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Integral Quadratic Forms and the Representation Type of an Algebra

(Lecture 3)

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INTEGRAL QUADRATIC FORMS AND THE REPRESENTATION TYPE OF AN ALGEBRA

LECTURE 3

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The Tits form of an algebra.

Let $A = kQ/I$ be a *triangular* algebra, that is, Q has no oriented cycles. Choose R a minimal set of generators of I, such that $R \subset \bigcup_{i,j \in Q_0} I(i,j)$. We have:

- $\dim_k \mathrm{Ext}^1_A(S_i, S_j) = \#$ arrows from i to j
- $r(i,j) = |R \cap I(i,j)|$ is independent of the choice of R
- $r(i,j) = \dim_k \mathrm{Ext}^2_A(S_i, S_j)$

The *Tits form* of A is the quadratic form

$$q_A \colon \mathbb{Z}^{Q_0} \to \mathbb{Z},$$

given by $q_A(v) = \sum_{i \in Q_0} v(i)^2 - \sum_{i \to j} v(i)v(j) + \sum_{i,j \in Q_0} r(i,j)v(i)v(j)$.

Proposition. *Assume $A = kQ/I$ is triangular. Let $z \in N^{Q_0}$. Then for any $X \in$ $mod_A(z)$.*

$$q_A(z) \geq \dim_k \operatorname{End}_A(X) - \dim_k \operatorname{Ext}^1_A(X, X).$$

Proof: Let $X \in \operatorname{mod}_A(z)$. The *local dimension* $\dim_X \operatorname{mod}_A(z)$ is the maximal dimension of the irreducible components of $\operatorname{mod}_A(z)$ containing X. By Krull's Hauptidealsatz, we have

$$\dim_X \operatorname{mod}_A(z) \geq \sum_{(i-j)\in Q_1} z(i)z(j) - \sum_{ij\in Q_0} r(i,j)z(i)z(j).$$

Therefore, we get the following inequalities,

$$\begin{aligned} q_A(z) &\geq \dim G(z) - \dim_X \operatorname{mod}_A(z) \geq \dim G(z) - \dim T_X \geq \\ &\geq \dim_k \operatorname{End}_A(X) - \dim_k \operatorname{End}^1_A(X, X). \end{aligned}$$

□

In 1975, Brenner observed certain connections between properties of q_A and the representation type of A. She wrote about her remarks: "...is written in the spirit of experimental science. It reports some regularities and suggests that there should be a theory to explain them".

Theorem. *Let $A = kQ/I$ be a triangular algebra.*
[Bongartz]: *If A is representation-finite, then q_A is weakly positive*
[de la Peña]: *If A is tame, then q_A is weakly non-negative*

Proof. In general, for $v \in \mathbb{N}^{Q_0}$

$$\dim \operatorname{mod}_A(v) \geq \sum_{i \to j} v(i)v(j) - \sum_{i,j \in Q_0} r(i,j)v(i)v(j)$$

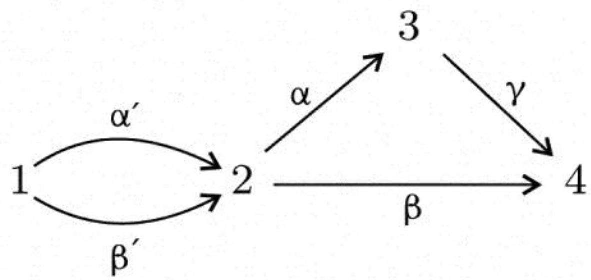
$$\dim G(v) = \sum_{i \in Q_0} v(i)^2$$

$$q_A(v) \geq \dim G(v) - \dim \operatorname{mod}_A(v)$$

If A is tame, then $q_A(v) \geq 0$.

If A is representation-finite, $\operatorname{mod}_A(v) = \bigcup_{i=1}^{m} G(v)X_i$ where $X_1, \ldots, X_m$ are representatives of the isoclasses of A-modules of $\mathbf{dim} = v$. Hence $\dim \operatorname{mod}_A(v) = \dim G(v)X_j = \dim G(v) - \dim \operatorname{Stab}_{G(v)} X_j \leq \dim G(v) - 1$ and $q_A(v) \geq 1$. □

Consider the algebra A given by the quiver



with relations $\gamma\alpha\alpha' = \beta\beta'$ and $\alpha\beta' = 0$. The Tits form q_A is

$$\begin{aligned} q_A(x) &= \sum_{i=1}^{4} x_i^2 - 2x_1x_2 - x_2x_3 - x_2x_4 - x_3x_4 + x_1x_3 - x_1x_4 \\ &= \left(x_1 - x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4\right)^2. \end{aligned}$$

and therefore (weakly) non-negative. We shall see later that A is wild.

Modules on preprojective components.

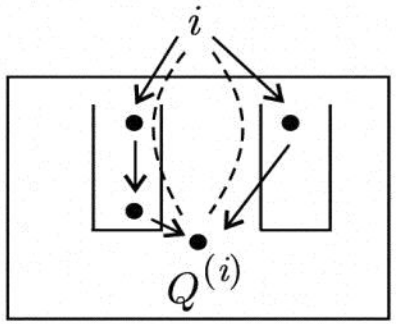
Recall that a component $\mathcal{P}$ of the Auslander-Reiten quiver Γ_A of A is called **preprojective** if it does not contain oriented cycles and for every $X \in \mathcal{P}$ there is a translate $\tau_A^n X$ which is projective. If $X \in \mathcal{P}$ and Y is an indecomposable such that $\mathrm{Hom}_A(Y, X) \neq 0$, then $Y \in \mathcal{P}$.

We give some examples of algebras with preprojective components:

a) Let $A = k[\Delta]$ be a hereditary algebra. Then Γ_A has a preprojective component $\mathcal{P}$, and the indecomposable projective modules form a slice.

b) **Tree algebras** have preprojective components (an algebra $A = k[Q]/I$ is a tree algebra if the underlying graph $|Q|$ of Q has no cycles). This is a particular case of the following situation.

c) An indecomposable projective P_i is said to have **separated radical** whenever the supports of any two non-isomorphic direct summands of *rad* P_i are contained in different components of the subquiver $Q^{(i)}$ of Q obtained by deleting all vertices in $[\to i] = \{j \in Q_0 : \{j \in Q_0 \colon j \rightsquigarrow i\}$. If for every vertex $i \in Q_0$, P_i has separated radical, then A satisfies the **separation condition**. Note that tree algebras satisfy the separation condition. If A satisfies the separation condition, then Γ_A has a preprojective component.



A representation-finite algebra A such that Γ_A is a preprojective component is said to be **representation-directed**.

Let Q' be a subquiver of Q, we say that Q' is **convex** in Q if Q' is path closed in Q (that is, whenever $i_0 \to i_1 \to ... \to i_m$ is a path in Q with $i_0, i_m \in Q'$ then $i_j \in Q'$ for $1 \leq j \leq m-1$).

Remark: Suppose that X is an indecomposable lying in a preprojective component $\mathcal{P}$ of Γ_A. Then supp X is convex in Q.

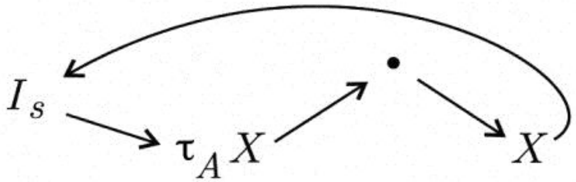
Proof: Supose that $i_1 \xrightarrow{\alpha_1} i_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{m-1}} i_m$ is a path in Q such that $X(i_1) \neq 0 \neq X(i_m)$ but $X(i_j) = 0$ for $2 \leq j \leq m-1$. Let I' be the ideal of $k[Q]$ generated by all paths of the form: $\epsilon\gamma$ with $\epsilon, \gamma \in Q_1$ where either $i_1 \xrightarrow{\gamma} i_2$ and ϵ starts at i_2 or γ ends at i_{m-1} and $i_{m-1} \xrightarrow{\epsilon} i_m$. Let $A' = k[Q]/(I + I')$. Then X is a A'-module and there is a chain of non-zero morphisms

$$X \longrightarrow I'_{i_m} \longrightarrow S_{m-1} \longrightarrow M_{i_{m-1}}^{i_{m-2}} \longrightarrow S_{m-2} \longrightarrow ... \longrightarrow S_{i2} \longrightarrow P'_{i1} \longrightarrow X$$

where M_i^j denotes the indecomposable module $k_i \to k_j$ and I'_{i_m} is the A'-module associated with i_m. Since $X \in \mathcal{P}$, this cycle should lie in $\mathcal{P}$. A contradiction. $\square$

Corollary. *Let X be a preprojective A-module. Then $q_A(\mathbf{dim}\, X) = 1$.*

Proof. We may assume that X is omnipresent in A. Then $p\dim_A X \leq 1$: otherwise there are non-zero maps a contradiction. Similarly, $gl\dim A \leq 2$. Hence $q_A(\mathbf{dim}\, X) =$



$\dim_k \mathrm{End}_A(X) - \dim_k \mathrm{Ext}^1_A(X, X) = 1.$ □

Lemma:
A weakly positive quadratic form $q : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ has only finitely many positive roots.

Proof. Consider q as a function $q : \mathbb{R}^n \longrightarrow \mathbb{R}$. By continuity $q(z) \geq 0$ in the positive cone $K = (\mathbb{R}^n)^+$. By induction on n, it can be shown that $q(z) > 0$ for any $0 \neq z \in K$. Let $0 < \gamma$ be the minimal value reached by q on $\{z \in K : \| z \| = 1\}$ (a compact set). Then a positive root z of q satisfies $\gamma \leq q\left(\frac{z}{\|z\|}\right) = \frac{1}{\|z\|^2}$, that is $\| z \| \leq \sqrt{1/\gamma}$. □

Theorem: [Bongartz]

Let $A = k[Q]/I$ be an algebra such that Q has no oriented cycles. Assume that Γ_A has a preprojective component. Then A is representation-finite if and only if the Tits form q_A is weakly positive. In that case, there is a bijection $X \mapsto \mathbf{dim}\, X$ between the isoclasses of indecomposable A-modules and the positive roots of q_A.

Proof: Assume that q_A is weakly positive. Let $\mathcal{P}$ be a preprojective component of Γ_A. Let $X \in \mathcal{P}$ then $\mathbf{dim} X$ is a root of q_A. Moreover, the map $X \to \mathbf{dim} X$, for $X \in \mathcal{P}$, is injective. Indeed, let $X, Y \in \mathcal{P}$ be such that $\mathbf{dim}\ X = \mathbf{dim}\ Y$. We may assume that X is omnipresent. Then, we get

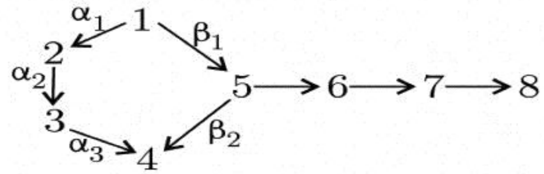
$$1 = q_A(\mathbf{dim}\ X) = \dim_k \operatorname{Hom}_A(X, Y) - \dim_k \operatorname{Ext}^1_A(X, Y).$$

In particular, $\operatorname{Hom}_A(X, Y) \neq 0$. By symmetry, $\operatorname{Hom}_A(Y, X) \neq 0$ and $X = Y$. It follows that $\mathcal{P}$ is a finite component of Γ_A and $\mathcal{P} = \Gamma_A$.

Finally, let $z \in \mathbb{N}^{Q_0}$ be a root of q_A. Then there is a module $X \in \operatorname{mod}_A(z)$ with the orbit $G(z)X$ of dimension $\dim\ G(z) - 1$. Since $\dim\ G(z)X = \dim\ G(z) - \dim \operatorname{End}_A(X)$, we obtain that $\operatorname{End}_A X = k$.

We give some **examples**.

a) The statement of (2.3) may be false if A has no preprojective component. Consider the algebras A_i given by the quiver Q with relations $I_i = \langle \rho_i \rangle$:

$Q :$ 

$$\begin{aligned} \rho_1 &= (\alpha_3\alpha_2\alpha_1 - \beta_2\beta_1) \\ \rho_2 &= \alpha_3\alpha_2\alpha_1 \end{aligned}$$

Clearly, they have the same Tits form

$$\begin{aligned} q &= \sum_{i=1}^{8} x_i^2 - x_1x_2 - x_2x_3 - x_3x_4 - x_1x_5 - x_4x_5 - x_5x_6 - x_6x_7 - x_7x_8 + x_1x_4 \\ &= \left(x_1 - \frac{1}{2}x_2 + \frac{1}{2}x_4 - \frac{1}{2}x_5\right)^2 + \frac{3}{4}\left(x^2 - \frac{2}{3}x_3 + \frac{1}{3}x_4 - \frac{1}{3}x_5\right)^2 + \\ &+ \frac{2}{3}\left((x_3 - \frac{1}{2}x_4 - \frac{1}{4}x_5\right)^2 + \frac{1}{2}\left(\frac{1}{2}(x_4 - \frac{1}{2}x_5\right)^2 + \frac{1}{2}(x_5 - x_6)^2 + \frac{1}{2}(x_6 - x_7)^2 + \\ &+ \frac{1}{2}(x_7 - x_8)^2 + \frac{1}{2}x_8^2. \end{aligned}$$

which is positive.

The algebra A_1 satisfies the separation condition and Bongartz theorem applies. The algebra A_2 is not representation-finite: *mod* A_2 contains the representations of the Euclidean quiver

$$\begin{array}{ccccccc} 3 \leftarrow 2 \leftarrow 1 \rightarrow & 5 & \rightarrow 6 \rightarrow 7 \rightarrow 8 \\ & \downarrow & \\ & 4 & \end{array}$$

Critical forms and critical algebras.

We recall some important facts of linear algebra

a) Let $A = (a_{ij})$ be an $n \times n$-matrix. Let $1 \leq i_1 < i_2 < ... < i_s \leq n$ and $1 \leq j_1 < j_2 < ... < j_s \leq m$. Form the $s \times s$-matrix

$$A \begin{pmatrix} i_1 & i_2 & ... & i_s \\ j_1 & j_2 & ... & j_s \end{pmatrix} = \begin{pmatrix} a_{i_1 j_1} & a_{i_1 j_2} & \cdots & a_{i_1 j_s} \\ & \vdots & & \\ a_{i_s a_1} & a_{i_s j_2} & \cdots & a_{i_s j_s} \end{pmatrix}$$

The determinant $det\ A \begin{pmatrix} i_1 ... i_s \\ j_1 ... j_s \end{pmatrix}$ is called a **minor** of A.

If $i_1 = j_1, ..., i_s = j_s$, then $A \begin{pmatrix} i_1 ... i_s \\ j_1 ... j_s \end{pmatrix}$ is called a **principal submatrix** and $det\ A \begin{pmatrix} i_1 ... i_s \\ j_1 ... j_s \end{pmatrix}$ a **principal minor**.

If $s = n - 1, \{i_1, ..., i_s\} = \{1, ..., \hat{i}, ..., n\}$ and $\{j_1, ..., j_s\} = \{1, ..., \hat{j}, ..., n\}$, then $A \begin{pmatrix} i_1 ... i_s \\ j_1 ... j_s \end{pmatrix}$ is denoted by $A^{i,j}$.

b) The matrix $ad(A)$ whose (i, j) entry is $(-1)^{i+j}\ detA^{(i,j)}$, is called the **adjoint matrix** of A. It has the property that $A\ ad(A) = (det\ A)\ E_n = \ ad(A)A$.

c) Let q be the quadratic form associated with a symmetrical real matrix A, that is $q(x) = \frac{1}{2}xAx^t$.

The form q is **positive** if and only if the determinants of the principal submatrices $A\begin{pmatrix}1\\1\end{pmatrix}, A\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}, ..., A\begin{pmatrix}12...n\\12...n\end{pmatrix} = A$ are positive, or equivalently, if all principal minors are positive.

The form q is **non-negative** if and only if all principal minors of A are non-negative $det\ A\begin{pmatrix}i_1...i_s\\j_1...j_s\end{pmatrix} \geq 0$ for all $1 \leq i_1 < i_2 < ... < i_s \leq n, x = 1, ..., n$.

d) **Perron-Frobenious theorem:** Let $A = (a_{ij})$ be a real matrix with $a_{ij} \geq 0$. Then for the **spectral radius** $\rho = \ max\ \{\| \lambda \|: \lambda$ is an eigenvalue of $A\}$, there is a vector y with non-negative coordinates such that $yA = \rho y$. Moreover, if $a_{ij} > 0$ for every i, j, then $0 < \rho$ and the coordinates of y are positive.

We say that an integral quadratic form $q(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^2 + \sum_{i<j} q_{ij}x_ix_j$ is a *unit form.*

Theorem: [Zeldich]
Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a unit form and let A be the associated symmetric matrix. The following are equivalent:

a) *q is weakly positive.*
b) *For each principal submatrix B of A either det $B > 0$ or ad (B) is not positive (that is, it has an entry ≤ 0).*

Proof: a) $\Rightarrow$ b): Let B be a principal submatrix of A. Suppose that $ad\,(B)$ is positive. Then there is a positive vector v and a number of $\rho > 0$ such that $v\ ad(B) = \rho v$. Then $0 < q(v) = vBv^t = \rho^{-1}\ ad(B)Bv^t = \rho^{-1}(det\ B)vv^t$. Thus $det\ B > 0$.

b) $\Rightarrow$ a). Let A be a $n \times n$-matrix satisfying (b). We show that q is weakly positive by induction on n.

Since property (b) is inhereted to principal submatrices, we can assume that the quadratic form, $q^{(i)}$ associated with each principal submatrix $A^{(i,i)}$ is weakly positive.

Claim: $q^{(i)}$ is positive, $1 \leq i \leq n$.

Assume that q is not weakly positive. Therefore, we get a vector $0 \ll y \in \mathbb{N}^n$ such that $q(y) \leq 0$.

In particular, every proper principal submatrix B of A has $det\ B > 0$. Since A is not positive, $det\ A \leq 0$. By hypothesis, $ad\ (A)$ is not positive. Suppose that the j-th row v of $ad(A)$ has some non positive coordinate. Therefore, there exists a number $\lambda \geq 0$ such that $0 \leq \lambda y + v$ is not omnipresent. Therefore

$$\begin{aligned} 0 < q(\lambda y + v) &= \lambda^2 q(y) + \lambda v A y^t + q(v) \leq \lambda(det\ A)y(j) + (det\ A)v(j) \\ &\leq (det\ A)(det\ A^{(j,j)}) \leq 0, \end{aligned}$$

since by the claim $q^{(j)}$ is positive. □

A unit form $q\colon \mathbb{Z}^n \to \mathbb{Z}$ is *critical* if q is not weakly positive but all its restrictions $q^{(i)}$ $(i = 1, \ldots, n)$ are weakly positive.

Corollary:
If q is critical, then the set

$$C_q = \{v \in \mathbb{Z}^n : v(i) \geq 0 \text{ and } v(j) < 0 \text{ for some } 1 \leq i, j \leq n \text{ and } q(v) = 1\}$$

is finite.

Theorem [Ovsienko].
Let q be a critical form. Then there exists a Euclidean quiver Δ and an invertible transformation qT of q such that $q_\Delta = qT$. In particular, q is non negative and there is a vector $0 \ll z \in \mathbb{Z}^n$ such that rad $q = \mathbb{Z}z$.

Proof: Since $n \geq 3, 0 < q(e_s \pm e_t) = 2 \pm a_{st}$. Choose $q' = qT$ an invertible transformation of q such that the set $C_{q'}$ has minimal cardinality.

Therefore, $q' = \sum_{i=1}^{n} x_i^2 + \sum_{i \neq j} a'_{ij} x_i x_j$ is critical and $-1 \leq a'_{ij} \leq 0$ for every pair i, j with $i \neq j$. Thus $q' = q_\Delta$ for some quiver Δ. Since q' is critical, Δ is Euclidean. Then $\operatorname{rad} q' = \mathbb{Z}u$ with $u \gg 0$ and $z = T^{-1}(u)$. □

Let $A = k[Q]/I$ be a k-algebra. We say that A is **minimal representation-infinite** it it is representation-infinite but every quotient A/AeA is representation-finite for any idempotent $0 \neq e$ of A.

A minimal representation-infinite algebra A with preprojective component is called **critical**. Observe that a preprojective component of a critical algebra contains all the indecomposable projective modules (and therefore is unique).

Lemma.
Let A be an algebra with a preprojective component containing all projective modules. If e is an idempotent of A, then A/AeA has preprojective components such that their union contains all indecomposable projective A/AeA-modules. □

Theorem [Happel-Vossieck].
Let $A = k[Q]I$ be an algebra with preprojective component. Assume that Q has al least 3 vertices. Then the following are equivalent:

(a) *A is critical;*
(b) *The Tits form q_A is critical;*
(c) *A is tame concealed.*

Proof: Let $\mathcal{P}$ be a preprojective component of Γ_A.

b) $\Rightarrow$ c): Assume that q_A is critical. Therefore, A is representation-infinite. A preprojective component $\mathcal{P}$ of Γ_A should contain all indecomposable projective modules. Moreover, this component $\mathcal{P}$ does not contain injective modules. Therefore, A is tilted.

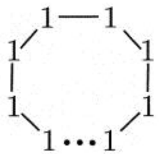
Assume that $A = \mathrm{End}_B(T)$ where $B = k[\Delta]$ is an hereditary algebra and B^T is a tilting module. Therefore the Euler forms χ_A and χ_B are equivalent. Since $gl.dim\ A \leq 2$, then $\chi_A = q_A$. Therefore, Δ is a tame quiver.

By a dual argument, A has a preinjective component with all injectives. Hence A is tame concealed. □

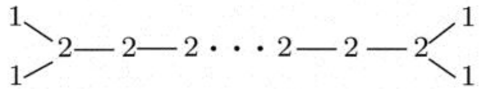
Theorem [Bongartz].

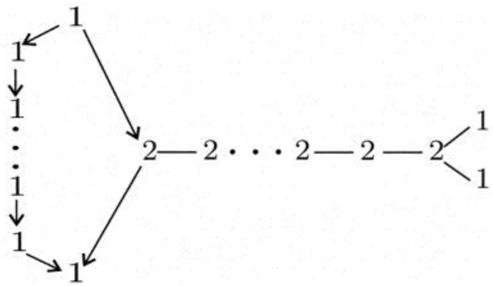
Let $A = k[Q]/I$ be an algebra with preprojective component. Then A is representation-finite if and only if there is no convex subalgebra A_0 of A such that A_0 is critical. □

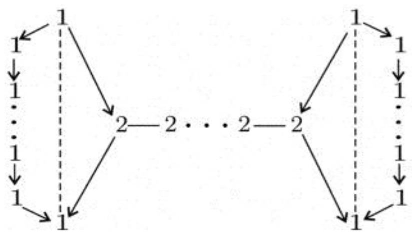
The frame of type $\tilde{\mathbb{A}}_n$:

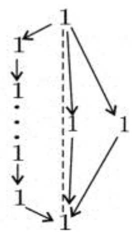


The frames of type $\tilde{\mathbb{D}}_n$:

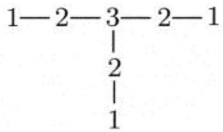


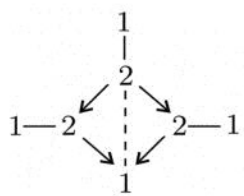


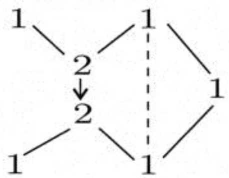


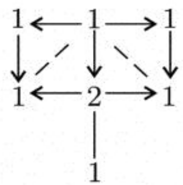


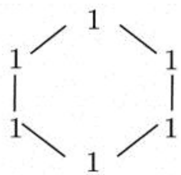
The frames of type $\tilde{\mathbb{E}}_6$:

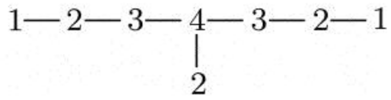


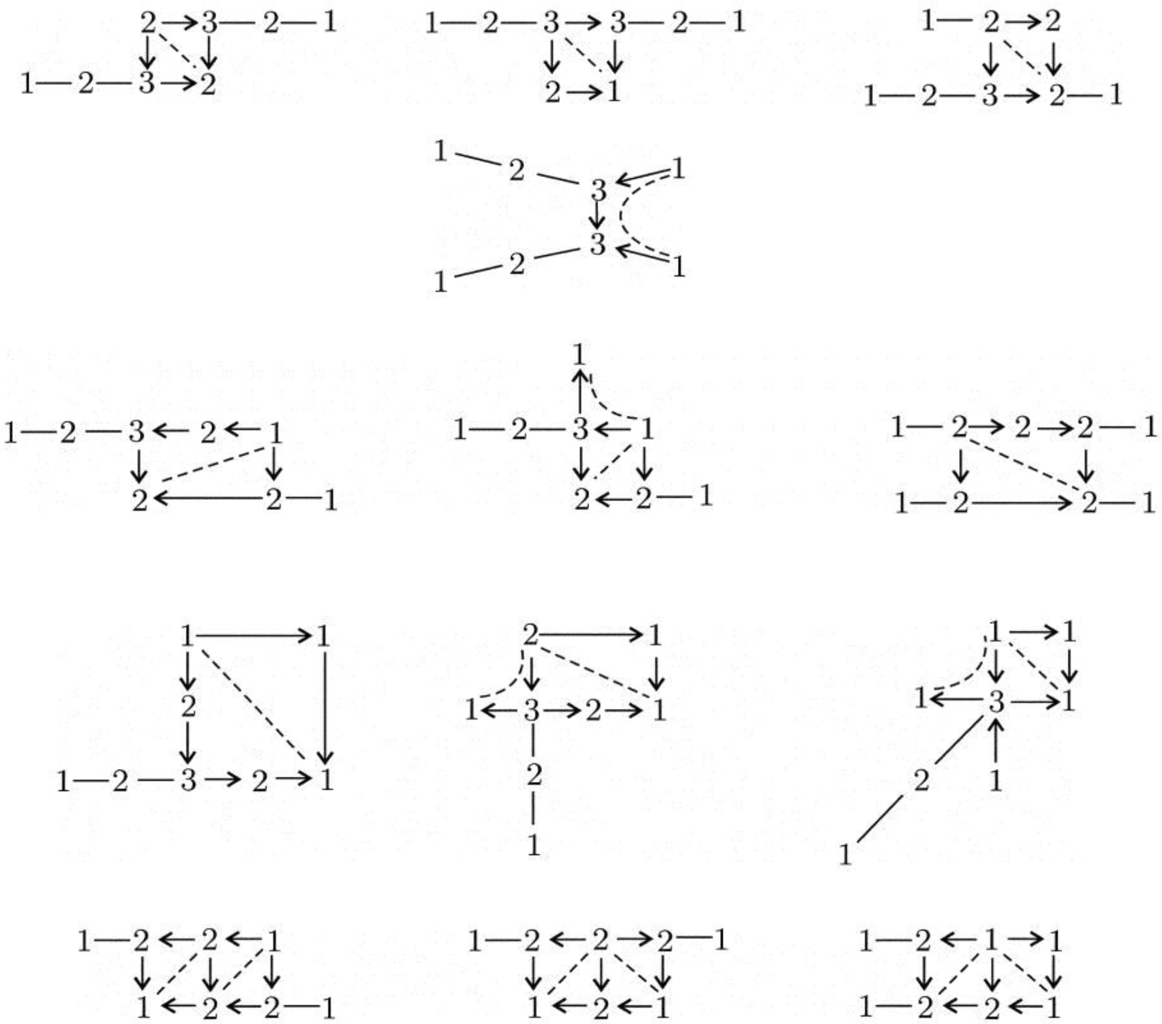


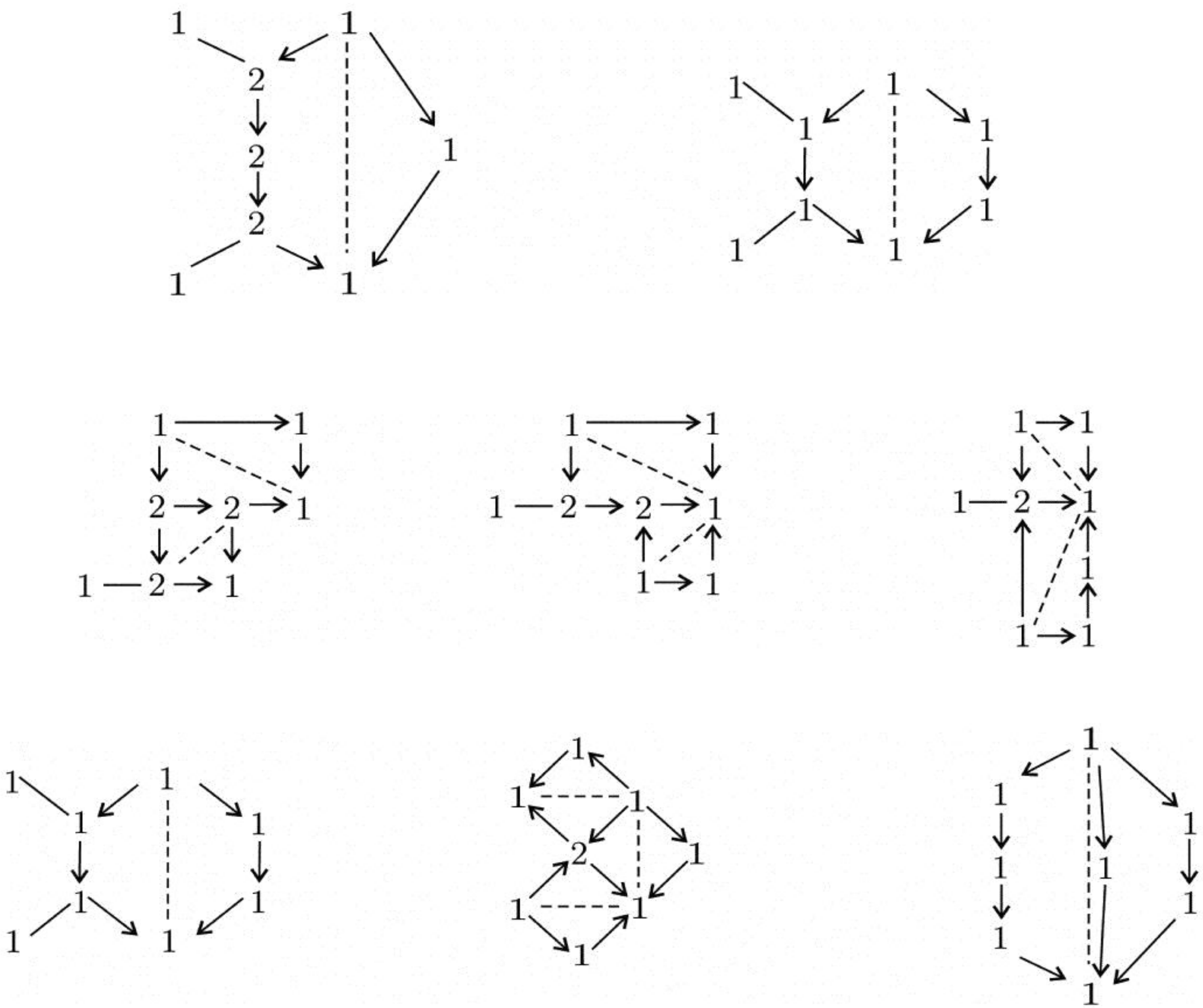






The frames of type $\tilde{\mathbb{E}}_7$: 





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