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Integral Quadratic Forms and the Representation Type of an Algebra

(Lecture 4)

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INTEGRAL QUADRATIC FORMS AND THE REPRESENTATION TYPE OF AN ALGEBRA

LECTURE 4

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Tame algebras and modules on tubes.

Standard tubes in Auslander-Reiten quivers.

Let A be a finite dimensional k-algebra. We say that an A-module X is a *brick* if $\mathrm{End}_A(X) = k$. In particular, a brick is always an indecomposable module.

We recall that two modules X_1, X_2 are said to be *orthogonal* if $\mathrm{Hom}_A(X_1, X_2) = 0 = \mathrm{Hom}_A(X_2, X_1)$.

Let $E_1, ..., E_s$ be a family of pairwise orthogonal bricks. Define $\varepsilon(E_1, ..., E_s)$ as the full subcategory of mod A whose objects X admit a filtration $X = X_0 \supset X_1 \supset ... \supset X_m = 0$ for some $m \in \mathbf{N}$, with X_i/X_{i+1} isomorphic to some E_j, for any $1 \le i \le n$.

Lemma. *The category* $\varepsilon = \varepsilon(E_1, ..., E_s)$ *is an abelian category, with* $E_1, ..., E_s$ *being the simple objects of* E. □

An abelian category ε is said to be *serial* provided any object in E has finite lenght and any indecomposable object in ε has a unique composition series.

Proposition. *Let $E_1, ..., E_s$ be pairwise orthogonal bricks in some module category* mod A. *Assume that (a) $\tau E_i \cong E_{i-1}$ for $1 \leq i \leq s$ with $E_0 = E_s$ and (b)* $\mathrm{Ext}^2_A(E_i, E_j) = 0$ *for all $1 \leq i, j \leq n$. Then $\varepsilon = \varepsilon(E_1, ..., E_s)$ is serial, it is a standard component of Γ_A of the form $\mathbf{ZA}_\alpha/(n)$.* □

With the notation of the Proposition above: we denote by $E_i[t]$ the unique module in the serial category E which has socle E_i and lenght t.

A family $\mathcal{T} = (T_\lambda)_{\lambda \in L}$ of the Auslander-Reiten quiver of an algebra A is a *standard stable tubular family* if each T_λ is a standard component of the form $\mathbf{ZA}_\infty/(n_\lambda)$ for some n_λ and for $\lambda \neq \mu$ the components T_λ and T_μ are orthogonal.

Corollary. *Let $\mathcal{T} = (T_\lambda)_{\lambda \in L}$ be a standard stable tubular family in the Auslander-Reiten quiver of A. Then the additive closure add $\mathcal{T}$ of $\mathcal{T}$ in* mod A *is an abelian category which is serial and is closed under extensions in* mod A. □

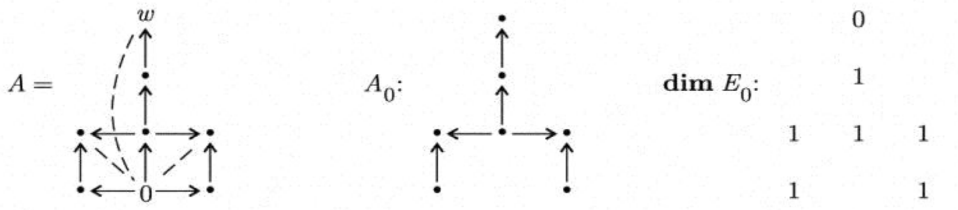
A standard stable tubular family $\mathcal{T} = (T_\lambda)_{\lambda \in L}$ is said to be *separating* if there are full subcategories $\mathcal{P}$ and $\mathcal{I}$ of mod A satisfying the following conditions:

(i) each indecomposable A-module belongs to one of $\mathcal{P}, \mathcal{T}$ or $\mathcal{I}$;

(ii) for modules $X \in \mathcal{P}, Y \in \mathcal{T}$ and $Z \in \mathcal{I}$ we have $\mathrm{Hom}_A(Z, Y) = 0 = \mathrm{Hom}_A(Z, X)$ and $\mathrm{Hom}_A(Y, X) = 0$.

(iii) each non zero morphism $f \in \mathrm{Hom}_A(X, Z)$ for indecomposable modules $X \in \mathcal{P}, Z \in \mathcal{I}$, factorizes through each component T_λ.

Example: Let A be the algebra given by the quiver with relations below

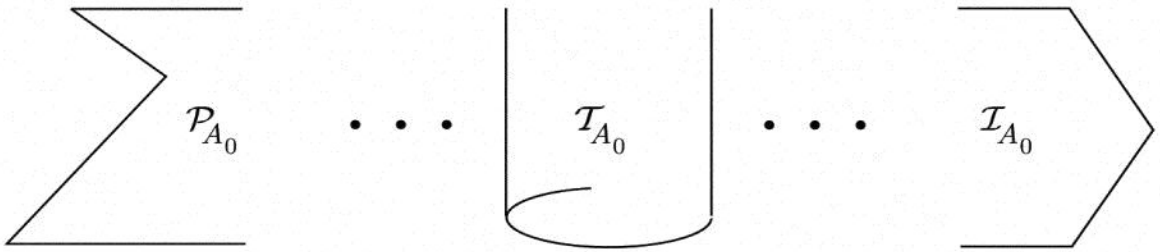


Then A is the *one-point extension* $A_0[E_0]$ as follows

$$A_0[E_0] = \begin{bmatrix} A_0 & E_0 \\ 0 & k \end{bmatrix}$$

with the usual matrix operations and where E_0 is considered as an $A_0 - k$-bimodule. Moreover rad $P_0 = E_0$.

The algebra A_0 is tame hereditary with an Auslander-Reiten quiver of the shape



where $\mathcal{P}_{A_0}$ is a preprojective component, $\mathcal{I}_{A_0}$ a preinjective component and $\mathcal{T}_{A_0}$ is a separating tubular family of tubular type $(2,3,3)$. In $\mathcal{T}_{A_0} = (T_\lambda)_\lambda$ almost all tubes are of rank one with a module on the mouth with

$$z_0 = \mathbf{dim}: \quad \begin{array}{ccc} & 1 & \\ & 2 & \\ 2 & 3 & 2 \\ 1 & & 1 \end{array}$$

The tubes of rank 2 and rank 3 have modules on the mouths with the unique indecomposable A_0-modules with the indicated dimension vectors:

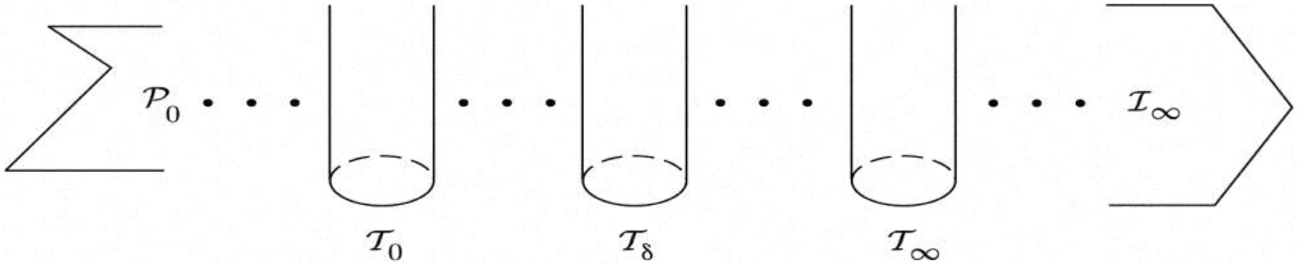
$$E_0: \quad \begin{array}{ccc} & 0 & \\ & 1 & \\ 1 & 1 & 1 \\ 1 & & 1 \end{array} \qquad E_1: \quad \begin{array}{ccc} & 1 & \\ & 1 & \\ 1 & 2 & 1 \\ 0 & & 0 \end{array}$$

$$X_0: \quad \begin{array}{ccc} & 0 & \\ & 1 & \\ 1 & 1 & 0 \\ 0 & & 0 \end{array} \qquad X_1: \quad \begin{array}{ccc} & 0 & \\ & 0 & \\ 1 & 1 & 1 \\ 1 & & 0 \end{array} \qquad X_2: \quad \begin{array}{ccc} & 1 & \\ & 1 & \\ 0 & 1 & 1 \\ 0 & & 1 \end{array}$$

$$Z_0: \quad \begin{array}{ccc} & 0 & \\ & 1 & \\ 0 & 1 & 1 \\ 0 & & 0 \end{array} \qquad Z_1: \quad \begin{array}{ccc} & 0 & \\ & 0 & \\ 1 & 1 & 1 \\ 0 & & 1 \end{array} \qquad Z_2: \quad \begin{array}{ccc} & 1 & \\ & 1 & \\ 1 & 1 & 0 \\ 1 & & 0 \end{array}$$

and where the Auslander-Reiten translation is given by $\tau_{A_0} E_1 = E_0$, $\tau_{A_0} X_i = X_{i-1}$ and $\tau_{A_0} Z_i = Z_{i-1}$ cyclically.

The structure of Γ_A is given as follows:



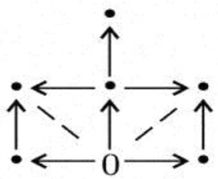
where $\mathcal{T}_0 = \bigvee_{\lambda \neq 2} T_\lambda \vee T_2[E_0]$ is the family of tubes $\mathcal{T}_{A_0}$ with the exception of the tube of rank 2 which appears now 'inserted' with the new projective at the extension vertex 0.

For each positive rational number $\delta = \frac{a}{b}, (a, b), \mathcal{T}_\delta$ is a separating family of tubes of tubular type $(3, 3, 3)$ with all homogeneous tubes but 2 of rank 3. The homogeneous tubes have modules on the mouths of vector dimension

$$az_0 + bz_\infty$$

when z_∞ is given by

A_∞ :



$$z_\infty : \begin{matrix} & 1 & \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{matrix} \qquad \mathbf{dim}\, E_\infty : \begin{matrix} & 1 & \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix}$$

Observe that A_∞ is tame concealed and $A = [E_\infty]A_\infty$ is a one-point coextension where the module E_∞ lies on a regular tube of Γ_{A_∞}. The algebra A is typical *tubular algebra* as defined by Ringel.

Proposition. *Let $\mathcal{T} = (T_\lambda)_\lambda$ be a standard separating tubular family for the module category mod A. Then*

a) *For almost every λ, the tube T_λ is homogeneous.*

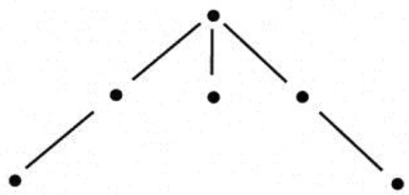
b) *Let T_λ be a homogeneous tube of the family $\mathcal{T}$. Let X be a module in the mouth of T_λ and $v = \mathbf{dim}\, X$. Then $q_A(v) = 0$.*

Proof of (b): Let X be a module in the mouth of a homogeneous tube T_λ in $\mathcal{T}$. Let B be the convex closure in A of $\cup \text{supp}\, X$ with $X \in T_\lambda$. Since B is convex in A and $gl\text{dim } B \leq 2$, then

$$q_A(\mathbf{dim}\ X) = q_B(\mathbf{dim}\ X) = \dim_k \text{End}_A(X) - \dim_k \text{Ext}^1_A(X, X).$$

Since T_λ is standard and $X \simeq \tau X$, then $\text{Ext}^1_A(X, X) \cong D\text{Hom}_A(X, \tau X)$ and we get $q_A\ (\mathbf{dim}\ X) = 0$. □

Notation: Let $\mathcal{T} = (T_\lambda)_\lambda$ be a standard separating stable tubular family in mod A. Let $r(\lambda)$ be the *period* (or *rank*) of the tube T_λ. Consider those $r(\lambda_1), \dots, r(\lambda_s)$ which are strictly bigger than 1 (finite number by (1.4)). We define the *star diagram* T_τ of the family $\mathcal{T}$ as the diagram with a unique ramification point and s branches of lengths $r(\lambda_1), \dots, r(\lambda_s)$. For example, the tame concealed algebra of tubular type $(2, 3, 3)$ has the star diagram depicted below.



Theorem. [Ringel, Lenzing-de la Peña]
Let $A = kQ/I$ be a k-algebra. Let n be the number of vertices of Q. Let $\mathcal{T} = (T_\lambda)_{\lambda \in L}$ be a standard separating stable sincere tubular family in mod A. Let $r(\lambda)$ be the rank of the tube T_λ. Then

$$\sum_{\lambda \in L} (r(\lambda) - 1) = n - 2.$$

Moreover, A is a tame algebra if and only if the star diagram $\mathbf{T}_r$ is a Dynkin or extended Dynkin diagram.

Tubes and isotropic roots of the Tits form.

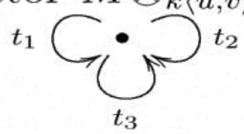
We say that a property P is satisfied by almost every indecomposable if for each $d \in \mathbf{N}$, the set of indecomposable A-modules of dimension d which do not satisfy P form a finite set of isomorphism classes. The following is a central fact about the structure of the representation-quiver Γ_A of a tame algebra A.

Theorem [Crawley-Boevey]
Let A be a tame algebra. Then almost every indecomposable lies in a homogeneous tube. In particular, almost every indecomposable X satisfies $X \simeq \tau X$.

Open problem: Is it true that an algebra is of tame type if and only if almost every indecomposable module belongs to a homogeneous tube?

Proposition. *Let A be an algebra such that almost every indecomposable lies in a standard tube. Then A is tame.*

Proof. Our hypothesis implies that almost every indecomposable X satisfies $\dim_k \mathrm{End}_A(X) \leq \dim_k X$. We show that this condition implies the tameness of A.

Indeed, assume that A is wild and let M be a $A-k\langle u,v\rangle$-bimodule which is finitely generated free as right $k\langle u,v\rangle$-module and the functor $M \otimes_{k\langle u,v\rangle}$-insets indecomposables. Consider the algebra B given by the quiver 

and with radical J satisfying $J^2 = 0$. Then there is a $A-B$-bimodule N such that N_B is free and $N \otimes_B - :$ mod $B \longrightarrow$ mod A is fully faithful. Therefore the composition $F = M \otimes_A (N \otimes_B -)$ is faithful and insets indecomposables. Moreover, $\dim_k FX \leq m \dim_k X$ for any $X \in$ mod B if we set $m = \dim_k(M \otimes_A N)$.

Consider also the functor H : mod $A \longrightarrow$ mod B sending X to the space $X' = X \oplus X$ with endomorphisms

$$X'(t_1) = \begin{bmatrix} 0 & X(w) \\ 0 & 0 \end{bmatrix}, X'(t_2) = \begin{bmatrix} 0 & X(v) \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad X'(t_3) = \begin{bmatrix} 0 & 1_X \\ 0 & 0 \end{bmatrix}$$

This functor insets indecomposables. For the simple $A-$modules X of dimension n, we get indecomposable $A-$modules $FH(X)$ with

$$\dim_k FH(X) \leq m \dim_k H(X) = 2mn$$

and

$$\dim_k \mathrm{End}_A(FH(X)) \geq \dim_k \mathrm{End}_B(H(X)) = n^2 + \dim_k \mathrm{End}_A(X) = n^2 + 1.$$

Let $A = kQ/I$ be a triangular algebra. In case A is tame, we would like to find the dimensions $z \in \mathbf{N}^{Q_0}$ where indecomposable modules X with $\mathbf{dim}\, X = z$ and X in a homogeneous tube exist.

Proposition. *Assume that A is tame and $q_A(z) = 0$. Then there is a decomposition $z = w_1 + ... + w_s$ with $w_i \in \mathbf{N}^{Q_0}$ and an open subset $\mathcal{U}$ of $\mathrm{mod}_A(z)$ satisfying:*

(a) $\dim \mathcal{U} = \dim\, \mathrm{mod}_A(z)$.

(b) *Every $X \in \mathcal{U}$ has an indecomposable decomposition $X = X_1 \oplus ... \oplus X_s$ such that $\dim X_i = w_i$ and the module X_i lies in the mouth of a homogeneous tube. Moreover, $\dim_k \mathrm{Hom}_A(X_i, X_j) = \delta_{ij} = \mathrm{Ext}^1_A(X_i, X_j)$ for $1 \leq i, j \leq s$.*

Hypercritical algebras.

Let $q = \sum_{i=1}^{n} x_i^2 + \sum_{i \neq j} a_{ij} x_i x_j$ be a unit form. Let M be the symmetric matrix associated with q.

Proposition:
The following are equivalent:

(a) *q is weakly non negative*
(b) *Every critical restriction q^I of q with v the positive generator of rad q^I, satisfies $v^0 M \geq 0$.*

Proof: a) $\Rightarrow$ b): Assume that q^I is critical and $v^0 M$ has its j-th component negative. Then $0 \leq 2v^0 + e_j \in \mathbb{Z}^n$ and $q(2v^0 + e_j) = 2v^0 M e_j^t + 1 < 0$

b) $\Rightarrow$ a): Assume q satisfies (b) but not (a). By induction, we may suppose that $q^{(i)}$ satisfies (a), $1 \leq i \leq n$. Let $0 \ll z$ be such that $q(z) < 0$. Let q^I be a critical restriction. Let v be the positive generator of *rad* q^I. We can find a number $a \leq 0$ such that $0 \leq z + av^0$ and $(z + av^0)(j) = 0$ for some $1 \leq j \leq n$. Then

$$0 \leq q^{(j)}(z + av^0) < av^0 M z^t \leq 0,$$

a contradiction. □

Corollary:
The unit form q is weakly non negative if and only if $0 \leq q(z)$ for every $z \in [0, 12]^n$.
□

Conjeture:

Let A be a good algebra. Then A is tame if and only if its Tits form q_A is weakly non negative.

Which are the good algebras?

Theorem [Brüstle]

Let A be tree algebra. Then A is tame if and only if q_A is weakly non-negative.

Proposition:

Let A be a tree algebra. Then q_A is weakly non negative if and only if A has no convex hypercritical subalgebras.

Let A be a tree algebra and assume that q_A is not weakly non negative. Therefore, there is a hypercritical restriction q_A^I. Then $I = J \cup \{x\}$ such that q_A^I is critical and $q_A(v^0, e_x) < 0$, where v is the positive generator of $radq_A^J$.

Hence, $q_A^J = q_B$, where B is a convex critical subalgebra of A. Since $q_A(v^0, e_x) < 0$, there is an arrow connecting x and A. Let C be the full (convex !) subcategory of A with vertices x and those of B. Therefore $q_A^I = q_C$ is hypercritical. □

We recall that a triangular algebra $A = kQ/I$ is said to satisfy the *separation condition* if every vertex $x \in Q_0$ has separated radical. The algebra A is *strongly simply connected* if every convex subcategory B of A satisfies the separation condition.

Theorem [Brüstle-de la Peña-Skowroński].
Let A be a strongly simply connected algebra, then the following are equivalent:

(a) *A is tame*

(b) *q_A is weakly non-negative*

(c) *A does not contain a full convex subcategory which is hypercritical.* □

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