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Hereditary Categories

Lectures 1 and 2

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1 Basic properties of hereditary categories

For the whole series k denotes a field, which later we will assume to be algebraically closed. We are going to investigate k-linear categories, k-categories for short, that are small abelian, Hom-finite, hereditary and satisfy Serre duality.

1.1 Abelianness

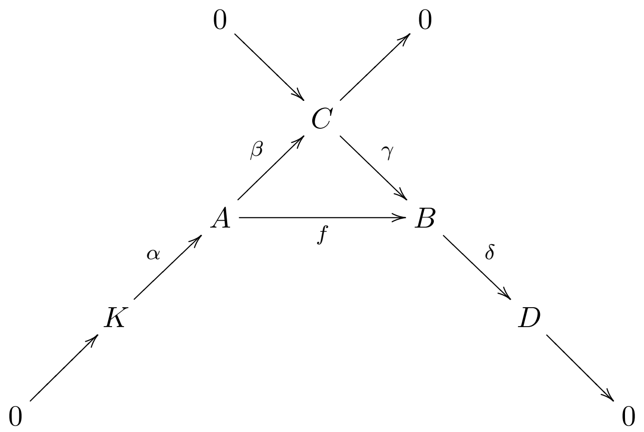
(H1) *$\mathcal{H}$ is an abelian k-linear category.*

Recall that k-linearity of $\mathcal{H}$ means that the morphism groups are k-vector spaces, and that composition $\mathrm{Hom}(Y,Z)\times\mathrm{Hom}(X,Y)\to\mathrm{Hom}(X,Z)$, $(g,f)\mapsto g\,f$, is k-bilinear for all objects X,Y and Z from $\mathcal{H}$.

Next we deal with the concept of abelianness: By definition a sequence of $0\to A\xrightarrow{u}B\xrightarrow{v}C\to 0$ is called **short exact** if for each object X from $\mathcal{H}$ the induced sequence $0\to\mathrm{Hom}(X,A)\xrightarrow{u\circ -}\mathrm{Hom}(X,B)\xrightarrow{v\circ -}\mathrm{Hom}(X,C)$ is exact and dually for each object Y of $\mathcal{H}$ the sequence $0\to\mathrm{Hom}(C,Y)\xrightarrow{-\circ v}\mathrm{Hom}(B,Y)\xrightarrow{-\circ u}\mathrm{Hom}(C,Y)$ is exact.

Abelianness of $\mathcal{H}$ now requests two things:

(1) For every every morphism $A \xrightarrow{f} B$ there exist two short exact sequences $0 \to K \xrightarrow{\alpha} A \xrightarrow{\beta} C \to 0$ and $0 \to C \xrightarrow{\gamma} B \xrightarrow{\delta} D \to 0$ such that f is obtained from the commutative diagram below:



(2) Further we request that $\mathcal{H}$ has **finite direct sums**, implying uniqueness of the additive structure, see Keller's course.

1.2 Hom-finiteness and the Krull-Schmidt property

Our next requirement (H2) is an important finiteness assumption:

(H2) *$\mathcal{H}$ is (skeletally) small and **Hom-finite**, that is, all morphism spaces* $\mathrm{Hom}(X, Y)$ *from $\mathcal{H}$ are finite dimensional over k.*

Properties (H1) and (H2) already implies that $\mathcal{H}$ is a Krull-Schmidt category, so that we are already in a setting close to representation theory.

Proposition 1.1 *Each abelian Hom-finite k-category $\mathcal{H}$ is a **Krull-Schmidt-category**, that is,*
(i) *Each indecomposable object from $\mathcal{H}$ has a local endomorphism ring.*
(ii) *Each object from $\mathcal{H}$ is a finite direct sum of indecomposable objects.*

Proof (i) Let E be the endomorphism ring of an indecomposable object U. Then E is a finite dimensional algebra with 0 and 1 as the only idempotents, since idempotents split in $\mathcal{H}$. It follows that E viewed as a right E-module is a finite dimensional indecomposable module over E, and hence — by fundamental properties of modules over finite dimensional algebras — has a local endomorphism ring $E = \mathrm{End}(E_E)$.

Concerning (ii), assume that X is any object of $\mathcal{H}$. Clearly, the number of summands s of any direct decomposition $X = X_1 \oplus \cdots \oplus X_s$ into non-zero summands is bounded by the k-dimension of $\mathrm{End}(X)$. Choosing s maximal, hence yields a decomposition of X into a finite number of indecomposable objects. $\square$

As for modules the **theorem of Krull-Schmidt** states that the decomposition $X = X_1 \oplus \cdots \oplus X_s$ of X into indecomposable objects is unique up to order and isomorphism of summands. In order to understand the category $\mathcal{H}$ it is therefore sufficient to understand the full subcategory ind-$\mathcal{H}$ of indecomposable objects of $\mathcal{H}$, which however is no longer abelian. Similarly, if we are interested in classifying objects, it suffices to do this for the indecomposable ones. Also the quiver $\Gamma_{\mathcal{H}}$ whose vertices consist of (a representative system of) indecomposable objects of $\mathcal{H}$, where the number of arrows from U to V is derived from $\mathrm{rad}_{\mathcal{H}}(X,X)/\mathrm{rad}^2_{\mathcal{H}}(X,Y)$ with X, Y from ind-$\mathcal{H}$ can now be formed.

1.3 Heredity and the Euler form

We call an abelian category $\mathcal{H}$ **hereditary** if the extensions $\mathrm{Ext}^n_{\mathcal{H}}(X,Y)$ vanish in degrees $n \geq 2$ for all objects X, Y from $\mathcal{H}$. It is equivalent to assume that all first extension functors $\mathrm{Ext}^1(X,-)$ and $\mathrm{Ext}^1(-,Y)$ are right exact, that is, send short exact sequences to right exact sequences. Our next requirement (H3) requests that $\mathcal{H}$ is hereditary.

(H3) *The category $\mathcal{H}$ is hereditary.*

Remark 1.2 (i) The most prominent example of a hereditary category is the category $\mathcal{H} = \text{mod-}\Lambda$ of all finite dimensional right modules over a finite dimensional **hereditary algebra** Λ. Recall that this means that all submodules of projective modules are again projective. For that it is sufficient to establish that all maximal submodules of indecomposable projective modules are again projective.

This method is used, in particular, to show that the path algebra $k[Q]$ of a finite quiver Q without oriented cycles is hereditary.

(2) If k is algebraically closed, assume conversely that $\mathcal{H} = \text{mod-}\Lambda$ with Λ finite dimensional hereditary. With Q the quiver of the full subcategory of indecomposable projectives, it follows that mod-Λ is equivalent to mod-$k[Q]$.

Thus the study of finite dimensional modules over finite dimensional hereditary algebras is reduced to study k-linear representations of quivers if $k = \bar{k}$.

Exercise. Assume Λ is a finite dimensional algebra. Prove that submodules of projective right Λ-modules are projective if and only if the same assertion holds for left modules.

[Hint: Show first that factor modules of injective right modules are injective, and then invoke duality $\mathrm{D} = \mathrm{Hom}_k(-, k)$ from left to right modules.]

The assumptions (H1), (H2) and (H3), introduced sofar, are very useful as we will see now.

Recall that the **Grothendieck group** $\mathrm{K}_0(\mathcal{H})$ of a small abelian category $\mathcal{H}$ is obtained by taking the free abelian group on the set of objects of $\mathcal{H}$ modulo all relations $A - B + C$ given by short exact sequences $0 \to A \to B \to C \to 0$ in $\mathcal{H}$. We denote the class of an object A in $\mathrm{K}_0(\mathcal{H})$ by $[A]$.

A linear form $\lambda : \mathrm{K}_0(\mathcal{H}) \to \mathbb{Z}$ on the Grothendieck group may thus be viewed as a mapping $\lambda : \mathcal{H} \to \mathbb{Z}$, defined on the set of objects of $\mathcal{H}$, which is additive on short exact sequences, that is, for each short exact sequence $0 \to A \to B \to C \to 0$ we assume $\lambda(B) = \lambda(A) + \lambda(C)$.

The next assertion gives a very useful construction of hereditary abelian categories from existing ones. Recall that a full subcategory $\mathcal{H}'$ of an abelian category $\mathcal{H}$ is called an **exact subcategory** if it is closed under kernels, cokernels and direct sums formed in $\mathcal{H}$. It is obvious that $\mathcal{H}'$ is again an abelian category, with the short exact sequences inherited from $\mathcal{H}$.

Proposition 1.3 *Let $\mathcal{H}$ be a small abelian hereditary category. For each $\lambda : \mathrm{K}_0(\mathcal{H}) \to \mathbb{Z}$ the full subcategory $\mathcal{H}(\lambda)$ consisting of all objects satisfying*

(i) $\lambda(X) = 0$,

(ii) We have $\lambda(X') \leq \lambda(X)$ for each subobject X' of X,

is an exact abelian subcategory of $\mathcal{H}$ which is closed under extensions. In particular, the category $\mathcal{H}(\lambda)$, called the ***subcategory controlled by*** *λ, is again hereditary abelian.*

Proof. We have to show that $\mathcal{H}(\lambda)$ is closed under kernels, cokernels and extensions. We show closedness under kernels, the other proofs are similar. So let $0 \to K \to A \xrightarrow{u} B$ be exact with A and B in $\mathcal{H}(\lambda)$. Let I be the image of u, such that $\eta : 0 \to K \to A \to I \to 0$ is exact. Since I is contained in B we have $\lambda(I) \leq 0$, on the other hand $\lambda(K) \leq 0$, so exactness of η shows $\lambda(I) \geq 0$. This proves $\lambda(I) = 0$; invoking η again we find $\lambda(K) = 0$ as claimed. We have established that $\mathcal{H}(\lambda)$ is an abelian category in its own right.

To see that $\mathcal{H}(\lambda)$ is hereditary, we use that for any two objects A and B from $\mathcal{H}(\lambda)$ we have $\mathrm{Ext}^1_{\mathcal{H}(\lambda)}(A,B) = \mathrm{Ext}^1_{\mathcal{H}}(A,B)$, since $\mathcal{H}(\lambda)$ is extension-closed in $\mathcal{H}$. Here, we use the interpretation of first extensions as equivalence classes of short exact sequences. It follows that the functors $\mathrm{Ext}^1_{\mathcal{H}(\lambda)}(A,-)$, with A in $\mathcal{H}(\lambda)$, are right exact, proving the claim. $\square$

The easiest way to produce appropriate linear forms on $\mathrm{K}_0(\mathcal{H})$ is by means of the **Euler form** which is the bilinear form $\langle -,- \rangle : \mathrm{K}_0(\mathcal{H}) \times \mathrm{K}_0(\mathcal{H}) \to \mathbb{Z}$ given on classes $[X]$ of objects X from $\mathcal{H}$ by the homological expression

$$\langle [X],[Y] \rangle = \dim_k \mathrm{Hom}(X,Y) - \dim_k \mathrm{Ext}^1(X,Y).$$

In order to make this work we have to assume that $\mathcal{H}$ is actually **Ext-finite**, meaning that all extension spaces $\mathrm{Ext}^n(X,Y)$ are finite dimensional. (Here, of course, only $n = 0,1$ matter.) Heredity of $\mathcal{H}$ and the long exact Hom-Ext-sequences induced by short exact sequences $0 \to X' \to X \to X'' \to 0$ and $0 \to Y' \to Y \to Y'' \to 0$ are used in order to see that a bilinear form with the above properties indeed exists. Exercise: Check this!

1.4 Strengthening heredity: Serre duality

For a finite dimensional hereditary algebra we have Auslander-Reiten duality $\mathrm{D}\,\mathrm{Ext}^1(X,Y) = \mathrm{Hom}(Y,\tau X)$, without passage to the stable category, see Lydia Angeleri's lectures, where τ denotes the functor $\mathrm{D}\,\mathrm{Tr} = \mathrm{D}\,\mathrm{Ext}^1(-,\Lambda)$. Note that τ annihilates exactly the projective modules. In order to get **some new phenomena** we assume that $\mathcal{H}$ satisfies the corresponding property where τ is now assumed to be an equivalence of $\mathcal{H}$.

(H3*) *(**Serre duality**) We assume the existence of an **equivalence** $\tau : \mathcal{H} \to \mathcal{H}$ and of natural isomorphisms*

$$\mathrm{Ext}^1(X,Y) \xrightarrow{\sim} \mathrm{D}\,\mathrm{Hom}(Y,\tau X)$$

for all objects X, Y from $\mathcal{H}$.

Two consequences of Serre duality are of major importance:

Proposition 1.4 *Assume that $\mathcal{H}$ is an abelian k-category which is Hom-finite and satisfies Serre duality. Then the following holds:*

(i) *$\mathcal{H}$ is an* ***Ext-finite hereditary*** *category without nonzero projectives or injectives.*

(ii) *$\mathcal{H}$ has* ***almost-split sequences*** *with τ acting as the* ***Auslander-Reiten translation****.*

That is, for each indecomposable object X there is an almost-split sequence $0 \to \tau X \to E \to X \to 0$.

Since τ is an equivalence, observe that by the above each indecomposable object also occurs as the left hand term of an almost-split sequence.

Proof (i): By Serre duality, the first extension functor $\mathrm{Ext}^1(X, -)$ is right exact as the dual of the left exact functor $\mathrm{Hom}(-, \tau X)$. Hence $\mathcal{H}$ is hereditary

Next assume that X is projective. It follows that $\mathrm{Ext}^1(X, -) = 0$, hence $\mathrm{Hom}(-, \tau X) = 0$ and thus τX and therefore X must be zero. A similar proof shows that $\mathcal{H}$ does not admit nonzero injectives.

(ii) Assume, simplifying, that k is **algebraically closed**. (The general proof is similar.) We are going to show that for each indecomposable object X from $\mathcal{H}$ there is an almost-split sequence $0 \to \tau X \to E \to X \to 0$. By Serre duality we have an isomorphism of functors $\mathrm{Ext}^1(X, -) \xrightarrow{\cong} \mathrm{D}\,\mathrm{Hom}(-, \tau X)$, yielding an isomorphism $\psi : \mathrm{Ext}^1(X, \tau X) \xrightarrow{\cong} \mathrm{D}\,\mathrm{End}(\tau X)$. Note that τX is indecomposable since X is indecomposable and τ is an equivalence. Hence $\mathrm{End}(\tau X)$ is a local ring with residue class field k, since k is algebraically closed.

This yields a natural k-linear form $\mathrm{End}(\tau X) \to k$, whose kernel is the radical of $\mathrm{End}(\tau X)$ and corresponding via ψ to a member μ_X from $\mathrm{Ext}^1(X, \tau X)$. It is now straightforward to show that $\mu_X : 0 \to \tau X \to E \to X \to 0$ is almost-split in $\mathcal{H}$. □

The following properties of the Euler form will be used over and over again. Note that properties (ii) and (iii) are only valid in a hereditary context.

Lemma 1.5 *Assume the above setting (H1), (H2) and (H3*) then we have:*
(i) *We have $\langle y, x \rangle = -\langle x, \tau y \rangle$ for all $x, y \in \mathrm{K}_0(\mathcal{H})$*
(ii) *If $\langle [X], [Y] \rangle > 0$ there is a non-zero morphism $X \to Y$.*
(iii) *If $\langle [X], [Y] \rangle < 0$ there is a non-split exact sequence $0 \to Y \to Z \to X \to 0$.*

PROOF. (*i*) It suffices to check this on classes of objects from $\mathcal{H}$, where it follows from Serre duality.

Properties (*ii*) and (*iii*) are an obvious consequence of the heredity of $\mathcal{H}$. □

1.5 Hereditary length categories

In order to get closer to the case of a module category mod-Λ, Λ finite dimensional over k, we assume additionally to (H1), (H2) and (H3') that $\mathcal{H}$ is a length category, that is, each object of $\mathcal{H}$ has finite length.

Example 1.6 [Jordan normal form] Let $k[X]$ be the polynomial ring in one variable over a field k. The category $\mathcal{H} = \text{mod}_0\text{-}k[X]$ of *finite dimensional* $k[X]$-modules is a hereditary abelian k-category which is Hom-finite and satisfies Serre duality with τ the identity functor.

If $k = \bar{k}$ the indecomposables have the form $S_\lambda^{[n]} = k[X]/(X-\lambda)^n$. Further each $S_\lambda^{[n]}$ is **uniserial**, that is, its submodules are linearly ordered and form a finite chain, here the chain $0 \subseteq S_\lambda^{[1]} \subseteq S_\lambda^{[2]} \subseteq \cdots S_\lambda^{[n]}$.

There are natural monomorphisms $\iota_n : S_\lambda^{[n]} \to S_\lambda^{[n+1]}$ and epimorphisms $\pi_n : S_\lambda^{[n+1]} \to S_\lambda^{[n]}$, unique up to scalars. The almost-split sequences have the form

$$0 \to S_\lambda^{[1]} \xrightarrow{\iota_1} S_\lambda^{[2]} \xrightarrow{\pi_n} S_\lambda^{[1]} \to 0$$

and

$$0 \to S_\lambda^{[n]} \xrightarrow{(\iota_n,\pi_n)} S_\lambda^{[n+1]} \oplus S_\lambda^{[n-1]} \xrightarrow{\pi_n} S_\lambda^{[n]} \to 0.$$

Hence the quiver $\Gamma_{\mathcal{H}}$ decomposes into a one-parameter family $(\mathcal{T}_\lambda)$ of homogeneous tubes (we have $\tau X \cong X$ for each member), indexed by the points λ of the affine line k, where $\mathcal{T}_\lambda$ consists of all $S_\lambda^{[n]}$, $n \geq 1$.

Our next result is closely related to the above example. It is an instance of **classification by homological algebra** — as opposed to other classification principles, like combinatorial ones (knitting!) or classification by linear algebra.

As for modules an object U of an abelian category is called **uniserial** if the subobjects of U are linearly ordered by inclusion and form a finite chain $0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_{\ell-1} \subseteq U_\ell = U$. If all indecomposables in an abelian length category $\mathcal{U}$ are uniserial we call $\mathcal{U}$ a **uniserial category**.

Theorem 1.7 (Gabriel) *Let $\mathcal{H}$ be a Hom-finite hereditary length category with Serre duality. Then $\mathcal{H}$ is* ***uniserial****.*

Moreover, ind-$\mathcal{H} = \coprod_{\lambda\in I}\mathcal{T}_\lambda$*, where the quiver of $\mathcal{T}_\lambda$ is of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^n)$, where $n = 0, 1, 2, \cdots$.*

Therefore the quiver of $\mathcal{H}$ decomposes into stable tubes, where for convenience $\mathbb{Z}\mathbb{A}_\infty$ is also viewed as a tube (of infinite period).

Proof. Let S and T be a simple objects. Then $\mathrm{Ext}^1(S,T)$ is nonzero if and only if $\mathrm{Hom}(T,\tau S)\neq 0$ hence, by Schur's lemma, $T\cong \tau S$, and in this case $\mathrm{Ext}^1(S,T)$ is one-dimensional. By a theorem of Gabriel [2] this implies that starting with any simple object S we obtain a chain of inclusions

$$S = U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots,$$

where each U_n is uniserial of length n and $U_{n+1}/U_n \cong \tau^n S$ and, moreover, all indecomposables in $\mathcal{H}$ are obtained this way. □

Further examples of hereditary length categories with Serre duality are given next. They show that the homological classification from the above theorem is also useful to classify in module categories.

Example 1.8 *Let $\Lambda = k[\circ \rightrightarrows \circ]$ be the* ***Kronecker algebra*** *or more generally a* ***tame hereditary algebra****.*

Then the full subcategory reg-Λ *of* mod-Λ *consisting of all modules without an indecomposable preprojective or preinjective module is an abelian hereditary length category with Serre duality. Hence* reg-$\Lambda = \coprod_{\lambda\in I}\mathcal{T}_\lambda$*, where each $\mathcal{T}_\lambda$ is a connected uniserial category whose associated quiver is a tube of finite period.*

One proves the above statement by establishing a linear form δ, called ***defect****, on* $\mathrm{K}_0(\Lambda) := \mathrm{K}_0(\text{mod-}\Lambda)$ *such that* reg-Λ *is the full subcategory of* mod-Λ *controlled by δ. (We will return to this later.) Then one applies Gabriel's theorem and uses that* $\mathrm{K}_0(\Lambda)$ *has finite rank.*

For instance for a ***Kronecker module*** $M = (M_0 \rightrightarrows M_1)$ *one uses the additive function given by* $\delta(M) = \dim_k M_0 - \dim_k M_1$.

Example 1.9 *Let $\mathcal{H}$ be the category of coherent sheaves over a smooth projective curve C. This is a small abelian, Hom-finite category with Serre duality. Then $\mathcal{H}_0$, the full subcategory of $\mathcal{H}$, consisting of all objects of finite length is an exact abelian subcategory of $\mathcal{H}$ which is closed under extensions and closed under τ, so satisfies (H1), (H2) and (H3).*

Again $\mathcal{H}_0$ is uniserial The claim follows from Gabriel's theorem.

2 The repetitive shape of the derived category

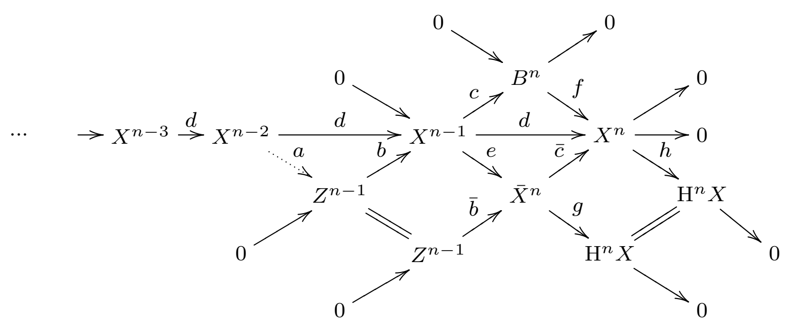
We recall that the **bounded derived category** of an abelian category $\mathcal{H}$ is obtained from the category of bounded complexes in $\mathcal{H}$ by formally inverting all quasi-isomorphisms, see Keller's course.

Theorem 2.1 *Let $\mathcal{H}$ be a hereditary abelian category. Then the bounded derived category $\mathrm{D}^b(\mathcal{H})$ is naturally equivalent to the repetitive category $\bigvee_{n\in\mathbb{Z}}\mathcal{H}[n]$, where each $\mathcal{H}[n]$ is a copy of $\mathcal{H}$, with objects written $X[n]$ for X in $\mathcal{H}$, and morphisms given by*

$$\mathrm{Hom}_{\mathrm{D}^b(\mathcal{H})}(X[n],Y[m]) = \mathrm{Ext}^{m-n}_{\mathcal{H}}(X,Y).$$

Here the expression $\bigvee_{n\in\mathbb{Z}}\mathcal{H}[n]$ has two meanings. First it stands for the additive closure $\mathrm{add}(\bigcup_{n\in\mathbb{Z}}\mathcal{H}[n])$ of the union of all $\mathcal{H}[i]$, and secondly it indicates that there are no nonzero morphisms backwards, that is, from $\mathcal{H}[n]$ to $\mathcal{H}[m]$ for $n > m$.

PROOF. Let X be a bounded complex, being zero in degrees $> n$. As usual let B^n (resp. Z^{n-1}) be the image (resp. the kernel) of $X^{n-1} \xrightarrow{d} X^n$. Let X' denote the complex obtained from X by replacing X^n by 0 and X^{n-1} by Z^{n-1}. We are going to show that X is quasi-isomorphic to $X' \oplus \mathrm{H}^n X[n]$ which by induction implies the claim. Since $\mathcal{H}$ is hereditary, the epimorphism $c : X^{n-1} \to B^n$, induced by d, induces an epimorphism $\mathrm{Ext}^1(\mathrm{H}^n X, X^{n-1}) \to \mathrm{Ext}^1(\mathrm{H}^n X, B^n)$. We thus obtain a commutative diagram with exact diagonals



hence an induced diagram of complexes

$$
\begin{array}{ccccccccccc}
X & & \cdots \to & X^{n-3} & \xrightarrow{d} & X^{n-2} & \xrightarrow{d} & X^{n-1} & \xrightarrow{d} & X^{n} & \to 0 \\
\alpha\uparrow & & & \| & & \| & & \uparrow (1,b) & & \bar{c}\uparrow & \\
\bar{X} & & \cdots \to & X^{n-3} & \xrightarrow{d} & X^{n-2} & \xrightarrow{\binom{0}{a}} & X^{n-1}\oplus Z^{n-1} & \xrightarrow{(e,0)} & \bar{X}^{n} & \to 0 \\
\downarrow\beta & & & \| & & \| & & \downarrow (0,1) & & \downarrow g & \\
X'\oplus \mathrm{H}^{n}X[n] & & \cdots \to & X^{n-3} & \xrightarrow{d} & X^{n-2} & \xrightarrow{a} & Z^{n-1} & \xrightarrow{0} & \mathrm{H}^{n}X & \to 0
\end{array}
$$

with quasi-isomorphisms α and β. The claim follows by induction on the number of non-vanishing cohomology groups. □

Remark 2.2 Let $\mathcal{H}$ satisfy (H1), (H2) and (H3).

(1) Each indecomposable of $\mathrm{D}^b(\mathcal{H})$ belongs to some $\mathcal{H}[n]$. Hence knowing the indecomposables of $\mathcal{H}$, we know the indecomposables of $\mathrm{D}^b(\mathcal{H})$, and of its full subcategories. Important for tilting!

(2) From the description of indecomposable objects it follows that $\mathrm{D}^b(\mathcal{H})$ is **Hom-finite** and further a **Krull-Schmidt category**. These assertions are true in a larger context, see Keller's course, but don't afford here any extra work.

(2) There are only non-zero morphisms from $\mathcal{H}[n]$ to $\mathcal{H}[m]$ if $m \in \{m, m+1\}$. This uses that there are no extensions of negative degree or of degree ≥ 2.

We thus arrive at the following visualization of the derived category

$\cdots$	$\mathcal{H}[-1]$	$\mathcal{H} = \mathcal{H}[0]$	$\mathcal{H}[1]$	$\cdots$

where morphisms between indecomposable objects only exist from left to right, and — again restricting to indecomposable objects — from a copy $\mathcal{H}[n]$ there exist only morphisms within $\mathcal{H}[n]$ or from $\mathcal{H}[n]$ to $\mathcal{H}[n+1]$.

3 Derived category and tilting

An object T from a hereditary abelian Hom-finite category $\mathcal{H}$ is called a **tilting object** if it satisfies the following two conditions:

(i) T has no self-extensions, that is, $\mathrm{Ext}^1(T,T) = 0$.

(*ii*) T generates $\mathcal{H}$ homologically, that is, any object X from $\mathcal{H}$ with $\mathrm{Hom}(T, X) = 0 = \mathrm{Ext}^1(T, X)$ must be the zero object.

(Note that since we do not assume the existence of projectives in $\mathcal{H}$ our definition of a tilting object differs from the one given in Keller's course.)

Theorem 3.1 *Assume $\mathcal{H}$ is a hereditary abelian Hom-finite category with Serre duality and T is a tilting object in $\mathcal{H}$ with endomorphism ring E. Then the following properties hold:*

(*i*) *The right derived functor $R\mathrm{Hom}(T, -)$ induces an equivalence between $\mathrm{D}^b(\mathcal{H})$ and $\mathrm{D}^b(\text{mod-}E)$.*

(*ii*) *The category mod-E of finite dimensional right E-modules is equivalent to the full subcategory $\mathcal{T} \vee \mathcal{F}[1]$ of $\mathrm{D}^b(\mathcal{H})$, where $\mathcal{T}$ consists of all objects X from $\mathcal{H}$ satisfying $\mathrm{Ext}^1(T, X) = 0$, and $\mathcal{F}$ consists of all objects Y of $\mathcal{H}$ such that $\mathrm{Hom}(T, Y) = 0$.*

(*iii*) $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0$, *and each object X of $\mathcal{H}$ is the extension term of a short exact sequence $0 \to X_{\mathcal{T}} \to X \to X_{\mathcal{F}} \to 0$ with $X_{\mathcal{T}}$ from $\mathcal{T}$ and $X_{\mathcal{F}}$ from $\mathcal{F}$. That is, the pair $(\mathcal{T}, \mathcal{F})$ is a torsion theory for $\mathcal{H}$.*

(*iv*) *The global dimension of E is at most two.*

In the module theoretic setting this is close to the **main theorem of tilting theory** due to Brenner-Buttler [1] and Happel-Ringel [3].

Proof. Since T is a tilting object in $\mathcal{H}$, assertion (*i*) follows as in Keller's course.

For the remaining assertions we identify $\mathrm{D}^b(\mathcal{H})$ and $\mathrm{D}^b(\text{mod-}E)$; in particular mod-$E$ then is a full subcategory of $\mathrm{D}^b(\mathcal{H})$, and consists exactly of those objects X of $\mathrm{D}^b(\mathcal{H})$ satisfying $\mathrm{Ext}^n(T, X) = 0$ for any nonzero integer n. Here, $\mathrm{Ext}^n(T, X)$ has to be interpreted as $\mathrm{Hom}_{\mathrm{D}^b(\mathcal{H})}(T, X[n])$, where $[n]$ stands for the **shift by n copies** in $\mathrm{D}^b(\mathcal{H})$.

First observe that indecomposable objects X from $\mathrm{D}^b(\mathcal{H})$, satisfying condition

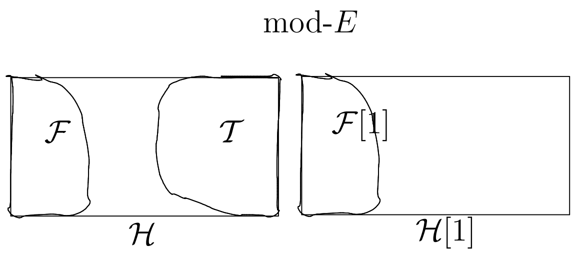
$$(*) \qquad \mathrm{Hom}_{\mathrm{D}^b(\mathcal{H})}(T, X[n]) = 0 \text{ for all nonzero } n$$

can only live in $\mathcal{H}$ or $\mathcal{H}[1]$: Assume that $X = Y[m]$ with Y from $\mathcal{H}$ and $m \geq 2$, then applying $(*)$ for $n = -m$ and $n = 1 - m$ yields $\mathrm{Hom}(T, Y) = 0 = \mathrm{Ext}^1(T, Y)$, and so the tilting condition (*ii*) implies that $Y = 0$, and hence $X = 0$. Similarly one shows that no nonzero object $X = Y[m]$ with $m < 0$ satisfies $(*)$.

Further the objects X from $\mathcal{H}$ satisfying $(*)$ are exactly those satisfying $\mathrm{Ext}^1(T,X) = 0$, and those from $\mathcal{H}[1]$ satisfying $(*)$ are exactly those objects $Y[1]$ with Y in $\mathcal{H}$ satisfying $\mathrm{Hom}(T,Y) = 0$. This proves (*ii*). We are skipping the proof of (*iii*).

Concerning (*iv*) we note that, with the above identifications, any two simple E-modules S and T are lying in $\mathcal{H}$ or in $\mathcal{H}[1]$. Hence $\mathrm{Ext}^3_E(S,T) = \mathrm{Hom}_{\mathrm{D}^b(\mathcal{H})}(S, T[3]) = 0$, since there are no nonzero morphisms from $\mathcal{H}[0]$ or $\mathcal{H}[1]$ to $\mathcal{H}[3]$ or $\mathcal{H}[4]$. This implies that $\mathrm{Ext}^n(-,-)$ vanishes in degrees $n \geq 3$ and proves the claim. □

We may visualize the tilting theorem above as follows:



Usually in this process, passing from $\mathcal{H}$ to mod-E, we will loose some indecomposables. This does not happen, however, if the endomorphism algebra $\mathrm{End}(E)$ is hereditary.

References

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