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Integral Quadratic Forms and the Representation Type of an Algebra

(Lecture 2)

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INTEGRAL QUADRATIC FORMS AND THE REPRESENTATION TYPE OF AN ALGEBRA

LECTURE 2

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2. The geometric approach.

2.1. Some elements of algebraic geometry.

We consider the affine space $V = k^n$ with the *Zariski topology*, that is, closed sets are of the form

$$Z(p_1, \dots, p_s) = \{v \in V : p_i(v) = 0\}$$

where $p_i \in k[t_1, \dots, t_n]$ is a polynomial in n indeterminates.

- $S \subset k[t_1, \dots, t_n]$, then $Z(S)$ is the zero set of S.
- $Z(S) = Z(\langle S \rangle) = Z(\sqrt{\langle S \rangle})$, where

$$\begin{aligned} \langle S \rangle &= \text{ideal of } k[t_1, \dots, t_n] \text{ generated by } S \\ \sqrt{I} &= (\text{radical of } I) = \{p \in k[t_1, \dots, t_n] : p^i \in I \text{ for some } i \in \mathbb{N}\} \end{aligned}$$

- $Z\left(\bigcup_{i \in I} S_i\right) = \bigcap_{i \in I} Z(S_i)$ and $Z(S \cdot S') = Z(S) \cup Z(S')$
- *Hilbert's basis theorem*: $\exists\, p_1, \dots, p_s \in S$ with $Z(S) = Z(p_1, \dots, p_s)$
- *Hilbert's Nullstellensatz*: $\{p \in k[t_1, \dots, t_n] : p \equiv 0 \text{ on } Z(S)\} = \sqrt{\langle S \rangle}$

  We say that $Z = Z(S)$ is an *affine variety* and $k[Z] = k[t_1, \dots, t_n]/\sqrt{\langle S \rangle}$ is its *coordinate ring*.

An affine variety $Z = Z(p_1, \ldots, p_s)$ is *reducible* if $Z = Z_1 \cup Z_2$ with proper closed subsets $Z_i \subset Z$. Otherwise Z is *irreducible*.

- There is a finite decomposition of any affine variety $Z = \bigcup_{i=1}^{s} Z_i$ into irreducible subsets $Z_i \subset Z$. If the decomposition is irredundant, we say that $Z_1, \ldots, Z_s$ are the *irreducible components* of Z.
- If Z is an irreducible variety, then the maximal length of a chain

$$\emptyset \neq Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_s = Z$$

is called the *dimension* of Z (=: $\dim Z$).

If $Z = \bigcup_{i=1}^{s} Z_i$ is an irreducible decomposition

$$\dim Z = \max_i \dim Z_i.$$

A map $\mu\colon Y \to Z$ between affine varieties is a *morphism* (a *regular map*), if $\mu^*\colon k[Z] \to k[Y]$, $p \mapsto p \circ \mu$ is well-defined. In fact, μ^* is a k-algebra homomorphism.

- Any morphism $\mu\colon Y \to Z$ is continuous.
- A map $\mu\colon Y \to Z$ is a morphism if and only if $\exists\, \mu_1, \ldots, \mu_m \in k[t_1, \ldots, t_n]$ such that $\mu(y) = (\mu_1(y), \ldots, \mu_m(y))$, $\forall\, y = (y_1, \ldots, y_n) \in Y \subset k^n$.

Proposition. *Let $\mu\colon Y \to Z$ be a morphism between irreducible affine varieties and assume μ is dominant (i.e. $\overline{\mu(Y)} = Z$). Then for every $z \in Z$ and every irreducible component C of $\mu^{-1}(Z)$ we have*

$$\dim C \geq \dim Y - \dim Z$$

with equality on a dense open set of Z.

In particular, if C is an irreducible component of $Z(p_1, \ldots, p_t) \subset k^n$, we have

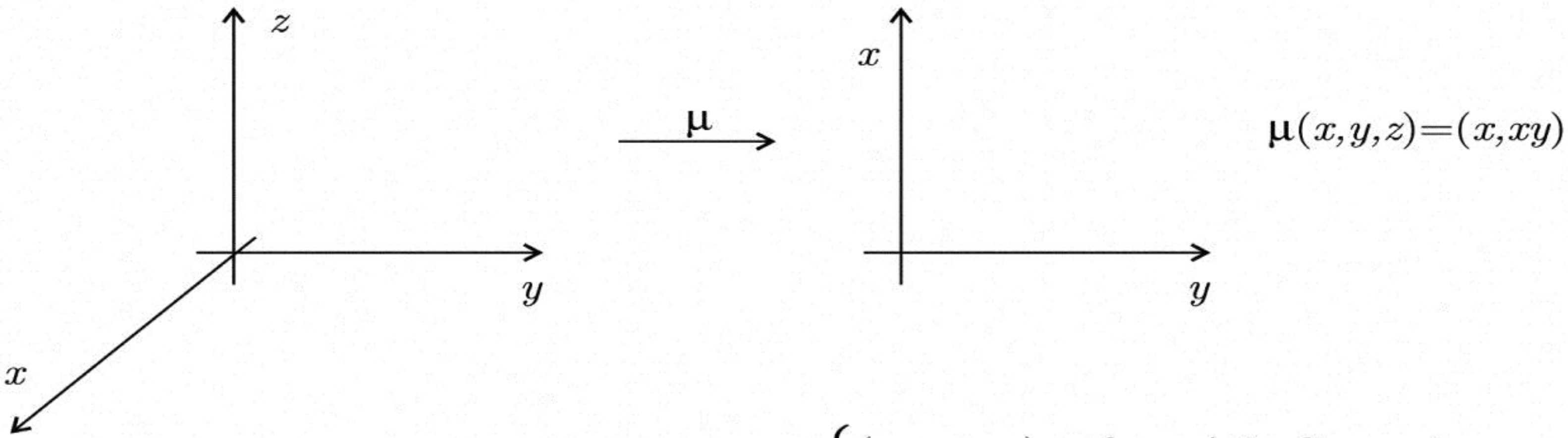
$$\dim C \geq n - t$$

A fundamental result is the following

Theorem (Chevalley) *Let $\mu\colon Y \to Z$ be a morphism between affine varities. Then the function*

$$y \mapsto \dim_y \mu^{-1}(\mu(y)) = \max\{\dim C \colon y \in C \text{ irreducible component of } \mu^{-1}(\mu(y))\}$$

is upper semicontinuous (that is, $d\colon Y \to \mathbb{N}$ has $\{y \in Y \colon d(y) < n\}$ open in Y, for all $n \in \mathbb{N}$).



$$\mu^{-1}(\mu(x_0, y_0, z_0)) = \mu^{-1}(x_0, x_0 y_0) = \begin{cases} (x_0, y_0, x) & \text{if } x_0 \neq 0,\ \dim = 1 \\ (0, y, z) & \text{if } x_0 = 0,\ \dim = 2 \end{cases}$$

A general morphism $\mu\colon Y \to Z$ is neither open nor closed, but $\mu(Y)$ is a finite union of locally closed subsets of Z.

A finite union of locally closed subsets of a variety Z is called a *constructible* subset.

Proposition. *If $\mu\colon Y \to Z$ is a morphism and $Y' \subset Y$ a constructible subset, then $\mu(Y')$ is also constructible.*

2.2. The main example: module varieties.

Let $A = kQ/I$ be a finite dimensional k-algebra and fix a finite set L of admissible generators of I. Let $z \in \mathbb{N}^{Q_0}$ be a dimension vector.

The *module variety* $\mathrm{mod}_A(z)$ is the closed subset, with respect to the Zariski topology, of the affine space $k^z = \prod\limits_{i \to j} k^{z(j)z(i)}$ defined by the polynomial equations given by the entries of the matrices

$$m_r = \sum_{i=1}^{t} \lambda_i m_{\alpha i1} \ldots m_{\alpha i s_i}, \text{ where } r = \sum_{i=1}^{t} \lambda_i \alpha_{i1} \ldots \alpha_{i s_i} \in L$$

and for each arrow $x \xrightarrow{\alpha} y$, m_α is the matrix of size $z(y) \times z(x)$.

$$m_\alpha = (X_{\alpha ij})_{ij}$$

where $x_{\alpha ij}$ are pairwise different indeterminates. We shall identify points in the variety $\mathrm{mod}_A(z)$ with representations X of A with vector dimension $\mathbf{dim}\,\mathbf{X} = z$.

Example: $A = kQ/I$ where $Q\colon \bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet$ and $I = \langle \alpha\beta \rangle$

$$\begin{pmatrix} x_{\alpha 11} & x_{\alpha 12} \\ x_{\alpha 21} & x_{\alpha 22} \end{pmatrix} \begin{pmatrix} x_{\beta 11} & x_{\beta 12} \\ x_{\beta 21} & x_{\beta 22} \end{pmatrix} = \begin{pmatrix} x_{\alpha 11}x_{\beta 11} + x_{\alpha 12}x_{\beta 21} & x_{\alpha 11}x_{\beta 12} + x_{\alpha 12}x_{\beta 22} \\ x_{\alpha 21}x_{\beta 11} + x_{\alpha 22}x_{\beta 21} & x_{\alpha 21}x_{\beta 12} + x_{\alpha 22}x_{\beta 22} \end{pmatrix}$$

$\mathrm{mod}_A(2,2,2) \subset k^{2\times 2} \times k^{2\times 2} = k^8$ defined by 4 equations.

The group $G(z) = \prod_{i \in Q_0} GL_{z(i)}(k)$ acts on k^z by conjugation, that is, for $X \in k^z$, $g \in G(z)$ and $x \xrightarrow{\alpha} y$, then $X^g(\alpha) = g_y X(\alpha) g_x^{-1}$.

By restriction of this action, $G(z)$ also acts on $\mathrm{mod}_A(z)$. Moreover, there is a bijection between the isoclasses of A-modules X with $\mathbf{dim}\, X = z$ and the $G(z)-$orbits in $\mathrm{mod}_A(z)$.

Given $X \in \mathrm{mod}_A(z)$, we denote by $G(z)X$ the $G(z)$-orbit of X. Then

$$\dim G(z)X = \ dim\ G(z) - \ \dim \mathrm{Stab}_{G(z)}(X),$$

where the *stabilizer* $\mathrm{Stab}_{G(z)}(X) = \{g \in G(z) : X^g = X\} = \ \mathrm{Aut}_A(X)$ is the group of automorphisms of X. As $\mathrm{Aut}_A(X)$ is an open subset of the affine variety $\mathrm{End}_A(X)$, then

$$\dim \mathrm{Stab}_{G(x)}(X) = \dim \mathrm{Aut}_A(X) = \dim \mathrm{End}_A(X).$$

Finally, we get

$$\dim G(z)X = \dim G(z) - \dim \mathrm{End}_A(X).$$

also that an orbit $G(z)X$ is *locally closed*, that is $G(z)X$ is open in the closure $G(z)X$ defined in $\mathrm{mod}_A(z)$. In particular, $G(z)X \backslash G(z)X$ is formed by the union of orbits of dimension strictly smaller than $G(z)X$.

Let $X, Y \in \text{mod}_A(z)$. If the orbit $G(z)Y$ is contained in $\overline{G(z)X}$, we say that Y is a *degeneration* of X.

Proposition. *Let $X \in \text{mod}_A(z)$. We have the following.*

(a) *Let $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ be an exact sequence. Then $X' \oplus X''$ is a degeneration of X.*

(b) *Consider the semisimple module $gr\ X = \bigoplus_{i \in Q_0} S_i^{z(i)}$, obtained as direct sum of the composition factors of X. Then $gr\ X$ is a degeneration of X.*

Proof.

(a) We may assume that X' is a submodule of X and $X'' = X/X'$. Then for each arrow $i \xrightarrow{\alpha} j$, we have

$$X(\alpha) = \begin{pmatrix} X'(\alpha) & f_\alpha \\ 0 & X''(\alpha) \end{pmatrix},$$

where $f_\alpha : X''(i) \longrightarrow X'(j)$. For each $\lambda \in k$, we may define the representation $X_\lambda \in \text{mod}_A(z)$, with

$$X_\lambda(\alpha) = \begin{pmatrix} X'(\alpha) & \lambda f_\alpha \\ 0 & X''(\alpha) \end{pmatrix}.$$

For $\lambda \neq 0$, we get $X_\lambda \simeq X$. Indeed,

$$g_\lambda = \begin{pmatrix} I_{z'(i)} & 0 \\ 0 & \lambda I_{z''(i)} \end{pmatrix}_i \in G(z)$$

satisfies that $X_\lambda^{g_\lambda} = X$. Therefore □

$$X' \oplus X'' = X_0 =\in \overline{G(z)X}.$$

Corollary. *The orbit $G(z)X$ is closed if and only if X is semisimple.*

Examples:

(a) Let $F = k\langle T_1, ..., T_m\rangle$ be the free algebra in m indeterminates. Let M be a $A - F-$bimodule which is free as right $F-$module.

Then the functor $M \otimes_F - : \text{mod}_F \longrightarrow \text{mod}_A$ induces a family of regular maps $f_M^n : \text{mod}_F(n) \to \text{mod}_A(nz)$ for some vector $z \in \mathbb{N}^{Q_0}$ and every $n \in \mathbb{N}$.

Indeed, for each vertex $i \in Q_0$, fix a basis of the free right $F-$module $M(i)$, set $z(i) = rk_F M(i)$. Then for an arrow $i \xrightarrow{\alpha} j$ in Q, $M(\alpha) : M(i) \longrightarrow M(j)$ is a $z(j) \times z(i)-$matrix with entries in F. Now, an element $\lambda = (\lambda_1, ..., \lambda_m) \in \text{mod}_F(n)$ determines an $F-$module N_λ with $N_\lambda(T_i) = \lambda_i, i = 1, ..., m$. Then

$$M \otimes_F N_\lambda(\alpha) : (k^{z(i)})^n \longrightarrow (k^{z(j)})^n$$

is the matrix $M(\alpha)(\lambda) = (M(\alpha)_{st}(\lambda_1, ..., \lambda_m))_{s,t}$. Therefore

$$f_M^n(\lambda) = (M(\alpha)_{st}(\lambda_1, ..., \lambda_m))_{s,t}$$

is the induced regular map.

(b) Let C be a finitely generated commutative $k-$algebra without nilpotent elements and $z \in \mathbb{N}^{Q_0}$. For any regular map $g : \text{mod}\,_C(1) \longrightarrow \text{mod}_A(z)$, there is a $A - C$-bimodule M which is free as right $C-$module and $rk_C(M)(i) = z(i)$, for each $i \in Q_0$, such that $g = f^1_M$.

Indeed, from Hilbert's theorem $C = k[\text{mod}_C(1)]$ is the affine algebra of regular functions on $\text{mod}_C(1)$. We define $M(i) = C^{z(i)}$, for $i \in Q_0$; for $i \xrightarrow{\alpha} j$ in Q, we put $M(\alpha)$ the matrix corresponding to $g(\alpha) : \text{mod}_C(1) \longrightarrow k^{z(j)z(i)}$. By (a), $f^1_M = g$.

(c) Consider the subset $\text{ind}_A(z)$ of $\text{mod}_A(z)$ $\text{ind}_A(z)$ is a constructible subset of $\text{mod}_A(z)$. Indeed, the set of pairs.

$\{(X, f) : X \in \text{ mod}_A(z), f \in \text{ End}_A(X) \text{ with } 0 \neq f \neq 1_X \text{ and } f^2 = 1_X\}.$

is a locally closed subset of $\text{mod}_A(z) \times k^{d^2}$, where $d = \sum_{i \in Q_0} z(i)$. The projection

$\pi_1 : \text{mod}_A(z) \times k^{d^2} \longrightarrow \text{mod}_A(z)$ is a regular map with image

$$\text{mod}_A(z) \backslash \text{ind}_A(z).$$

(d) Let $z \in \mathbb{N}^{Q_0}$. Let C be an irreducible component of $\text{mod}_A(z)$. A decomposition $z = w_1 + ... + w_s$ with $w_i \in \mathbb{N}^{Q_0}$ determines a constructible subset

$$C(w_1, ..., w_s) = \{X \in C : X = X_1 \oplus ... \oplus X_s \text{ with } X_i \in \text{ ind}_A(w_i)\}$$

in C. We say that $(w_1, \ldots, w_s)$ is a *generic decomposition* in C if $C(w_1, ..., w_s)$ contains an open and dense subset of C.

Proposition. *Let C be an irreducible component of* $\mathrm{mod}_A(z)$, *then there exists a unique generic decomposition* $(w_1, \dots, w_s)$ *in* C. *Moreover, there exists an irreducible component* C_i *of* $\mathrm{mod}_A(w_i)$ *such that the generic decomposition in* C_i *is* (w_i) *and the following inequality holds:*

$$\dim G(z) - \dim C \geq \sum_{i=1}^{s} (\dim G(w_i) - \dim C_i).$$

Proof: For each decomposition $z = z_1 + ... + z_t$ with $z_i \in \mathbf{N}^{Q_0}$ we get a regular map

$$\varphi_{z_1 \dots z_i} : G(z) \times \mathrm{mod}_A(z_1) \times ... \times \mathrm{mod}_A(z_t) \longrightarrow \mathrm{mod}_A(z), (g, (X_i)_i) \longmapsto (\oplus_{i=1}^{t} X_i)^g.$$

Since $\mathrm{ind}_A(z_i) = \{Y \in \mathrm{mod}_A(z_i) : Y$ is indecomposable$\}$ is constructible in $\mathrm{mod}_A(z_i)$, then

$$\mathrm{ind}_A(z_1, ..., z_t) = \varphi_{z_1, \dots z_t}(G(z) \times \mathrm{ind}_A(z_1) \times ... \times \mathrm{ind}_A(z_t))$$

is constructible in $\mathrm{mod}_A(z)$. Moreover, $\mathrm{mod}_A(z) = \cup\{\mathrm{ind}_A(z_1, ..., z_t) : \sum z_i = z\}$. There is a decomposition $z = w_1 + \cdots + w_s$ such that C equals the closure of the intersection $\mathrm{ind}_A(w_1, \dots, w_s) \cap C$. There is an open dense subset U_C of C contained in $\mathrm{ind}_A(w_1, ..., w_s)$. Thus $z = w_1 + ... + w_s$ is generic in C. The unicity is clear. □

2.3. The tangent space.

Suppose $V \subset k^n$ is defined by certain polynomials $f(T_1, ..., T_n)$. For $x \in V$, define

$$d_x f = \sum_{i=1}^{n} \frac{\partial f}{\partial T_i}(x)(T_i - x_i)$$

the *derivative of* f *at the point* x. Then the *tangent space* of V at x is the linear variety $T_x(V)$ in the k^n defined by the vanishing of all $d_x f$ as $f(T)$ ranges over the polynomials in the radical ideal $\mathcal{I}(V)$ defining V.

There are more algebraic ways to define tangent spaces: let $R = k[V]$ be the affine algebra associated with V and M_x be the maximal ideal of R vanishing at x. Since R/M_x can be identified with k and M_x is a finitely generated R-module, then then R/M_x-module M_x/M_x^2 is a finite dimensional k-vector space.

Then $(M_x/M_x^2)^*$ the dual space over k may be identified with $T_x(V)$.

Some facts and examples:

(a) Let $x \in V$ and C_x be any irreducible component of X containing x. Then we have $\dim_k T_x(V) \geq \dim C_x$. If equality holds, x is called a *simple point of* V. If all points of V are simple, we say that V is *smooth*. An important fact:

- the simple points of V form an open dense subset of V.

(b) Consider the variety $\text{mod}_A(z)$ as a topological space. The orbit $G(z)X$ of a point $X \in \text{mod}_A(z)$ is a smooth space. Indeed, given two points x, y in the orbit, there is an element g of the group $G(z)$ such that $y = gx$. The regular map $\ell_g : G(z)X \longrightarrow G(z)X$ given as right multiplication by g, induces a linear isomorphism $T\ell_g : T_x(G(z)X) \longrightarrow T_y(G(z)X)$. Therefore x is a simple point of the orbit if and only if so is y. Thus (a) implies that $G(z)X$ is smooth.

The following is an important result:

Theorem. (Voigt) *Let* $X \in \operatorname{mod}_A(z)$.

Consider $T_X(G(z)X)$ *as a linear subspace of* $T_X(\operatorname{mod}_A(X))$. *Then there exists a natural linear monomorphism*

$$T_X(\operatorname{mod}_A(X))/T_X(G(z)X) \hookrightarrow \operatorname{Ext}^1_A(X,X).$$

(b) *Assume that* X *satisfies* $\operatorname{Ext}^2_A(X,X)=0$. *Then the linear morphism*

$$T_X(\operatorname{mod}_A(X))/T_X(G(z)X) \xrightarrow{\sim} \operatorname{Ext}^1_A(X,X).$$

is an isomorphism.

We will observe several consequences:

(a) For any $X \in \operatorname{mod}_A(z)$, let C_X be an irreducible component of $\operatorname{mod}_A(z)$ containing X. Then

$$\begin{aligned} \dim_k \operatorname{Ext}^1_A(X,X) &\geq \dim_k T_X(\operatorname{mod}_A(z)) - \dim_k T_X(G(z)X) \\ &\geq \dim C_X - \dim G(z)X \\ &= \dim C_X - \dim G(z) + \dim_k \operatorname{End}_A(X). \end{aligned}$$

Hence,

$$\dim G(z) - \dim C_X \geq \dim_k \operatorname{End}_A(X) - \dim \operatorname{Ext}^1_A(X,X)$$

(b) The inclusion above is not always an isomorphism, as the following simple example shows:

Let $A = k[T]/(T^2)$. Consider the simple module $S \in \operatorname{mod}_A(1)$. Then $\operatorname{mod}_A(1) = G(1)S = \{S\}$ and T_S $(\operatorname{mod}_A(1))$ is trivial. On the other hand $\operatorname{Ext}^1_A(S,S)$ has dimension 1.

2.4. Exercises.

(1) Let $X \in \mathrm{mod}_A(z)$. Then $G(z)X$ is open if and only if $T_X(\mathrm{mod}_A(z)) = T_X(G(z)X)$.

(2) Let $n \in \mathbb{N}$, the function

$$e^n\colon \mathrm{mod}_A(z) \longrightarrow \mathbb{N}, \qquad x \mapsto \dim_k \mathrm{Ext}^n_A(X, X)$$

is upper semicontinuous.

(3) Up to isomorphism, there are only finitely many modules X with $\mathbf{dim} X = z$ and satisfying $\mathrm{Ext}^1_A(X, X) = 0$.

3. Tame algebras and varieties.

Proposition. *The following conditions are equivalent:*

(T_0): *A is tame.*

(T_1) : *For each* $z \in \mathbf{N}^{Q_0}$*, there is a constructible subset* C *of* $\mathrm{mod}_A(z)$ *satisfying* dim $C \leq 1$ *and* $\mathrm{ind}_A(z) \subset G(z)C$.

(T_2) : *For each* $z \in \mathbf{N}^{Q_0}$*, if* C *is a constructible subset of* $\mathrm{ind}_A(z)$ *intersecting each orbit of* $G(z)$ *in at most one point, then* dim $C \leq 1$.

Proof: $(T_0) \Longrightarrow (T_1)$: Let $z \in N^{Q_0}$. Let $M_1, ..., M_s$ be the $A - k[t]$-bimodules such that M_i is a free finitely generated $k[t]-$module and any $X \in \mathrm{ind}_A(z)$ is isomorphic to $M_i \otimes_{k[t]} S$ for some i and some simple $k[t]$-module S. Therefore, the functor $M_i \otimes_{k[t]} (-)$ induces a regular map $f_i : \mathrm{mod}_{k[t]}(1) \longrightarrow \mathrm{mod}_A(z), i = 1, ...s$.

The set

$$C = \bigcup_{i=1}^{s} (\text{Im } f_i \cap \text{ ind}_A(z))$$

is a constructible subset of $\text{ind}_A(z)$ with $\dim C \leq 1$ and $G(z)C = \text{ind}_A(z)$.

$(T_2) \implies (T_0)$: Assume that A is not tame. Then by the tame-wild dichotomy, the algebra A is *wild*. That is, there exists a $A - k\langle u, v\rangle-$bimodule M which is free finitely generated as right $k\langle u, v\rangle-$module and such that the functor $M \otimes_{k\langle x,y\rangle} (-) :$ $\text{mod}_{k\langle u,v\rangle} \longrightarrow \text{mod}_A$ insets indecomposable modules.

Let $z \in N^{Q_0}$, where $z(x)$ is the rank of the free $k\langle u, v\rangle$-module $M(x)$. We get an induced regular map $f_M : \text{mod}_{k\langle u,v\rangle}(1) \longrightarrow \text{mod}_A(z)$. By definition, Im f_M is a constructible subset of $\text{ind}_A(z)$ intersecting each orbit in at most one point. Moreover, f_M is injective and theferefore dim Im $f_M = 2$. □

Corollary. *An algebra can not both tame and wild.*

□

Proposition. *Let $A = kQ/I$ be a tame algebra. Then for every $z \in \mathbf{N}^{Q_0}$,*

$$\dim \operatorname{mod}_A(z) \leq \dim G(z)$$

Proof: By (1.4), it is enough to show that $\dim G(z) - \dim C \geq 0$, for an irreducible component C of $\operatorname{mod}_A(z)$

Since A is tame, we may choose a $A - k[t]$-bimodule M which is free as right $k[T]$-module and the following map is dominant

$$\varphi : G(z) \times \operatorname{Im} f^1_M \longrightarrow C, \qquad (g, X) \longmapsto X^g.$$

Let $X \in \operatorname{Im} \varphi$ be such that $\dim \varphi^{-1}(X) = \dim G(z) - \dim C + \dim \operatorname{Im} f^1_M$ and $(g, Y) \in \varphi^{-1}(X)$. Then the regular map

$$\operatorname{Aut}_A(Y) \longrightarrow \varphi^{-1}(X), \quad h \longmapsto (hg, Y)$$

is injective. Therefore,

$$0 \leq \dim \operatorname{Aut}_A(Y) - 1 \leq \dim G(z) - \dim C$$

□

Example: Unfortunately, the converse of the above results are not true.

Let $A_m = k[\alpha_1, ..., \alpha_m]/(\alpha_i\alpha_j : 1 \leq i \leq j \leq m)$ with $m \geq 3$. We will calculate dim $\text{mod}_{Am}(n)$.

We get

$$\dim \text{mod}_{A_m}(n) = \begin{cases} \left(\frac{m+1}{4}\right) n^2 & \text{if } n \text{ even} \\ \left(\frac{m+1}{4}\right)(n^2 - 1) & \text{if } n \text{ odd.} \end{cases}$$

If $m = 3$, then dim $\text{mod}_{A_3}(n) \leq n^2$, showing that the converse of the above Proposition fails.