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Integral Quadratic Forms and the Representation Type of an Algebra

(Lecture 1)

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INTEGRAL QUADRATIC FORMS AND THE REPRESENTATION TYPE OF AN ALGEBRA

LECTURE 1

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Plan.

Lecture 1: Examples of tame and wild algebras

Lecture 2: Geometric approach
- varieties of modules
- tame algebras and varieties

Lecture 3: The Tits (quadratic) form
- Tits form of representation finite and tame algebras
- Combinatorial criteria

Lecture 4: The Tits form and the structure of the Auslander-Reiten quiver of an algebra

k = algebraically closed field

A = finite associative dimensional k-algebra with 1.

We shall assume that A is basic, hence $A = kQ/I$ where kQ = path algebra associated to the quiver Q.

mod_A = category of finite-dimensional left A-modules.

> *Aim:* Study of mod_A:
>
> description of indecomposable A-modules.
>
> use of combinatorial invariants (the Tits quadratic form)

Motivation: Let Δ be a quiver without oriented cycles and consider the associated hereditary algebra $A = k\Delta$. We assume Δ is connected.

Let $\Delta_0 = \{1, \dots, n\}$ be the vertices of Δ and

$$M_\Delta = (m_{ij}) \text{ the } \textit{incidence matrix} \text{ of } \Delta,$$

$$m_{ij} = \begin{cases} 2, & \text{if } i = j \\ -\# \text{ edges between } i \text{ and } j, & \text{if } i \neq j \end{cases}$$

$V^+ = \{v \in V : v(i) \geq 0, \forall i\}$ positive cone

Lemma. $M_\Delta^{-1}(V^+) \cap \partial V^+ = \{0\}$.

Proof. Assume that $0 \neq y \in M_\Delta^{-1}(V^+) \cap \partial V^+$.

By the connectivity of Δ we find an edge i —— j such that $y(i) > 0$ and $y(j) = 0$. Then

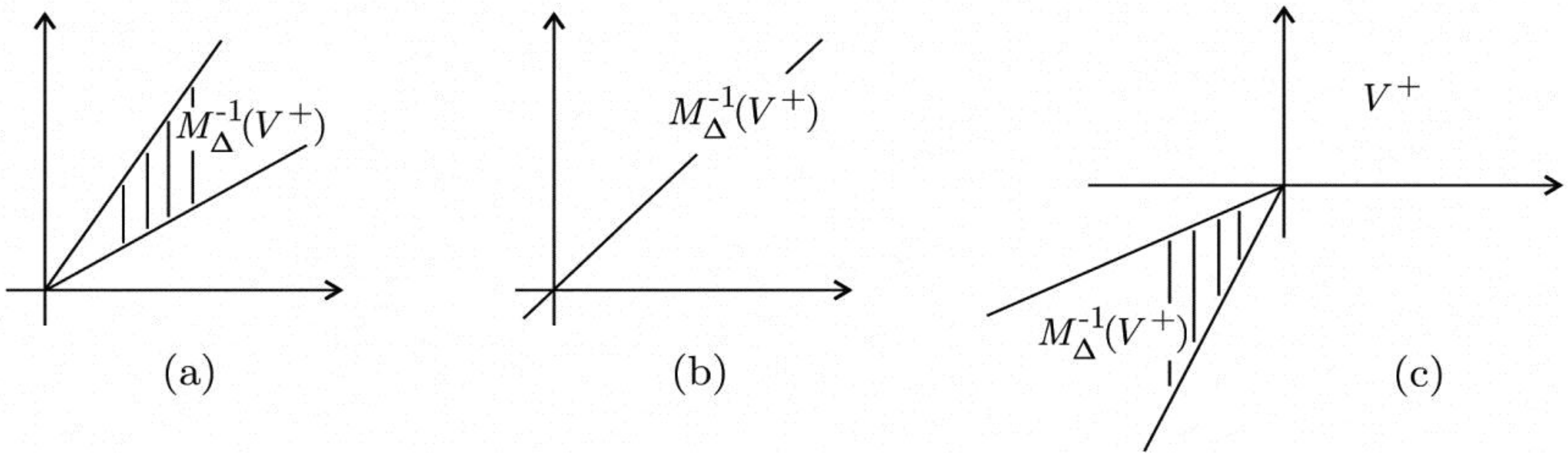
$$\begin{aligned} 0 \leq M(y)(j) &= \sum_k m_{jk} y(k) = m_{jj} y(j) + m_{ji} y(i) + \\ &\quad + \sum_{k \neq i,j} m_{jk} y(k) \leq m_{ji} y(i) < 0, \end{aligned}$$

a contradiction. □

Proposition. *The matrix M_Δ satisfies one and only one of the properties:*

(a) $M_\Delta^{-1}(V^+) \subset V^+$
(b) $M_\Delta^{-1}(V^+) = \mathbb{R}u$ *for some* $u \gg 0$. *In this cases* $M_\Delta(u) = 0$
(c) $M^{-1}(V^+) \cap V^+ = \{0\}$

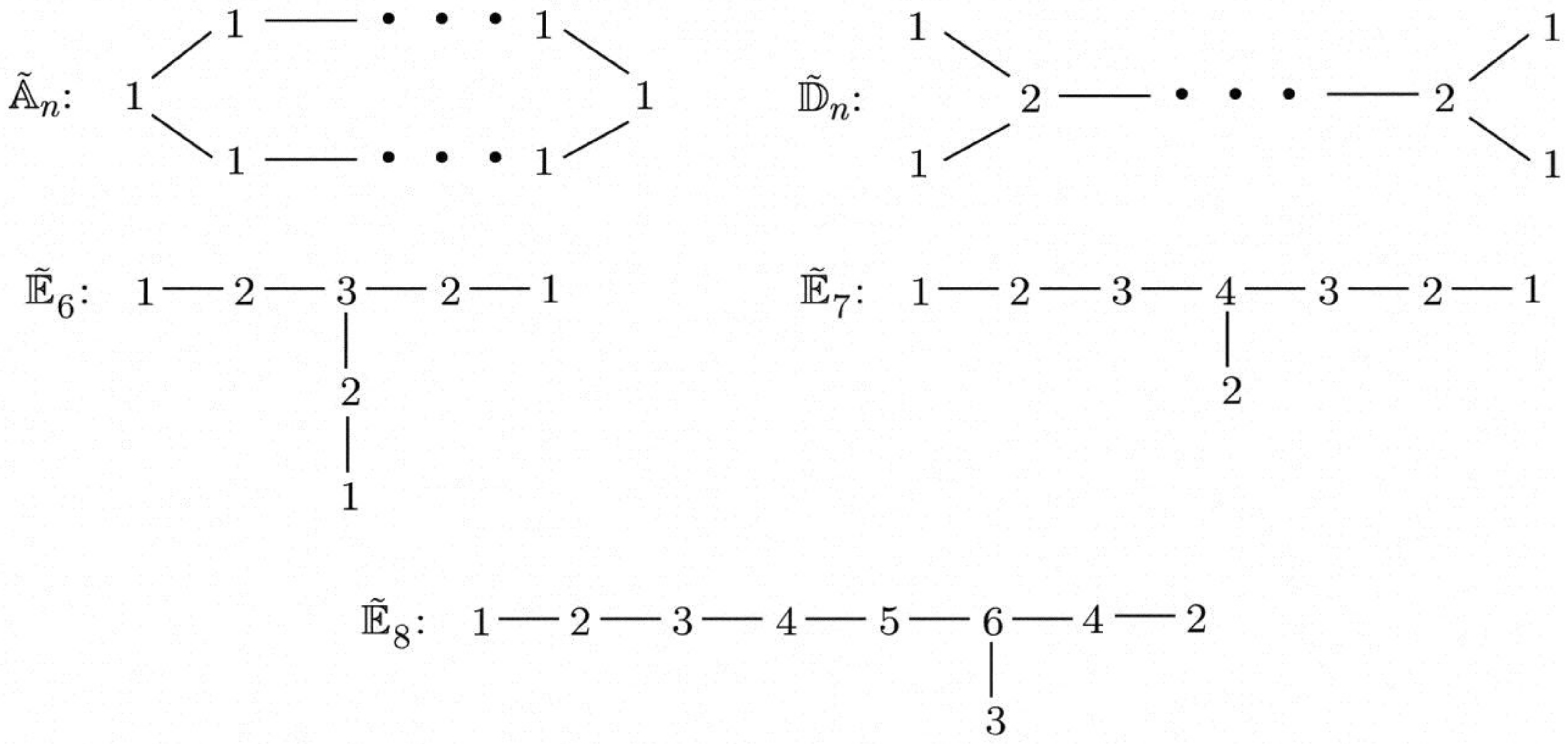
This can be illustrated for $n = 2$:



(a) (b) (c)

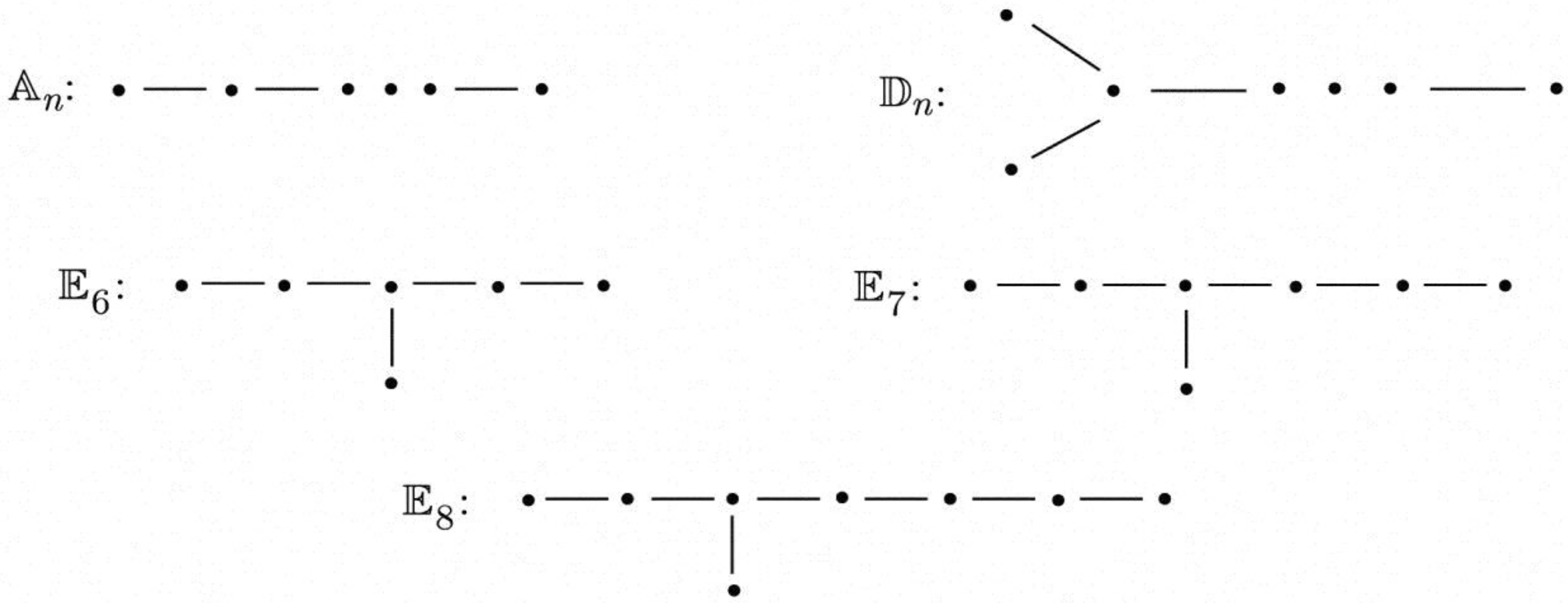
Three type of quives. *Classification*:

(1) Δ of type (b), $u \gg 0$ with $M_\Delta(u) = 0$. Then $|\Delta|$ is one of the following



(Euclidean diagrams).

(2) Δ of type (a) if and only if $|\Delta|$ does not any diagram of type (a). Hence $|\Delta|$ is one of the following (Dynkin diagrams).



(3) If Δ is of type (a) and $A = k\Delta$ the corresponding algebra, then there is a bijection between the isoclasses of indecomposable A-modules and the positive roots of the quadratic form $q_A(v) = vM_\Delta v^t$, given by

$$[X] \mapsto \mathbf{dim}\, X = (\dim_k X(i))_{i\in\Delta_0}.$$

1. Tame-wild dichotomy.

Local algebras.

(1) Observe that the algebra $A = k[x]/(x^n)$ admits only finitely many indecomposable modules, up to isomorphism. We say that A is *representation-finite*.

Indeed, a module $M \in \mathrm{mod}_A$ is a nilpotent matrix, hence M is equivalent to

$$J_{n_1} \oplus \cdots \oplus J_{n_s}$$

where J_i is the $i \times i$ matrix

$$\begin{bmatrix} 0 & & & \Large 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ \Large 0 & & 1 & 0 \end{bmatrix}$$

with $n_i \leq n$. If M is indecomposable, $M \cong J_s$, for some $s \leq n$.

(2) Consider the infinite-dimensional k-algebra $k[x]$.

Let $M \in \text{mod}_{k[x]}$, then M is a $n \times n$ matrix.

Let $\chi(x) = \det(xI_n - M)$ be the characteristic polynomial of M. Then M is equivalent to

$$J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_s}(\lambda_s)$$

where $\chi(x) = (x - \lambda_1)^{n_1} \dots (x - \lambda_s)^{n_s}$ is the decomposition of $\chi(x)$ in linear factors ($k = \bar{k}$!) and $J_{n_i}(\lambda_i)$ is the $n_i \times n_i$ Jordan block

$$J_{n_i}(\lambda_i) = \begin{bmatrix} \lambda_i & & & \text{\Large 0} \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ \text{\Large 0} & & 1 & \lambda_i \end{bmatrix}$$

Consider the $k[x] - k[x]$-bimodule given by the $n \times n$ matrix

$$J_n(x) = \begin{bmatrix} x & & & \text{\Large 0} \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ \text{\Large 0} & & 1 & x \end{bmatrix}$$

Let $S_\lambda = k[x]/(x - \lambda)$ be a (one-dimensional) simple $k[x]$-module. Then

$$J_n(x) \otimes_{k[x]} S_\lambda = J_n(\lambda).$$

Therefore, the indecomposable $k[x]$-modules of dimension n are isomorphic to modules in the image of the functor

$$J_n(x) \otimes_{k[x]} - : \text{mod}_{k[x]}(1) \to \text{mod}_{k[x]}.$$

(3) The free algebra $k\langle x,y\rangle$ has a 'problematic' behaviour.

Proposition. *Let B be any finitely generated k-algebra, then there exists a fully faithful functor $F\colon \mathrm{mod}_B \to \mathrm{mod}_{k\langle x,y\rangle}$.*

Proof. Let $b_1,\dots,b_s$ be a system of generators of B. Define the $k\langle x,y\rangle - B$-bimodule M as $M_B = B^{s+2}$ and the structure of left $k\langle x,y\rangle$-module given by the $(s+2)\times(s+2)$-matrices

$$
{}_xM = \begin{bmatrix} 0 & 1 & & & \text{\Large 0} \\ & 0 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ \text{\Large 0} & & & & 0 \end{bmatrix}
\qquad
{}_yM = \begin{bmatrix} 0 & & & & & \text{\Large 0} \\ 1 & 0 & & & & \\ b_1 & 1 & & & & \\ & \ddots & \ddots & & & \\ & & \ddots & 1 & 0 & \\ \text{\Large 0} & & & b_s & 1 & 0 \end{bmatrix}
$$

We set $F = M\otimes_B\colon \mathrm{mod}_B \to \mathrm{mod}_{k\langle x,y\rangle}$.

Exercise: check that F is full and faithful. □

This means that the representation theory of $k\langle x,y\rangle$ is as complicated as the representation theory of any other algebra.

We say that an algebra A is *wild* if there is a functor $F\colon \mathrm{mod}_{k\langle x,y\rangle} \to \mathrm{mod}_A$ which preserves indecomposable modules and iso-classes. We shall say that the functor F *insets* indecomposable modules.

(4) *Example*: Let p be a prime number ≥ 3. Assume k has characteristic p. The group algebra $A = k[\mathbb{Z}_p \times \mathbb{Z}_p]$ is wild.

Proof. Let $\varphi\colon k[u,v] \to A$, $x \mapsto g-1$, $y \mapsto h-1$, where $\mathbb{Z}_p \times \mathbb{Z}_p = \langle g \rangle \times \langle h \rangle$. Then $A \cong k[u,v]/\ker\varphi = k[u,v]/(u^p, v^p)$.

Moreover $k[u,v]/(u^p,v^p) \twoheadrightarrow k[u,v]/(u,v)^3 = k[u,v]/(u^3,v^3,uv^2,vu^2) =: B$. It is enough to show that B is wild.

Consider the B–$k\langle x,y\rangle$-bimodule M defined as $M_{k\langle x,y\rangle} = k\langle x,y\rangle^4$ and the structure as B-module defined by the matrices

$$ {}_uM = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & x & y & 0 \end{bmatrix} \qquad {}_vM = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & x & 0 \end{bmatrix} $$

Exercise: check that ${}_BM$ is well defined and

$$ M \otimes_{k\langle x,y\rangle} -\colon \operatorname{mod}_{k\langle x,y\rangle} \to \operatorname{mod}_B $$

insets indecomposable modules. □

Return to hereditary algebras.

(1) The indecomposable modules over the quiver algebra A:

$$\bullet \rightrightarrows \bullet$$

were classified by Weierstrass and Kronecker:

$$k^n \underset{\begin{bmatrix} 0 & | & \\ \vdots & | & I_n \\ 0 & | & \end{bmatrix}}{\overset{\begin{bmatrix} & | & 0 \\ I_n & | & \vdots \\ & | & 0 \end{bmatrix}}{\rightrightarrows}} k^{n+1} \qquad\qquad k^{n+1} \underset{\begin{bmatrix} 0 & \cdots & 0 \\ - & - & - \\ & I_n & \end{bmatrix}}{\overset{\begin{bmatrix} & I_n & \\ - & - & - \\ 0 & \cdots & 0 \end{bmatrix}}{\rightrightarrows}} k^n$$

(preprojective representation) (preinjective representation)

$$R_n(\lambda): \; k^n \underset{J_n(\lambda)}{\overset{I_n}{\rightrightarrows}} k^n \qquad\qquad R_n(\infty): \; k^n \underset{I_n}{\overset{J_n(0)}{\rightrightarrows}} k^n$$

(regular representations)

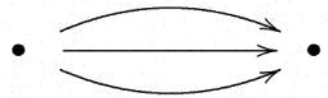
Let M_n be the $A - k[x]$-bimodule

$$k[x]^n \underset{J_n(x)}{\overset{I_n}{\rightrightarrows}} k[t]^n$$

then $M_n \otimes_{k[t]} k[t]/(t-\lambda) \cong R_n(\lambda)$.

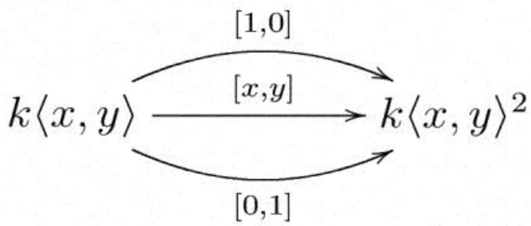
(2) After the work of Dlab-Ringel we know that for the hereditary algebra $A = k\Delta$ with $|\Delta|$ and Euclidean diagram, for any vector dimension $v \in \mathbb{N}^{\Delta_0}$, there exists an $A - k[t]$-bimodule M_v such that almost any indecomposable A-module X with $\mathbf{dim}\, X = v$ is isomorphic to $M_v \otimes_{k[t]} S_\lambda$ for some $\lambda \in k$.

(3) Consider the hereditary algebra B associated to the quiver



Then B is wild.

Proof. Consider the $B - k\langle x, y\rangle$-bimodule M given by



Exercise: $M \otimes_{k\langle x,y\rangle} -\colon \mathrm{mod}_{k\langle x,y\rangle} \to \mathrm{mod}_B$ insets indecomposable modules. $\square$

Definition: An algebra A is *tame* if for every $n \in \mathbb{N}$ there is a finite family of $A - k[t]$-bimodules $M_1, \ldots, M_{t(n)}$ with the following properties:

(i) M_i is finitely generated free as a right $k[t]$-module;

(ii) almost every indecomposable left A-module X with $\dim_k X = n$ is isomorphic to a module of the form $M_i \otimes_{k[t]} S_\lambda$ for some $\lambda \in k$.

We say that an algebra A is *wild* if there is a functor $F\colon \mathrm{mod}_{k\langle x,y\rangle} \to \mathrm{mod}_A$ which preserves indecomposable modules and iso-classes. We shall say that the functor F *insets* indecomposable modules.

Dichotomy Theorem of Drozd: Every finite dimensional k-algebra is either tame or wild.