellular algebra is a heredity chain then we have a hereditary algebra. But if this cell chain is not a hereditary chain, could we claim that the given algebra has no heredity chain any more ? that is, is the given algebra not quasi-hereditary with any chain of ideals ? We will give an answer to this question later.

Finally, we mention the following

Proposition 1.10 *If $f : R \longrightarrow S$ is a homomorphism of commutative rings. If A is a cellular algebra over R with cell datum (Λ, M, C, i), then the scalar extension $S \otimes_R A$ of A is a cellular algebra over S with the cell basis $\{1_S \otimes_R C^\lambda_{U,V} \mid \lambda \in \Lambda, U, V \in M(\lambda)\}$ and the involution $i(s \otimes_R C^\lambda_{U,V}) = s \otimes_R i(C^\lambda_{U,V})$ for $s \in S$.*

This proposition enable one simultaneously to consider the representation theory over different ground rings. This may be useful for passing representation theory from characteristic zero to positive characteristic.

1.4 Examples

Examples 1. Let A be the direct sum of matrix algebras over a field k. We take the usual matrix transpose as an involution of A. Then A is a cellular algebra with respect to the usual matrix units as a cellular basis. Another simple example of cellular algebra is $A := k[x]/(x^{n+1})$, where $k[x]$ is the polynomial algebra over k with x a variable. Here we take identity as the involution and Λ the opposite order of the usual poset $\{0, 1, ..., n\}$. The finite index set $M(i)$ just consists one element, say i. The cellular basis is $\{\bar{x}^n, \bar{x}^{n-1}, ..., \bar{x}, \bar{1}\}$, where $\bar{x}$ stands for the image of x under the canonical map from $k[x]$ to A.

Examples 2. A is given by the quiver

$$\Delta: \quad 1 \underset{\rho'}{\overset{\rho}{\rightleftarrows}} 2 \underset{\delta'}{\overset{\delta}{\rightleftarrows}} 3$$

with relations

$$\rho\delta = \delta'\rho' = \rho'\rho - \delta\delta' = 0.$$

This algebra is symmetric and has the following regular representation

$$\begin{array}{ccccc} 1 & & 2 & & 3 \\ 2 & \oplus\ 1 & & 3\ \oplus & 2 \\ 1 & & 2 & & 3 \end{array}.$$

For this algebra we define an involution i by fixing each vertices, and interchanging δ with δ', ρ with ρ'. We may display a basis, and use Proposition 1.5 to check if we do have a cell chain or not. The details is as follows:

$$\rho\rho' \quad ; \quad \begin{array}{cc} e_1 & \rho \\ \rho' & \rho'\rho \end{array} \quad ; \quad \begin{array}{cc} e_2 & \delta \\ \delta' & \delta'\delta \end{array} \quad ; \quad e_3$$

Examples 3. If (A, i) and (B, j) are cellular algebras over a field k, then the tensor product $A \otimes_k B$ of A and B is cellular with respect to the involution $i \otimes_k j$.

Examples 4. Let A be a finite dimensional algebra over a field k. We denote by $T(A)$ the **trivial extension** of A, which has the underlying vector space $A \oplus DA$, where $DA = Hom_k(A, k)$ is viewed as an A-A-bimodule, and $T(A)$ has multiplication given by

$$(a + f)(b + g) = ab + fb + ag \quad a, b \in A; f, g \in DA.$$

It is a symmetric algebra with non-degenerate bilinear form $< a + f, b + g >= f(b) + g(a)$.

If there is an involution i on A, then we may define an involution ϵ on the trivial extension $T(A)$ by $a + f \mapsto i(a) + i \circ f$, where $i \circ f$ stands for the composition of the linear map i with the function f.

Let A be a cellular algebra with respect to an involution i. Then the trivial extension $T(A)$ of A by $D(A)$ is a cellular algebra with respect to the involution ϵ defined below.

Proof. If N is a subset of A, we denote by $N^{\perp}$ the set $\{f \in DA \mid f(N) = 0\}$. Since A is a cellular algebra with respect to i, there is an i-invariant decomposition: $A = J'_1 \oplus J'_2 \oplus \cdots \oplus J'_m$ such that if we define $J_j = \oplus_{l=1}^{j} J'_l$ for all j then the chain $0 \subset J_1 \subset J_2 \subset \cdots \subset J_m = A$ is a cell chain for A. Thus we have a decomposition of $T(A)$:

$$T(A) = D(J'_m) \oplus D(J'_{m-1}) \oplus \cdots \oplus D(J'_1) \oplus J'_1 \oplus J'_2 \oplus \cdots \oplus J'_m$$

Let $I_j := J'_{j+1} \oplus J'_{j+2} \oplus \cdots \oplus J'_m$. Then $A = J_j \oplus I_j$ and $J_j^\perp \simeq D(I_j)$. Note that $J_j^\perp$ and $D(I_j)$ are in fact isomorphic as $A-A$-bimodules. So the decomposition of $T(A)$ gives rise a chain of ideals of $T(A)$:

$$J_m^\perp = 0 \subset J_{m-1}^\perp = D(I_{m-1}) \subset J_{m-2}^\perp = D(I_{m-2}) \subset \cdots \subset J_1^\perp = D(I_1) \subset DA$$

The subquotient $J_{j-1}^\perp/J_j^\perp$ is isomorphic to $D(J'_j)$ as A-A-bimodules. Thus it is a cell ideal in $T(A)/J_j^\perp$. This implies that the decomposition of $T(A)$ satisfies all conditions in the definition of cellular algebras. Thus $T(A)$ is a cellular algebra with respect to the involution ϵ.

Examples 5. For any field k, the group algebras of symmetric groups Σ_n on n letters are cellular algebras with the cell datum:
$\Lambda =$ the set of the partitions of n with the dominant order;
$M(\lambda) =$ the set of standard Young tableaux of shape λ;
$C^\lambda_{P,Q} = w \in \Sigma_n$, where λ, P, Q are uniquely determined by w.
$i =$ the involution induced by $w \mapsto w^{-1}$.
To prove this is cellular basis one needs to know the Robinson-Schensted correspondence. For the details of the proof (or more general case of Hecke algebras or Ariki-Koike Hecke algebra) of type A_n we refer to [12].

Finally, we mention that the algebra corresponding a block of the category $\mathcal{O}$ in the representation theory of Lie algebras are quasi-hereditary cellular algebras.

2 Representation theory of cellular algebras

The main purpose of this section is to understand the simple modules of an arbitrary cellular algebras.

2.1 Simple modules and bilinear forms

Let (A, i) be a cellular algebra with cell datum (Λ, M, C, i). For each $\lambda \in \Lambda$, there is a **cell module** $W(\lambda)$ with a k-basis $\{C_S \mid S \in M(\lambda)\}$, the module structure is given by

$$aC_S = \sum_{T \in M(\lambda)} r_a(T, S) C_T,$$

where the coefficients $r_a(T, S)$ are the same as in Definition 1.1.

The proof of Proposition 1.3 and Proposition 1.6 shows that the set of standard modules coincides with the set of cell modules.

We have also a right cell module $i(W(\lambda))$ which is defined dually. We denote by $\nabla(\lambda)$ the module $Di(W(\lambda))$.

Note that by the proof of Proposition 1.3 the set of standard modules coincides with the set of cell modules.

For a cell module $W(\lambda)$ one can define a bilinear form $\Phi_\lambda : W(\lambda) \times W(\lambda) \to k$ by

$$C^\lambda_{S,S} C^\lambda_{T,T} \equiv \Phi_\lambda(C_S, C_T) C^\lambda_{S,T}$$

modulo the ideal generated by all basis elements C^μ_{UV} with upper index μ less than λ. We denote this ideal by $J^{<\lambda}$.

Let $\Lambda_0 = \{\lambda \in \Lambda \mid \Phi_\lambda \neq 0\}$. We shall see that this set parameterizes the simple modules.

Proposition 2.1 *The bilinear form Φ_λ is symmetric, that is, $\Phi_\lambda(x, y) = \Phi_\lambda(y, x)$ for all $x, y \in W(\lambda)$. Moreover, for any $x, y \in W(\lambda)$, and $a \in A$, we have*

$$\Phi_\lambda(i(a)x, y) = \Phi_\lambda(x, ay).$$

Proof. We just show the last statement. The first statement is easy, we leave it to the reader.

We may write Φ_λ as a matrix $\Phi_\lambda = (\Phi_\lambda(C_S, C_T))$ and define the matrix $r_a = (r_a(S,T))$ for $a \in A$ (see Definition 1.1). We shall prove that the two matrix satisfies that $(r_{i(a)})^T \Phi_\lambda = \Phi_\lambda r_a$. Once this is proved, then the last statement is just the matrix form of $\Phi_\lambda(i(a)x, y) = \Phi_\lambda(x, ay)$ with respect to the canonical basis of $W(\lambda)$. Take $S, T \in M(\lambda)$ and $a \in A$, then we have

$$C^\lambda_{S,S} a C^\lambda_{T,T} \equiv C^\lambda_{S,S} \sum_U r_a(U,T) C^\lambda_{U,T} \equiv \sum_U r_a(U,T)\Phi_\lambda(C_S, C_U) C^\lambda_{S,T} \pmod{J^{<\lambda}}.$$

This shows that the coefficient of $C^\lambda_{S,T}$ is the (S,T)-entry in the matrix $\Phi_\lambda r_a$. Similarly, we calculate the $C^\lambda_{S,S} a C^\lambda_{T,T}$ by $(C^\lambda_{S,S} a) C^\lambda_{T,T}$ and find that the coefficient of $C^\lambda_{S,T}$ is the (S,T)-entry in the matrix $r^T_{i(a)} \Phi_\lambda$. This finishes the proof.

Let $\mathrm{rad}(\lambda) = \{x \in W(\lambda) \mid \Phi_\lambda(x,y) = 0 \text{ for all } y \in W(\lambda)\}$. Then we have the following result.

Proposition 2.2 *Suppose $\lambda \in \Lambda$. Then (1) $rad(\lambda)$ is an A-submodule of $W(\lambda)$.*

(2) If $\Phi_\lambda \neq 0$, then $W(\lambda)/rad(\lambda)$ is absolutely simple.

(3) If $\Phi_\lambda \neq 0$, then $rad(\lambda)$ is the radical of $W(\lambda)$.

Proof. (1) can be checked using Proposition 2.1.

For (2) we refer to [12]. Now we use (2) to show (3). Note that for λ minimal, the proof of Proposition 1.6 shows that $\Delta(\lambda)$ (which is isomorphic to $W(\lambda)$) is just Ae with e a primitive idempotent, thus indecomposable with simple top. Now (3) follows from this fact and (2). Inductively we can show (3) for general λ.

Theorem 2.3 [*12*] *Let A be a cellular algebra with the cell datum (Λ, M, C, i). Suppose $W(\lambda)$, $\Phi(\lambda)$ and Λ_0 are defined as above.*

(1) The set $\{L(\lambda) := W(\lambda)/rad(\lambda) \mid \Phi_\lambda \neq 0\}$ is a complete set of non-isomorphic absolutely simple A-modules.

(2) The following are equivalent for A to be semisimple:

(a) All cell modules are simple and pairwise non-isomorphic;

(b) The form Φ_λ is non-degenerate (that is, $rad(\lambda) = 0$) for each $\lambda \in \Lambda$.

Proof. (1) is a consequence of Proposition 2.2 and 1.6.

Thus, for cellular algebras, the following questions can be answered in some extent:

(1) How to determine the isomorphism classes of simple modules ?
(2) How to determine the dimension of simple modules ?
(3) How to determine the blocks of the algebra A ?
(4) How to determine the Cartan matrix of A ?
(5) When is A semisimple ?

The precise answers to these questions are reduced to linear algebra and the detailed analysis of the individual case. For example, if we know the linear form Φ_λ in details, we may know the dimension of simple module $L(\lambda)$, this equals $|M(\lambda)| - \dim_k \mathrm{rad}(\lambda)$.

2.2 Cartan matrices

We recall the notion of a **Cartan matrix** in the following abstract sense (which coincides with the one used in group theory if A is the group algebra of a finite group over a splitting field). Denote the simple A–modules by $L(1), \dots, L(m)$ and their projective covers by $P(1), \dots, P(m)$. The entries $c_{j,h}$ of the Cartan matrix C_A are the composition multiplicities $[P(j) : L(h)]$. The determinant of C_A is called the Cartan determinant.

Let A be a cellular k-algebra with cell datum (Λ, M, C, i), and k a field. For $\lambda \in \Lambda$ and $\mu \in \Lambda_0$, we define $d_{\lambda\mu}$ to be the multiplicity of the simple module $L(\mu)$ in $W(\lambda)$, and D to be the matrix $(d_{\lambda\mu})_{\lambda\in\Lambda, \mu\in\Lambda_0}$; it is called the **decomposition matrix**. Usually, the matrix D is not a square matrix.

The following is true for cellular algebras.

Theorem 2.4 *(1) D is upper unitriangular, that is, $d_{\lambda\mu} = 0$ unless $\lambda \leq \mu$, and $d_{\lambda\lambda} = 1$ for $\lambda \in \Lambda_0$.*

(2) $C_A) = D^T D$.

Proof. It follows from Proposition 1.6 that $\mathrm{End}_A(\Delta(\lambda)) \simeq k$ if $\lambda \in \Lambda_0$. Thus k is a splitting field for A, in this case the multiplicity of a simple module L corresponding a primitive idempotent e of a module M is $\dim_k eM$. Thus $d_{\lambda\lambda} = 1$ (see the proof of Proposition 1.6 and use induction). Suppose we have a cell chain

$$(*) \quad 0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$$

of ideals in A. Then simple modules correspond to the indices l with $J_l^2 \nsubseteq J_{l-1}$. If e is a primitive idempotent such that $e\Delta(j) \neq 0$, then e is not in J_{l-1}, this implies that the index j corresponding to e is bigger than or equal to l, thus $\Delta(l)$ has composition factors $L(\lambda)$ with $\lambda \geq l$.

To see the Cartan matrix of A is the mentioned form, we just calculate the entry $[Ae_j : L(l)]$. It follows from the chain $(*)$ that we have a chain for Ae_j:

$$0 = J_0 e_j \subset J_1 e_j \subset J_2 e_j \subset \cdots \subset J_n e_j = Ae_j.$$

This shows that $J_\mu / J_{\mu-1} \simeq \Delta(\mu) \otimes_k i(\Delta(\mu))$ for all μ. From this we have

$$\begin{aligned}[Ae_j : L(l)] = & \sum_\mu [\Delta(\mu) : L(l)] \dim_k i(\Delta(\mu)) e_j \\ = & \sum_\mu [\Delta(\mu) : L(l)] \dim_k i(i(e_j)\Delta(\mu)) \\ = & \sum_\mu [\Delta(\mu) : L(l)][\Delta(\mu) : L(j)] \\ = & \sum_\mu d_{\mu,l} d_{\mu,j} = \sum_\mu (D^T)_{j\mu} D_{\mu,l}.\end{aligned}$$

Here we use the fact that $Ae_j \simeq Ai(e_j)$ for each primitive idempotent e_j, where i is the involution. $\square$

Theorem 2.5 *The Cartan matrix of a cellular algebra is positive definite. Its determinant equals one if and only if the length n of a cell chain is equal to the number m of simple modules.*

Before we give the proof, we recall a fact on symmetric real matrices in linear algebra. The transpose of a matrix X will be denoted by X^T.

Proposition 2.6 *Let X be a positive definite matrix, Y a positive semidefinite matrix and Z a 'square root' of X, that is, $Z^2 = X$ and $Z = Z^T$ and Z is positive definite. Then the matrix $U = Z^{-1} Y Z^{-1}$ is positive semidefinite and has the same eigenvalues as the matrix $V = X^{-1} Y$.*

Proof. The square root Z of the matrix X always exists. The assertion follows from $U = ZVZ^{-1}$. $\square$

Proof of Theorem 2.5. Suppose we have a cell chain

$$(*) \quad 0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$$

of ideals in A. We may rearrange the rows of the decomposition matrix D such that D is of the form $\begin{pmatrix} D_1 \\ D_2 \end{pmatrix}$, where both D_1 and D_2 are integer matrices and D_2 (whose rows correspond to those indices l such that $J_l^2 \not\subset J_{l-1}$) is a square matrix. In case $n = m$ we consider D_1 as a matrix of size 0×0. We note that D equals D_2 if and only if n equals m.

We have that $C := C(A) = (D_1^T, D_2^T) \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = D_1^T D_1 + D_2^T D_2$. Denote by C_1 the product $D_2^T D_2$. The unitriangularity of D_2 gives $det(C_1) = 1$. Hence $\det(C) = \det(C_1^{-1} C) = \det(I + C_1^{-1} D_1^T D_1)$ where I denotes the identity matrix. Clearly, C_1 is positive definite and $D_1^T D_1$ is positive semidefinite. Of course, $D_1^T D_1$ is zero if and only if $n = m$. This shows that C is a positive definite matrix.

If $n = m$, then $\det(C) = 1$. Conversely, suppose $\det(C) = 1$. If we take a square root C_2 of the positive definite matrix C_1, then we can show that $C_1^{-1} D_1^T D_1$ has the same eigenvalues as the matrix $C_2^{-1} D_1^T D_1 C_2^{-1}$. Note that C_2 is a symmetric matrix. Thus $C_2^{-1} D_1^T D_1 C_2^{-1}$ is symmetric and its eigenvalues are non-negative real numbers. This implies that the eigenvalues of $C_1^{-1} D_1^T D_1$ is non-negative real numbers. It follows from $\det(C) = 1$ that all eigenvalues of $C_1^{-1} D_1^T D_1$ are zero. So all eigenvalues of $D_1^T D_1$ are zero., that is, $D_1 = 0$, and therefore $n = m$.

2.3 Indecomposable projective modules

First, we mention the following observation which follows from the cell chain.

Proposition 2.7 *Let A be a cellular algebra with the set of standard modules $\Delta := \{\Delta(j) \mid j \in \Lambda\}$. Then each projective A-module has a Δ-filtration.*

For an involution i on an algebra A, there is a duality i (we use the same letter) between A-mod and mod-A. Let * denote the self-duality $Di : A\text{-mod} \longrightarrow A\text{-mod}$.

Proposition 2.8 *Let A be a cellular algebra with cell datum (Λ, M, C, i).*
(1) $Ae \simeq Ai(e)$ for each primitive idempotent element $e \in A$.
(2) For each simple module S, $S^* \simeq S$.
(3) $Ext_A^i(S, T) \simeq Ext_A^i(T, S)$ for any simple module S and T.

Thus the quiver of a cellular algebra has a symmetric shape, that is, between two vertices v and u the arrows from u to v equals the number of arrows from v to u.

3 Homological aspects of cellular algebras

We have seen that the simple modules of a cellular algebra can be parameterized by using the bilinear forms. Thus for such a class of algebras the simple modules can be determined in some sense. But the story of cellular algebras does not end at this point, it goes further. That is, cellular algebras can be used to investigate homological properties and to distinguish finite global dimension from infinite global dimension. This is the main content in this section.

3.1 Definition of global dimension and two examples

Let A be a finite-dimensional k-algebra over a field k. We denote by A-mod the category of all finitely generated left A-modules.

Let X be in A-mod.

Definition 3.1 *If there is an exact sequence*

$$o \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

with P_j projective in A-mod, and if n is minimal with respect to this property, then we say that the projective dimension of X is n, denoted $pd_A X = n$; otherwise, we say that the projective dimension of X is infinity, denoted by $pd_A X = \infty$

The global dimension of A is:

$$gl.dim(A) = sup\{pd_A X \mid X \in A - mod\}.$$

To calculate global dimension, Maurice Auslander shew that

Proposition 3.2 *For a finite-dimensional algebra A,*

$$gl.dim(A) = sup\{pd_A X \mid X \text{ is simple}\}$$

The Example 2 of 1.4 shows a self-injective cellular algebra, thus its global dimension is infinite. The Cartan matrix of this algebra is of the form $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$, its determinant is 4. On the other hand, the following example shows that cellular algebras may have finite global dimension.

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, shift left, "\alpha"] & \underset{2}{\bullet} \arrow[l, shift left, "\alpha'"] \end{tikzcd} \qquad \alpha\alpha' = 0$$

We can check that this is a cellular algebra (an involution is given by fixing vertices and interchange α with α'). The Cartan matrix of this algebra is $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, its determinant is 1. By a simple calculation we see that the global dimension of this algebra is finite.

Thus a natural question rises: What makes the cellular algebras have different global dimension ?

3.2 Global dimensions and Cartan determinants

Global dimension is one of the interesting homological invariants. We have seen that cellular algebras may have finite or infinite global dimension. How to distinguish the cellular algebras of finite global dimension from the ones of infinite global dimension ? Usually, the determinant of a Cartan matrix can not say much about the homological property of a given algebra. However, for cellular algebras the situation is surprisingly better than any other one. We have the following

Theorem 3.3 *Let k be a field and A a cellular k–algebra (with respect to an involution i). Then the following are equivalent:*

(a) Some cell chain of A (with respect to some involution, possibly different from i) is a heredity chain as well, i.e. it makes A into a quasi–hereditary algebra.

(a') There is a cell chain of A (with respect to some involution, possibly different from i) whose length equals the number of isomorphism classes of simple A–modules.

(b) A has finite global dimension.

(c) The Cartan matrix of A has determinant one.

(d) Any cell chain of A (with respect to any involution) is a heredity chain.

Proof. (a) $\Longleftrightarrow$ (a'). Consequence of 1.9 and the proof of 2.5.

(a) $\Longrightarrow$ (b). This is clear since quasi-hereditary algebras have always finite global dimension.

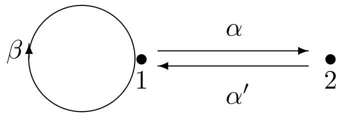
(b)$\Longrightarrow$ (c). This follows from 2.5 together with the fact that the Cartan determinant of an algebra with finite global dimension has determinant ± 1. (A simple proof of this fact: Consider the free abelian group generated by the isomorphism classes $[L(\lambda)]$ of simple modules $L(\lambda)$. The Cartan matrix C_A of an algebra A is just the matrix representing the composition factors of the isomorphism classes $[P_j]$ of indecomposable projective module P_j. In case of finite global dimension, each $[L(i)]$ can be expressed as a linear combination of these $[P_j]$. Thus the Cartan matrix has an inverse over $\mathbb{Z}$, this implies that $\det(C_A)$ must be an unit in $\mathbb{Z}$.)

(c)$\Longrightarrow$ (d). This is a consequence of the second statement of 2.5.

(d)$\Longrightarrow$ (a). This is trivial. $\square$

So, if you find a cell chain $0 \subset J_1 \subset \cdots \subset J_1 = A$ of a cellular algebra A is not a heredity chain, that is, $J_l^2 \subseteq J_{l-1}$ for some l, then this algebra is of infinite global dimension.

Remark. Theorem 2.5 may be wrong outside the class of cellular algebras. Let us look at the following example:



$$\beta^2 = \beta\alpha = \alpha\alpha' = 0$$
$$\beta\alpha = \alpha'\alpha = \beta\alpha = 0$$

The Cartan matrix of this algebra is $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, its determinant is 1, and gl.dim$(A) = \infty$.

The following is a special case:

Proposition 3.4 *Let k be a field and A a cellular k–algebra (with respect to an involution i). Then the algebra A is semisimple if and only if all eigenvalues of the Cartan matrix of A are rational numbers and the Cartan determinant is 1.*

Before we start with the proof of Proposition 3.4, let us first prove the following lemma in the linear algebra.

Lemma 3.5 *Let C be a positive definite (symmetric) matrix with non-negative integers as its entries. If det$(C) = 1$ and all eigenvalues of C are rational numbers, then C is in fact the identity matrix.*

Proof. Since C is a positive definite matrix with non-negative entries and $\det(C) = 1$, we know that all eigenvalues of C are positive integers and their product is 1. This means that all eigenvalues λ_i of C are equal to 1. Let $C = (c_{ij})$ be of order n. Then $\sum_i^n c_{ii} = \sum_i \lambda_i = n$. Hence $c_{ii} = 1$ for all i. Since C is positive definite, every principal submatrix is positive definite, too. In particular, for any pair i and j, the principal submatrix

$$\begin{pmatrix} 1 & c_{ij} \\ c_{ji} & 1 \end{pmatrix}$$

is positive definite. This yields that $1 - c_{ij}^2 > 0$ and $c_{ij} = 0$. Thus the Cartan matrix C of A is an identity matrix. ■

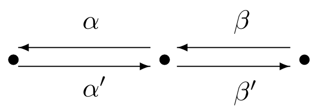
Proof of Proposition 3.4. It is obvious that (1) implies (2). The implication from (2) to (1) follows now from Lemma 4.13 immediately.■

Problem 1. Given a prime p, classify cellular algebras with the properties that the Cartan determinant equals p and that all eigenvalues of the Cartan matrix are integers.

Proposition 3.6 *Let $S(n)$ denote the set of partitions λ of n such that the product of any two parts of λ is a square . Let $M(n)$ be the set of all symmetric matrices C over the natural numbers such that the spectrum of C is $\{\mu_1 = n+1, \mu_2 = ... = \mu_m = 1\}$. Then the cardinalities of $S(n)$ and $M(n)$ are the same. In particular, if $s_1(n)$ stands for the number of partitions in $S(n)$ whose parts are coprime, then the number of the Cartan matrices (up to congruence) of indecomposable cellular algebras with the properties in Problem 1 is $\sum_{d|p-1} s_1((p-1)/d)$.*

For the details of the proof of the above proposition we refer to [44].

As an exercise, prove that the following algebra is not cellular with respect to any involution:



$$\alpha\alpha' = \beta'\beta; \quad \alpha'\alpha = \beta\beta' = 0.$$

3.3 Cohomology of cell modules and finiteness of global dimension

In this section we shall prove that the cohomology of cell modules can be used to determine the finite global dimension.

Let us mention the following result on cell modules

Proposition 3.7 *For any ideal J in a k–algebra A, the following two assertions are equivalent:*
(I) $J^2 = 0$,
(II) $Tor_2^A(A/J, A/J) \simeq J \otimes_A J$.

Proof. Applying $J \otimes_A -$ to the exact sequence $0 \to J \to A \to A/J \to 0$ produces the exact sequence

$$0 \to Tor_1^A(J, A/J) \to J \otimes_A J \to J \otimes_A A \to J \otimes_A A/J \to 0$$

Now, the last term equals J/J^2. Hence, if $J^2 = 0$, the first two terms must be isomorphic. Dimension shift then proves that (II) is valid. Conversely, again by dimension shift, condition (II) implies that the first two terms, hence also the last two terms are isomorphic, thus (I) is valid. ■

Now assume A has a nilpotent cell ideal, say J, which is isomorphic to $\Delta \otimes_k i(\Delta)$, where i is an involution on A and Δ is a left ideal inside J.

Since J is isomorphic to $\Delta \otimes_k i(\Delta)$ we get an isomorphism of k vector spaces $J \otimes_A J \simeq \Delta \otimes_k (i(\Delta) \otimes_A \Delta) \otimes_k i(\Delta)$, thus the Tor space in the previous proposition will be quite large, provided $i(\Delta) \otimes_A \Delta$ is not zero. But the latter space is the k–dual of $\mathrm{Hom}_A(\Delta, \mathrm{Hom}_k(i(\Delta), k))$ which is non–zero since it contains the map $\Delta \to \mathrm{top}(\Delta) \simeq \mathrm{socle}(\mathrm{Hom}_k(i(\Delta), k)) \to \mathrm{Hom}_k(i(\Delta), k)$.

Corollary 3.8 *Let J be a nilpotent cell ideal in the k–algebra A. Then the space $Tor_2^A(A/J, A/J)$ is not zero.*

The following is true for an idempotent cell ideal J.

Lemma 3.9 *Let J be a non-nilpotent cell ideal in an algebra A. Then gl.dim$(A) < \infty$ if and only if gl.dim$(A/J) < \infty$.*

Proof. Let $B = A/J$, and suppose $J \simeq Ae \otimes_k i(Ae)$ with e an idempotent (see Proposition 1.6). Then ${}_AJ$ and J_A is projective.

Let $m = \mathrm{gl.dim}(A) < \infty$. Let ${}_BX$ be an B-module. As an A-module, there is an exact sequence

$$0 \longrightarrow P_m \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0.$$

Since J_A is projective and $JX = 0$, we have the following exact commutative diagram:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & JP_m & \longrightarrow & \cdots & \longrightarrow & JP_0 & \longrightarrow & JX = 0 & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_m & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \| & & \\
0 & \longrightarrow & P_m/JP_m & \longrightarrow & \cdots & \longrightarrow & P_0/JP_0 & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

Clearly, the modules P_j/JP_j are projective A/J-module, thus $\mathrm{pd}_B X \leq m$.

Conversely, let $n = \mathrm{gl.dim}(B) < \infty$. For any ${}_AX$ we have an exact sequence $0 \longrightarrow JX \longrightarrow X \longrightarrow X/JX \longrightarrow 0$ with $JX \simeq \Delta \otimes_k eX$ thus $\mathrm{pd}({}_AX) \leq \mathrm{pd}({}_AX/JX)$. For the B-module X/JX we have an exact seqence

$$0 \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow X/JX \longrightarrow 0,$$

with Q_j projective B-modules. Note that the exact sequence $0 \longrightarrow J \longrightarrow A \longrightarrow B \longrightarrow 0$ shows that $\mathrm{pd}({}_AB) \leq 1$. Thus $\mathrm{pd}({}_AQ_j) \leq 1$ for all j. By induction we can show that $\mathrm{pd}_A(X/JX) \leq n+1$. ■

Recall that given an algebra A with an involution i, we may define a self-duality * on A-mod by putting $X^* = Di(M)$, where the right A-module structure of $i(X)$ is given by the involution on the left side, that is, $m \cdot a := i(a)m$ for $m \in X$ and $a \in A$.

Now let us prove the following characterization of quasi-heredity.

Theorem 3.10 *For a cellular algebra A the following are equivalent:*
(1) A has finite global dimension (that is, A is quasi-hereditary);
(2) $Ext^1_A(W(\lambda), (W(\mu))^*) = 0$ for all $\lambda, \mu \in \Lambda$.

Proof. If A has finite global dimension, then, by Theorem 3.3, any cell chain will have the length equal to the number of the simple modules. this implies that each cell ideal produced from the cell chain is non-nilpotent. thus A has a finite global dimension by Lemma 3.9.

Conversely, assume that (2) holds. For the given cell datum, we have a cell chain

$$0 \subset J_1 \subset J_2 \subset ... \subset J_m = A.$$

Note that the cell modules are obtained from the sections of this chain. If J_1 is a heredity ideal in A, then, by induction on the length of the cell chain, we can show that A is a quasi-hereditary algebra. Since J_1 is either a heredity ideal or $J_1^2 = 0$ by 1.6, the remaining case to be considered is the latter one, i.e. when $J_1^2 = 0$. We shall prove that this is impossible unless $J_1 = 0$. Let $J = J_1$ and $B = A/J$. Then $J \cong W \otimes_k i(W)$, where W is a left cell module and $i(W)$ is the right cell module. This means that J_A is a direct sum of copies of $i(W)$. The canonical exact sequence

$$0 \longrightarrow J \longrightarrow A \longrightarrow B \longrightarrow 0$$

shows that $J \otimes_A J \cong \mathrm{Tor}_2^A(B, B) \cong \mathrm{Tor}_1^A(\oplus i(W), B)$ by 3.7. From the definition of the cell chain, we may assume that $J_j/J_{j-1} \cong W(j) \otimes_k i(W(j))$, where $W(j)$ is the cell module for all j and $W = W(1)$. Now it follows from the canonical isomorphism $\mathrm{DExt}^j_A(X, Y) \cong \mathrm{Tor}_j^A(DY, X)$ that $\mathrm{Tor}_j^A(i(W(s)), W(t)) \cong \mathrm{Ext}^j_A(W(t), Di(W(s))) = 0$ for $j = 1$ and all s, t. Now we apply $i(W) \otimes_A -$ to the exact sequences

$$0 \longrightarrow J_j/J \longrightarrow J_{j+1}/J \longrightarrow W(j+1) \otimes_k i(W(j+1)) \longrightarrow 0,$$

with $j = 2, 3, ..., m-1$, and we get that $\mathrm{Tor}_1^A(i(W), B) = 0$. Thus $J \otimes_A J = 0$. However, Corollary 3.8 says that if J is not zero, then $J \otimes_A J$ is never zero. Hence we must have J to be zero , and the proof is finished.

Now let us consider the second cohomology groups of cell modules. Comparing with the homological definition of quasi-hereditary algebras, the following question arises naturally.

Question. Let A be a cellular algebra. Are the following statements equivalent:
(1) A is quasi-hereditary;
(2) $\mathrm{Ext}^1_A(W(\Lambda), W(\Lambda)^*) = 0$;
(3) $\mathrm{Ext}^2_A(W(\Lambda), W(\Lambda)^*) = 0$.

Our answer to this question is the following theorem.

Theorem 3.11 *Let A be a cellular algebra with cell modules $W(\lambda)$. Then A is quasi-hereditary if and only if $Ext^2_A(W(\Lambda), W(\Lambda)^*) = 0$.*

Proof. The " only if " part can be proved similarly as we did in theorem 3.10. We need only to show the " if " part. Let

$$0 \subset J_1 \subset J_2 \subset ... \subset J_m = A$$

be a cell chain which produces the cell modules $W(\Lambda)$. It follows from $\mathrm{Ext}^2_A(W(\Lambda), W(\Lambda)^*) = 0$ that $\mathrm{Ext}^1_A(J_j, W(\Lambda)^*) = \mathrm{Ext}^2_A(A/J_j, W(\Lambda)^*) = 0$ since A/J_j has a $W(\Lambda)$-filtration. Now we show that J_1 is a heredity ideal. If this is done, then we can use induction to get the desired statement.

Since a cell ideal J is either a heredity ideal or $J^2 = 0$, what we have to do is just to exclude the case $J_1^2 = 0$. Now the proof is similar to that of 3.10. The condition that $\mathrm{Ext}^1_A(J_j, W(\Lambda)^*) = 0$ can be interpreted as $\mathrm{Tor}^A_1(i(W(\Lambda)), J_j) = 0$ by the canonical isomorphism $\mathrm{DExt}^j_A(X, Y) \cong \mathrm{Tor}^A_j(DY, X)$. Since as a right A-module A/J_1 has an $i(W(\Lambda))$-filtration, we know that $\mathrm{Tor}^A_1(A/J_1, J_1) = 0$. Suppose that J_1 is non-zero with $J_1^2 = 0$. Then we have

$$J_1 \otimes_A J_1 \cong \mathrm{Tor}^A_2(A/J_1, A/J_1) \cong \mathrm{Tor}^A_1(A/J_1, J_1) = 0.$$

This implies that J_1 must be zero, a contradiction. Thus J_1 must be a heredity ideal in A. This finishes the proof.

Some improvement of Theorem 3.10 can be found in [5]. Let us mention the following question.

Remark. If there is a set of cell modules $\{W(\lambda) \mid \lambda \in \Lambda\}$ such that $\mathrm{Ext}^j_A(W(\lambda), W(\mu)^*) = 0$ for all $\lambda, \mu \in \Lambda$ and for some fixed $j \geq 3$, one may use the cell chain to show that $\mathrm{Ext}^{j-1}_A(W(\lambda), W(\mu)^*) = 0$ for all $\lambda, \mu \in \Lambda$. In this way, we see that the above result is true for higher cohomologies.

4 Applications to diagram algebras in mathematics and physics

Since the introduction of the cellular algebras, many classes for algebras from mathematics and physics are found to be cellular. In this section we introduce a few well-known classes of algebras appearing in mathematics and physics: The Temperley-Lieb algebras, Brauer algebras and partition algebras as well as Birman-Wenzl algebras. The cellularity of these algebras is given in details only for partition algebras. The other algebras can be proved by a similar idea.

Let us first give a short list of cellular algebras.

- The group algebras of symmetric groups, or dihedral groups;
- The Hecke algebra of type A and B;
- The Ariki-Koike Hecke algebra of type A;
- The Birman-Wenzl algebras;
- The q-Schur algebras of type A;
- The cyclotomic q-Schur algebras
- The Temperley-Lieb algebras;
- The cyclotomic Temperley-Lieb algebras;
- The Brauer algebras
- The cyclotomic Brauer algebras;
- The partition algebras;
- The G-vertex coloured algebras
- The blob algebras;
- The Jones algebras;
- The block algebras of the category $\mathcal{O}$ in Lie algebras.

4.1 One strategy for proving cellularity

To prove an algebra is cellular, one canonical way is to use the definitions. The following lemma provides another possibility to verify the cellularity of a given algebra, and describes also the structure of a general cellular algebra.

Lemma 4.1 *Let A be an algebra with an involution i. Suppose there is a decomposition*

$$A = \bigoplus_{j=1}^{m} V_j \otimes_k V_j \otimes_k B_j \quad \textit{(direct sum of vector space)}$$

where V_j is a vector space and B_j is a cellular algebra with respect to an involution σ_j and a cell chain $J_1^{(j)} \subset \cdots \subset J_{s_j}^{(j)} = B_j$ for each j. Define $J_t = \bigoplus_{j=1}^{t} V_j \otimes_k V_j \otimes_k B_j$. Assume that the restriction of i on $V_j \otimes_k V_j \otimes_k B_j$ is given by $w \otimes v \otimes b \longmapsto v \otimes w \otimes \sigma_j(b)$. If for each j there is a bilinear form $\phi_j : V_j \otimes_k V_j \longrightarrow B_j$ such that $\sigma_j(\phi_j(w,v)) = \phi_j(v,w)$ for all $w, v \in V_j$ and that the multiplication of two elements in $V_j \otimes V_j \otimes B_j$ is governed by ϕ_j modulo J_{j-1}, that is, for $x, y, u, v \in V_j$ and $b, c \in B_j$, we have

$$(x \otimes y \otimes b)(u \otimes v \otimes c) = x \otimes v \otimes b\phi_j(y,u)c$$

modulo the ideal J_{j-1}, and if $V_j \otimes V_j \otimes J_l^{(j)} + J_{j-1}$ is an ideal in A for all l and j, then A is a cellular algebra.

Proof. A direct proof of this lemma reads as follows. Since

$$J_1^{(j)} \subset \cdots \subset J_{s_j}^{(j)} = B_j, \quad j = 1, \cdots, m$$

is a cell chain for the given cellular algebras B_j, we can check that the following chain of ideals in A satisfies all conditions in Definition 1.2:

$$\begin{array}{ll} & V_1 \otimes V_1 \otimes J_1^{(1)} \subset \cdots \subset V_1 \otimes V_1 \otimes J_{s_1}^{(1)} \subset V_1 \otimes V_1 \otimes B_1 \oplus V_2 \otimes V_2 \otimes J_1^{(2)} \\ \subset & V_1 \otimes V_1 \otimes B_1 \oplus V_2 \otimes V_2 \otimes J_2^{(2)} \subset \cdots \subset V_1 \otimes V_1 \otimes B_1 \oplus V_2 \otimes V_2 \otimes B_2 \\ \subset & \cdots \subset \bigoplus_{j=1}^{m-1} V_j \otimes V_j \otimes B_j \oplus V_m \otimes V_m \otimes J_1^{(m)} \subset \cdots \\ \subset & \bigoplus_{j=1}^{m-1} V_j \otimes V_j \otimes B_j \oplus V_m \otimes V_m \otimes J_{s_m}^{(m)} = A. \end{array}$$

Now we take a fixed non–zero element $v_j \in V_j$ and suppose that $\alpha : J_t^{(j)} \longrightarrow \Delta_t^{(j)} \otimes i(\Delta_t^{(j)})$ is the bimodule isomorphism in the definition of the cell ideal $J_t^{(j)}$. Define

$$\beta : V_j \otimes V_j \otimes J_t^{(j)} \longrightarrow (V_j \otimes v_j \otimes \Delta_t^{(j)}) \otimes (v_j \otimes V_j \otimes \otimes i(\Delta_t^{(j)}))$$

$$u \otimes v \otimes x \longmapsto \sum_l (u \otimes v_j \otimes x_l) \otimes (v_j \otimes v \otimes y_l),$$

where $u, v \in V_j, x \in J_t^{(j)}$ and $\alpha(x) = \sum_l x_l \otimes y_l$. Then one can verify that β makes the corresponding diagram in the definition of cell ideals commutative. Hence $V_j \otimes v_j \otimes \Delta_t^{(j)}$ is a standard module for A, and $V_j \otimes V_j \otimes J_t^{(j)}$ is a cell ideal in the corresponding quotient of A. Thus A is a cellular algebra.

4.2 Temperley-Lieb algebras

The Temperley-Lieb algebras were first introduced in 1971 in the paper [35] where they were used to study the single bond transfer matrices for the Ising model in statistic mechanics. Later they were independently found by Jones when he characterized the algebras arising from the tower construction of semisimple algebras in the study of subfactors in mathematics. Their relationship with knot theory comes from their role in the definition of the Jones polynomial.

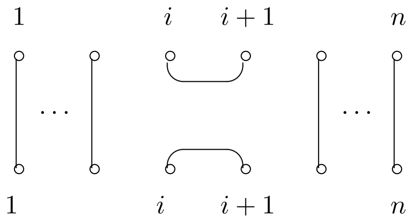
Suppose R is a commutative ring with 1, and $\delta \in R$. Let n be a natural number.

Definition 4.2 *The* **Temperley-Lieb algebra** $TL_n(\delta)$ *(or TL_n for simplicity) is the associative algebra over R with generators* 1 (the identity), $e_1, \ldots, e_{n-1}$ *subject to the following conditions:*

$$\begin{aligned}
&(1) \quad e_ie_je_i = e_i \ \text{ if } \ |j-i| = 1,\\
&(2) \quad e_ie_j = e_je_i \ \text{ if } \ |j-i| > 1,\\
&(3) \quad e_i^2 = \delta e_i \ \text{ for } \ 1 \leqslant i \leqslant n-1,
\end{aligned}$$

It was shown that $TL_n(\delta)$ can be described by planar diagrams,or TL-diagrams,that is, a TL-diagram D is a diagram with $2n$ vertices $\{1, 2, ..., 2n\}$ and n edges such that each vertices belongs exactly to one edge and that there are not any two edges $\{i < j\}$ and $\{k < l\}$ with $i < k < j < l$.

We may represent a TL-diagram D in a rectangle of the plane, where there are n numbers $\{1, 2, .., n\}$ on the top row from left to right, and there are another n numbers $\{1, 2, ..., n\}$ on the bottom row again from left to right; and if two numbers belong to the same edge we draw a line between the vertices, the planar condition means that inside the rectangle there are not any edges crossing each other. For example, e_i can be presented as follows:



The multiplication of two such diagrams D_1 and D_2 is a diagram obtained by concatenation of the two diagrams and dropping all close cycles, together with a coefficient $\delta^{n(D_1,D_2)}$, where $n(D_1, D_2)$ recalls the number of closed cycles which were dropped.

In [12] Temperley-Lieb algebras are proved to be cellular. As a consequence of Theorem 3.3, we can show the following extensions of results of Westbury [38] and of Graham and Lehrer [12].

Proposition 4.3 *Let $TL_n(\delta)$ be a Temperley–Lieb algebra over a field k. Then $T_n(\delta)$ is quasi–hereditary if and only if $\delta \neq 0$ or n odd.*

In fact, if we allow edges to cross each other then we get algebras which are called Brauer algebras.

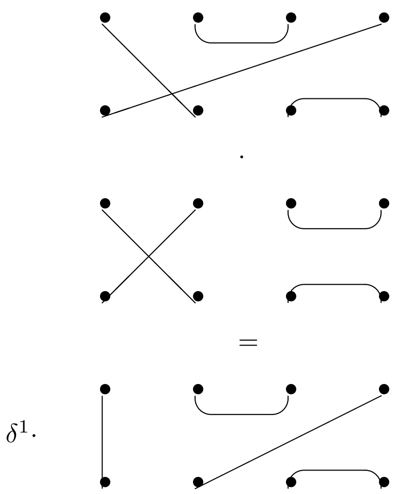
4.3 Brauer algebras

Let k be any field and n a natural number. Let V be the vector space k^n on which the group $GL_n(k)$ acts naturally, say on the left. Then $Gl_n(k)$ also acts (diagonally) on the r–fold tensor product $V^{\otimes r}$ for any natural number r. On this space, also the symmetric group Σ_r acts on the right, by place permutations. The two actions centralise each other. In particular, the endomorphism ring $\text{End}_{kGl_n(k)}(V^{\otimes r})$ is a quotient of the group algebra $k\Sigma_r$. This setup is called Schur–Weyl duality. Richard Brauer's starting point for defining 'Brauer algebras' was the following question: Which algebra shall replace $k\Sigma_r$ in this setup if we replace $GL_n(k)$ by either its orthogonal or its symplectic subgroup (in the latter case, of course, n has to be even)? He defined such an algebra by generators and relations, and he also gave an equivalent definition via diagrams. This definition is a special case of the following one, where the parameter δ has to be chosen as a positive or negative integer (for orthogonal or symplectic groups, respectively).

Definition 4.4 *Fix a commutative noetherian domain k, an element $\delta \in k$ and a natural number r. Then the* **Brauer algebra** $\mathbf{B}_k(r, \delta)$ *is a k–vector space having a basis consisting of diagrams of the following form: a diagram contains $2n$ vertices, n of them called 'top vertices' and the other n called 'bottom vertices' such that the set of vertices is written as a disjoint union of n*

subsets each of them having two elements; these subsets are called 'edges'. Two diagrams x and y are multiplied by concatenation, that is, the bottom vertices of x are identified with the top vertices of y, thus giving rise to edges from the top vertices of x to the bottom vertices of y, hence defining a diagram z. Then $x \cdot y$ is defined to be $\delta^{m(x,y)}z$ where $m(x,y)$ counts those connected components of the concatenation of x and y which do not appear in z, that is, which neither contain a top vertex of x nor a bottom vertex of y.

Let us illustrate this definition by an example. We multiply two elements in $B_k(4,\delta)$:



In the literature, the Brauer algebra sometimes is called Brauer centraliser algebra, a term which we will not use, since it is slightly misleading. In fact, in Brauer's original setup, the endomorphism algebra of $V^{\otimes r}$ in general is just a quotient of the Brauer algebra.

Brauer algebra $B_n(\delta)$ can be described by generators and relations:
Generartors: $1(identity), e_1, \cdots, e_{n-1}; \sigma_1, \cdots, \sigma_{n-1}$.
Relations:

$$\begin{aligned}
&(TL1) \quad e_ie_je_i = e_i \;\; if \;\; |j-i| = 1,\\
&(TL2) \quad e_ie_j = e_je_i \;\; if \;\; |j-i| > 1,\\
&(TL3) \quad e_i^2 = \delta e_i \;\; for \;\; 1 \leqslant i \leqslant n-1,\\
&(B1) \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\\
&(B2) \quad \sigma_i\sigma_j = \sigma_j\sigma_i \;\; if \;\; |j-i| > 1,\\
&(S) \quad \sigma_i^2 = 1 \;\; for \; all \;\; i,\\
&(BT1) \quad \sigma_ie_i = e_i\sigma_i = e_i \;\; for \; all \;\; i,\\
&(BT2) \quad \sigma_ie_j = e_j\sigma_i \;\; if \;\; |j-i| > 1,\\
&(BT3) \quad \sigma_ie_{i+1}e_i = \sigma_{i+1}e_i\\
&(BT4) \quad e_{i+1}e_i\sigma_{i+1} = e_{i+1}\sigma_i.
\end{aligned}$$

The cellularity of Brauer algebras was proved first by Graham and Lehrer in [12]. A basis free approach is given in [24]. Using the cellularity, we can prove the following

Theorem 4.5 *Let k be any field, fix $\delta \in k$ and denote by $B(r,\delta)$ the Brauer algebra on $2r$ vertices and with parameter δ.*
Then $B(r,\delta)$ is quasi–hereditary if and only if
(1) δ is not zero or r is odd; and
(2) the characteristic of k is either zero or strictly bigger than r.

This extends previous results by Graham and Lehrer; they proved the 'if'–part in [12], 4.16. and 4.17.

4.4 Partition algebras

In this section we recall the definition of partition algebras and some basic facts from [28] which are needed in this paper.

Let M be a finite set. We denote by E_M the set of all equivalent relations on, or equivalently all partitions of the set M:

$$\begin{aligned} E_M := \quad & \{\rho = \big((M_1)(M_2)\cdots(M_i)\cdots\big) \mid \emptyset \neq M_i \subset M, \cup_i M_i = M, \\ & M_i \cap M_j = \emptyset\,(i \neq j)\} \end{aligned}$$

For example, we take $M = \{1, 2, 3\}$, then

$$E_M = \{(123), (1)(23), (12)(3), (13)(2), (1)(2)(3)\}.$$

If $\rho = ((M_1)\cdots(M_s))$, we define $|\rho|$ to be the number of the equivalence classes of ρ. If we call each M_j in ρ a part of ρ, then $|\rho|$ is the number of the parts of $|\rho|$.

Note that there is a partial order on E_M : if ρ_1 and ρ_2 are two elements in E_M, we say by definition that ρ_1 is smaller than or equal to ρ_2 if and only if each part of ρ_1 is a subset of a part of ρ_2. With this partial order, E_M is a lattice.

if $\mu \in E_M$ and $\nu \in E_N$, then we define $\mu \cdot \nu \in E_{M\cup N}$ is the smallest ρ in $E_{M\cup N}$ such that $\mu \cup \nu \subset \rho$.

We are mainly interested in the case

$$M = \{1, 2, \cdots, n, 1', 2', \cdots, n'\}.$$

Note that E_M depends only upon the cardinality $|M|$ of M. So we sometimes write E_{2n} for E_M. To formulate our definitions, we denote by M' the set $\{1', 2', \cdots, n', 1'', 2'', \cdots, n''\}$.

Definition 4.6 *Let*

$$f : E_M \times E_M \longrightarrow \mathbf{Z}$$

be such that $f(\mu, \nu)$ is the number of parts of $\mu\cdot\nu \in E_{M\cup M'}$ (note that $|M \cup M'| = 3n$) containing exclusively elements with a single prime.

For example, in case $n = 3$, $((123)(1'2')(3')) \cdot ((1')(2'3')(1'')(2'')(3'')) = ((123)(1'2'3')(1'')(2'')(3''))$ and $f(\mu, \nu) = 1$.

Definition 4.7 *Let*

$$C : E_M \times E_M \longrightarrow E_M$$

be such that $C(\mu, \nu)$ is obtained by deleting all single primed elements of $\mu \cdot \nu$ (discarding the $f(\mu, \nu)$ empty brackets so produced), and replacing all double primed elements with single primed ones.

The partition algebra $P_n(q)$ is defined in the following way.

Definition 4.8 *(see [28]) Let k be a field and $q \in k$. We define a product on E_M:*

$$E_M \times E_M \longrightarrow E_M$$

$$(\mu, \nu) \longmapsto \mu\nu = q^{f(\mu,\nu)} C(\mu, \nu)$$

This product is associative. Let $P_n(q)$ denote the vector space over k with the basis E_M. Then, by linear extention of the product on E_M, the vector space $P_n(q)$ becomes a finite dimensional algebra over k with the above product. We call this algebra $P_n(q)$ the **partition algebra**.

If we take $B_M = \{\rho \in E_M \mid \text{ each part of } \rho \text{ has exactly two elements of } M\}$ and define the product of two elements in B_M in the same way as in 4.8, then the subspace $D_n(q)$ of $P_n(q)$ with the basis B_M becomes a finite dimensional algebra. This is just the Brauer algebra. Similarly, if we take $P_M = \{\rho \in B_M \mid \rho \text{ is planar}\}$, then we get the Temperley–Lieb algebra $TL_n(q)$ with the basis P_M and the product 4.8. The word "planar" means that if we think of the basis elements diagrammatically, then there are no edges crossing each other in the diagram (see [12]).

For an element $\mu \in P_n(q)$, we define $\#^P(\mu)$ to be the maximal number of distinct parts of μ containing both primed and unprimed elements of M, over the E_M basis elements with non-zero coefficients in μ.

The following fact is true in $P_n(q)$.

Lemma 4.9 *For $\mu, \nu \in P_n(q)$, we have*

$$\#^P(\mu\nu) \leq min\{\#^P(\mu), \#^P(\nu)\}.$$

Given a partition $\rho \in E_M$, if we interchange the primed element j' with unprimed element j, then we get a new partition of Ml let us denote this new partition by $i(\rho)$. Then i extends by linearity to $P_n(q)$.

For example, if $n = 4$ and $\rho = \big((12)(341'2')(3'4')\big)$ then $i(\rho) = \big((1'2')(3'4'12)(34)\big)$.

Lemma 4.10 *The linear map i is an anti-automorphism of $P_n(q)$ with $i^2 = id$.*

Proof. Clearly, the map i is k–linear with $i^2 = id$. It remaind to check that $i(\mu\nu) = i(\nu)i(\mu)$ holds true for all $\mu, \nu \in E_M$. However, this follows immediately from the graphical realization of the product in $P_n(q)$ (see [29]), or from a verification of the above equation for the products of two generators of $P_n(q)$ displayed in [28].

Theorem 4.11 *The partition algebra $P_n(q)$ is a cellular algebra.*

The proof of this theorem is based on a series of lemmas. We keep the notation introduced in the previous sections. Recall that E_n denotes the set of all partition of $\{1, 2, \cdots, n\}$.

For each $l \in \{0, 1, \cdots, n\}$, we define a vector space V_l which has as a basis the set

$$\begin{aligned}\mathcal{S}_l = \quad & \{(\rho, S) \mid \rho \in E_n, |\rho| \geq l, \quad S \text{ is a subset} \\ & \text{of the set of all parts of } \rho \quad \text{with} \quad |S| = l\}\end{aligned}$$

(Note that in [28] this set is denoted by $S_n(l)$).

If $\rho \in E_n$, we may write ρ in a standard way: Suppose $\rho = \big((M_1) \cdots (M_s)\big)$, we write each M_i in such a way that $M_i = (a_1^{(i)} a_2^{(i)} \cdots a_{t_i}^{(i)})$ with $a_1^{(i)} < a_2^{(i)} < \cdots < a_{t_i}^{(i)}$. If $a_1^{(1)} < a_1^{(2)} < \cdots < a_1^{(s)}$, then we say that ρ is written in standard form. It is clear that there is only one standard form for each ρ. We may also introduce an order on the set of all parts of ρ by saying that $M_j < M_k$ if and only if $a_1^{(j)} < a_1^{(k)}$.

If $N \subset M$ and $\rho \in E_M$, we denote by $r_N(\rho)$ the partition of $M \backslash N$ obtained from ρ by deleting all elements in N from the parts of ρ, and by $d_N(\rho)$ the set of parts of ρ which do not contain any element in N. Finally, we denote by Σ_r the symmetric group of all permutations on $\{1, 2, \cdots, r\}$ and by $k\Sigma_r$ the corresponding group algebra over the field k.

Lemma 4.12 *Each element $\rho \in E_M$ can be written uniquely as an element of $V_l \otimes V_l \otimes k\Sigma_l$ for a natural number $l \in \{0, 1, \cdots, n\}$.*

Proof. Take a partition $\rho \in E_M$, we define $x := r_{\{1',2',\cdots,n'\}}(\rho) \in E_n$. If we identify the set $\{1', 2', \cdots, n'\}$ with $\{1, 2, \cdots, n\}$ by sending j' to j, then $y := r_{\{1,2,\cdots,n\}}(\rho)$ lies in E_n. Let S_ρ be the set of parts of ρ containing both primed and unprimed elements. Then $|S_\rho| = \#^P(\rho)$. Now let S be the set of those parts of x which are obtained from elements of S_ρ by deleting the numbers contained in $\{1', 2', \cdots, n'\}$. Similarly, we get a subset T of the set of all parts of y. It is clear that both S and T contain $l(= |S_\rho|)$ elements. Now if we write $S = \{S_1, \cdots, S_l\}$ and $T = \{T_1, T_2, \cdots, T_l\}$ such that $S_1 < S_2 < \cdots < S_l$ and $T_1 < T_2 < \cdots < T_l$, we may define a permutation $b \in \Sigma_l$ by sending j to k if there is a part $Y \in S_\rho$ containing both S_j and T'_k, where $T'_k = \{a' \mid a \in T_k\}$. Since x, y and b are uniquely determined by ρ in a standard form, we can associate with the given ρ a unique element

$$(x, S) \otimes (y, T) \otimes b.$$

Obviously, (x, S) and (y, T) belong to V_l and $b \in \Sigma_l$. Conversely, each element $(x, S) \otimes (y, T) \otimes b$ with $(x, S), (y, T) \in \mathcal{S}_l$ and $b \in \Sigma_l$ corresponds to a unique partition $\rho \in E_M$. This finishes the proof of the lemma.

For example, for $\rho = \big((1232'3')(41')(54')(5')\big)$, we have $x = \big((123)(4)(5)\big), y = ((1)(23)(4)(5)), S = \{(123), (4), (5)\}, T = \{(1), (23), (4)\}$ and $b = (12) \in \Sigma_3$.

Now we want to define a bilinear form $\phi_l : V_l \otimes V_l \longrightarrow k\Sigma_l$. Let (ρ, S) be in $\mathcal{S}_l$. We may assume that $S = \{S_1, \cdots, S_l\}$ with $S_1 < S_2 < \cdots < S_l$. We define

$$\phi_l : V_l \otimes V_l \longrightarrow k\Sigma_l$$

by sending $(x, S) \otimes (y, T)$ to zero if there are i and j with $1 \le i, j \le l$ and $i \ne j$ and there is a part of $x \cdot y \in E_n$ containing both S_i and S_j, or dually there are i and j with $1 \le i, j \le l$ and $i \ne j$ and there is a part of $x \cdot y \in E_n$ containing both T_i and T_j, or there is a number $1 \le i \le l$ and a part of $x \cdot y$ containing only S_i, or dually there is a number $1 \le i \le l$ and a part of $x \cdot y$ containing only T_i, and to $q^{|d_{S\vee T}(x\cdot y)|} b \in k\Sigma_l$ in other case, where $S \vee T$ stands for the union of all parts of S and T, and b is defined as follows: Since for each i there is a unique part of $x \cdot y$ containing both S_i and a unique part T_j, we define b to be the permutation taking i to j. Thus $b \in \Sigma_l$. We denote this b by $p_l(x, S; y, T)$. If we extend ϕ_l by linearity to the whole space $V_l \otimes V_l$, then we have the following lemma.

Lemma 4.13 *The map $\phi_l : V_l \otimes_k V_l \longrightarrow k\Sigma_l$ is a bilinear form.*

The multiplication of two elements in $P_n(q)$ is given by the following two lemmas.

Lemma 4.14 *Let μ, ν be in E_M. If $\mu = (u, R) \otimes (x, S) \otimes b_1 \in V_l \otimes V_l \otimes k\Sigma_l$ and $\nu = (y, T) \otimes (v, Q) \otimes b_2 \in V_l \otimes V_l \otimes k\Sigma_l$, then*

$$\mu\nu = (u, R) \otimes (v, Q) \otimes b_1 \phi_l((x, S), (y, T)) b_2$$

modulo $J_{l-1} = \bigoplus_{j=0}^{l-1} V_j \otimes V_j \otimes k\Sigma_j$.

Proof. By the definition of the multiplication in $P_n(q)$ and the definition of $d_{S\vee T}(x \cdot y)$, we know that $f(\mu, \nu) = |d_{S\vee T}(x \cdot y)|$. Hence it is sufficient to show that the element $(u, R) \otimes (v, Q) \otimes b_1 \phi_l((x, S), (y, T)) b_2$ just presents the element $q^{f(\mu,\nu)} C(\mu, \nu)$ in $P_n(q)$ modulo J_{l-1}.

If $\phi_l((x, S), (y, T)) = 0$, then, by the definiton of ϕ_l, we see that $\#^P(\mu\nu) < l$. This implies that $C(\mu, \nu) \in J_{l-1}$. Now assume that $\phi_l\big((x, S), (y, T)\big) = q^{|d_{S\vee T}(x\cdot y)|} b$, where b is defined as above. Now we have to show that $(u, R) \otimes (v, Q) \otimes b_1 b b_2$ presents the element $C(\mu, \nu)$. Indeed, by the definition of ϕ_l, we have obviously that $r_{\{1',2',\cdots,n'\}}(C(\mu, \nu)) = u \in E_n$ and that $r_{\{1,2,\cdots,n\}}(C(\mu, \nu)) = v \in E_n$ if we identify j' with j for $1 \le j \le n$. Note that there is only l distinct parts of $x \cdot y$, sayiny $P_1, P_2, \cdots, P_l$, containing a single S_i and a single T_{ib}. Hence there is a part in $C(\mu, \nu)$ which contains both $R_{ib^{-1}}$ and S_i. Since T_i and Q_{ib_2} are contained in the same part of ν, we see finally that $R_{ib_1^{-1}}$ and Q_{ibb_2} are contained in the same part of $C(\mu, \nu)$. Hence $C(\mu, \nu)$ is presented by $(u, R) \otimes (v, Q) \otimes b_1 b b_2$. This finishes the proof.

Lemma 4.15 *Let l and m be two natural numbers with $l < m$. Take $\alpha = (u,R) \otimes (x,S) \otimes b \in V_m \otimes V_m \otimes k\Sigma_m$ with $b \in \Sigma_m$ and $\beta = (y,T) \otimes (v,Q) \otimes c \in V_l \otimes V_l \otimes k\Sigma_l$ with $c \in \Sigma_l$. If $\alpha\beta = q^{|d_{S\vee T}(x\cdot y)|}(w,F) \otimes (z,G) \otimes d$, then*

(1) if $|F| = l$, then $(z,G) = (v,Q), d = d'c$, and (w,F) and $d' \in \Sigma_l$ do not depend on c.

(2) if $|F| < l$, then for any $c_1 \in \Sigma_l$ there holds $\alpha\big((y,T) \otimes (v,Q) \otimes c_1\big) \in J_{l-1}$.

Proof. (1) If $|F| = l$, then $|G| = l$. Since G is always obtained from Q, we infer that (z,G) must be (v,Q). Hence d is also of the desired form. The other assertions follow immediately from the definition of the multiplication of two basis elements in $P_n(q)$.

(2) This is trivial since c and c_1 can be considered as two bijections from T to Q. If there is a part of $x \cdot y$ containing more than one elements of T, then we always have $\alpha\big((y,T)\otimes(v,Q)\otimes c_1\big) \in J_{l-1}$ for any $c_1 \in \Sigma_l$. The proof is finished.

There is, of course, a dual version of the above lemma, in which the case of $\beta\alpha$ is considered. By Lemma 4.12, we may identify E_M with $\bigcup_{l=0}^{n} \mathcal{S}_l$. Then we have the following fact.

Lemma 4.16 *$J_t := \sum_{j=0}^{t} V_j \otimes V_j \otimes k\Sigma_j$ is an ideal of $P_n(q)$.*

This follows from Lemma 4.9 and Lemma 4.14. The following lemma is a consequence of definitions and Lemma 4.12.

Lemma 4.17 *If $\mu = (x,S) \otimes (y,T) \otimes b$ with $(x,S),(y,T) \in \mathcal{S}_l$ and $b \in \Sigma_l$, then $i(\mu) = (y,T) \otimes (x,S) \otimes b^{-1}$.*

Note that the bilinear form ϕ_l is not symmetric, but we have the following fact.

Lemma 4.18 *Let $i : k\Sigma_l \longrightarrow k\Sigma_l$ be the involution on $k\Sigma_l$ defined by $\sigma \longmapsto \sigma^{-1}$ for all $\sigma \in \Sigma_l$. Then $i\phi_l(v_1,v_2) = \phi_l(v_2,v_1)$ for all $v_1, v_2 \in V_l$.*

Proof. We may assume that $v_1 = (x,S)$ and $v_2 = (y,T)$. If $\phi_l(v_1,v_2) = 0$, then it follows from the definition of ϕ_l and $x \cdot y = y \cdot x$ that $\phi_l(v_2,v_1) = 0$. Hence we assume now that $\phi_l(v_1,v_2) \neq 0$. In this case, if S_i and T_{ib} with $b = p_l(x,S;y,T)$ are contained in the same part of $x \cdot y$, then T_i and $S_{ib^{-1}}$ are contained also in the same part of $y \cdot x$. Thus $p_l(y,T;x,S) = b^{-1}$. This shows that $i\phi_l(v_1,v_2) = \phi_l(v_2,v_1)$. The proof is finished.

Now we are in the position to prove our main result.

Proof of the Theorem. Put $J_{-1} = 0, \Sigma_0 = \{1\}$ and $B_l = k\Sigma_l$. Then the partition algebra has a decomposition

$$P_n(q) = V_0 \otimes_k V_0 \otimes_k B_0 \oplus \cdots \oplus V_l \otimes_k V_l \otimes_k \otimes B_l \oplus \cdots \oplus V_n \otimes_k V_n \otimes_k B_n.$$

Note that B_r is a cellular algebra with respect to the involution $\sigma \longmapsto \sigma^{-1}$ for $\sigma \in \Sigma_r$ (see [12]). According to Lemma 4.14 and Lemma 4.15, the chain displayed in the proof of Lemma 4.1 is a chain of ideals in $P_n(q)$. Hence, by the lemmas in this section, the above decomposition satisfies all conditions in Lemma 4.1. Thus the algebra $P_n(q)$ is a cellular algebra.

For partition algebras we have

Theorem 4.19 *Let k be any field, fix $\delta \in k$ and denote by $P(r,\delta)$ the partition algebra on $2r$ vertices and with parameter δ.*

Then $P(r,\delta)$ is quasi–hereditary (or equivalently, of finite global dimension) if and only if δ is not zero and the characteristic of k is either zero or strictly bigger than r.

Martin [28] had shown this in case of characteristic zero and $\delta \neq 0$. In [40], the "if part" of 4.19 is proved.

5 Notes on cellularity of semigroup algebras

Recently, the cellularity of semigroups are investigated by J.East [9] and Wilox in [39], respectively. In this section we summarize some of the developments in this direction.

A semigroup S is called an inverse semigroup if each element $s \in S$ has a unique inverse s^{-1} such that $ss^{-1}s = s$ and $s^{-1}ss^{-1} = s^{-1}$. This is equivalent to saying that S is inverse if and only if $s \in sSs$, and idempotents of S commute. A semigroup is called regular if for each $x \in S$ there is a $y \in S$ with $x = xyx$.

Suppose that the group algebra of the maximal subgroup S is cellular, and that cell data of these group algebras are compatible.Then it is proved in [9] that the semigroup algebra of an inverse semigroup over a commutative ring is cellular. This result is extended by Wilcox to certain twisted semigroup algebras, namely the product in semigroup algebra is changed by a bilinear form with certain associative restrictions. The new product of two elements from the semigroup S is a scalar (giving by the linear form) of the original product of the two elements. The corresponding semigroup algebra is called the twisted semigroup algebra. In this way Jeast's result is extended.

We should note that partition algebras, Temperley-Lieb algebras and Brauer algebras listed at the beginning of this Chapter are twisted semigroup algebras.

Thus we can get cellular algebras from semigroups. The follwoing question seems to be of interest.

Question 2. Find a general machinery to construct cellular algebras directly from quiver and relations. For example, the dual extension of a directed algebra is always quasi-hereditary cellular [41].

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