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Representations of Quivers

Michael Barot
Universidad Nacional Autonoma de Mexico - UNAM
Instituto de Matematicas
Mexico City D.F., Mexico

REPRESENTATIONS OF QUIVERS

M. BAROT

The study of representations of quivers was initiated by P. Gabriel in the early seventies of the past century. By now, a number of remarkable connections to other algebraic topics have been discovered, in particular to Lie algebras, Hall algebras and quantum groups and more recently to cluster algebras. We describe briefley the contents of the five lectures. In the first lecture the principal notions of quivers and their representations are introduced, in the second the classification problem is studied in three examples, the third lecture is concerned with morphisms between representations, the forth with properties of the representations which do not depend on the orientation of the quiver and the fifth and last lecture with the connection of representations of quivers to finite-dimensional modules over algebras.

These notes were written for a course with the same title at the *Advanced School on Representation Theory and Related Topics*, at the Abdus Salam International Centre for Theoretical Physics in January 2006, in Trieste, Italy. As the course was planned to be the most basic one, a major emphasis was put to maintain things as simple and elementary as possible, while at the same time it was intended to reach some of the milestones of representation theory. There were two guiding rules: the notes should be completely self-contained (of course, this one was missed) and the simple and intuitive special case would have to be favored over the general and tricky one.

Also, since this version was finished only days before the meeting started, it almost certainly contains some misprints and misleading formulations. You should therefore be encouraged to point out any possible improvement without pity to me.

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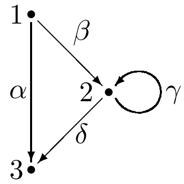
1. The objects we study

1.1. **Quivers.** Roughly speaking, a ***quiver*** is an oriented graph, and to be more precise, we explicitly allow multiple arrows between two vertices and do not reduce to graphs with attached labels expressing the multiplicity. Formally, a quiver is a quadruple $Q = (Q_0, Q_1, s, t)$, where Q_0 is the set of ***vertices***, Q_1 is the set of ***arrows*** and s, t are two maps $Q_1 \to Q_0$, assigning the ***starting vertex*** and the ***terminating vertex*** or ***end vertex*** for each arrow. The quiver is ***finite*** if both sets Q_0 and Q_1 are finite. For arrows α with $s(\alpha) = i$ and $t(\alpha) = j$, we usually write. $\alpha : i \to j$.

Example 1.1. The following picture shows the quiver Q, where $Q_0 = \{1, 2, 3\}$, $Q_1 = \{\alpha, \beta, \gamma, \delta\}$ and $s(\alpha) = s(\beta) = 1$, $s(\gamma) = s(\delta) = 2$ and $t(\alpha) = t(\delta) = 3$, $t(\beta) = t(\gamma) = 2$.

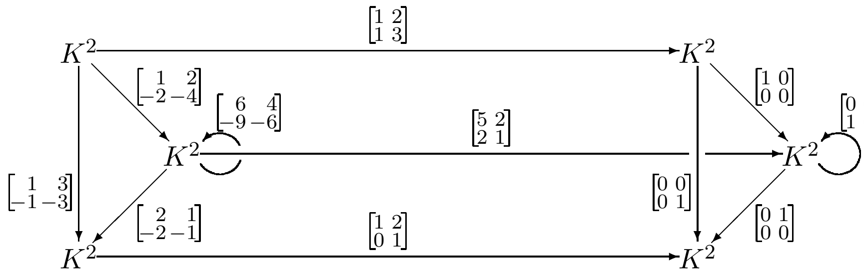


1.2. **Representations.** A ***representation*** of a given quiver Q over a field K assigns to each vertex i of Q a vector space $V(i)$ and to each arrow $\alpha : i \to j$ a K-linear map $V(\alpha) : V(i) \to V(j)$. More formally, a representation is a pair of families $V = \Big((V(i))_{i \in Q_0}, (V(\alpha))_{\alpha \in Q_1} \Big)$. We do not make explicit reference to the ground field K, if no confusion can arise.

Example 1.2. For any quiver there exists the ***zero representation***, which assigns to each vertex the zero space (and consequently to each arrow the zero map).

In this notes, we will always restrict to ***finite-dimensional*** representations, that is, those representations V for which the ***total dimension*** $\dim_K V = \sum_{i \in Q_0} \dim_K V(i)$ is finite. We say that two representations are ***isomorphic*** if they define the same vector spaces and linear maps up to some base change. More precisely, if V and W are the two representations of Q, then V is isomorphic to W if and only if there exists a familiy of linear, bijective maps $\varphi(i) : V(i) \to W(i)$ such that for any arrow $\alpha : i \to j$ we have $W(\alpha) = \varphi(j)V(\alpha)\varphi(i)^{-1}$. In that case, the family $\varphi = (\varphi(i))_{i \in Q_0}$ is called an ***isomorphism*** from V to W and we shall denote this by $\varphi : V \xrightarrow{\sim} W$. We write $V \simeq W$ if V and W are isomorphic.

Example 1.3. Let Q be the quiver of Example 1.1. Then the following picture shows two representations, one on the left hand side and one on the right hand side whereas the three horizontal maps (given in terms of matrices) define an isomorphism, as can be verified directly.



1.3. **Morphisms.** The notion of a ***morphism*** arises as a generalization of an isomorphism, where the condition of bijectivity is dropped. So, a morphism from a representation V to a representation W (both of the same quiver Q − since there is no natural way to define no morphisms between representations of different quivers), is a familiy $\varphi = (\varphi(i))_{i\in Q_0}$ of linear maps $\varphi(i) : V(i) \to W(i)$ such that for any arrow $\alpha : i \to j$ we have $W(\alpha)\varphi(i) = \varphi(j)V(\alpha)$. This condition is equivalent to the commutativity of the following diagram.

$$\begin{array}{ccc} V(j) & \xrightarrow{\varphi(i)} & W(j) \\ {\scriptstyle V(\alpha)}\downarrow & & \downarrow{\scriptstyle W(\alpha)} \\ V(i) & \xrightarrow[\varphi(j)]{} & W(i) \end{array} \tag{1.1}$$

We write $\varphi : V \to W$ to denote the fact that φ is a morphism from V to W.

Example 1.4. For any representation V of Q there is always the ***identity morphism*** $\mathbf{1}_V : V \to V$ defined by the identity maps $(\mathbf{1}_V)(i) : V(i) \to V(i)$, for any vertex i of Q.

It is straightforward that the set of morphisms $\varphi : V \to W$ is a vector space over K, in particular there always exists the ***zero morphism***, also denoted by 0, given by the family of zero maps. Also, two morphisms $\varphi : U \to V$ and $\psi : V \to W$ can be ***composed*** to a morphism $\psi\varphi = (\psi(i)\varphi(i))_{i\in Q_0} : U \to W$. This composition is bilinear, that is, whenever the sums and compositions are defined, we have

$$(a'\psi' + a''\psi'')\varphi = a'\psi'\varphi + a''\psi''\varphi \quad \text{for all } a', a'' \in K \text{ and}$$
$$\psi(b'\varphi' + b''\varphi'') = b'\psi\varphi' + b''\psi'\varphi'' \quad \text{for all } b', b'' \in K.$$

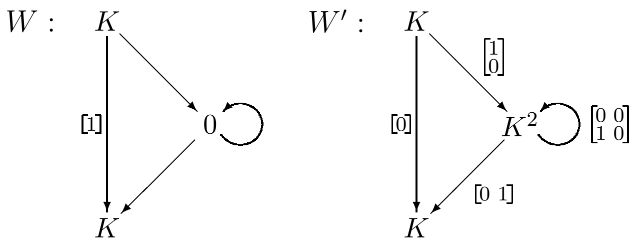
The vector space of morphisms $V \to W$ will be denoted by $\mathrm{Hom}_Q(V, W)$. Notice that $\mathrm{End}_Q(V) = \mathrm{Hom}_Q(V, V)$ is a K-***algebra***, that is, a ring R together with an injective ring-homomorphism from K to the center of R. Any element of $\mathrm{End}_Q(V)$ is called an ***endomorphism*** of V and $\mathrm{End}_Q(V)$ itself the ***endomorphism algebra*** of V.

Example 1.5. If Q is the quiver of Example 1.1 and V the representation on the right in Example 1.3 and W the representation with $W(1) = W(2) = 0$ and $W(3) = k$, then there is only the zero morphism from V to W, but the space of morphisms form W to V is two-dimensional, as any linear map $f : W(3) \to V(3)$ gives rise to a morphism $\varphi : W \to V$ with $\varphi(3) = f$.

Notice that the isomorphisms of representations are exactly the ***invertible*** morphisms, that is, those morphisms $\varphi : V \to W$ for which there exists a morphism $\psi : W \to V$ such that $\psi\varphi = \mathbf{1}_V$ and $\varphi\psi = \mathbf{1}_W$.

1.4. **Indecomposables.** If V and W are representations of a quiver Q then we can define a new representation, called the ***direct sum*** of V and W, and denoted by $V \oplus W$, by putting $(V \oplus W)(i) = V(i) \oplus W(i)$ for each vertex i and $(V \oplus W)(\alpha) = V(\alpha) \oplus W(\alpha)$ (defined componentwise) for each arrow α of Q.

Example 1.6. Let Q be the quiver of Example 1.1 and V be one of the two isomorphic representations in example 1.3. Then V is isomorphic to $W \oplus W'$, where W and W' are given as follows.



A representation of Q is called ***indecomposable*** if it is not isomorphic to the direct sum of two non-zero representations (that is, neither of them is the zero representation). Any non-zero representation can be ***decomposed*** into a finite direct sum of indecomposables, that is, for any $V \neq 0$ there exist indecomposable representations $W_1, \ldots, W_t$ such that $V \simeq W_1 \oplus \ldots \oplus W_t$. This follows easily by induction on the total dimension $\dim_K V$.

Theorem 1.7 (Krull-Remak-Schmidt)**.** *The indecomposables appearing in the decomposition are unique up to order and isomorphism. More precisely, if $V_1, \ldots, V_s, W_1, \ldots, W_t$ are indecomposable representrations such that $V_1 \oplus \ldots \oplus V_s \simeq W_1 \oplus \ldots \oplus W_t$, then $s = t$ and there exists a permutation π (of s elements) such that $V_i \simeq W_{\pi(i)}$ for any $1 \leq i \leq s$.*

The proof is given in section 1.6. If V and W are two non-zero representations and $\psi_V : V \xrightarrow{\sim} V_1 \oplus \ldots \oplus V_s$ and $\psi_W : W \xrightarrow{\sim} W_1 \oplus \ldots \oplus W_t$ are decompositions into indecomposables, then any morphism $\varphi : V \to W$ can be written uniquely as a matrix of morphisms between the indecomposable summands, where the entry in the i-th row and j-th column is the composition $\varphi_{ij} : V_i \to W_j$ of the canonical inclusion $\iota_i : V_i \to \bigoplus_{a=1}^s V_a$ with $\psi_W \varphi \psi_V^{-1} : \bigoplus_{a=1}^s V_a \to \bigoplus_{b=1}^t W_b$ and the canonical projection $\pi_j : \bigoplus_{b=1}^t W_b \to W_j$. Notice that ι_i and π_j are morphisms of representations.

1.5. **Fitting's Lemma.** The following result is very useful in many situations. Recall that an algebra (or more generally, a ring) A is ***local*** when its subset of non-invertible elements is closed under adition, and also recall that an element $a \in A$ is called ***nilpotent*** if $a^n = 0$ for some $n > 0$.

Proposition 1.8 (Fitting's Lemma)**.** *If V a non-zero, finite dimensional representation of Q, then V is indecomposable if and only if* $\mathrm{End}_Q(V)$ *is local. When the field K is algebraically closed, this happens if and only if any endomorphism of V can be written as sum of a nilpotent endomorphism with a multiple of the identity.*

Proof. Since $\dim_K V$ is finite, for any endomorphism φ of V, there exists a natural number n such that $\varphi^n(V) = \varphi^{n+1}(V)$. Set $\psi = \varphi^n$. Then we have $V \simeq \psi(V) \oplus W$, where $W(i) = \psi(i)^{-1}(0)$. Hence, if V is indecomposable, we must have $\psi(V) = 0$ or $W = 0$. If $W = 0$ then $\psi(i)$ is bijective and therefore $\varphi(i)$ is bijective for all i. Hence φ is invertible. If $\psi(V) = 0$ then φ is nilpotent. This shows that any endomorphism is either invertible or nilpotent.

Now, the composition (in any order) of any endomorphism with a nilpotent one is clearly non-invertible and hence nilpotent. Also, for ψ nilpotent, $\mathbf{1}_V - \psi$ is invertible (its inverse is $\mathbf{1}_V + \psi + \psi^2 + \psi^3 + \ldots$, which is a finite sum since ψ is nilpotent). To show that the sum of any two non-invertible endomorphisms is non-invertible again, take two nilpotent endomorphisms ψ and ψ'. Then $\psi + \psi'$ is nilpotent, since otherwise it would be invertible and then we could find η such that $\mathbf{1}_V = \eta\psi + \eta\psi'$, which is impossible since $\mathbf{1}_V - \eta\psi = \eta\psi'$ would be both invertible and nilpotent. We conclude that the set of non-invertible endomorphisms of V is closed under adition.

Conversely, assume that V is not indecomposable, say $\eta : V \xrightarrow{\simeq} V' \oplus V''$, with V' and V'' both non-zero. Then the endomorphisms $e' =$

$\eta^{-1}\begin{bmatrix}\mathbf{1}_{V'} & 0\\ 0 & 0\end{bmatrix}\eta$ and $e'' = \eta^{-1}\begin{bmatrix}0 & 0\\ 0 & \mathbf{1}_{V''}\end{bmatrix}\eta$ are both ***idempotents***, that is $e'e' = e'$ and $e''e'' = e''$. Now, e' and e'' are both non-invertible but $\mathbf{1}_V = e' + e''$. So $\mathrm{End}_Q(V)$ is not local.

Now assume that K is algebraically closed. If V is indecomposable, then for any endomorphism φ, we can certainly find a scalar $a \in K$ such that $\varphi - a\mathbf{1}_V$ is not invertible and therefore nilpotent. This shows that any endomorphism of V can be written in the form $a\mathbf{1}_V + \psi$ with $a \in K$ and ψ nilpotent. If V decomposes, then the idempotents e', e'' defined as above cannot be written in this form. □

1.6. **Proof of the Theorem 1.7.** Let $\varphi : V \to W$ be an isomorphism, $V = V_1 \oplus \ldots \oplus V_s$ and $W = W_1 \oplus \ldots \oplus W_t$ decompositions into indecomposables. Then φ can be written as a matrix $\varphi = (\varphi_{ji})_{j=1}^{s}{}_{i=1}^{t}$, where $\varphi_{ji} : V_i \to W_j$. Similarly write $\psi = \varphi^{-1}$ as matrix and observe that $\mathbf{1}_{V_1} = (\psi\varphi)_{11} = \sum_{l=1}^{t} \psi_{1l}\varphi_{l1}$. One summand on the right must then be invertible, since $\mathrm{End}_A(V_1)$ is local. We assume without loss of generality that $\psi_{11}\varphi_{11}$ is invertible. But, since both representations, V_1 and W_1, are indecomposable, we have that both morphisms, ψ_{11} and φ_{11}, are invertible.

We now exchange the given isomorphism φ by $\varphi' = \alpha\varphi\beta$, where α and β are given as

$$\alpha = \begin{bmatrix}\mathbf{1}_{W_1} & 0 & \cdots & 0\\ -\varphi_{21}\varphi_{11}^{-1} & \mathbf{1}_{W_2} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ -\varphi_{t1}\varphi_{11}^{-1} & 0 & \cdots & \mathbf{1}_{W_t}\end{bmatrix}, \ \beta = \begin{bmatrix}\mathbf{1}_{V_1} & -\varphi_{11}^{-1}\varphi_{12} & \cdots & -\varphi_{11}^{-1}\varphi_{1s}\\ 0 & \mathbf{1}_{V_2} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{1}_{V_s}\end{bmatrix}$$

and observe that $\varphi' = \begin{bmatrix}\varphi_{11} & 0\\ 0 & \Phi\end{bmatrix}$, where $\Phi : \bigoplus_{i=2}^{s} V_i \to \bigoplus_{j=2}^{t} W_j$. Since φ' is bijective, so must be Φ, which is thus an isomorphism again. The result follows now by induction. □

2. The classification problem

2.1. **The problem.** Given the fact that any representation can be decomposed into indecomposable representations uniquely (up to isomorphism and order) and that any morphism then can be viewed as a matrix between the indecomposables, one is tempted to think that all what remains to do is to understand what indecomposables there are and what morphisms between those indecomposables can exist for the given quiver Q. This is the classification problem. The part concerned with representations can be stated as follows: find a complete list of pairwise non-isomorphic indecomposable representations.

We will solve this classification problem for certain, well chosen examples, but we also stress that this is not the whole story of representation theory, not even close.

2.2. **A first example.** We consider a linearly ordered quiver (called ***linear*** quiver, for short) as shown in the following picture

$$Q: \underset{1}{\bullet} \xrightarrow{\alpha_1} \underset{2}{\bullet} \xrightarrow{\alpha_2} \underset{3}{\bullet} \longrightarrow \cdots \longrightarrow \underset{n-2}{\bullet} \xrightarrow{\alpha_{n-2}} \underset{n-1}{\bullet} \xrightarrow{\alpha_{n-1}} \underset{n}{\bullet}$$

and will try to solve the classification problem, that is, we will try to determine a complete set of pairwise non-isomorphic indecomposable representations for Q. For that, let V be an indecomposable representation of Q.

Step 1. *If $V(\alpha_i)$ is not injective then $V(j) = 0$ for $j > i$.* Suppose first that $V(\alpha_1), \ldots, V(\alpha_{i-1})$ are all injective, but $V(\alpha_i)$ is not. Then we set $W(i) = \operatorname{Ker} V(\alpha_i)$ and inductively $W(j) = V(\alpha_j)^{-1}(W(j+1))$ for $j = i-1, i-2, \ldots, 1$. Let $S(1)$ be a supplement of $W(1)$ (that is, a subspace of $V(1)$ such that $V(1)$ is the internal direct sum of $W(1)$ and $S(1)$) and for $j = 1, \ldots, i-1$, inductively define $S(j+1)$ to be a supplement of $W(j+1)$ such that $V(\alpha_j)(S(j)) \subseteq S(j+1)$ (this is possible since $V(\alpha_j)$ is injective and $V(\alpha_j)(S(j)) \cap W(j+1) = 0$).

We thus see that V decomposes into $W \oplus V'$, where

$$\begin{aligned} W &= W(1) \to \ldots \to W(i) \to 0 \to \ldots \to 0) \text{ and} \\ V' &= S(1) \to \ldots \to S(i) \to V(i+1) \to \ldots \to V(n). \end{aligned}$$

Since V is indecomposable and $W(i) \neq 0$ the summand V' must be zero.

Step 2. *If $V(\alpha_j)$ is not surjective then $V(h) = 0$ for all $h \leq j$.* This is proved very similarly to Step 1.

Step 3. *The representation V is isomorphic to*

$$[j, i]: \quad 0 \to \ldots \to 0 \to K \xrightarrow{1_K} \ldots \xrightarrow{1_K} K \to 0 \to \ldots \to 0,$$

where the first occurrence of K happens in place $1 \leq j$ and the last in place $i \leq n$.

If all maps $V(\alpha_h)$ $(1 \leq h < n)$ are injective, let $i = n$. Otherwise, let i be minimal such that $V(\alpha_i)$ is not injective. If all maps $V(\alpha_h)$ $(1 \leq h < i)$ are surjective, let $j = 1$. Otherwise let $j \leq i$ be maximal such that $V(\alpha_{j-1})$ is not surjective. Hence the maps $V(\alpha_h)$ are bijective for $j \leq h < i$ and by Step 1 and 2, we have $V(h) = 0$ for $h < j$ and $h > i$.

Therefore, V is isomorphic to

$$0 \to \ldots \to 0 \to K^d \xrightarrow{\mathbf{1}_d} \ldots \xrightarrow{\mathbf{1}_d} K^d \to 0 \to \ldots \to 0,$$

where $\mathbf{1}_d$ denotes the identity matrix of size $d \times d$. Hence V is isomorphic to d copies of $[j, i]$ and since V is indecomposable, we must have $d = 1$.

Step 4. *The representations $[j, i]$ are indecomposable and pairwise non-isomorphic.* The indecomposablity follows easily by contradiction. Suppose that $V := [j, i]$ decomposes, say $V = W \oplus W'$ with $W \neq 0$ and $W' \neq 0$. Then $\dim_K V \geq 2$ and hence $j < i$. Assume that W is the representation with $W(j) \simeq K$ (otherwise switch W and W') and let $h \geq j$ be minimal such that $W(h) = 0$. Since $W' \neq 0$ we must have $h < i$ and $W'(h) \simeq K$. Therefore, we have $(W \oplus W')(\alpha_{h-1}) = 0$ in contradiction to $V(\alpha_{h-1}) = 1_K$.

Clearly, $V = [j, i]$ is isomomorphic to $V' = [j', i']$ if and only if $i = i'$ and $j = j'$ since you easily can find a vertex h for which $\dim_K V(h) \neq \dim_K V'(h)$ in case $(j, i) \neq (j', i')$.

Hence we have found a complete list of pairwise non-isomorphic indecomposables. There are $\frac{n(n+1)}{2}$ indecomposables, up to isomorphism, and therefore Q is ***representation-finite*** or ***of finite representation type***.

2.3. **A second example.** Here we consider a much simpler quiver $\mathcal{L}_1$, which has only one vertex and one arrow, which then must be a ***loop***, that is, it starts and ends at the same vertex. A representation of this quiver is just a vector space V together with a linear endomorphism $f : V \to V$. Choosing a base of V and writing f in this base, we get a square matrix F and therefore an isomorphic representation (K^m, F), where $m = \dim_K V$. Two such representations (K^m, F) and (K^n, G) are isomorphic if and only if $m = n$ and F and G are conjugate.

We know from linear algebra that, *if the field K is algebraically closed*, then there exists a normal form under conjugation (the ***Jordan normal form***): the matrix F is conjugate to a ***Jordan matrix***, which we conveniently write as

$$J_{m_1,\mu_1} \oplus J_{m_2,\mu_2} \oplus \ldots \oplus J_{m_t,\mu_t},$$

that is, it is a block matrix, whose non-diagonal blocks are zero and its diagonal blocks are ***Jordan blocks*** of size $m \times m$

$$J_{m,\mu} = \begin{bmatrix} \mu & 1 & & & \\ & \mu & 1 & & \\ & & \ddots & \ddots & \\ & & & \mu & 1 \\ & & & & \mu \end{bmatrix}$$

with $\mu \in K$, where the two big circles indicate that everything there is zero. For our representation, this means that (K^m, F) decomposes as follows

$$(K^m, F) \simeq (K^{m_1}, J_{m_1,\mu_1}) \oplus \ldots \oplus (K^{m_t}, J_{m_t,\mu_t}).$$

If the field K is not algebraically closed, then there also exists a normal form, but it is more complicated.

The representations $R_{m,\mu} = (K^m, J_{m,\mu})$ are therefore good candidates for our list of indecomposables. In fact, they are all the indecomposables up to isomorphism and pairwise non-isomorphic. Hence they give the complete list we are looking for. To see this, we assume that $\varphi : R_{m,\mu} \to R_{n,\nu}$ is a morphism, given by a matrix T of size $n \times m$. From $TJ_{m,\mu} = J_{n,\nu}T$ we obtain

$$\mu T + TJ_{m,0} = \nu T + J_{n,0}T. \tag{2.2}$$

If $\mu \neq \nu$ then look at the position $(n, 1)$, that is in the n-th row and first column of (2.2), we see that $\mu T_{n,1} = \nu T_{n,1}$ and therefore $T_{n,1} = 0$. Next, we look at the position $(n-1, 1)$. We get $\mu T_{n-1,1} = \nu T_{n-1,1} + T_{n,1} = \nu T_{n-1,1}$ so $T_{n-1,1} = 0$. Inductively, we see that $T_{i,1} = 0$ for all i. Then we repeat this argument for the second (third, forth and so on) column, starting form the bottom to the top and get that any column of T is zero. So, for $\mu \neq \nu$, we get $\mathrm{Hom}_Q(R_{m,\mu}, R_{n,\nu}) = 0$ for any m and any n. This already shows that for $\mu \neq \nu$ the representations $R_{m,\mu}$ and $R_{n,\nu}$ are non-isomorphic.

Now, we shall see that $R_{m,\mu}$ is indecomposable. So, assume that $\mu = \nu$. Then we can simplify the equation (2.2) and get $TJ_{m,0} = J_{n,0}T$. If we look at the last row on both sides, we get $T_{n,j} = 0$ for all $j < m$. Then we look at the row $n-1$ and get that $T_{n-1,j} = 0$ for $j < m-1$ and $T_{n,m} = T_{n-1,m-1}$. Inductively, we get $T_{ij} = 0$ for $i > j$ and $T_{i-1,j-1} = T_{ij}$ for $1 < i \leq n$ and $1 < j \leq m$. Hence $\mathrm{Hom}_Q(R_{m,\mu}, R_{n,\mu})$ is $|m-n|+1$ dimensional. In particular, if $m = n$, the morphism φ is of the form

$$\varphi = T_{1,1}\mathbf{1} + T_{1,2}\gamma_m + T_{1,3}\gamma_m^2 + \ldots + T_{1,n}\gamma_m^{n-1}$$

where γ_m is given by the nilpotent matrix $J_{m,0}$. Therefore we can write $\varphi = T_{1,1}\mathbf{1} + \psi$, where ψ is nilpotent. In particular, φ is invertible if and only if $T_{1,1} \neq 0$, and the sum of two non-invertible endomorphisms is again non-invertible. By Fitting's Lemma (Proposition 1.8), this is enough to ensure the indecomposability of the representation $R_{m,\mu}$. Therefore

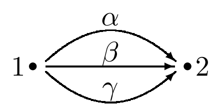
$$R_{m,\mu}, \qquad \text{with } \mu \in K \text{ and } m \in \mathbb{N}_1 = \{1, 2, 3, \ldots\},$$

is a complete list of pairwise non-isomorphic indecomposables. This list is infinite, and we call therefore the quiver Q to be ***representation infinite***, or of ***infinite representation type***.

Observe that in each dimension there is a ***one-parameter family*** of indecomposables, that is a family indexed by the field of pairwise non-isomorphic indecomposables.

2.4. **The phenomenon of wildness.** We have already seen two examples and we shall consider one more to complete the picture. In the third example, we consider the ***three-Kronecker*** quiver, which has two vertices and three arrows, all of them starting in one vertex and ending in the other vertex, as shown in the following picture.

(2.3)



In this case we consider the following representations $V_{\lambda,\mu}$, for which $V_{\lambda,\mu}(1) = K^2$, $V_{\lambda,\mu}(2) = K$ and

$$V_{\lambda,\mu}(\alpha) = [1\ 0], \quad V_{\lambda,\mu}(\beta) = [0\ 1] \ \text{ and } \ V_{\lambda,\mu}(\gamma) = [\lambda\ \mu].$$

Suppose that there is a morphism $\varphi : V_{\lambda,\mu} \to V_{\rho,\sigma}$, given by matrices T and U such that

$$U\,[1\ 0] = [1\ 0]\,T, \quad U\,[0\ 1] = [0\ 1]\,T, \ \text{ and } \ U\,[\lambda\ \mu] = [\rho\ \sigma]\,T.$$

Since U is of size 1×1, say $U = [u]$ with $u \in K$, from the first two equations we get $u\mathbf{1}_2 = \mathbf{1}_2 T = T$. Hence we infer from the last equation that either $u = 0$ and $T = 0$ or $u \neq 0$, in which case $\lambda = \rho$ and $\mu = \sigma$.

This shows that the representations $V_{\lambda,\mu}$ are pairwise non-isomorphic, and that any endomorphism of $V_{\lambda,\mu}$ is a multiple of the identity morphism. Hence $V_{\lambda,\mu}$ is indecomposable by Fitting's Lemma (Proposition 1.8).

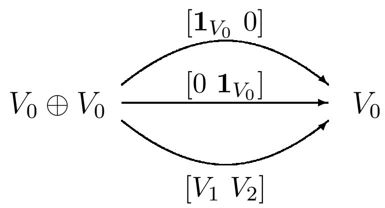
Thus we found a ***two parameter family*** of pairwise non-isomorphic indecomposables. For a representation theorist, this is a point where it is clear that it is hopeless to try to write down a complete list of all indecomposables. The three-Kronecker quiver is called ***wild***, a terminology which becomes clearer by the following result:

Proposition 2.1. *If we could solve the classification problem for the quiver Q as in* (2.3) *then we could solve it for any quiver Q'.*

Proof. The proof is done in several steps, in which we shall consider the quivers $\mathcal{L}_t$ for $t \geq 2$, which have one vertex and t loops $\alpha_1, \ldots, \alpha_t$ starting and ending at the one vertex 0. To simplify notation, for a representation V of $\mathcal{L}_t$, set $V_0 = V(0)$ and $V_i = V(\alpha_i)$ for $i = 1, \ldots, t$.

Step 1. *If we could solve the classification problem for Q then also for $\mathcal{L}_2$.*

For any representation V of $\mathcal{L}_2$, we get a representation $F(V)$ of the quiver Q as follows.



that is, $F(V)(1) = V_0 \oplus V_0$, $F(V)(2) = V_0$ and $F(V)(\alpha) = [\mathbf{1}_{V_0}\ 0]$, $F(V)(\beta) = [0\ \mathbf{1}_{V_0}]$ and $F(V)(\gamma) = [V_1\ V_2]$. Furthermore, for any morphism $\varphi \in \mathrm{Hom}_{\mathcal{L}_2}(V, W)$ a morphism $F(\varphi) \in \mathrm{Hom}_Q(F(V), F(W))$ is obtained by $F(\varphi)(1) = \varphi \oplus \varphi$ and $F(\varphi)(2) = \varphi$.

Conversely, if $\psi : F(V) \to F(W)$ is a morphism, then we have

$$\begin{aligned}\psi(1)\,[\mathbf{1}_{W_0}\ 0] &= [\mathbf{1}_{V_0}\ 0]\,\psi(2),\\ \psi(1)\,[0\ \mathbf{1}_{W_0}] &= [0\ \mathbf{1}_{V_0}]\,\psi(2) \quad \text{and}\\ \psi(1)\,[W_1\ W_2] &= [V_1\ V_2]\,\psi(2).\end{aligned}$$

From the first two equations, we infer that $\psi(1) = \psi(2) \oplus \psi(2)$ and from the last that $\psi(2)W_i = V_i\psi(2)$ for $i = 1, 2$, that is $\psi(2) : V \to W$ is a morphism of representations of $\mathcal{L}_2$ and $\psi = F(\psi(2))$. It is now easy to see that the map $\varphi \mapsto F(\varphi)$ is linear and satisfies $F(\varphi_2\varphi_1) = F(\varphi)_2 F(\varphi)_1$. Hence $\mathrm{End}_{\mathcal{L}_2}(V)$ and $\mathrm{End}_Q(F(V))$ are always isomorphic and thus it follows from Proposition 1.8 that V is indecomposable if and only if $F(V)$ is indecomposable and two representations V and W of $\mathcal{L}_2$ are isomorphic if and only if $F(V)$ and $F(W)$ are isomorphic representations of Q. Hence the classification problem of $\mathcal{L}_2$ is included in the classification problem of Q.

Step 2. *If we could solve the classification problem for Q then also for $\mathcal{L}_t$ for any $t \geq 2$.*

For a representation V of $\mathcal{L}_t$ define a representation $G(V)$ of $\mathcal{L}_2$ by setting $G(V)_0 = (V_0)^{t+1}$ and

$$G(V)_1 = \begin{bmatrix} 0 & \mathbf{1}_{V_0} & & & \bigcirc \\ & 0 & \mathbf{1}_{V_0} & & \\ & & \ddots & \ddots & \\ \bigcirc & & & 0 & \mathbf{1}_{V_0} \\ & & & & 0 \end{bmatrix}, \qquad G(V)_2 = \begin{bmatrix} 0 & V_1 & & & \bigcirc \\ & 0 & V_2 & & \\ & & \ddots & \ddots & \\ \bigcirc & & & 0 & V_t \\ & & & & 0 \end{bmatrix}.$$

For any $\varphi \in \mathrm{Hom}_{\mathcal{L}_t}(V, W)$, we define $G(\varphi) : G(V) \to G(W)$ by $G(\varphi) = \varphi \oplus \varphi \oplus \ldots \oplus \varphi$ ($t{+}1$ times). Conversely, let $\psi \in \mathrm{Hom}_{\mathcal{L}_2}(G(V), G(W))$ be a morphism. Write ψ in blocks and consider first $\psi G(V)_1 = G(W)_1 \psi$. With similar arguments as when we studied morphisms between indecomposable representations of $\mathcal{L}_1$, we obtain that there exists some linear maps $f_0, f_1, \ldots, f_t : V_0 \to W_0$ such that

$$\psi = \begin{bmatrix} f_0 & f_1 & \cdots & f_t \\ & \ddots & \ddots & \vdots \\ \bigcirc & & f_0 & f_1 \\ & & & f_0 \end{bmatrix}.$$

Now, consider the second equation $\psi G(V)_2 = G(W)_2 \psi$. We get $f_0 V_i = W_i f_0$ for all $i = 1, \ldots, t$ and some other equations (to be precise, we get $f_{j-i-1} V_{j-1} = W_i f_{j-i-1}$ for $1 \leq i \leq j \leq t$). Hence $f_0 : V \to W$ is a mophism of representations of $\mathcal{L}_2$. Also, ψ is invertible if and only if f_0 is invertible. Therefore $V \simeq W$ if and only if $G(V) \simeq G(W)$. Moreover, any endomorphism $\psi \in \mathrm{End}_{\mathcal{L}_2}(G(V))$ can be written as $\psi = G(f_0) + \psi'$, where $f_0 \in \mathrm{End}_{\mathcal{L}_t}(V)$ and ψ' is a nilpotent endomorphism of $G(V)$. Hence $\mathrm{End}_{\mathcal{L}_t}(V)$ is local if and only if $\mathrm{End}_{\mathcal{L}_2}(G(V))$ is local. We conclude that the classification problem of $\mathcal{L}_t$ is included in the classification problem of $\mathcal{L}_2$ and therefore Step 2 follows from Step 1.

Step 3. *If we could solve the classification problem for Q then also for any quiver Q'.*

To simplify notations, assume that $Q'_0 = \{1, \ldots, n\}$ and that $Q'_1 = \{\beta_1, \ldots, \beta_r\}$ with $\beta_j : s_j \to e_j$. Then let $t = n + s$ and for any representation V of Q' define a representation $H(V)$ of $\mathcal{L}_t$ as follows. Set $H(V)_0 = V(1) \oplus \ldots \oplus V(n)$. For $1 \leq i \leq n$ let $H(V)_i$ be the block matrix, whose only non-zero block is $\mathbf{1}_{V(i)}$ at position (i, i). For $1 \leq j \leq r$

let $H(V)_{n+j}$ be the block matrix, whose only non-zero block is $V(\beta_i)$ at position (e_i, s_i).

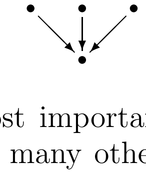
For $\varphi \in \text{Hom}_{Q'}(V,W)$, we define $H(\varphi) \in \text{Hom}_{\mathcal{L}_t}(H(V), H(W))$ by $H(\varphi) = \varphi(1) \oplus \ldots \oplus \varphi(n)$. And if $\psi : H(V) \to H(W)$ is a morphsim we get immediatly from $\psi H(V)_i = H(W)_i \psi$ for $1 \leq i \leq n$ that $\psi = \psi(1) \oplus \ldots \oplus \psi(n)$ for some linear maps $\psi(i) : V(i) \to V(i)$. But then, the equations $\psi H(V)_{n+j} = H(W)_{n+j}\psi$ show that the family $(\psi(i))_{i \in Q'_0}$ is a morphism $V \to W$ of representations of Q. Again, H is linear on the morphism spaces and commutes with the composition. Hence we get that V is indecomposable if and only if $H(V)$ is indecomposable and two representations of Q' are isomorphic if and only if their images under H are isomorphic. Hence the classification problem of Q' is included in the classification problem of $\mathcal{L}_t$ and therefore Step 3 follows from Step 2. □

2.5. **The representation types.** We have encountered three different situations:

- A *finite* list of indecomposables as in the case of the linear quivers.
- An *infinite but complete* list of indecomposables containing in each dimension only finitely many *one-parameter family* of pairwise non-isomorphic indecomposables. This was the case for the quiver $\mathcal{L}_1$.
- A *two parameter family* of pairwise non-isomorphic indecomposable representations in some dimension, as in the three Kronecker problem.

These are the basic three ***representation types*** which occur in representation theory. The second and third case are of ***infinte representation type***, but there is no general agreement on the term ***tame***, since some authors prefer to use it to refer only to the second case above (and hence always assume that the quiver is representation infinite) while others include the representation-finite quivers when they speak of tame quivers.

Exercise 2.2. What representation type has the following quiver?



Classifying is one of the most important problems in representation theory, but clearly there are many others. We used only linear algebra, which is simple and powerful enough to do the job. However, you should have noticed that we lack some general strategy. Furthermore, we gained no structural insight and got just a plain list of indecomposables. There are better methods, very effective when applied to examples because they are somehow "self-correcting", which means that mistakes tend to surface quickly. These "better methods" are based on the Auslander-Reiten theory, to be explained in another course.

3. Morphisms

3.1. **Radical morphisms.** The additional structure for these lists of indecomposables is given by the morphisms. We start with some simple facts and therefore fix a quiver Q.

A linear map f is injective if and only if $fg = fg'$ always implies $g = g'$. Now, a morphism $f : V \to W$ of representations of Q satisfies that $fg = fg'$ implies $g = g'$ if and only $f(i)$ is injective for every vertex $i \in Q_0$ and we say then that f itself is ***injective***. This can be easily seen by considering $\operatorname{Ker} f$, a representation of Q, called the ***kernel*** of f, given by $(\operatorname{Ker} f)(i) = \operatorname{Ker}(f(i))$ for each vertex $i \in Q_0$ and the maps $(\operatorname{Ker} f)(\alpha)$ are induced by $V(\alpha)$ for each arrow $\alpha \in Q_1$. Notice that $\operatorname{Ker} f$ is a ***subrepresentation*** of V, that is the inclusion $\iota : \operatorname{Ker} f \to V$, given by the family of inclusion $\operatorname{Ker} f(i) \subseteq V(i)$, is a morphism. Now, if f is injective then for the morphisms ι and 0 from $\operatorname{Ker} f$ to V we have $f\iota = f0$ and therefore $\iota = 0$. This means that $\operatorname{Ker} f$ is the zero representation and hence $f(i)$ is injective for any i. The converse is straightforward.

Similarly, we define the subrepresentation $\operatorname{Im} f$ of W called the ***image*** of f and see that $gf = g'f$ always implies $g = g'$ if and only if $f(i)$ is surjective for any vertex $i \in Q_0$ if and only if $\operatorname{Im} f = W$, in which case, we call f ***surjective***. Hence by definition, f is an isomorphism if and only if f is injective and surjective.

Lemma 3.1. *The composition of two non-isomorphisms between indecomposables is a non-isomorphism again.*

Proof. Assume that U, V and W are indecomposables and that $f : U \to V$ and $h : V \to W$ are such that hf is an isomorphism. Then set $g = (hf)^{-1}h : V \to U$, so $gf = \mathbf{1}_U$. Then f is injective, $U \simeq \operatorname{Im} f$ and we have $V = \operatorname{Im} f \oplus \operatorname{Ker} g$. Indeed, if $f(u) \in \operatorname{Ker} g$ then $u = gf(u) = 0$ showing $\operatorname{Im} f \cap \operatorname{Ker} g = 0$. On the other hand, for any $v \in V$, we have $v - fg(v) \in \operatorname{Ker} g$ and therefore $v = f(g(v)) + (v - fg(v)) \in \operatorname{Im} f + \operatorname{Ker} g$. Since V is indecomposable and $\operatorname{Im} f \simeq U \neq 0$ we must have $\operatorname{Ker} g = 0$

and therefore $\operatorname{Im} f = V$ and both, f and g are isomorphisms. Therefore also h is an isomorphism. □

A morphism $f : V \to W$ between representations is called ***radical*** if, written as a matrix of morphisms between indecomposables, no entry is an isomorphism. The set of radical morphisms from U to V shall be denoted by $\operatorname{rad}_Q(U, V)$.

Proposition 3.2. *The sets* $\operatorname{rad}_Q(U, V)$ *are vector spaces and the composition (in any order) of a radical morphism with any morphism is radical again.*

Proof. If U and V are non-isomorphic indecomposable representations, then $\operatorname{rad}_Q(U, V) = \operatorname{Hom}_Q(U, V)$, which is a vector space. If $U = V$ is indecomposable then the radical morphisms $U \to U$ are precisely the nilpotent morphisms and are therefore closed under sums and scalar multiples, by Fitting's Lemma (Proposition 1.8). If $f : V \to U$ is an isomorphism, then $g : U \to V$ is radical if and only if $fg : U \to U$ is nilpotent, hence $\operatorname{Hom}_Q(U, f) : Hom_Q(U, V) \to \operatorname{Hom}_Q(U, U), g \mapsto fg$ is a linear bijection, which maps $\operatorname{rad}_Q(U, V)$ into $\operatorname{rad}_Q(U, U)$, a vector space. The first assertion follows now by decomposing into indecomposables.

Let $T \xrightarrow{f} U \xrightarrow{g} V \xrightarrow{h} W$ be morphisms with g radical. By Lemma 3.1, if T, U, V and W are indecomposables then gf and hg are also radical. Otherwise decompose them into indecomposables and write the morphisms as matrices. Then any entry of gf and hg is a sum of compositions of a radical morphism with some other morphism between indecomposables and hence radical by the above. □

3.2. Irreducible morphims.

We calculate the radical morphisms between indecomposable representations in an example, namely the representation finite case we studied in 2.2. For that, let us adopt the notation established there.

Suppose that $\psi : [j, i] \to [j', i']$ is a morphism and set $\psi_j = \psi(j)$ to simplify notation. For each arrow α_i in Q, we obtain a commutative square. If we omit the cases where the commutativity always is garanteed, the following three cases remain.

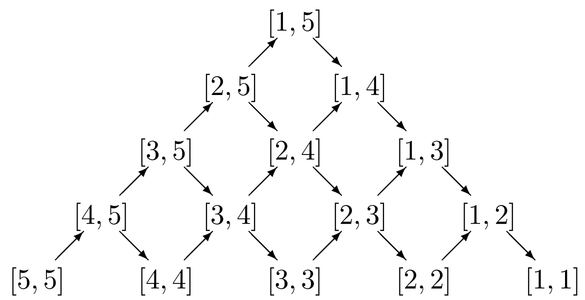
$$\begin{array}{ccc} K & \xrightarrow{1_K} & K \\ {\scriptstyle\psi_h}\downarrow & & \downarrow{\scriptstyle\psi_{h+1}} \\ K & \xrightarrow[1_K]{} & K \end{array} \qquad \begin{array}{ccc} K & \xrightarrow{1_K} & K \\ {\scriptstyle\psi_h}\downarrow & & \downarrow{\scriptstyle\psi_{h+1}} \\ 0 & \xrightarrow[0]{} & K \end{array} \qquad \begin{array}{ccc} K & \xrightarrow{0} & 0 \\ {\scriptstyle\psi_h}\downarrow & & \downarrow{\scriptstyle\psi_{h+1}} \\ K & \xrightarrow[1_K]{} & K \end{array}$$

It is now easy to see that we get the following description for the morphism spaces.

$$\mathrm{Hom}([j,i],[j',i']) = \begin{cases} K\,\gamma_{j,i}^{j',i'}, & \text{if } j' \leq j \leq i' \text{ and } j \leq i' \leq i \\ 0, & \text{else,} \end{cases}$$

where $\gamma = \gamma_{j,i}^{j',i'} : [j,i] \to [j',i']$ is the morphism with $\gamma(h) = 1_K$ for each $\max(j,j) \leq h \leq \min(i,i')$. Notice that the maps $\gamma_{j,i}^{j',i'}$ behave multiplicatively, that is $\gamma_{j',i'}^{j'',i''}\gamma_{j,i}^{j',i'} = \gamma_{j,i}^{j'',i''}$, if $j'' \leq j \leq i''$ and $j \leq i'' \leq i$. Therefore there are some "shortest" radical morphisms, namely $\gamma_{j,i}^{j-1,i}$ (if $1 < j$) and $\gamma_{j,i}^{j,i-1}$ (if $j < i$).

(3.4)



Notice that the whole diagram is commutative. Observe that in the picture (3.4), the morphisms going up are all injective and the morphisms going down are all surjective. But there is much more structure inside of it as we shall see soon.

For any two representations U and V, we define the ***square-radical*** $\mathrm{rad}_Q^2(U,V) = \sum_W \mathrm{rad}_Q(W,V)\,\mathrm{rad}_Q(U,W)$ to be the subspace of $\mathrm{rad}_Q(U,V)$ generated by all possible compositions $U \to W \to V$ of two radical morphisms. More generally, for $i \geq 2$, we define inductively the i^{th} power of the radical $\mathrm{rad}_Q^i(U,V) = \sum_W \mathrm{rad}_Q(W,V)\,\mathrm{rad}_Q^{i-1}(U,W)$ and finally, the ***infinite radical*** $\mathrm{rad}_Q^\infty(U,V) = \bigcap_{i\geq 1} \mathrm{rad}_Q^i(U,V)$ to be the set of morphisms which can be written as compositions of arbitrarily many radical morphisms.

A morphism $f \in \mathrm{rad}_Q(U,V)$ is called ***irreducible*** if $f \notin \mathrm{rad}_Q^2(U,V)$ (this is the formal definition for a morphism to be "shortest among the radical morphisms").

Exercise 3.3. Any irreducible morphism (between indecomposable representations) is either injective or surjective but not both. Compare this general result with the example above studied above (3.4).

Exercise 3.4. A morphism $f : U \to V$ between indecomposables is irreducible if and only if it is not an isomorphism and whenever $\eta = \psi\varphi$

for some morphisms $\varphi : V \to U$ and $\psi : U \to W$ (where U is not necessarily indecomposable) then φ is a ***section*** (that is there exists $\varphi' : U \to V$ with $\varphi'\varphi = \mathbf{1}_V$) or ψ a ***retraction*** (that is there exists $\psi' : W \to U$ with $\psi\psi' = \mathbf{1}_U$).

3.3. **The Auslander-Reiten quiver.** Let us recall some notions from homological algebra. A representation P of a quiver Q is ***projective*** if any surjective morphism to P is a retraction. We will study the projective representations in more detail in section 3.4 below. A representation I is called ***injective*** if any injective morphism starting in I is a section.

A sequence of morphisms $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \to \ldots \to X_{t-1} \xrightarrow{f_{t-1}} X_t$ is called ***exact*** if for each $1 \leq i \leq t-1$ we have $\operatorname{Im} f_i = \operatorname{Ker} f_{i-1}$. An exact sequence $0 \to X \to Y \to Z \to 0$ is called a ***short exact sequence***. Call two short exact sequences $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ and $0 \to X \xrightarrow{f'} Y' \xrightarrow{g'} Z \to 0$ ***equivalent*** if there exists an isomorphism $\eta : Y \to Y'$ such that $\eta f = f'$ and $g = g'\eta$. Since one always can choose $Y' = K^{\dim Y}$, the equivalence classes form a set denoted by $\operatorname{Ext}^1_Q(Z, X)$ (it can be even seen that these sets are finite dimensional vector spaces).

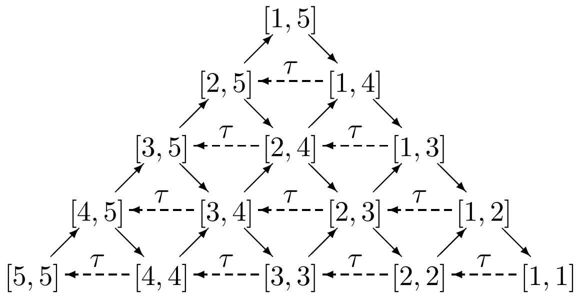
We say that a short exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ ***splits*** if g is a retraction. It is easily seen that this happens if and only if f is a section. An ***almost split sequence*** is a short exact sequence $0 \to W \xrightarrow{f} E \xrightarrow{g} V \to 0$ for which V and W are indecomposable and f is a ***source map***, that is f is not a section and any non-section $f' : W \to E'$ ***factors through*** f, and g is a ***sink map***, that is, g not a retraction and and any non-retraction $g' : E' \to V$ factors through g. Notice that an almost split sequence cannot be split.

The following result is the main theorem of the Auslander-Reiten theory, it shows that there is a very rich structure for the irreducible morphisms.

Theorem 3.5 (Auslander-Reiten)**.** *For any quiver Q, there exists a bijective map τ from the indecomposable non-projective representations to the indecomposable non-injective representations such that for each non-projective indecomposable V there exists an almost split sequence $0 \to \tau V \to E \to V \to 0$.*

The map τ is called the ***Auslander Reiten translate***. We show the Auslander-Reiten translate in (3.4):

(3.5)



Given a quiver Q, we define a new quiver Γ_Q, which has as vertices some chosen representatives of the isomorphism classes of indecomposable representations of Q (even if we cannot classify them) and for the arrows $U \to V$ chosen radical morphisms $f_1, \dots, f_d$ which are mapped to a base of $\operatorname{rad}_Q(U, V)/\operatorname{rad}_Q^2(U, V)$ under the canonical projection. This quiver is called the ***Auslander-Reiten quiver*** of Q. Some authors prefer not to choose representatives in order to make the assignation $Q \mapsto \Gamma_Q$ unique; they take the isomorphism classes of indecomposables as vertices of Γ_Q and take abstract symbols for the arrows.

Now, Γ_Q decomposes into components in a straightforward way: two objects X and Y lie in the same component if there is a "unoriented path" of irreducible morphisms between them, more precisely, if there are objects $X = Z_0, Z_1, \dots, Z_{t-1}, Z_t = Y$ such that for each $i = 1, \dots, t$ there is an arrow (an irreducible morphism) $Z_{i-1} \to Z_i$ or an arrow $Z_i \to Z_{i-1}$.

Although Γ_A encodes many information about the representations, it is not all information in the representation-infinite case since morphisms in the infinte radical remain unseen.

Corollary 3.6. *Let V be a non-projective indecomposable representation and*

$$0 \to \tau V \xrightarrow{a=[a_1 \ \dots \ a_t]^{\mathrm{tr}}} \bigoplus_{i=1}^{t} E_i \xrightarrow{b=[b_1 \ \dots \ b_t]} V \to 0$$

be an almost split sequence where $E_1, \dots, E_t$ are indecomposables. Then the morphisms a_i (resp. b_i) can be chosen as arrows in Γ_Q, and if done so, then they are all arrows starting in τV (resp. ending in V).

Proof. First note that the morphism a_i is radical, since otherwise it would be invertible and then a would be a section. Next, let $f : W =$

$\tau V \to X$ be an irreducible morphism into an indecomposable representation X. Then f is certainly not a section and hence factors over a, therefore there are morphisms $f_i' : E_i \to X$ such that $f = \sum_{i=1}^t f_i' a_i$. If all morphisms f_i' are radical, we could infer that $f \in \operatorname{rad}_Q^2(W, X)$ in contradiction to the irreducibility of f. Hence at least one f_j' is invertible. If we denote by $J \subseteq \{1, \dots, t\}$ the subset of indices for which $E_j \simeq X$ (to simplify notations, we shall assume $E_j = X$), we can write $f = (\sum_{j \in J} \lambda_j a_j) + f''$, where $f'' \in \operatorname{rad}_Q^2(W, X)$ and $\lambda_j \in K$ for $j \in J$. This shows that $\{a_j \mid j \in J\}$ is mapped to a generating set under the canonical projection to $\operatorname{rad}_Q(L, X)/\operatorname{rad}_Q^2(L, X)$.

Suppose that $\sum_{j \in J} \lambda_j a_j \in \operatorname{rad}_Q^2(L, X)$ for some scalars $\lambda_j \in K$. We express then $a_J = \sum_{j \in J} \lambda_j a_j$ in two different ways. First, $a_J = \rho f$ for some retraction $\rho : \bigoplus_{i=1}^t E_i \to \bigoplus_{j \in J} E_j$ and second, $a_J = gf$ for two radical morphisms g and $f = f'a$. Hence $(\rho - gf')a = 0$ and there is a morphism $h : M \to X$ such that $\rho - gf' = hb$. Since g and b are radical, we have that $\rho = gf' + hb$ is radical and simultaneously a retraction, a contradiction unless $\rho = 0$. Hence $a_J = 0$ and we see that a_j for $j \in J$ are linearly independent.

The statements for the morphism b are proved similarly. □

3.4. **Projective and injective representations.** Let Q be a quiver. A ***path of length*** l is a $(l + 2)$-tuple

$$w = (j|\alpha_l, \alpha_{l-1}, \dots, \alpha_2, \alpha_1|i) \tag{3.6}$$

which satisfies the following: its first and last entry are vertices and the other entries are arrows satisfying $s(\alpha_1) = i$, $t(\alpha_i) = s(\alpha_{i+1})$ for $i = 1, \dots, l-1$ and $t(\alpha_l) = j$. We allow $l = 0$ but then require that $j = i$. We extend the functions s and t in the obvious way: $s(w) = i$ and $t(w) = j$ if w is the path above. The ***composition*** of two paths $v = (i|\alpha_l, \dots, \alpha_1|h)$ and $w = (j|\beta_m, \dots, \beta_1|i)$ is then obviously declared as $wv = (j|\beta_m, \dots, \beta_1, \alpha_l, \dots, \alpha_1|h)$. Notice that we write the composition just in the same way as functions, which is not at all standard in the literature, but rather a question of the taste of each author. Since the case $l = 0$ is allowed in (3.6), we have one ***identity path*** $(i||i)$ for each vertex i.

For any representation V and any path $w = (j|\alpha_l, \dots, \alpha_1|i)$ of positive length we define $V(w) = V(\alpha_l) \dots V(\alpha_1) : V(i) \to V(j)$ and set $V(e_i) = \mathbf{1}_{V(i)}$. By induction on the length of the paths we get that a morphsim $f : V \to W$ always satisfies $f_h V(w) = W(w) f_i$ for any path $w \in KQ(i, h)$.

Let $KQ(j,i)$ be the vector space with basis all paths form i to j and extend the composition bilinearly. Also, for an arrow $\alpha : j \to h$ we denote by

$$KQ(i,\alpha) : KQ(i,j) \to KQ(i,h), w \mapsto \alpha w$$

the composition with α on the left. Whereas, for an arrow $\beta : h \to i$, the composition with β on the right is denoted by

$$KQ(\beta,j) : KQ(i,j) \to KQ(h,j), w \mapsto w\beta.$$

If we fix i, we therefore get two representations $P_i = KQ(i,?)$ and $I_i = \mathrm{D}\,K(?,i)$, where D always denotes dualization over the ground field K. Notice that all the representations P_i for $i \in Q_0$ are finite dimensional if and only if there is no ***oriented cycle*** in Q, that is a path of positive length which starts and ends in the same vertex. Since we want to deal only with finite dimensional representations we shall from now on assume that Q has no oriented cycle. The representations P_i and I_i are very important as we shall see now.

Exercise 3.7. Verify that for Q the linear quiver with n vertices we have $P_i \simeq [i,n]$ and $I_i \simeq [1,i]$. Observe also that in the picture (3.4), the representations P_i form the left border of the triangle and the representations I_i form the right border.

Lemma 3.8 (Yoneda's Lemma)**.** *For any representation V, the maps* $\mathrm{Hom}(P_i,V) \to V(i), f \mapsto f_i(e_i)$, *and* $\mathrm{Hom}(V,I_i) \to \mathrm{D}\,V(i), f \mapsto (x \mapsto (f_i(x))(e_i))$, *are bijections.*

Proof. Let us first verify that these maps make sense. Clearly, for $f \in \mathrm{Hom}_Q(P_i,V)$, we have $f_i : P_i(i) \to V(i)$ and since $e_i = (i||i) \in P_i(i) = KQ(i,i)$, we get $f_i(e_i) \in V(i)$. Similarly, if $f \in \mathrm{Hom}_Q(V,I_i)$ then $f_i : V(i) \to I_i(i) = \mathrm{D}\,KQ(i,i)$. So for any $x \in V(i)$, we get a linear map $f_i(x) : KQ(i,i) \to K$ and therfore $f_i(x)(e_i) \in K$.

Let $f \in \mathrm{Hom}_Q(P_i,V)$ and $x = f_i(e_i) \in V(i)$. We will show that f is completely determined by x, since $f_h(w) = f_h KQ(i,w)(e_i) = V(w)f_i(e_i) = V(w)(x) \in V(h)$ for any path $w \in KQ(i,h)$. Conversely, if $x \in V(i)$, we get a morphism $f \in \mathrm{Hom}_Q(P_i,V)$ by setting $f_h(w) = V(w)(x)$ for any h and any path $w \in KQ(i,h)$. The result for I_i is proved similarly. □

Lemma 3.9. *If Q has no oriented cycle then $\{P_i \mid i \in Q_0\}$ is a complete set of pairwise non-isomorphic representations, which are projective and indecomposable.*

Proof. Each P_i is indecomposable by Proposition 1.8, since $\operatorname{End}_Q(P_i) = \operatorname{Hom}_Q(P_i, P_i) \simeq K$ is local.

For an indecomposable projective representation V set $A = \bigoplus_{i \in Q_0} P_i^{d_i}$, where $d_i = \dim_K V(i)$ and observe that $\operatorname{Hom}_Q(A, V)$ contains a surjective morphism f (for each i, choose a base $v_1^{(i)}, \dots, v_{d_i}^{(i)}$ of $V(i)$ and define $f_j^{(i)} : P_i \to V$ by $f_j^{(i)}(e_i) = v_j^{(i)}$ using Yoneda's Lemma; then take $f_j^{(i)}$ as the entries of f as a matrix). Since V is projective, such an f is a retraction, that is, there exists $g : V \to A$ such that $fg = \mathbf{1}_V$. But then $A \simeq \operatorname{Ker} gf \oplus \operatorname{Im} gf \simeq Kergf \oplus V$. By Theorem 1.7, we get that V is isomorphic to P_i for some i.

On the other hand, for $i \neq j$, it is not possible that $KQ(i,j) \neq 0$ and $KQ(j,i) \neq 0$ since Q has no oriented cycle. Hence we have $\operatorname{Hom}_Q(P_i, P_j) = 0$ or $\operatorname{Hom}_Q(P_j, P_i) = 0$. In any case, P_i can not be isomorphic to P_j. □

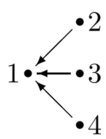
3.5. **Heredity.** By the above classification, we also get the following remarkable result.

Proposition 3.10. *Let Q be a quiver with no oriented cycle. A subrepresentation of a projective representation is projective again.*

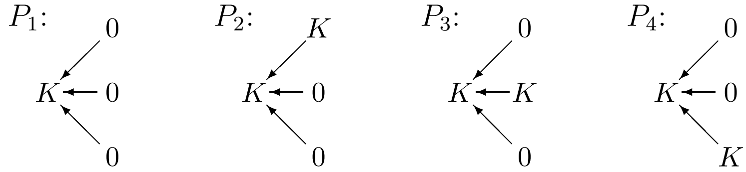
Proof. Also, if $V \subseteq P_i$ is a subrepresentation with $V(i) \neq 0$ then we have $V(i) = P_i(i)$ and therefore we must have $V = P_i$ since for any h and any path $w \in KQ(i,h)$, we have for the inlusion morphism $\iota : V \to P_i$ that $\iota_h(w) = \iota_h V(w) = KQ(i,w)\iota_i = KQ(i,w)$. But the subrepresentation V with $V(i) = 0$ and $V(h) = P_i(h)$ is isomorphic to $\bigoplus_{\alpha : i \to j} P_j$. Hence we get the following result. □

Exercise 3.11. If V is indecomposable and $f : V \to P_i$ is a non-zero morphism then $V \simeq P_j$ and if f is additionaly irreducible then f is given by the composition with some arrow $\alpha : i \to j$ on the right.

3.6. **Knitting with dimension vectors.** Now, we are ready to show how one can construct such a picture directly. Let us take an example which we have not considered yet, namely the quiver of exercise 2.2.

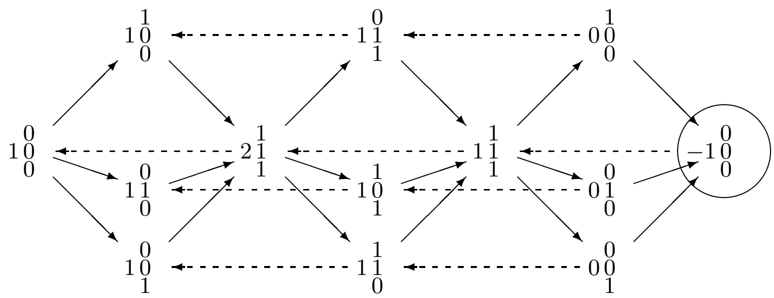


We start by calculating the left border, namely the projective indecomposables representations P_i, they are isomoprhic to:



From Lemma 3.8, we get that there is a morphism $P_1 \to P_i$ for $i = 2, 3, 4$ and these are all morphisms (up to scalar multiples) between the indecomposable projective representations which are not isomorphisms. Hence they must be irreducible by Exercise 3.11. Since P_1 is not injective we can consider the indecomposable representation $V = \tau^{-1}P_1$ and know that there exists an almost split sequence $0 \to P_1 \to E \to \tau^{-1}P_1 \to 0$. Now, $P_2 \oplus P_3 \oplus P_4$ must be a direct summand of E, but also no other summand can occur, since otherwise it would be non-projective and we would have an almost split sequence $0 \to \tau E \to P_1 \oplus F \to E \to 0$ and consequently a non-zero morphism $\tau E \to P_1$ which implies that τE is projective (Exercise 3.11). Hence, we must have $E = P_2 \oplus P_3 \oplus P_4$ by Corollary 3.6 and can calculate $\tau^{-1}P_1$ as quotient $(P_2 \oplus P_3 \oplus P_4)/P_1$, which must be indecomposable.

Instead of working with the representations themselves, we prefer to consider their ***dimension vectors***, that is, for a representation V, we consider the tuple $\underline{\dim}V = (\dim_K V(i))_{i \in Q_0}$. In our example, we have $\underline{\dim}V \in \mathbb{Z}^4$, but we write them conveniently not as a single column or row, but represent the shape of the quiver, for instance we have $\underline{\dim}\tau^{-1}P_1 = \begin{smallmatrix} & 1 \\ 2 & 1 \\ & 1 \end{smallmatrix}$. In a similar way, we can calculate $\underline{\dim}\tau^{-1}P_i = \underline{\dim}\tau^{-1}P_1 - \underline{\dim}P_i$ for $i = 2, 3, 4$ and then proceed to calculate $\underline{\dim}\tau^{-2}P_2 = \sum_{i=2}^{4} \underline{\dim}\tau^{-1}P_i - \underline{\dim}\tau^{-1}P_1 = \begin{smallmatrix} & 1 \\ 1 & 1 \\ & 1 \end{smallmatrix}$. This procedure is known as ***knitting***, and if we proceed with it, we get the following picture.



But of course, something is wrong with $\underline{\dim}\tau^{-3}P_1$ (the one in a circle) since it has a negative entry and therefore can not come from a representation. The only possible conclusion is that the reprentation $\tau^{-2}P_1$ is injective. Indeed, we have $\underline{\dim}\tau^{-2}P_1 = \underline{\dim}I_1$.

3.7. **Finite component.** As you can see, knitting with dimension vectors is really easy and tremendously powerful: it enabled us in the example to produce easily all indecomposable representations, since for each dimension vector appearing only one possible choice (up to isomorphism) for an indecomposable representation is possible.

But hold on, the previous is not clear at all, since we do not know that we indeed found *all* indecomposables. For there could be many undetected components out there. The following result will help us out.

Theorem 3.12 (Auslander)**.** *Suppose that the quiver Q is connected. If there exists a connected component C of Γ_Q which is finite then $C = \Gamma_Q$, in particular, Q is representation finite.*

Before we can enter the proof, we prove a useful tool.

Lemma 3.13. *For any indecomposable representation V of Q there exists n such that* $\mathrm{rad}_Q^n(V, V) = 0$.

Proof. Recall that $\mathrm{rad}_Q(V, V)$ is the subspace of nilpotent endomorphisms of V and write $R_i = \mathrm{rad}_Q^i(V, V)$ to simplify notations. The descending chain of finite-dimensional vector spaces $R_1 \supseteq R_2 \supseteq R_3 \supseteq \dots$ necesarily must get stationary at some point, say $R_n = R_{n+1}$. If $R_n = 0$ we are done.

So, suppose that $R_n \neq 0$. This means that any $f \in R_n$ can be written as $f = \sum_h r_h f_h$ for some $f_h \in R_n$ and some $r_h \in R_1$. However, it is possible to take always the same elements f_h, for instance by choosing a base of R_n. Let $F = \{f_1, \dots, f_t\} \subset R_n$ be a minimal set of elements such that any $f \in R_n$ can be written as $f = \sum_{h=1}^t r_h f_h$ for some $r_h \in R_1$.

Write $f_1 = \sum_h r_h f_h$ and hence $(1 - r_1)f_1 = \sum_{h=2}^t r_h f_h$. Since r_1 is nilpotent, $1 - r_1$ is invertible (the inverse is $1 + r_1 + r_1^2 + r_1^3 + \dots$) and therefore $r'_h = (1 - r_1)^{-1} r_h \in R_1$ for $2 \leq h \leq t$. But then $f_1 = \sum_{h=2}^t r'_h f_h$ and hence the set F was not minimal, a contradiction. □

Proof of Theorem 3.12. Denote by $V_1, \dots, V_m$ the (indecomposable) representations in C and let N be minimal such that $\mathrm{rad}_Q^N(V_i, V_i) = 0$ for all $1 \leq i \leq m$ (here Lemma 3.13 is used to ensure the existence of such an N). Then we must have that any composition of radical morphisms

$$V_{i_1} \xrightarrow{f_1} V_{i_2} \xrightarrow{f_2} V_{i_3} \to \dots \to V_{i_{d-1}} \xrightarrow{f_{d-1}} V_{i_d}$$

is zero for $d = m(N+1)$, since at least one V_j is repeated at least $N+1$ times and the composition of the corresponding subsequence (from the first occurrence to the $(N+1)$-th occurrence of V_j) is zero.

We show that this implies that for any $X \notin C$ and any $Y \in C$, we have $\mathrm{Hom}_Q(X, Y) = 0$. For if $f : X \to Y$ is a non-zero morphism, then it would factor through the sink map $\pi_Y : E \to Y$ ending in Y. Therefore there is at least one indecomposable direct summand $Y_1 \in C$ of E and radical morphisms $f_1 : X \to Y_1$ and $g_1 : Y_1 \to Y$ such that $g_1 f_1 \neq 0$. Iterating the argument for f_1, we obtain an indecomposable $Y_2 \in C$ and morphisms $f_2 : X \to Y_2$, $g_2 : Y_2 \to Y_1$ such that $g_1 g_2 f_2 \neq 0$. Hence $\mathrm{rad}^2(Y_2, Y) \neq 0$. Inductively, we find representation $Y_n \in C$ such that $\mathrm{rad}^n(Y_n, Y) \neq 0$ for any n, in contradiction to $\mathrm{rad}^d(Y_d, Y) = 0$. Similarly, we have $\mathrm{Hom}_Q(Y, X) = 0$ for any $Y \in C$ and any $X \notin C$.

Now, for a fixed $Y \in C$ there exists a projective indecomposable representation P admitting a non-zero morphism $P \to Y$. Hence P lies in C. Since Q is connected, inductively all other projective indecomposable representations belong to C. But then again, for any indecomposable representation X there exists a non-zero morphism $P \to X$ for some projective indecomposable representation and therefore X lies in C. □

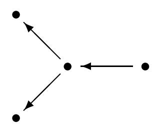
3.8. **Preprojective component.** A connected component C of the Auslander-Reiten quiver Γ_Q of Q is called ***preprojective*** (some authors prefer to use the term ***postprojective***) if each τ-orbit contains a projective indecomposable (and consequently there are only finitely many τ-orbits) and there are no cyclic paths in C. Dually, a component without cyclic paths in which each τ-orbit contains an injective and is called ***preinjective*** (nobody calls them "postinjective").

There are as many preprojective components in Γ_Q as there are connected components of Q and knitting of a preprojective component is always successful: starting with some simple projective in it, either the knitting goes on forever in each τ-orbits or in each τ-orbit an injective indecomposable is reached and then the component is finite and hence $C = \Gamma$ following Proposition 3.12.

Proposition 3.14. *If C is a preprojective or preinjective component of Γ_Q then for each indecomposable $X \in C$ we have $\mathrm{End}_A(X) \simeq K$ and $\mathrm{Ext}^1_A(X, X) = 0$.*

Proof. With a similar argument as in the proof of Theorem 3.12, we see that $\mathrm{Hom}_Q(X, Y) = 0$ whenever Y "lies to the left of X in C", meaning that there is no path in C from X to Y. As a consequence, we have $\mathrm{Hom}_Q(X, E) = 0$ if E is the middle term of the Auslander-Reiten sequence stopping at X and therefore $\mathrm{rad}_Q(X,X) = 0$ by Corollary 3.6, which in turn implies $\mathrm{End}_Q(X) \simeq K$. Moreover, if $0 \to X \xrightarrow{f} E \xrightarrow{g} X \to 0$ is a short exact sequence with $X \in C$ then g must be a retraction and therefore the sequence splits and its class is zero in $\mathrm{Ext}^1_Q(X,X)$. □

Exercise 3.15. Knit the preprojective component for the quiver



and compare the dimension vectors for the found indecomposable representations with the dimension vectors occuring in the Example 2.2.

4. Indepence of orientation

In this chapter, we want to classify those quivers which are representation finite. We shall see that the representation type does not depend on the orientation of the quiver, that is, if we reverse some of the arrows, we get a quiver of the same representation type. We shall need some tools before we are ready to understand the reason for this. At some points of this chapter, we will not argue very formally but provide a rather intuitive approach.

For the following we fix a quiver Q with no oriented cycle and, for the sake of simplicity, assume that $Q_0 = \{1, \dots, n\}$ is its set of vertices.

4.1. **Group action in a vector space.** Fix a dimension vector $d \in \mathbb{N}^n$. Then each representation V with dimension vector d is isomorphic to a representation of the form $M = \big((K^{d_i})_{i\in Q_0}, (M(\alpha))_{\alpha\in Q_1}\big)$, where, for each arrow $\alpha : i \to j$, $M(\alpha)$ is a matrix defining a linear map $K^{d_i} \to K^{d_j}$, that is $M(\alpha) \in K^{d_j \times d_i}$.

Thus in order to study the representations with that particular dimension vector, we can look at the vector space

$$\mathrm{rep}(Q,d) = \prod\nolimits_{(\alpha:i\to j)\in Q_1} K^{d_j\times d_i}, \tag{4.7}$$

whose elements we still call ***representations*** (of dimension d). Two representations $M, N \in \mathrm{rep}(Q,d)$ are isomorphic if and only if there exist a family $(f_i : K^{d_i} \to K^{d_i})_{i\in Q_0}$ of invertible linear maps, such that for every arrow $\alpha : i \to j$ in Q we have $N(\alpha)f_i = f_j M(\alpha)$. We can state this in a slightly different way. Let

$$\mathrm{GL}(Q,d) = \prod\nolimits_{i\in Q_0} \mathrm{GL}(d_i) \tag{4.8}$$

and define a group action of $\mathrm{GL}(Q,d)$ on $\mathrm{rep}(Q,d)$ by

$$g \cdot M = (g_j M_\alpha g_i^{-1})_{(\alpha:i\to j)\in Q_1}. \tag{4.9}$$

Then M and N are isomorphic if and only if they lie in the same orbit under the action of $\mathrm{GL}(Q,d)$. The orbit of M is denoted by $\mathcal{O}(M)$.

Now, let us look at the dimensions of $\mathrm{rep}(Q,d)$ and $\mathrm{GL}(Q,d)$. Clearly

$$\dim \mathrm{rep}(Q,d) = \sum\nolimits_{(\alpha:i\to j)\in Q_1} d_i d_j.$$

Now, $\mathrm{GL}(Q,d)$ is not a vector space, but an open set (for the ***Zariski topology***) in $G = \prod_{i\in Q_0} K^{d_i\times d_i}$ defined by a single polynomial inequality $\det f_1 \cdot \det f_2 \cdot \ldots \cdot \det f_n \neq 0$. Clearly, if $f \in \mathrm{GL}(Q,d)$, then there exists a ball $\{g \in G \mid \|f-g\| \leq \varepsilon\}$ completely contained in $\mathrm{GL}(Q,d)$ (here, we are thinking $K = \mathbb{R}$ or $\mathbb{C}$, and considering the euclidean norm $\|h\|$ for $h \in G$). Hence *locally* $\mathrm{GL}(Q,d)$ has dimension

$$\dim \mathrm{GL}(Q,d) = \sum\nolimits_{i\in Q_0} d_i^2.$$

To be a bit more formal: here, $\mathrm{GL}(Q,d)$ is an affine variety and $\dim \mathrm{GL}(Q,d)$ is its dimension as variety.

Now define $\chi_Q : \mathbb{Z}^n \to \mathbb{Z}$ by

$$\chi_Q(d) = \sum\nolimits_{i\in Q_0} d_i^2 - \sum\nolimits_{(\alpha:i\to j)\in Q_1} d_i d_j.$$

This is a cuadratic form, which satisfies

$$\chi_Q(d) = \dim \mathrm{GL}(Q,d) - \dim \mathrm{rep}(Q,d) \tag{4.10}$$

for each $d \in \mathbb{N}^n$.

4.2. **Quadratic forms.** For two representations X and Y of Q, we define

$$\langle X, Y\rangle_Q = \dim_K \mathrm{Hom}_Q(X,Y) - \dim_K \mathrm{Ext}^1_Q(X,Y) \in \mathbb{Z}.$$

This form has many names (depending on what generalization one is thinking of): it may be called ***homological form*** or ***Euler form*** or also ***Tits form*** or ***Ringel form***.

Lemma 4.1. *The number $\langle X,Y\rangle_Q$ depends only on the dimension vectors $x = \underline{\dim} X$ and $y = \underline{\dim} Y$, that is, there exists a bilinear form $b(-,-) : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$, which satisfies $\langle X,Y\rangle_Q = b(\underline{\dim} X, \underline{\dim} Y)$ for all representations X and Y.*

Proof. We already know that for any representation X there exists a projective representation P and a surjective morphism $f : P \to X$. By Proposition 3.10 $P' = \operatorname{Ker} f$ is projective again and hence we get a short exact sequence $0 \to P' \xrightarrow{\iota} P \xrightarrow{f} X \to 0$. Hence, by the fundamental theorem of homological algebra, we get a long exact sequence

$$0 \to \operatorname{Hom}_Q(X,Y) \to \operatorname{Hom}_Q(P,Y) \to \operatorname{Hom}_Q(P',Y) \to \operatorname{Ext}^1_Q(X,Y) \to 0$$

since $\operatorname{Ext}^1_Q(P,X) = 0$ (P is projective). By Lemma 3.9 we can assume that $P = \bigoplus_{i=1}^n P_i^{d_i}$ and $P' = \bigoplus_{i=1}^n P_i^{d'_i}$ for some $d_i, d'_i \in \mathbb{N}$.

The vectors $\underline{\dim}P_1, \dots, \underline{\dim}P_n$ are linearly independent since Q has no oriented cycle (order the vertices in such a way that $KQ(i,j) = 0$ if $i > j$, then the matrix $C_Q = (\dim_K KQ(i,j))_{ij}$ is upper triangular and has as i-th row vector exactly $\underline{\dim}P_i$). Therefore, it follows from $\underline{\dim}X = \underline{\dim}P - \underline{\dim}P' = \sum_{i=1}^n (d_i - d'_i)\underline{\dim}P_i$ that the numbers $c_i = d_i - d'_i$ are uniquely determined by $\underline{\dim}X$, namely $c = C_Q^{-1}\underline{\dim}X$.

The alternating sum of the dimension in the long exact sequence yields

$$\begin{aligned}
\langle X, Y\rangle_Q &= \dim_K \operatorname{Hom}_Q(P,Y) - \dim_K \operatorname{Hom}_Q(P',Y)\\
&= \sum\nolimits_{i=1}^n c_i \dim_K Y(i)\\
&= c^{\mathrm{tr}}\underline{\dim}Y\\
&= (\underline{\dim}X)^{\mathrm{tr}} C_Q^{-\mathrm{tr}}\underline{\dim}Y
\end{aligned}$$

and hence we can set $b(x,y) = x^{\mathrm{tr}} C_Q^{-\mathrm{tr}} y$ to get the result. $\square$

By abuse of notation, we shall denote $\langle x, y\rangle_Q = b(x,y)$ for $x, y \in \mathbb{Z}^n$.

The matrix C_Q is called ***Cartan matrix*** of Q.

Proposition 4.2. *For any representation V of Q, we have $\langle V, V\rangle_Q = \chi_Q(\underline{\dim}V)$.*

Proof. Denote by S_i the ***simple*** representation at the vertex i (it satisfies $S_i(I) = K$, $S_i(j) = 0$ for $j \neq i$ and $S_i(\alpha) = 0$ for any arrow α). By the above lemma, we have

$$\langle S_i, S_j\rangle = \dim_K \operatorname{Hom}_Q(S_i, S_j) - \dim_K \operatorname{Ext}^1_Q(S_i, S_j).$$

Now, $\dim_K \operatorname{Hom}_Q(S_i, S_i) = 1$ and $\dim_K \operatorname{Hom}_Q(S_i, S_j) = 0$ for $i \neq j$ and for all i, j we have that $\dim_K \operatorname{Ext}^1_Q(S_i, S_j)$ is the number of arrows from j to i. Hence be the above lemma, we get $\langle V, V\rangle_Q = \sum_{i,j} v_i v_j \langle S_i, S_j\rangle_Q = \sum_i v_i v_j - \sum_{\alpha: i\to j} v_i v_j$, which by definition equals $\chi_Q(\underline{\dim}V)$. $\square$

We will need also the following result.

Proposition 4.3. *For any positive definite quadratic form $q : \mathbb{Z}^n \to Z$ each fibre $q^{-1}(a)$ is finite.*

Proof. Let $q : \mathbb{Z}^n \to \mathbb{Z}, q(x) = \sum_{i=1}^n q_{ii}x_i^2 + \sum_{i<j} q_{ij}x_ix_j$ be positive definite. Denote by $q_\mathbb{Q} : \mathbb{Q}^n \to \mathbb{Q}$ and $q_\mathbb{R} : \mathbb{R}^n \to \mathbb{R}$ the obvious extensions of q. We show that $q_\mathbb{R}$ is also positive definite by contradiction. So let $x \in \mathbb{R}^n$, $x \neq 0$ be such that $q_\mathbb{R}(x) \leq 0$. Since $q_\mathbb{Q}(\frac{a}{b}) = \frac{1}{b^2}q(a)$, we see that $q_\mathbb{Q}$ is also positive definite and by continuity we have $q_\mathbb{R}(x) \geq 0$. Thus x is a global minimum of the function $q_\mathbb{R}$ and hence $(*)$ $\frac{\partial q_\mathbb{R}}{\partial x_i}(x) = 0$ for all $i = 1, \ldots, n$. But this means that x satisfies the system of linear equations $(*)$ non-trivially. But since all coefficients in $(*)$ are rational there exists also a non-trivial solution in $\mathbb{Q}^n$, a contradiction.

So $q_\mathbb{R}(x) > 0$ for any non-zero $x \in \mathbb{R}^n$ and therefore $\frac{1}{q_\mathbb{R}}$ is well defined on the compact $\mathrm{S}^1 = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ and takes there a global minimum m. Hence we have for all x with $q_\mathbb{R}(x) = a$ that $\frac{a}{\|x\|^2} = \frac{q_\mathbb{R}(x)}{\|x\|^2} = q_\mathbb{R}(\frac{x}{\|x\|}) \geq m$ and hence $\|x\| \leq \sqrt{\frac{a}{m}}$. Therefore there can only be finitely many vectors in $q_\mathbb{R}^{-1}(a) \cap \mathbb{Z}^n = q^{-1}(a)$. □

Any vector $x \in \mathbb{Z}^n$ such that $q(x) = 1$ is called a ***root*** of q.

4.3. **Dimension of the Orbits.** For a representation $M \in \operatorname{rep}(Q, d)$, we consider the surjective map

$$\mathrm{GL}(Q, d) \to \mathcal{O}(M), g \mapsto g \cdot M. \tag{4.11}$$

The fibre of M under this map is the ***stabilizer subgroup*** $\mathrm{GL}(Q, d)_M = \{g \in \mathrm{GL}(Q, d) \mid g \cdot M = M\}$ which has dimension

$$\dim \mathrm{GL}(Q, d)_M = \dim \mathrm{GL}(Q, d) - \dim \mathcal{O}(M). \tag{4.12}$$

This is the second crucial equation. We try to understand it very informally as before and we will be thinking that $K = \mathbb{R}$ or $\mathbb{C}$. Clearly, the identity $\mathrm{id} = (\mathbf{1}_{K^{d_i \times d_i}})_i$ belongs to the stabilizer. If we look on a small enough neighbourhood of id (like an open ball) then in some directions $g \cdot M$ may be equal to M, whereas in other directions this will not hold. A little bit more precise: for any $A \in G = \prod_{i \in q_0} K^{d_i \times d_i}$, the matrix $(\mathrm{id} + \varepsilon A)$ lies in $\mathrm{GL}(Q, d)$ for ε small enough and in that case $(\mathrm{id} + \varepsilon A) \cdot M$ has entries which are quadratic polynomials in ε, that is $(\mathrm{id} + \varepsilon A) \cdot M = M + \varepsilon A' + \varepsilon^2 A''$ for some matrices A' and A''. Now, the entries of A' depend linearly on A and therefore

$$A \mapsto \lim_{\varepsilon \to 0} \frac{(\mathrm{id} + \varepsilon A) \cdot M - M}{\varepsilon} = A'$$

is a linear function $\varphi : G = \prod_{i\in q_0} K^{d_i\times d_i} \to \mathrm{rep}(Q,d)$. The kernel of φ consists of the directions for which M is fixed under the action. Hence $\dim \mathrm{GL}(Q,d) = \dim_K \ker\varphi$ and on the other hand $\dim \mathcal{O}(M) = \dim \mathrm{Im}\,\varphi$ hence the result follows from the classical linear algebra formula since $\dim_K G = \dim \mathrm{GL}(Q,d)$.

We notice that $\mathrm{GL}(Q,d)_M$ always contains the one-dimensional subgroup consisting of the scalar multiples of the identity. Therefore $\dim \mathrm{GL}(Q,d)_M \geq 1$.

4.4. **Gabriel's Theorem.** We have now gathered enough material to be able to prove the following result, which is one of the founding stones of representation theory of algebras. A vector $x \in \mathbb{Z}^n$ is called ***positive*** if $x_i \geq 0$ for all i and $x \neq 0$. For a quadratic form q, we denote by $P(q)$ the set of positive roots of q. Also recall that we denote by Γ_Q the Auslander-Reiten quiver of Q.

Theorem 4.4 (Gabriel)**.** *Let Q be a quiver and denote by χ_Q its homological form. Then Q is representation-finite if and only if χ_Q is positive definite. Moreover, in that case, the function*

$$\Psi : (\Gamma_Q)_0 \to P(\chi_Q), X \mapsto \underline{\dim} X$$

is bijective.

Proof. First assume that χ_Q is not positive definite. Then there exists a non-zero vector d such that $\chi_Q(d) \leq 0$. Write $d = d^+ - d^-$ with $d_i^\pm \geq 0$ and $d_i^+ d_i^- = 0$ for any i. Then $0 \geq \chi_Q(d) = \chi_Q(d^+) + \chi_Q(d^-) - \sum q_{ij} d_i^+ d_j^-$, where the sum runs over all i, j such that $d_i^+ > 0$ and $d_j^- > 0$. Since $q_{ij} \leq 0$, we have thus that this sum is not negative and consequently $\chi_Q(d^+) \leq 0$ with $d^+ \neq 0$ or $\chi_Q(d^-) \leq 0$ with $d^- \neq 0$. In any case, there exists a *positive* vector d such that $\chi_Q(d) \leq 0$.

Hence by (4.10) and (4.12), we have $\dim \mathrm{rep}(Q,d) \geq \dim \mathrm{GL}(Q,d) = \dim \mathcal{O}(M) + \dim \mathrm{GL}(Q,d)_M > \dim \mathcal{O}(M)$ and hence there can be no orbit in $\mathrm{rep}(Q,d)$ with the same dimension than $\mathrm{rep}(Q,d)$ itself and consequently there must exist infinitely many orbits. That is, Q is not representation finite. This argument is usually referred to as the Tits-argument.

Now, suppose that χ_Q is positive definite. We will show that Q is representation finite. Observe that we can assume that Q is connected since the general case follows then easily. Since Q is hereditary it has a preprojective component C and any indecomposable representation $V \in C$ satisfies $\dim \mathrm{End}_Q(V) = 1$ and $\mathrm{Ext}^1_Q(V,V) = 0$ by Proposition

3.14. Hence, $\underline{\dim}V$ is a root of χ_Q. Now, let $V, W \in C$ be two indecomposable representations with the same dimension vector d. Then $1 = \chi_Q(d) = \langle \underline{\dim}V, \underline{\dim}W\rangle = \dim_K \operatorname{Hom}_Q(V,W) - \dim_K \operatorname{Ext}^1_Q(V,W)$ shows that $\operatorname{Hom}_Q(V,W) \neq 0$. Similarly, we have $\operatorname{Hom}_Q(W,V) \neq 0$ and since there is no cycle in C, we must have $V = W$. Thus it follows from Proposition 4.3, that this preprojective component is finite, and hence by Theorem 3.12 that Q is representation finite.

Having settled this, we assume now that Q is representation finite and χ_Q positive definite. Then the preprojective component C is the whole Auslander-Reiten quiver and hence $\underline{\dim}M$ is a positive χ_Q-root for any indecomposable representation and, as we have seen Ψ is injective since no two non-isomorphic indecomposable representations can have the same dimension vector.

Conversely, suppose that d is a positive χ_Q-root. Since there are only finitely many indecomposable representation in C, there are only finitely many combinations in which these representations can sum up to a representation V with $\underline{\dim}V = d$. Hence there are only finitely many orbits in $\operatorname{rep}(Q,d)$. One of them, say $\mathcal{O}(V)$, must necessarily satisfy $\dim \mathcal{O}(V) = \dim \operatorname{rep}(Q,d)$. But then it follows from $\dim \operatorname{rep}(Q,d) = \dim \operatorname{GL}(Q,d) - \chi_Q(d) = \dim \operatorname{GL}(Q,d)_V + \mathcal{O}(V) - 1$ that $\dim \operatorname{GL}(Q,d)_V = 1$, that is, V is indecomposable. Hence the function Ψ is surjective. $\square$

4.5. **Positive definite unit forms.** The previous result turns our atention to the positive definite forms, which we want to study in more detail here. Observe first, that the quadratic forms $\chi = \chi_Q$ for any representation-finite quiver Q satisfy $\chi_Q(e_i) = 1$ since $e_i = \underline{\dim}S_i$ is the dimension vector of an indecomposbale (even simple) representation.

Hence, if we write

$$\chi(x) = \sum_{i=1}^{n} \chi_{ii} x_i^2 + \sum_{i<j} \chi_{ij} x_i x_j$$

we have $\chi_{ii} = 1$. Such a form is called ***unit form***. Moreover, we can express the coefficients χ_{ij} for $i \neq j$ as follows

$$\begin{aligned}
\chi_{ij} &= \chi(e_i + e_j) - \chi(e_i) - \chi(e_j) \\
&= \langle e_i + e_j, e_i + e_j\rangle_Q - \langle e_i, e_i\rangle_Q - \langle e_j, e_j\rangle_Q \\
&= \langle e_i, e_j\rangle_Q + \langle e_j, e_i\rangle_Q \\
&= -\dim_K \operatorname{Ext}^1_Q(S_i, S_j) - \dim_K \operatorname{Ext}^1_Q(S_j, S_i),
\end{aligned}$$

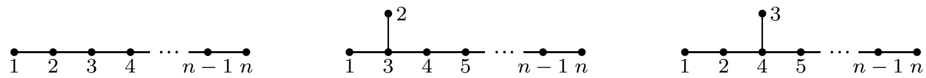
since $\mathrm{Hom}_Q(S_i, S_j)$ and $\mathrm{Hom}_Q(S_j, S_i)$ are both zero. Therefore $\chi_{ij} \leq 0$ for $i \neq j$ and $|\chi_{ij}|$ is the number of arrows between the two vertices i and j.

Theorem 4.4 shows therefore that representation-finiteness does not depend on the orientation, even more, the dimension vectors for which an indecomposbale representation may exist (and which is then unique up to isomorphism) are given independently of the orientation by the positive roots of χ_Q.

In general, we associate to a unit form $q : \mathbb{Z}^n \to \mathbb{Z}$, $q(x) = \sum_i x_i^2 + \sum_{i<j} q_{ij} x_i x_j$ a bigraph $\mathrm{B}(q)$, whose vertices are $1, \ldots, n$ and which has $|q_{ij}|$ ***dotted*** (resp. ***full***) edges between the vertices i and j if $q_{ij} \geq 0$ (resp. $q_{ij} < 0$).

The following graphs are called (***simply laced***) ***Dynkin diagrams***.

$\mathbb{A}_n$ $(n \geq 1)$: $\mathbb{D}_n$ $(n \geq 4)$: $\mathbb{E}_n$ $(n = 6, 7, 8)$:



If Q is a quiver whose underlying graph is Δ the we shall say that χ_Q has ***Dynkin type*** Δ.

Corollary 4.5. *The quiver Q is representation finite if and only if each connected component of the underlying graph of Q (obtained by forgetting the orientation of the arrows) is a Dynkin diagram.*

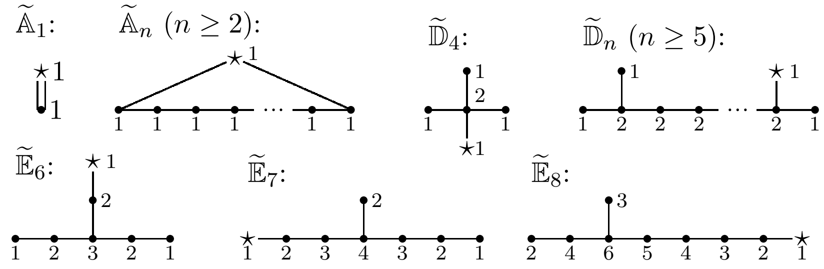
Proof. Let q be a unit form with $q_{ij} \leq 0$ for $i \neq j$. By Theorem 4.4 we only have to show that q is positive definite if and only if $\mathrm{B}(q)$ is a disjoint union of Dynkin diagrams. This can be done directly by the Lagrange algorithm, for instance, if $\mathrm{B}(q) = \mathbb{D}_n$ then

$$\begin{aligned} q(x) =& x_1^2 - x_1x_3 + x_2^2 - x_2x_3 + x_3^2 - x_3x_4 + x_4^2 - \ldots - x_{n-1}x_n + x_n^2 \\ =& (x_1 - \tfrac{1}{2}x_3)^2 + (x_2 - \tfrac{1}{2}x_3)^2 + \tfrac{1}{2}(x_3 - x_4)^2 + \ldots \\ & + \tfrac{1}{2}(x_i - x_{i+1})^2 + \ldots + \tfrac{1}{2}(x_{n-1} - x_n)^2 + \tfrac{1}{2}x_n^2, \end{aligned}$$

which shows that q is positive definite. The other cases are handled similarly. This shows that any quiver whose underlying graph is a Dynkin diagram is representation finite.

Now, assume that q is positive definite. Then $|q_{ij}| \leq 1$ for all $i \neq j$, since otherwise $q(e_i + e_j) \leq 0$. Next, we observe $\mathrm{B}(q)$ cannot contain an induced ***extended Dynkin diagrams*** (by some authors also called

Euclidean diagrams), i.e. a subgraph of the following list.



(4.13)

The reason is simple: if q is a unit form such that $\mathrm{B}(q)$ is one of the diagrams above, then $q(v) = 0$ for the non-zero vector v, which is defined by the numbers by the vertices above.

But a connected graph B, which does not contain an extended Dynkin diagram must be a Dynkin diagram: it must be a tree since no $\widetilde{\mathbb{A}}_n$ is contained. The degree of each vertex is less or equal than three since it does not contain $\widetilde{\mathbb{D}}_4$. The graph B is a ***star***, that is it has at most one vertex i of degree 3, since it does not contain $\widetilde{\mathbb{D}}_n$ for $n \geq 5$. Hence it is $\mathbb{A}_n$ or else a star with three arms of length $r \leq s \leq t$ (the length of an arm is counted by the vertices involved, including i). In the latter case we have $r = 2$ since B does not contain $\widetilde{\mathbb{E}}_6$; and $s \leq 3$ since B does not contain $\widetilde{\mathbb{E}}_7$; that means that $B = \mathbb{D}_n$ or else $B = \mathbb{E}_n$ for some $n \leq 8$ since it does not contain $\widetilde{\mathbb{E}}_8$. This finishes the proof. □

Remark 4.6. Readers which are familiar with the theory of Lie algebras should note that the roots of the positive definite unit forms χ_Q form a root system, where all roots have the same length. Therefore only the types $\mathbb{A}_n$, $\mathbb{D}_n$ and $\mathbb{E}_n$ occur in the Coxeter diagram, which is just the underlying graph of the quiver Q, as we have seen above.

The following table shows the number of positive roots of q for each Dynkin diagram $\Delta = \mathrm{B}(q)$.

$\Delta(q)$	$\mathbb{A}_n$	$\mathbb{D}_n$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$
$\lvert P(q)\rvert$	$\frac{n(n+1)}{2}$	$(n-1)n$	36	63	120

Since we know the indecomposable representations of the linearly oriented quiver, we can easily verify that there are $\frac{n(n+1)}{2}$ positive roots in case $\mathrm{B}(q) = \mathbb{A}_n$.

Exercise 4.7. Choose some orientations for the edges of $\mathbb{D}_6$ to get a quiver Q and knit the preprojective component (which is the whole

Auslander-Reiten quiver). Verify that there are 30 positive roots. Analize the roots and show that there are at least $(n-1)n$ positive roots of q if $\mathrm{B}(q) = \mathbb{D}_n$.

Exercise 4.8. Chose some orientation for the edges of $\mathbb{E}_8$ and verify that there are precisely 120 positive roots of q with $\mathrm{B}(q) = \mathbb{E}_8$. Count the positive roots x for which $x_8 = 0$ (resp. $x_7 = x_8 = 0$) to get that there are 63 (resp. 36) positive roots of q with $\mathrm{B}(q) = \mathbb{E}_7$ (resp. $\mathrm{B}(q) = \mathbb{E}_6$).

4.6. **Coxeter reflections and reflections by transposition.** We want to show a generalization of Gabriel's Theorem, namely the Theorem of Kac (Theorem 4.13) and prove part of it in the case that Q has no oriented cycle. Therefore we need some preparations, which we will provide in this section.

A vertex i of a quiver Q is called a ***source*** if no arrow of Q ends in i and it is called a ***sink*** if no arrow starts at i. If i is a sink, let $\sigma_i Q$ be the quiver which is obtained from Q by reversing the orientation of all arrows ending at i. Clearly, i is then a source of $\sigma_i Q$ and we can recover Q from $\sigma_i Q$ by the inverse construction denoted by σ_i^{-1}. Since we will consider different quivers having the same set of vertices $\{1, \dots, n\}$, we denote the functions $Q_1 \to Q_0$ which assign to an arrow its starting (resp. termination) vertex in Q by s_Q (resp. t_Q). For any vertex j let $Q_1(j \to) = \{\alpha \in Q_1 \mid s_Q(\alpha) = j\}$ and $Q_1(\to j) = \{\alpha \in Q_1 \mid t_Q(\alpha) = j\}$.

Assume now that i is a sink of a quiver Q. Then for any representation V of Q we define a representation $\mathrm{F}_i V$ of $Q' = \sigma_i Q$ as follows. Let $\mathrm{F}_i V(j) = V(j)$ for each vertex $j \neq i$ and $\mathrm{F}_i V(i) = \operatorname{Ker} V_{\to i}$, where $V_{\to i}$ is the following map

$$V_{\to i} : \bigoplus_{\alpha \in Q_1(\to i)} V(s_Q(\alpha)) \xrightarrow{[\cdots V(\alpha) \cdots]} V(i).$$

For $\beta \in Q_1'(i \to)$, the map $\mathrm{F}_i V(\beta) : \mathrm{F}_i V(i) \to \mathrm{F}_i V(t_{Q'}(\beta))$ is given by the inclusion $\operatorname{Ker} V_{\to i} \subseteq \bigoplus_{\alpha \in Q_1(\to i)} V(s_Q(\alpha))$ whereas for any other arrow β we have $\mathrm{F}_i V(\beta) = V(\beta)$.

In the following we want to describe the effect of this ***Coxeter reflection*** F_i on the dimension vectors. In order to do this, we define the ***reflection***

$$\rho_i : \mathbb{Z}^n \to \mathbb{Z}^n, \rho_i(x) = x - (x, e_i)_Q e_i, \tag{4.14}$$

where $(x,y)_Q = \langle x,y\rangle_Q + \langle y,x\rangle_Q$ and $\langle x,y\rangle_Q = x^{\mathrm{tr}}(C_Q^{-\mathrm{tr}}y$, being C_Q the Cartan matrix of Q. Since i is sink of Q, there is no loop at i and therefore $\rho_i(e_i) = -e_i$.

Lemma 4.9. *For any representation V for which $V_{\to i}$ is surjective we have $\underline{\dim}\,\mathrm{F}_i\,V = \rho_i(\underline{\dim}V)$.*

Proof. Clearly, for any vertex $j \neq i$ we have $(\underline{\dim}\,\mathrm{F}_i\,V)_j = (\underline{\dim}V)_j = (\rho_i(\underline{\dim}V))_j$. For $j = i$ we calculate $(\rho_i(\underline{\dim}V))_i = \dim_K V(i) - \langle\underline{\dim}V, e_i\rangle_Q = \dim_K V(i) - \sum_{j=1}^n \dim_K V(j)(e_j,e_i)$. Clearly $(e_i,e_i) = 2$ and for $j \neq i$ we know that $-(e_j,e_i) \geq 0$ counts the number of arrows between the two vertices i and j (in either direction). Hence $(\rho_i(\underline{\dim}V))_i = -\dim_K V(i) + \sum_{\alpha\in Q_1(\to i)} \dim_K V(s_Q(\alpha))$ is the dimension of $\mathrm{Ker}\,V_{\to i} = \mathrm{F}_i\,V(i)$. □

Remark 4.10. *If d is the dimension vector of some indecomposable representation V and $\rho_i(d) \notin \mathbb{N}^n$ where i is a sink of Q then $d = e_i$.*

Proof. An indecomposable representation V of Q either is isomorphic to S_i or satisfies that $V_{\to i}$ is surjective (otherwise V decomposes $V \simeq V' \oplus S_i^m$, where $m = \dim\mathrm{Coker}\,V_{\to i}$, and both summands are non-zero). But if $V_{\to i}$ is surjective then by Lemma 4.9, $\rho_i(d)$ is the dimension vector of the representation $\mathrm{F}_i\,V$, hence $\rho_i(d) \in \mathbb{N}^n$. Hence the result. □

For $d \in \mathbb{N}^n$, denote by $\mathrm{iso}_Q(d)$ the set of isomorphism classes of indecomposable representations of Q with dimension vector d. If i is a sink of Q, denote by $\mathrm{iso}_Q(d)_{\to i}$ the subset of $\mathrm{iso}_Q(d)$ given by those representations V for which $V_{\to i}$ is surjective. Similarly, if j is a source denote by $\mathrm{iso}_Q(d)_{j\to}$ the subset of $\mathrm{iso}_Q(d)$ given by those representations W for which

$$W_{j\to} : W(j) \xrightarrow{\begin{bmatrix}\vdots\\ W(\alpha)\\ \vdots\end{bmatrix}} \bigoplus_{\alpha\in Q_1(j\to)} W(t_Q(\alpha))$$

is injective (here $Q_1(j\to) = \{\alpha \in Q_1 \mid s_Q(\alpha) = j\}$).

Lemma 4.11. *If i is a sink of Q then F_i induces a bijective map from $\mathrm{iso}_Q(d)_{\to i}$ to $\mathrm{iso}_{\sigma_i Q}(\rho_i(d))_{i\to}$ which maps indecomposables to indecomposables.*

Proof. First of all, if $\varphi : V \to V'$ is an isomorphism of representations, then define $\psi(j) = \varphi(j)$ for all $j \neq i$ and let $\psi(i)$ be the unique map which turns the following diagram with exact rows commutative.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker} V_{\to i} & \xrightarrow{\text{incl}} & \bigoplus_{\alpha} V(s_Q(\alpha)) & \xrightarrow{V_{\to i}} & V(i) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi(i)} & & \downarrow{\scriptstyle \bigoplus_{\alpha} \varphi(s_Q(\alpha))} & & \downarrow{\scriptstyle \varphi(i)} & & \\
0 & \longrightarrow & \operatorname{Ker} V'_{\to i} & \xrightarrow[\text{incl}]{} & \bigoplus_{\alpha} V'(s_Q(\alpha)) & \xrightarrow[V'_{\to i}]{} & V'(i) & \longrightarrow & 0
\end{array}$$

Now, if V decomoses, say $V \simeq V' \oplus V''$ then $\operatorname{Ker} V_{\to i} \simeq \operatorname{Ker} V'_{\to i} \oplus \operatorname{Ker} V''_{\to i}$ and it is easy to see that $\mathrm{F}_i V \simeq \mathrm{F}_i V' \oplus \mathrm{F}_i V''$. The result follows now from the observation that there is an inverse construction, which satisfies the analogous properties. Namely, for a representation W of $\sigma_i Q$, we define a new representation $\mathrm{F}_i^{-1} W$ of Q by setting $\mathrm{F}_i^{-1} W(i) = \operatorname{Coker} W_{i\to}$ and $\mathrm{F}_i^{-1} W(j) = W(j)$ for all $j \neq i$. □

We shall need a second construction, which also assigns to each representation of Q a representation of $\sigma_i Q$ in case i is a sink of Q. This construction will not change the dimension vectors and is best explained if we reduce to representations given by matrices. Remember that each d-dimensional representations of Q is isomorphic to an element of $\operatorname{rep}(Q,d) = \prod_{\alpha \in Q_1} K^{d_{t_Q(\alpha)} \times d_{s_Q(\alpha)}}$. Similarly, if j is a source of Q, denote by $\operatorname{rep}(Q,d)_{j\to}$ the subset of $\operatorname{rep}(Q,d)$ given by those representations W for which $W_{j\to}$ is injective.

Given $V \in \operatorname{rep}(Q,d)$ define a new representation $\mathrm{T}_i V \in \operatorname{rep}(\sigma_i Q, d)$ by the following rule:

$$\mathrm{T}_i V(\beta) = \begin{cases} V(\beta) & \text{for } \beta \in \mathbb{Q}'_1 \text{ with } s_{Q'}(\beta) \neq i, \\ V(\beta)^{\mathrm{tr}} & \text{for } \beta \in \mathbb{Q}'_1 \text{ with } s_{Q'}(\beta) = i. \end{cases}$$

Call $\mathrm{T}_i V$ the representation obtained from V ***by transposition at*** i.

Lemma 4.12. *Let i be a sink of the quiver Q. Then T_i induces a bijection from $\operatorname{rep}(Q,d)_{\to i}$ to $\operatorname{rep}(\sigma_i Q, d)_{i\to}$ which maps isomorphic representations to isomorphic representations and indecomposable representations to indecomposable representations. Furthermore, its inverse satisfies the same properties.*

Proof. Let $V, V' \in \operatorname{rep}(Q,d)_{\to i}$ and assume that there exists an isomoprhism $\varphi : V \to V'$. Then set $\psi(j) = \varphi(j)$ for any $j \neq i$. Since $(\mathrm{T}_i V')_{i\to}$ is injective, the bijection $\bigoplus_{\alpha \in Q'_1(i\to)} \varphi(t(\alpha))$ induces a linear map $\psi(i) : V(i) \to V'(i)$ such that

$$\bigoplus_{\alpha \in Q'_1(i\to)} \varphi(t_{Q'}(\alpha)) \circ (\mathrm{T}_i V)_{i\to} = (\mathrm{T}_i V')_{i\to} \circ \psi(i).$$

Since the left hand side is injective, so must be $\psi(i)$ and hence $\psi(i) : V(i) \to V'(i)$ is even bijective since $\dim_K V(i) = \dim_K V'(i)$. This shows that $\psi : \mathrm{T}_i V \to \mathrm{T}_i V'$ is an isomorphism of representations of Q'.

If V decomposes, then let $V' \simeq V$ be a representation such that the matrices are given in block diagonal form. Then clearly $\mathrm{T}_i V'$ decomposes as well and therfore also $\mathrm{T}_i V$ since $\mathrm{T}_i V \simeq \mathrm{T}_i V'$. This shows that V is indecomposable if and only if $\mathrm{T}_i V$ is indecomposable. □

4.7. **Kac's Theorem.** Call a vertex i of Q ***loop-free*** if there is no arrow in Q which starts and ends in i. For any loop-free vertex j define the reflection as in (4.14). The ***Weyl group*** W is the subgroup of $\mathrm{GL}_n(\mathbb{Z})$ generated by these reflections.

Kac's Theorem shows for which dimension vectors indecomposable representations exist and these dimension vectors will be called "roots" again, but now they are defined differently as in the context of Gabriel's Theorem. By definition, the set of roots splits into two classes: the ***real roots*** are those vectors which may be obtained from a canonical base vector e_i for some loop-free vertex i by applying some $w \in W$. The ***imaginary roots*** are those which can be obtained by applying elements of the Weyl group to the ***fundamental region*** F. By definition, F consist of those non-zero vectors $x \in \mathbb{N}^n$ which have connected ***support*** (the support is the subquiver of Q induced by those vertices i for which $x_i > 0$) and for which $(x, e_i) \leq 0$ for all $i = 1, \dots, n$.

Theorem 4.13 (Kac)**.** *Let Q be a quiver.*

(i) *If V is an indecomposable representation of Q then $\underline{\dim} V$ is a root.*
(ii) *If x is a positive real root then there exists (up to isomorphism) exactly one indecomposable representation V with $\underline{\dim} V = x$.*
(iii) *If x is a positive imaginary root and the field K is algebraically closed then there exists infinitely many pairwise non-isomorphic representations V with $\underline{\dim} V = x$.*

We shall give an elementary proof of (i) and (ii) in case Q has no oriented cycle, in particular no loop.

Let Q be a quiver without oriented cycle, then the paths define a partial order in Q. Hence there must be at least one source and one sink. If i is a sink of Q then the quiver $\sigma_i Q$ again has no oriented cycle, as is easily seen by contradiction. So, we can iterate this construction. Let

ΣQ be the set of all quivers of the form

$$Q' = \sigma_{i_t}\sigma_{i_{t-1}} \dots \sigma_{i_1} Q$$

where i_1 is a sink of Q and inductively i_a is a sink of $\sigma_{i_{a-1}} \dots \sigma_{i_1} Q$.

Lemma 4.14. *For any vertex i of Q there exists a quiver $Q' \in \Sigma Q$ such that i is a sink of Q'.*

Proof. Let $I_1(Q)$ be the set of sinks of Q and $I_2(Q)$ the set of sinks of the quiver $Q \setminus I_1(Q)$, which is obtained from Q by removing all vertices of $I_1(Q)$ and all arrows ending in such vertices. Iteratively define the "a-th sink-slice" $I_a(Q)$ of Q to be the set of sinks of $Q \setminus (I_1(Q) \cup \ldots \cup I_{a-1}(Q))$. Observe that there can be now arrows in Q between two vertices of $I_a(Q)$.

Let a be such that $i \in I_a(Q)$. The proof is now done by induction on a. For $a = 1$ we have that i is a sink and we can take $Q' = Q$. So let $a > 0$. Now order the vertices in I_1 arbitrarily, say $I_1(Q) = \{i_1, \dots, i_r\}$. Since there are no arrows between any two vertices of $I_1(Q)$ we get that σ_{i_h} is a sink of $\sigma_{i_{h-1}} \dots \sigma_{i_1} Q$ for any $1 \le h \le r$. Hence $Q' = \sigma_{i_r} \dots \sigma_{i_1} Q \in \Sigma Q$ is a quiver for which $i \in I_{a-1}(Q')$ and therefore the result follows by induction. □

Exercise 4.15. If Q has no oriented cycle the each $Q' \in \Sigma Q$ has no oriented cycle and $\Sigma Q' = \Sigma Q$.

Denote by $\mathrm{ind}_Q(d)$ the set of isomorphism classes of indecomposable representations of Q with dimension vector d.

Proposition 4.16. *Let Q be a quiver without oriented cycle and $Q' \in \Sigma Q$. Assume $Q_0 = \{1, \dots, n\}$. Then for any $d \in \mathbb{N}^n$ and any element $w \in W$ such that $wd \in \mathbb{N}^n$ there exists a bijection $\mathrm{ind}_Q(d) \to \mathrm{ind}_{Q'}(wd)$.*

Proof. The proof is done in several steps.

Step 1. *The assertion is true in case $Q' = \sigma_i Q$ for some sink i of Q and $w = \mathrm{id}$.* If d is a multiple of the canonical base vector e_i then either $d = e_i$ and $\mathrm{ind}_Q(d)$ and $\mathrm{ind}_{Q'}(d)$ consist precisely of the class of S_i or d is a proper multiple of e_i and both sets $\mathrm{ind}_Q(d)$ and $\mathrm{ind}_{Q'}(d)$ are empty.

So assume that d is not a multiple of e_i. Then for every indecomposable representation V of Q of dimension d the map $V_{\to i}$ is surjective (since otherwise V decomposes $V = V' \oplus S_i^m$, where $m = \dim \mathrm{Coker}\, V_{\to i}$, and both summands are non-zero, in contradiction to the indecomposability of V). Similarly for any indecompoasable representation W of Q' of dimension d the map $W_{i\to}$ is injective. Hence the result follows from Lemma 4.12.

Step 2. *The assertion is true in case* $w = \mathrm{id}$. This follows by induction from Step 1.

Step 3. *The assertion is true in case* $Q' = Q$ *and* $w = \rho_i$. Let $Q'' \in \Sigma Q$ be such that i is a sink of Q''. By Step 2 we know that there are bijections $\mathrm{ind}_Q(d) \xrightarrow{\sim} \mathrm{ind}_{Q''}(d)$ and $\mathrm{ind}_{\sigma_i Q''}(\rho_I d) \xrightarrow{\sim} \mathrm{ind}_{Q'}(\rho_I d)$. Hence it remains to see that there is a bijection $\mathrm{ind}_{Q''}(d) \xrightarrow{\sim} \mathrm{ind}_{\sigma_i Q''}(\rho_i d)$. This follows from Lemma 4.11, since d can not be a multiple of e_i (otherwise $\rho_i(d) \notin \mathbb{N}^n$).

Step 4. *The assertion is true if* $\mathrm{ind}_Q(d)$ *is not empty.* Write $wd = \rho_{i_a}\rho_{i_{a-1}}\dots\rho_{i_1}d$ with t minimal. Set $w^{(b)} = \rho_{i_b}\dots\rho_{i_1}$ and $d^{(b)} = w^{(b)}d$. Inductively for $b = 1, 2, \dots$ we can apply Step 2 as long as $d^{(b)} \in \mathbb{N}^n$ we get from Step 3 that there is a bijection between $\mathrm{ind}_Q(d)$ and $\mathrm{ind}_Q(d)$ and $\mathrm{ind}_Q(d^{(b)})$.

If $d^{(b)} \notin \mathbb{N}^n$ for some b then choose b_1 minimal with that property. Since $\mathrm{ind}_Q(d^{(b_1-1)})$ is not empty, it follows from Remark 4.10 that $d^{(b_1-1)} = e_{j_1}$ for some j_1 and then $d^{(b_1)} = -e_{j_1}$. Set $x^{(b_1+1)} = \rho_{i_{b_1+1}}d^{(b_1-1)} = -d^{(b_1+1)}$ and for $b > b_1+1$ define inductively $x^{(b)} = \rho_{i_b}x^{(b-1)} = -d^{(b)}$. The sequence $x^{(b_1+1)}, x^{(b_1+2)}, \dots$ starts therefore with some positive vector and for $b = b_1, b_1+1, \dots$ we can apply Step 3 as long as $x^{(b)}$ is positive and get a bijection $\mathrm{ind}_Q(x^{(b)}) \xrightarrow{\sim} \mathrm{ind}_Q(d^{(b_1-1)}) \xrightarrow{\sim} \mathrm{ind}_Q(d)$. Since $x^{(t)} = -wd$ is not positive, there exists a minimal b_2 such that $x^{(b_2)} \notin \mathbb{N}^n$. Once again by Remark 4.10, we have $x^{(b_2-1)} = e_{j_2}$ for some j_2 and $x^{(b_2)} = -e_{j_2}$. Therefore, we have $d^{(b_2)} = -x^{(b_2)} = e_{j_2} = x^{(b_2-1)}$ and hence

$$wd = \rho_{i_t}\dots\rho_{i_{b_2+1}}\rho_{i_{b_2-1}}\dots\rho_{i_{b_1+1}}\rho_{i_{b_1-1}}\dots\rho_{i_1}d$$

which contradicts the minimality of t. This shows that $d^{(b)} \in \mathbb{N}^n$ for all $1 \le b \le t$ and therefore Step 4 follows.

Step5. *The general case.* If $\mathrm{ind}_Q(d)$ is not empty then the result is proved by Step 4. If $\mathrm{ind}_Q(d)$ is empty then $\mathrm{ind}_{Q'}(wd)$ must be empty too, since otherwise we could switch the roles of d and wd and get a contradiction. This finishes the proof of the proposition. □

Remark 4.17. The previous result states for $w = \mathrm{id}$ a certain independence on the orientation, which is not complete, that is, there are quivers Q and $Q' \notin \Sigma Q$ which have the same underlying graph. For example, if the underlying graph is just one (chordless) cycle, then $Q' \in \Sigma Q$ if and only if $m(Q') = m(Q)$, where $m(Q)$ counts the number of clockwise oriented arrows or the number of counterclockwise oriented arrows, whichever is smaller. This is in accordance with representation theory, as the "category of representations", see 5.2, of Q and Q' are "essentially the same" (which means here that the derived categories are equivalent) if and only if $m(Q) = m(Q')$.

For $x, y \in \mathbb{Z}^n$ we define a partial order by setting $x \leq y$ if $x_i \leq y_i$ for all i. Define $x < y$ if $x \leq y$ and $x \neq y$.

Proof of (i) and (ii) of Kac's Theorem in case Q has no oriented cycle. Let V be an indecomposable representation of Q and $d = \underline{\dim} V$. If there exists a vertex i such that $(d, e_i)_Q > 0$ then $\rho_i d < d$. Clearly $\rho_i d$ is a root if and only if d is a root and in case $\rho_i d \in \mathbb{N}^n$, we know by Proposition 4.16 that there exists an indecomposable representation W with $\underline{\dim} W = \rho_i d$.

Hence we can assume that either $(d, e_i) \leq 0$ for all i or that whenever $(d, e_i) > 0$ then $\rho_i d \notin \mathbb{N}^n$. In the first case d is in the fundamental region and hence an imaginatry root, in the second case $d = e_i$ by Remark 4.10 (the requirement for i to be a sink is not relevant by Step 2) and d is a real root. This shows (i).

Converesly, if d is a real root then we can write $d = we_i$ for some $w \in W$ and by Proposition 4.16, we have $\mathrm{ind}_Q(d) \simeq \mathrm{ind}_Q(e_i)$. So (ii) follows since we know that there exists precisely one indecomposable representation of dimension e_i. □

Remark 4.18. *For (iii) we prove only a very small portion. Supoose that the fundamental region is not empty and that $K = \mathbb{R}$ or $K = \mathbb{C}$. Then let d be an imaginary root which is minimal in the partial order in $\mathbb{Z}^n$. Then there are infinitely many pairwise non-isomorphic indecomposables of dimension d.*

Proof. Since $\dim \mathrm{GL}(Q, d) - \dim \mathrm{rep}(Q, d) = \chi_Q(d) \leq 0$ we have that for any $V \in \mathrm{rep}(Q, d)$ the orbit $\mathcal{O}(V)$ has dimension $\dim \mathcal{O}(V) = \dim \mathrm{GL}(Q, d) - \dim \mathrm{GL}(Q, d)_V \leq \dim \mathrm{rep}(Q, d) - 1$ which is smaller that the dimension of the enveloping space. Therefore there must be infinitely many orbits. But by our choice of d and (ii) there can only be finitely many orbits of decomposable representations. □

Remark 4.19. The advantage of this proof is that it does not make any assumptions on the field K. Statements (i) and (ii) are true for any field, whereas for (iii) some assumption has to be made.

Remark 4.20. *Gabriel's Theorem follows quite easily from what we proved of Kac's Theorem.*

Proof. If χ_Q is positive definite then the fundamental region must be empty (any vector x in the fundamental region satisfies $\chi_Q(x) \leq 0$, since $2\chi_Q(x) = (x,x)_Q = \sum_i x_i(x,e_i)_q)$). Hence we already get the one-to-one correspondence between roots (in the sense of Kac's Theorem) and the isoclasses of indecomposable representations. Since $\chi_Q(\rho_i d) = \chi_Q(d)$, we obtain that any root d belongs to the finite set $\chi_Q^{-1}(1)$.

If conversely Q is representation finite then there can only be finitely many roots. But then χ_Q must be positive definite, since otherwise there exists a non-zero vector $d \in \mathbb{N}^n$ for which $\chi_Q(d) \leq 0$. If $(d, e_i) > 0$ then again $\rho_i d < d$ but now $\rho_i d < 0$ is not possible. So after finitely many steps we get a vector in the fundamental region, and by Remark 4.18, Q is not representation finite, a contradiction. □

Remark 4.21. *Every real root is either positive or negative.*

Proof. This follows by the argument used in Step 4 in the proof of Proposition 4.16, together (ii). □

5. Connection with modules over algebras

In this last chapter the relationship of representations over a quiver with modules over algebras is inspected.

5.1. **The path algebra.** Recall that we denoted by $KQ(j,i)$ the vector space with basis all paths form i to j. Now we can define the ***path algebra*** KQ of the quiver Q to be

$$KQ = \bigoplus_{i,j \in KQ} KQ(j,i)$$

as vector space and define a multiplication on KQ by extending the composition of paths bilinearly, setting $vw = 0$ whenever $s(v) \neq t(w)$. We have then that KQ is an associative ring. It has a unit if and only if Q_0 is finite and then $1_{KQ} = \sum_{i \in Q_0}(i||i)$. Moreover, by $\lambda \mapsto \sum_{i \in Q_0} \lambda(i||i)$, the field K is mapped into the center of KQ. Note that in general KQ is not commutative, in the example of the linear quiver not unless $n = 1$. Observe also, that KQ is finite-dimensional if and only if Q is a finite quiver which has no oriented cycle.

Notice that in the example of the linearly ordered quiver, there is a path from i to j, and then up to scalar multiples just one path, if and only if $i \leq j$. Moreover, the algebra KQ is isomorphic to the lower triangular matrices of size $n \times n$, under the mapping induced by

$$(j|\alpha_{j-1}, \ldots \alpha_i|i) \mapsto \mathbf{E}^{(ji)},$$

where $\mathbf{E}^{(ji)}$ is the $n \times n$-matrix, whose unique non-zero entry equals 1 and sits in the j-th row and i-th column.

5.2. **Modules over the path algebra.** Assume that Q is a finite quiver. Given any representation V of Q, we can define a left KQ-module

$$V' = \bigoplus_{i \in Q_0} V_i$$

by defining the multiplication V'_w with a path $w = (j|\alpha_l, \dots \alpha_1|i)$ on a family $(v_h)_{h \in Q_0}$ as the family having $V(\alpha_l) \dots V(\alpha_1)(v_i)$ in the j-th coordinate and zero elsewhere. Notice that V' is always finite-dimensional, since Q is finite and by definition a representation has finite-dimensional vector spaces attached to each vertex. Conversely, given a finite-dimensional left KQ-module M, we define $M(i) = e_i M = \{e_i m \mid m \in M\}$. We have then $M = \bigoplus_{i \in Q_0} M(i)$ and can easily define a representation by setting $M(\alpha) : M(i) \to M(j), e_i m \mapsto (j|\alpha|i)m$ for any arrow $\alpha : i \to j$ in Q.

If $\varphi : V \to W$ is a morphism of representations, then we define $\varphi' = \bigoplus_{i \in Q_0} \varphi_i : V' \to W'$, which is a homomorphisms of KQ-modules. Conversely, any homomorphism $\psi : M \to N$ of finite-dimensional KQ-modules gives rise to a morphism of representations by $\psi(i) : M(i) \to N(i), e_i m \mapsto \psi(e_i m) = e_i \psi(m)$. The direct sum of KQ-modules correspond to the direct sum of representations and therefore their indecomposables correspond one-to-one (up to isomorphism).

Example 5.1. There are 15 indecomposable A-modules if A is the algebra of lower triangular 5×5-matrices, since $A \simeq KQ$, where Q is the linear quiver with 5 vertices.

5.3. **The categorical language.** We have seen that dealing with representations and mophisms of representations of a quiver Q amounts to the same thing as dealing with modules and homomorphimss of modules over the path algebra KQ.

In categorical language: the ***categories*** $\operatorname{rep} Q$ of representations of Q and $\operatorname{mod} KQ$ are ***equivalent***. A category $\mathcal{C}$ is a class of ***objects*** together with sets of ***morphisms*** $\mathcal{C}(X,Y)$ for each pair of objects (X,Y) and composition maps $\mathcal{C}(Y,Z) \times \mathcal{C}(X,Y) \to \mathcal{C}(X,Z), (g,f) \mapsto g \circ f$ which are associative and admit ***identity*** morphisms $1_X \in \mathcal{C}(X,X)$ for each object X. We write $X \in \mathcal{C}$ to express the fact that X is an object of the category $\mathcal{C}$. Two objects $X, y \in \mathcal{C}$ are ***isomorphic*** if there exist morphisms $f \in \mathcal{C}(X,Y)$ and $g \in \mathcal{C}(Y,X)$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$.

Example 5.2. The representations of a quiver Q form a category where the morphism sets are denoted by $\mathrm{Hom}_Q(V, W)$ for $V, W \in \mathrm{rep}\, Q$. For a finite-diemsnional algebra A, the finite-dimensional (left) A-modules form a category $\mathrm{mod}\, A$, where the morphisms are called homomorphisms and form sets denoted by $\mathrm{Hom}_A(M, N)$ for $M, N \in \mathrm{mod}\, A$.

If $\mathcal{C}$ and $\mathcal{D}$ are two categories, a functor $F : \mathcal{C} \to \mathcal{D}$ assigns to each object $X \in \mathcal{C}$ an object $FX \in \mathcal{D}$ and to each morphism $g \in \mathcal{C}(X, Y)$ a morphism $Fg \in \mathcal{D}(FX, FY)$, in such a way that $F1_X = 1_{FX}$ for each object $X \in \mathcal{C}$ and $F(g \circ f) = FG \circ Ff$ for any $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$. The functor F is an ***equivalence*** if for any two objects $X, Y \in \mathcal{C}$ the map $\mathcal{C}(X, Y) \to \mathcal{D}(FX, FY), g \mapsto Fg$ is bijective and every object of $\mathcal{D}$ is isomoprhic to FX for some object $X \in \mathcal{C}$.

Exercise 5.3. Show that if F is an equivalence, then two non-isomorphic objects of $\mathcal{C}$ are mapped to two non-isomorphic objects of $\mathcal{D}$.

Two categories $\mathcal{C}$ and $\mathcal{D}$ are called ***equivalent*** if there exists an equivalence $F : \mathcal{C} \to \mathcal{D}$.

Thus, we can rephrase our findings of Section 5.2 in the following terms.

Proposition 5.4. *The category* $\mathrm{rep}\, Q$ *of representation of a quiver without oriented cycle is equivalent to the category* $\mathrm{mod}\, KQ$ *of finite-dimensional (left)* KQ*-modules.*

5.4. **Morita equivalence.** Two finite dimensional algebras A and B are called ***Morita equivalent*** if the categories $\mathrm{mod}\, A$ and $\mathrm{mod}\, B$ are equivalent.

Example 5.5. Let A be a finite-dimensional algebra and $B = A^{2\times 2}$ be the algebra of 2×2-matrices with entries in A. Then A and B are Morita equivalent. Indeed, for $M \in \mathrm{mod}\, A$ define $FM = M \oplus M$ as B-module where the multplication is given by the matrix multiplication. Then for any $g \in \mathrm{Hom}_A(M, N)$ we define $Fg = g \oplus g \in \mathrm{Hom}_B(FM, FN)$. It is straightforward to see that F defines a functor. Clearly, $\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_B(FM, FN)$ are injective. So, let $h \in \mathrm{Hom}_B(FM, FN)$ and define for $i, j \in \{1, 2\}$ the linear maps $h_{ij} = \pi_j h \iota_i : M \to N$, where $\pi_j : M \oplus M \to M$ is the canoncial projection and $\iota_i : M \to M \oplus M$ is the canonical inclusion.

Let $\mathbf{E}^{(ji)} \in B$ be the 2×2-matrix, whose unique non-zero entry equals 1_A and sits in the j-th row and i-th column. Since h is a B-module homomorphism, we have $h(\mathbf{E}^{(11)}m) = \mathbf{E}^{(11)}h(m) = (h_{11}m_1 + h_{12}m_2, 0)$ on one hand and $h(\mathbf{E}^{(11)}m) = h(m_1, 0) = (h_{11}m_1, h_{21}m_1)$ on the other.

Therefore $h_{12} = h_{21} = 0$ and infer then from $h(\mathbf{E}^{(12)}m) = \mathbf{E}^{(12)}h(m)$ that $h_{11} = h_{22}$ and hence $h = Fh_{11}$.

Given a finite-dimensional algebra A, consider it as a left module over itself and decompose it into indecomposables $A = \sum_{i=1}^{n} P_i$, which are projective as is easily seen. The algebra A is called ***basic*** if P_i is not isomorphic to P_j for $i \neq j$. The next result shows that if we are only interested in the module category, then we can reduce our atention to basic algebras. But first some general considerations. Therefore assume that we have given some decomposition $A = \bigoplus_{i=1}^{n} P_i$ into indecomposable A-modules and denote by $\pi_j : A \to P_j$ the canonical projection and by $\iota_j : P_j \to A$ the canoncial inclusion, which are homomorphisms of A-modules.

Then $e_j = \iota_j \pi_j(1_A)$ is an ***idempotent***, that is, it satisfies $e_j^2 = e_j$ (this follows from $\pi_j \iota_j = 1_{P_j}$). Moreover, these idempotents are pairwise ***orthogonal***, that is $e_i e_j = 0$ for $i \neq j$ (this follows from $\pi_i \iota_j = 0$) and finally each e_j is a ***primitive*** idempotent, that is, it is non-zero and cannot be written as a sum of two non-zero orthogonal idempotents (this follows from the indecomposability of P_j).

Observe that $\iota_j : P_j \to Ae_j$ is an isomorphism. For any A-module M, we have that $\mathrm{Hom}_A(Ae_j, M) \to e_j M, \psi \mapsto \psi(e_j)$ is a bijection whose inverse maps $e_j m$ to the homomorphism $ae_j \mapsto ae_j m$ (in analogy to Lemma 3.8), again called Yoneda Lemma.

Proposition 5.6. *Every finite dimensional algebra is Morita-equivalent to a basic finite-dimensional algebra.*

Proof. Let A be a finite-dimensional algebra and $A = \bigoplus_{i=1}^{n} Ae_i$ the decomposition into indecomposables. If A is not basic, renumber the summands Ae_i such that $Ae_1 \simeq Ae_2$.

Let $e = e_2 + \dots e_n = 1 - e_1$ and set $B = eAe = \{eae \mid a \in A\}$ which is clearly closed under addition and multiplication and $1_B = e$ since e is an idempotent. We will show that A and B are Morita-equivalent. Clearly, this implies the result by induction on the number of summands in the decomposition of A.

For an A-module M define $FM = eM = \{em \mid m \in M\}$. This is a B-module, since for any $b = eae \in B$ and any $x \in FM$ we have $bx = e(ax) \in eM$. Also, if $f \in \mathrm{Hom}_A(M, N)$ define $Ff \in \mathrm{Hom}_B(FM, FN)$ by $Ff(em) = f(em) = ef(m) \in eN$. It is easily verified that F is a functor.

Now, an isomorphism $\varphi : Ae_1 \to Ae_2$ corresponds to an element $a_{12} = e_1a_{12}e_2$ and its inverse φ^{-1} to an element $a_{21} = e_2a_{21}e_1$ and they satisfy $a_{12}a_{21}e_1e_1$ and $a_{21}a_{12}e_2 = e_2$.

By Yoneda's Lemma, we have $e_1M \simeq e_2M$ as vector spaces and the left multiplications $\mu_i : A \times e_iM \to M$ for $i = 1, 2$ are ***coupled*** in the following sense

$$\mu_1(a, e_1m) = \mu_2(aa_{12}, e_2a_{21}e_1m).$$

This shows that we can define an A-module structure on $M' = e_2M \oplus eM$ from the B-module structure on eM such that $M' \simeq M$ as A-modules. Therefore any object in $\operatorname{mod} B$ is isomorphic to an object FM for some $M \in \operatorname{mod} A$.

Any homomorphism $f \in \operatorname{Hom}_A(M, N)$ induces linear maps $f_i : e_iM \to e_iN, e_im \mapsto f(e_im) = e_if(m)$. But in our situation, the map f_1 is completely determined by f_2 since we have $f_1(e_1m) = f(a_{12}a_{21}e_1m) = a_{12}f_2(e_2a_{21}e_1m)$. Now, it is easy to see that the map $\operatorname{Hom}_A(M, N) \to \operatorname{Hom}_B(FM, FN), f \mapsto Ff$ is bijective. □

5.5. **Quotients of a path algebra.** Notice that the path algebra KQ of a quiver Q is ***graded***, that is $KQ = \bigoplus_{n\geq 0}(KQ)_n$ as vector spaces, where $(KQ)_n$ is the subspace of KQ generated by the paths of length n and the multiplication in KQ induces bilinear maps $(KQ)_n \times (KQ)_m \to (KQ)_{n+m}$.

An ideal I of KQ is called ***admissible*** if $(KQ)_N \subseteq I \subseteq (KQ)_2$ for some N.

Theorem 5.7 (Gabriel)**.** *Every basic finite dimensional algebra A is isomorphic to the quotient KQ/I of a path algebra of some quiver Q modulo an admissible ideal I of KQ.*

Proof. Let $e_1, \dots, e_n$ be a set of pairwise orthogonal primitive idempotents. Then define a category $\mathcal{C}$ whose objects are $1, \dots, n$ and whose morphism spaces are $\mathcal{C}(i, j) = e_jAe_i = \{e_jae_i \mid a \in A\}$. Since the idempotents are primitive, the algebras $\mathcal{C}(i, i)$ are local and therefore i is indecomposable in $\mathcal{C}$. Since A is basic, the objects of $\mathcal{C}$ are pairwise non-isomorphic.

Since there are only finitely many objects, all of which have a local endomorphism algebra, we conclude as in the proof of Lemma 3.13, that if $\operatorname{rad}^n_{\mathcal{C}}(i, j) = \operatorname{rad}^{n+1}_{\mathcal{C}}(i, j)$ then $\operatorname{rad}^n_{\mathcal{C}}(i, j) = 0$ and that for some, sufficiently large N we have $\operatorname{rad}^N_{\mathcal{C}}(i, j) = 0$ for all i, j.

Now, let Q be the quiver of $\mathcal{C}$, that is, the objects of Q are the objects of $\mathcal{C}$, that is the vertices $1, \ldots, n$ and there are $\dim_K \left(\operatorname{rad}_{\mathcal{C}}(i,j)/\operatorname{rad}^2_{\mathcal{C}}(i,j)\right)$ arrows from i to j in Q labeled by some chosen irreducible morphisms $\alpha_1, \ldots, \alpha_{d_{ij}} \in \mathcal{C}(i,j) = e_j A e_i$ which are linearly independent modulo $\operatorname{rad}^2_{\mathcal{C}}(i,j)$.

Then we get a homomoprhism of algebras $\lambda : KQ \to A$ which maps the arrow $i \to j$ labeled by α to $\alpha \in e_j A e_i \subseteq A$ and the identity morphism $1_i = (i||i)$ to e_i. Clearly, the map λ is surjective. Let I be the kernel of λ. Then $A \simeq KQ/I$. Since the morphisms which label the arrows are linearly independent, the restriction of λ to $(KQ)_0 \oplus (KQ)_1$ is injective and therefore $I \subseteq (KQ)_2$. On the other hand, we must have $(KQ)_N \subseteq I$ if N is as above with $\operatorname{rad}^N_{\mathcal{C}}(i,j) = 0$ for all i, j. This shows that I is admissible. □

Remark 5.8. *The quiver Q is uniquely determined by A.*

Proof. It is easy to prove that for any idempotent e, the module Ae is projective and that $\operatorname{Hom}_A(Ae, M) \to eM$, $f \mapsto f(e)$ is a bijection (this is the general version of Yoneda's Lemma). From this it follows that Ae is indecomposable if and only if eAe is local, which happens if and only if e is primitive. Hence the number of vertices of Q equals the number of isoclasses of indecomposable projective A-modules, which clearly does not depend on any choice.

Let $Ae_1, \ldots, Ae_n$ be representatives of the isoclasses of indecomposable projective A-modules. Then the number of arrows $i \to j$ in the quiver Q equals the dimension of the space $\operatorname{rad}_A(Ae_i, Ae_j)/\operatorname{rad}^2_A(Ae_i, Ae_j)$, which again is uniquely determined by A. □

An algebra (or more generally a ring) A is called ***hereditary*** if every submodule of a projective module is projective.

Proposition 5.9. *A finite dimensional algebra is hereditary if and only if it is Morita equivalent to a path algebra of some quiver without oriented cycle.*

Proof. We already know by Proposition 3.10 that any path algebra over a quiver without oriented cycle is hereditary.

Suppose now that A is an algebra which is basic, finite dimensional and hereditary. Write $A = KQ/I$, where Q is the quiver of A and I an admissible ideal of KQ. We have to show that $I = 0$. Let $Ae_1, \ldots, Ae_n$ be a complete set of pairwise non-isomorphic indecomposable projective A-modules. Then $Ae_i = KQ(i, ?)/Ie_i =: P_i$.

Let $f : P_i \to P_j$ be a non-zero morphism. Then $\operatorname{Im} f \subseteq P_j$ is a a projective submodule since A is hereditary. Hence the surjective morphism $P_i \to \operatorname{Im} f$ is a retraction and therefore $\operatorname{Im} f$ a direct summand of P_i. Since P_i is indecomposable, we have that $P_i \simeq \operatorname{Im} f$ and f is injective. Thus, if f is not an isomorphism then $\dim_K P_i < \dim_K P_j$. This implies that Q cannot contain an oriented cycle and therefore KQ is finite-dimensional.

Let R_i be the submodule of P_i generated by the classes of the paths of positivce length. Since I is admissible, $I \subseteq (KQ)_2$ and therefores, for each arrow $i \to j$, there exists an injective morphism $\alpha_* : P_j \to P_i$ which sends the class of a path γ to the class of $\gamma\alpha$. Recall that $Q_1(i \to) = \{\alpha \in Q_1 \mid s(\alpha) = i\}$ and observe that $g = \big[\cdots \alpha_* \cdots\big] : \bigoplus_{\alpha \in Q_1(i\to)} P_{t(\alpha)} \to R_i$ is surjective, but if any direct summand of the domain, say $P_{t(\alpha)}$, is dropped, then the resulting morphism g' is not surjective anymore, since the class of α cannot be in the image of g'. This shows that $R_i \simeq \bigoplus_{\alpha \in Q_1(i\to)} P_{t(\alpha)}$.

Now it follows by induction on $\dim_K P_i$ that $P_i = KQ(i, ?)$, that is $Ie_i = 0$. Hence $I = 0$. □

5.6. **Modules over other algebras.**

Proposition 5.10. *Let A be an algebra, I an ideal of A and $B = A/I$. Then $\operatorname{mod} B$ is a full subcategory of $\operatorname{mod} A$.*

Proof. Let $\pi : A \to B$ be the canonical projection, which is a homomorphism of algebras. Then any B-module M can be viewed as an A-module, where the multiplication is given by $a \cdot m = \pi(a)m$. If $f \in \operatorname{Hom}_B(M, N)$ then for all $a \in A$ we have $f(a \cdot m) = f(\pi(a)m) = \pi(a)f(m) = a \cdot f(m)$, which shows that $f \in \operatorname{Hom}_A(M, N)$. Hence we get a functor $\operatorname{incl} : \operatorname{mod} B \to \operatorname{mod} A$, which is injective on objects and morphisms since π is surjective.

It only remains to be seen that for any two B-modules M and N, each $g \in \operatorname{Hom}_A(M, N)$ satisfies $g((a + I)m) = g(a \cdot m) = a \cdot g(m) = (a + I)g(m)$ and therefore defines a homomorphism of B-modules. □

Apply this to the situation where $A = KQ$ and I is an admissible ideal. We know from Proposition 5.4 that the KQ-modules can be understood as representations of Q. Therefore it is quitre natural to ask which such representations correspond to $B = KQ/I$-modules? The answer is quite simple, we just need some vocabulary to formulate it.

Let I be an (adimissible) ideal of KQ and write $I(i,j) = I \cap KQ(i,j)$. Each $\gamma \in I(i,j)$ is a linear combination of paths which start in i and end in j, say $\gamma = \sum_{h=1}^{t} \lambda_h \gamma_h$, where $\lambda_h \in K$ and $\gamma_h = (j|\alpha_{hm_h}, \ldots, \alpha_{h1}|i)$ is a path from i to j. Then we can define

$$V(\gamma) = \sum_{h=1}^{t} \lambda_h V(\alpha_{hm_h}) \ldots V(\alpha_{h1}) : V(i) \to V(j)$$

A representation V of the quiver ***satisfies*** the ideal I if for each i, j and each $x \in I(i,j)$ we have $V(x) = 0$.

Let $\operatorname{rep}_I Q$ be the full subcategory of $\operatorname{rep} Q$ given by all representations which satisfy the ideal I.

Proposition 5.11. *Let Q be a finite quiver and I be an admissible ideal of KQ. Then the category* $\operatorname{mod}(KQ/I)$ *is equivalent to the category* $\operatorname{rep}_I Q$.

Proof. A KQ-module M is a KQ/I-module precisely when $IM = 0$. This happens under the equivalence given in Section 5.2 exactly when the corresponding representation satisfies I. □

Exercise 5.12. Let Q be the linear quiver with 5 points and $I \subset KQ$ the ideal generated by all paths of length ≥ 3. Determine the Auslander-Reiten quiver of $B = KQ/I$.

As it seems, all what we have to understand properly are representations of quivers in order to understand finite dimensional modules over any finite dimensional algebra. However we should warn here that this could be misleading since in many cases the quiver of some interesting algebra is wild, whereas the algebra itself is tame.

Bibiography

For general theory of rings and modules you might look at

- **F. Anderson and K. Fuller:** *Rings and categories of modules.* Graduate Texts in Mathematics, 13. Springer-Verlag, New York, 1992.

The following books give an introduction to representation theory each with a different emphasis (the first on Auslander-Reiten theory, the second on the representation finite case and the third on tame, in particular tubular, algebras and on the use of quadratic forms):

- **M. Auslander, I. Reiten and S. Smalø:** *Representation theory of Artin algebras.* Cambridge Studies in Advanced Mathematics, 36. Cambridge University Press, Cambridge, 1997.
- **P. Gabriel and A. V. Roĭter:** *Representations of finite-dimensional algebras.* With a chapter by B. Keller. Encyclopaedia. Math. Sci., 73, Algebra VIII, Springer-Verlag, Berlin, 1992.
- **C. M. Ringel:** *Tame algebras and integral quadratic forms.* Lecture Notes in Mathematics, 1099. Springer-Verlag, Berlin, 1984.

For readers interested in geometrical aspects in representation theory, we recommend one of the following texts.

- **H. Kraft:** *Geometric methods in representation theory.* Representations of alegbras (Puebla 1980), pp. 180-258, Lecture Notes in Mathematics, 944. Springer-Verlag, Berlin - New York, 1982.
- **W. Crawley-Boevey:** *Geometry of representations of algebras.* Available from his homepage (`http://www.amsta.leeds.ac.uk/~pmtwc/`).

Proofs of Kac's Theorem can be found for example in the following texts (and also a sketch of it is contained in the above preprint by W. Crawley-Boevey):

- **V. Kac:** *Root systems, representations of quivers and invariant theory.* Invariant theory, pp. 74-108. Lecture Notes in Mathematics 996, Springer-Verlag, Berlin 1983.
- **H. Kraft and Ch. Riedtmann:** *Geometry of representations of quivers.* Representations of algebras (Durham, 1985), pp. 109-145. London Math. Soc. Lecture Notes Series, 116. Cambridge University Press, Cambridge, 1986.

We mention two more instructive intruductions, the first to the Auslander-Reiten theory, which includes a description of the knitting technique and the second on the use of quadratic forms and geometrical methods.

- **P. Gabriel:** *Auslander-Reiten sequences and representation-finite algebras.* Representation theory I (Proc. Workshop, Carleton Univ. Ottawa, Ont. 1979), pp. 1-71. Lecture Notes in Mathematics 831. Springer-Verlag, Berlin 1980.
- **J. A. de la Peña:** *Quadratic forms and the representation type of an algebra.* Preprint E90-003, Bielefeld, 1990.