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Periodicity in Representation Theory of Algebras

Andrzej Skowronski
Nicholas Copernicus University
Mathematics & Informatics Department
Torun, Poland

Strada Costiera 11, 34014 Trieste, Italy - Tel. +39 040 2240 111; Fax +39 040 224 163 - sci_info@ictp.it, www.ictp.it

PERIODICITY IN REPRESENTATION THEORY OF ALGEBRAS

Andrzej Skowroński

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I. Selfinjective algebras

II. Periodicity of modules and algebras

III. Periodicity of finite groups

IV. Periodicity of tame symmetric algebras

V. Periodicity and hypersurface singularities

I. Selfinjective algebras

K algebraically closed field

A finite dimensional K-algebra

$\operatorname{mod} A$ category of finite dimensional right A-modules

A^{op} opposite algebra of A, $a * b = ba$

$\operatorname{mod} A^{\mathrm{op}} = A$-mod category of finite dimensional left A-modules

$D = \operatorname{Hom}_K(-, K)$

$$\operatorname{mod} A \underset{D}{\overset{D}{\rightleftarrows}} \operatorname{mod} A^{\mathrm{op}} \quad \text{duality}$$

$D \cdot D \cong_{\operatorname{mod} A}$, $\quad D \cdot D \cong 1_{\operatorname{mod} A^{\mathrm{op}}}$

$$1_A = \sum_{i=1}^{n_A} \sum_{j=1}^{m_A(i)} e_{ij}$$

e_{ij} pairwise orthogonal primitive idempotents

$e_{ij}A \cong e_{ij'}A$ for all $j, j' \in \{1, \ldots, m_A(i)\}$
$e_{ij}A \not\cong e_{i'j}A$ for $i \neq i'$

We will abbreviate $e_i = e_{i1}$ for $i \in \{1, \ldots, n_A\}$

$P_i = e_i A$, $1 \leq i \leq n_A$, complete set of pairwise nonisomorphic indecomposable projective right A-modules

$I_i = D(Ae_{i1})$, $1 \leq i \leq n_A$, complete set of pairwise nonisomorphic indecomposable injective right A-modules

A is **basic** if $m_A(i) = 1$ for all $i \in \{1, \ldots, n_A\}$

In general, consider the basic idempotent of A

$$e = \sum_{i=1}^{n_A} e_{i1} = \sum_{i=1}^{n_A} e_i$$

$A^{\mathsf{b}} = eAe$ **basic algebra** of A

$\operatorname{mod} A \underset{-\otimes_{A^{\mathsf{b}}} A}{\overset{(-)e}{\rightleftarrows}} \operatorname{mod} A^{\mathsf{b}}$ equivalence of categories

A is **Morita equivalent** to A^{b}

proj A category of projective modules in mod A

proj $A = \text{add}\{P_1, P_2, \ldots, P_n\}$

inj A category of injective modules in mod A

inj $A = \text{add}\{I_1, I_2, \ldots, I_n\}$

$$\text{proj}\, A \underset{D}{\overset{D}{\rightleftarrows}} \text{inj}\, A^{\text{op}}$$

$$\text{inj}\, A \underset{D}{\overset{D}{\rightleftarrows}} \text{proj}\, A^{\text{op}}$$

Proposition. *The following are equivalent for an algebra A:*

(1) A_A *is injective.*

(2) proj $A =$ inj A.

(3) proj $A^{\text{op}} =$ inj A^{op}.

(4) ${}_AA$ *is injective.*

A is **selfinjective** if A_A and ${}_AA$ are injective.

A selfinjective $\Rightarrow$ $e_{11}A, e_{21}A, \ldots, e_{n1}A$ complete set of pairwise nonisomorphic indecomposable injective right A-modules

Hence, there exists a permutation ν of $\{1, \ldots, n_A\}$, called the **Nakayama permutation** such that

$$\operatorname{top} e_{i1}A \cong \operatorname{soc} e_{\nu(i)1}A \text{ for all } i \in \{1, \ldots, n_A\}$$

Theorem (Nakayama, 1941). *An algebra A is selfinjective if and only if there exists a permutation ν of $\{1, \ldots, n_A\}$ such that $\operatorname{top} e_{i1}A \cong \operatorname{soc} e_{\nu(i)1}A$ for all $i \in \{1, \ldots, n_A\}$.*

$D(A) = \operatorname{Hom}_K(A, K)$ is an A-A-bimodule

$(af)(b) = f(ba)$, $\quad (fa)(b) = f(ab)$

for $a, b \in A$, $f \in D(A)$.

$D(A)_A$ injective cogenerator in $\operatorname{mod} A$

${}_AD(A)$ injective cogenerator in $\operatorname{mod} A^{\text{op}}$

Theorem (Brauer, Nesbitt, Nakayama, 1937–1939). *The following statements are equivalent for an algebra A:*

(1) *There exists a nondegenerate K-bilinear form $(-,-) : A \times A \to K$ such that $(a, bc) = (ab, c)$ for all $a, b, c \in A$.*

(2) *There exists a K-linear form $\varphi : A \to K$ such that $\ker \varphi$ does not contain nonzero right ideal of A.*

(3) *There exists an isomorphism $\theta : A_A \to D(A)_A$ of right A-modules.*

(4) *There exists a K-linear form $\varphi' : A \to K$ such that $\ker \varphi'$ does not contain nonzero left ideal of A.*

(5) *There exists an isomorphism $\theta' : {}_AA \to {}_AD(A)$ of left A-modules.*

Proof. (1) ⇒ (2) Let $(-,-) : A \times A \to K$ be a nondegenerate associative K-bilinear form. Define K-linear map $\varphi : A \to K$ by

$$\varphi(a) = (a, 1) = (1, a) \text{ for } a \in A.$$

Let I be a right ideal of A such that $\varphi(I) = 0$. Take $a \in I$. Then $(a, A) = (aA, 1) = \varphi(aA) = 0$ implies $(a, -) = 0$, and so $a = 0$. Hence $I = 0$.

(2) ⇒ (1), (3) Let $\varphi : A \to K$ be a K-linear map such that $\varphi(I) \neq 0$ for any nonzero right ideal I of A. Define K-bilinear form $(-,-) : A \times A \to K$ by

$$(a, b) = \varphi(ab) \text{ for all } a, b \in A.$$

Observe that

$$(a, bc) = \varphi(a(bc)) = \varphi((ab)c) = (ab, c),$$

for $a, b, c \in A$. Let $a \in A$. If $(a, -) = 0$ then $\varphi(aA) = (a, A) = 0$ implies $a = 0$. Assume $(-, a) = 0$. Then $(a, -) = 0$, and hence $a = 0$.

Indeed, consider a K-linear basis $a_1, \ldots, a_m$ of A. Then $a = \sum_{i=1}^{m} \lambda_i a_i$ for some $\lambda_1, \ldots, \lambda_m \in K$, and, for any $j \in \{1, \ldots, m\}$, we have $0 = (a_j, a) = \sum_{i=1}^{n} \lambda_i (a_j, a_i)$, or equivalently

$$\left[(a_j, a_i)\right] \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{bmatrix} = 0.$$

Taking the transpose, we get

$$[\lambda_1, \ldots, \lambda_m] \left[(a_i, a_j)\right] = 0,$$

or equivalently $0 = \sum_{i=1}^{n} \lambda_i (a_i, a_j) = (a, a_j)$ for any $j \in \{1, \ldots, m\}$. Hence $(a, -) = 0$, as required. Therefore $(-, -)$ is a nondegenerate associative K-bilinear form, and (1) holds.

For (3), define the K-linear map

$$\theta = \theta_\varphi : A \to D(A) = \mathrm{Hom}_K(A, K)$$

such that $\theta(a)(b) = \varphi(ab)$, for $a, b \in A$. Then θ is a homomorphism of right A-modules: for $a, b, c \in A$, we have $\theta(ab)(c) = \varphi((ab)c) = \varphi(a(bc)) = \theta(a)(bc) = (\theta(a)b)(c)$, and hence

$\theta(ab) = \theta(a)b$. Moreover, θ is a monomorphism, because, for $a \in A$, $\theta(a) = 0$ implies $\varphi(aA) = \theta(a)(A) = 0$, and hence $aA = 0$, and consequently $a = 0$, by the condition (2). Since $\dim_K A = \dim_K D(A)$, we conclude that θ is an isomorphism of right A-modules.

(3) $\Rightarrow$ (2) Assume that $\theta : A \to D(A)$ is an isomorphism of right A-modules. Define the K-linear map $\varphi = \varphi_\theta = \theta(1) \in D(A)$. Let I be a right ideal of A such that $\varphi(I) = 0$. Then, for any $a \in A$, we have $aA \subseteq I$, and hence we obtain $0 = \varphi(aA) = \theta(1)(aA) = (\theta(1)a)(A) = \theta(a)(A)$ and hence $a = 0$, because θ is and isomorphism of right A-modules. Hence $I = 0$, and (2) holds.

In a similar way, we prove the equivalences (1) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (5). □

An algebra A statisfying one of the equivalent conditions (1)–(5) is called a **Frobenius algebra**.

The class of Frobenius algebras coincides with the class of algebras for which the left and the right regular representations are equivalent, introduced in **1903 by Frobenius**.

A Frobenius $\Rightarrow$ A selfinjective
($A_A \xrightarrow{\sim} D(A)_A \Rightarrow A_A$ is injective)

A basic, selfinjective $\Rightarrow$ A Frobenius

In particular, every selfinjective algebra A is Morita equivalent to a Frobenius algebra, namely its basic algebra A^{b}.

In general, we have the following

Theorem (Nakayama 1939). *Let A be a selfinjective algebra. Then A is a Frobenius algebra if and only if, for the Nakayama permutation $\nu = \nu_A$ of A, we have $m_A(i) = m_A(\nu(i))$ for all $i \in \{1, \ldots, n_A\}$.*

Example. $\Lambda = KQ/I$ where Q is the quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$$

$I = \langle \alpha\beta, \beta\alpha \rangle$. Then Λ is a basic, connected selfinjective algebra with $\operatorname{rad}^2 \Lambda = 0$. Moreover,

$$\Lambda = e_1 A \oplus e_2 \Lambda$$

Take $A = \Lambda(2,1) = \operatorname{End}_\Lambda(e_1\Lambda \oplus e_1\Lambda \oplus e_2\Lambda)$. Then A is a 9-dimensional **selfinjective non-Frobenius algebra** exhibited already by Nakayama.

Hence, the class of Frobenius algebras is not closed under Morita equivalences. The class of selfinjective algebras is the smallest class of algebras containing the Frobenius algebras and closed under Morita equivalences.

An important class of Frobenius algebras is formed by the symmetric algebras.

Theorem (Brauer, Nesbitt, Nakayama, 1937–1941). *The following statements are equivalent for an algebra A:*

(1) *There exists a nondegenerate symmetric K-bilinear form $(-,-) : A \times A \to K$.*

(2) *There exists a K-linear form $\varphi : A \to K$ such that $\varphi(ab) = \varphi(ba)$ for all $a, b \in A$, and $\ker \varphi$ does not contain nonzero one-sided ideal of A.*

(3) *There exists an isomorphism $\theta : {}_AA_A \to {}_AD(A)_A$ of A-A-bimodules.*

Proof. This follows from the proof of the characterizations of Frobenius algebras. □

An algebra A satisfying one of the equivalent conditions (1)–(3) is called a **symmetric algebra**.

Let A be a finite dimensional Frobenius K-algebra and $(-,-) : A \times A \to K$ a nondegenerate associative K-bilinear form. Then there exists a unique K-algebra isomorphism

$$\nu_A : A \to A$$

such that $(a,b) = (b, \nu_A(a))$ for all $a, b \in A$, called the **Nakayama automorphism** of A.

We will see later that ν_A induces the Nakayama permutation of A.

Moreover, $\nu_A = \mathrm{id}_A$ if A is symmetric.

Theorem (Nakayama, 1939). *Let A be a selfinjective algebra. Then* $\mathrm{soc}({}_A A) = \mathrm{soc}(A_A)$. *In particular,* $\mathrm{soc}(A) := \mathrm{soc}({}_A A) = \mathrm{soc}(A_A)$ *is an ideal of A.*

Two selfinjective algebras A and Λ are said to be **socle equivalent** if the factor algebras $A/\mathrm{soc}(A)$ and $\Lambda/\mathrm{soc}(\Lambda)$ are isomorphic.

Examples.

(1) Let $A = K[X]/(X^n), n \geq 1$, be a truncated polynomial algebra. Then A is a commutative local K-algebra with $A \cong D(A)$ as A-A-bimodules, and hence A is symmetric. **Therefore, every finite dimensional commutative K-algebra is a symmetric algebra.**

(2) G finite group, $A = KG$ **group algebra**

$$A = \left\{ \sum_{g \in G} \lambda_g g \ \mid \ \lambda_g \in K \right\}$$

$\dim_K A = \mid G \mid$

$(-,-) : A \times A \to K$

$$\left(\sum_{g \in G} \lambda_g g, \sum_{h \in H} \mu_h h \right) = \sum_{g \in G} \lambda_g \mu_{g^{-1}}$$

symmetric, associative, nondegenerate K-bilinear form. Hence, $A = KG$ is a **symmetric algebra**.

(3) A arbitrary finite dimensional K-algebra

$\mathsf{T}(A) = A \ltimes D(A)$ **trivial extension** of A by the A-A-bimodule $D(A)$

$\mathsf{T}(A) = A \oplus D(A)$ as K-vector space

$$(a, f)(a', f') = (aa', af' + fa')$$

for $a, a' \in a$, $f, f' \in D(A)$

$\mathsf{T}(A)$ is a **symmetric algebra**,

$$\dim_K \mathsf{T}(A) = 2 \dim_K A.$$

$(-,-) : \mathsf{T}(A) \times \mathsf{T}(A) \to K$

$((a, f), (a', f')) = f(a') + f'(a)$

for $a, a' \in A$, $f, f' \in D(A)$, is a symmetric, associative, nondegenerate, K-bilinear form for $\mathsf{T}(A)$.

Observe that $D(A) = 0 \oplus D(A)$ is a two-sided ideal of $\mathsf{T}(A)$ and $A = \mathsf{T}(A)/D(A)$.

The class of symmetric algebras is closed under Morita equivalences (A is symmetric $\iff$ A^{b} is symmetric)

(4) For $\lambda \in K \setminus \{0\}$, let $A_\lambda = KQ/I_\lambda$, where

$$Q: \ \alpha \circlearrowright \bullet \circlearrowleft \beta \ ,$$

$I = \langle \alpha^2, \beta^2, \alpha\beta - \lambda\beta\alpha \rangle$. Then A_λ is a 4-dimensional local Frobenius algebra. But

$$A_\lambda \textbf{ is symmetric } \iff \lambda = 1.$$

Indeed, let $a = \alpha + I_\lambda$, $b = \beta + I_\lambda$. Then $1, a, b, ab = \lambda ba$ is a basis of A_λ over K.

Define $\varphi_\lambda : A_\lambda \to K$ by

$$\varphi(1) = \varphi(a) = \varphi(b) = 0, \quad \varphi(ab) = 1$$

$\ker \varphi$ does not contain nonzero right (left) ideal of A_λ

For $\lambda = 1$, $\varphi = \varphi_1$ has the property $\varphi(xy) = \varphi(yx)$ for all $x, y \in A_1$.

For $\lambda \neq 1$, A_λ is not symmetric. Assume

$$\psi : A \to K, \quad \psi(xy) = \psi(yx) \text{ for all } x, y \in A_\lambda,$$

$\ker \psi$ does not contain nonzero one-sided ideal of A_λ. Then $Kab = Kba$ is a nonzero ideal and hence

$$0 \neq \psi(ba) = \psi(ab) = \psi(\lambda ba) = \lambda\psi(ba) \Rightarrow \lambda = 1.$$

Finite dimensional Hopf algebras

A K-vector space A is a K-**algebra** if and only if there are K-linear maps

$$m : A \otimes_K A \to A \quad \text{and} \quad \eta : K \to A$$

multiplication **unit**

such that the following diagrams are commutative

$$\begin{array}{ccc} A \otimes_K A \otimes_K A & \xrightarrow{1 \otimes m} & A \otimes_K A \\ \downarrow{\scriptstyle m \otimes 1} & & \downarrow{\scriptstyle m} \\ A \otimes_K A & \xrightarrow[m]{} & A \end{array}$$

$$\begin{array}{ccccc} K \otimes_K A & \xrightarrow{\eta \otimes 1} & A \otimes_K A & \xleftarrow{1 \otimes \eta} & A \otimes_K K \\ & \searrow{\scriptstyle \cong} & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle \cong} & \\ & & A & & \end{array}$$

Dually, a K-vector space C is a K-**coalgebra** if there are K-linear maps

$$\Delta : C \to C \otimes_K C \quad \text{and} \quad \varepsilon : C \to K$$

comultiplication **counit**

such that the following diagrams are commutative

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes_K C \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle 1\otimes\Delta} \\ C \otimes_K C & \xrightarrow[\Delta\otimes 1]{} & C \otimes_K C \otimes_K C \end{array}$$

$$\begin{array}{ccccc} K \otimes_K C & \xleftarrow{\varepsilon\otimes 1} & C \otimes_K C & \xrightarrow{1\otimes\varepsilon} & C \otimes_K K \\ & {\scriptstyle\cong}\nwarrow & \uparrow{\scriptstyle\Delta} & \nearrow{\scriptstyle\cong} & \\ & & A & & \end{array}$$

A K-vector space H is a K-**bialgebra** if there are K-linear maps $m : H \otimes_K H \to H$, $\eta : K \to H$, $\Delta : H \to H \otimes_K H$ and $\varepsilon : H \to K$ such that

(1) (H, m, η) is a K-algebra

(2) (H, Δ, ε) is a K-coalgebra

(3) Δ, ε are homomorphisms of K-algebras

If $H = (H, m, \eta, \Delta, \varepsilon)$ is a bialgebra over K then we have the **convolution map**

$$* : \text{Hom}_K(H,H) \times \text{Hom}_K(H,H) \to \text{Hom}_K(H,H)$$

which assigns to $f, g \in \text{Hom}_K(H,H)$ the composition

$$f * g : H \xrightarrow{\Delta} H \otimes_K H \xrightarrow{f \otimes g} H \otimes_K H \xrightarrow{m} H$$

Then a bialgebra $H = (H, m, \eta, \Delta, \varepsilon)$ over K is a **Hopf algebra** if there exists a K-linear map

$$s : H \longrightarrow H \quad \textbf{antipode}$$

such that $s * \text{id}_H = \eta\varepsilon = \text{id}_H * s$. Then $\text{Hom}_K(H,H)$ has a group structure with the multiplication $*$, the unit $\eta\varepsilon$, and the inverse $^{-1}$ given by $f^{-1} = fs$, for $f \in \text{Hom}_K(H,H)$.

Examples. (1) The group algebra KG of a finite group G is a Hopf algebra with the comultiplication Δ, the counit ε and the antipode s given by

$$\Delta(g) = g \otimes g,\ \varepsilon(g) = 1,\ s(g) = g^{-1}, \text{ for } g \in G.$$

(2) Let $H = (H, m, \eta, \Delta, \varepsilon, s)$ be a finite dimensional Hopf algebra over K. Then the dual space $H^* = \mathrm{Hom}_K(H, K)$ is again a Hopf algebra $H^* = (H^*, \Delta^*, \varepsilon^*, m^*, \eta^*, s^*)$ with

$$\Delta^* : H^* \otimes_K H^* \xrightarrow{\sim} (H \otimes_K H)^* \xrightarrow{\Delta^*} H^*,$$

$$\varepsilon^* : K = K^* \longrightarrow H^*,$$

$$m^* : H^* \xrightarrow{\Delta^*} (H \otimes_K H)^* \xrightarrow{\sim} H^* \otimes_K H^*,$$

$$\eta^* : H^* \longrightarrow K^* = K,$$

$$s^* : H^* \longrightarrow H^*.$$

For an antipode s of a Hopf algebra H, we have $s(xy) = s(y)s(x)$ for $x, y \in H$ and $s(1) = 1$.

Theorem (Radford, 1976). *An antipode s of a finite dimensional Hopf algebra H has a finite order. Then s is an antiisomorphism of the algebra H.*

Let $H = (H, m, \eta, \Delta, \varepsilon, s)$ be a Hopf algebra over K. Then the set

$$\int_H^r = \left\{ x \in H \;\middle|\; xh = \varepsilon(h)x \text{ for all } h \in H \right\}$$

is called the space of **right integrals of** H

Theorem (Larson-Sweedler, 1969). *Let H be a finite dimensional Hopf algebra over K. Then the following statements hold*

(1) $\dim_K \int_H^r = 1$ *and* $\dim_K \int_{H^*}^r = 1$.

(2) *For $\varphi \in \int_{H^*}^r \setminus \{0\}$, the K-bilinear form*

$$(-,-) : H \times H \to K$$

such that $(a,b) = \varphi(ab)$ for $a, b \in H$, is nondegenerate and associative.

In particular, H is a **Frobenius algebra**.

Let H be a finite dimensional Hopf algebra over K. Then there exists a homomorphism of K-algebras

$$\xi : H \longrightarrow K \quad \textbf{modular function on } H$$

such that $hx = \xi(h)x$ for all $h \in H, x \in \int_H^r$.

Consider the convolution map

$$\xi * \mathrm{id}_H : H \xrightarrow{\Delta} H \otimes_K H \xrightarrow{\xi \otimes \mathrm{id}_H} K \otimes_K H \xrightarrow{\sim} H$$

Theorem (Fischman-Montgomery-Schneider, 1997). *Let H be a finite dimensional Hopf algebra over K. Then the following statements hold:*

(1) $\nu_H = (\xi * \mathrm{id}_H) \cdot s^{-2}$ *is the Nakayama automorphism of the Frobenius algebra H, that is, $(a, b) = (b, \nu_H(a))$ for all $a, b \in H$.*

(2) ν_H *has finite order dividing $2 \dim_K H$.*

Example. Let $H = KG$ be the group algebra of a finite group G. Then $\int_H^r = Kt$, where $t = \sum_{g \in G} g$

$\xi = \varepsilon : H \to K, \quad s^2 = \mathrm{id}_H$

$\xi * \mathrm{id}_H = \varepsilon * \mathrm{id}_H = \mathrm{id}_H$

$\nu_H = (\xi * \mathrm{id}_H)s^{-2} = \mathrm{id}_H$

This is correct because KG is a symmetric algebra.

Example. Let $n \geq 2$ and λ be a primitive n-th root of unity (hence $\mathrm{char}\, K \not| \, n$). Let

$$H = H_{n^2}(\lambda) = K\langle g, x\rangle/(g^n - 1, x^n, xg - \lambda gx)$$

Then $H_{n^2}(\lambda)$ is an n^2-dimensional Hopf algebra, with K-basis $\{g^i x^j \mid 0 \leq i, j \leq n-1\}$, and the comultiplication Δ, counit ε and antipode s given by

$$\Delta(g) = g \otimes g, \;\; \Delta(x) = g \otimes x + x \otimes 1$$

$$\varepsilon(g) = 1, \quad \varepsilon(x) = 0$$

$$s(g) = g^{-1}, \quad s(x) = -g^{-1}x$$

$H_{n^2}(\lambda)$ is called the **Taft algebra**.

The Taft algebra is neither commutative nor cocommutative (For $n = 2$, $H_4(\lambda)$ is the 4-dimensional **Sweedler's algebra**)

Since $s^2(x) = \lambda x$, $s^2(g) = g$, s has order $2n$. Further, $\int_H^r = Kt$, where

$$t = (\sum_{m=0}^{n-1} \lambda^{-m} g^m) x^{n-1}$$

The modular function $\xi : H \to K$ is given by

$$\xi(g) = \lambda, \quad \xi(x) = 0$$

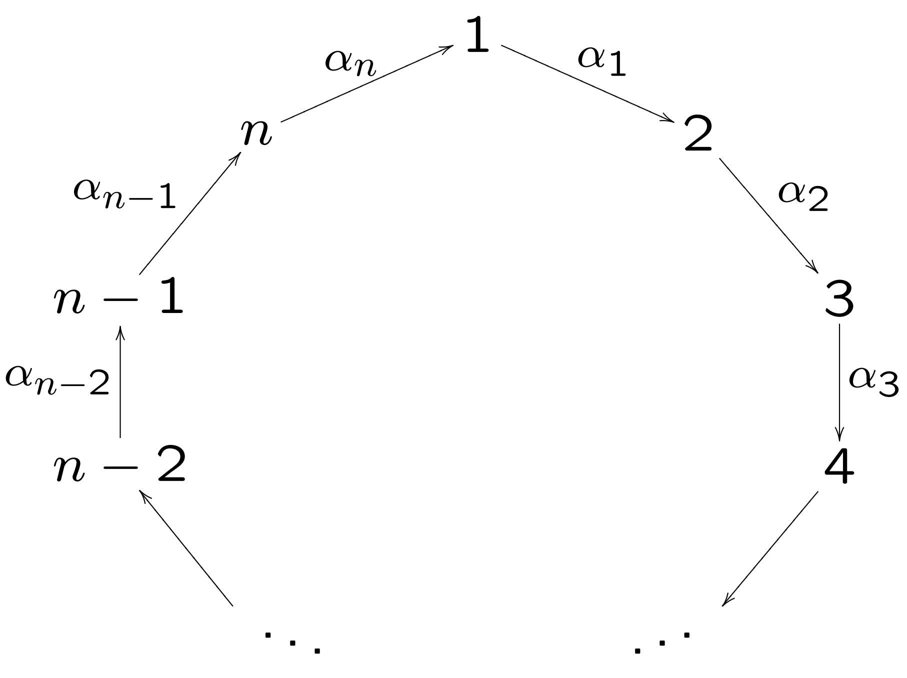
Then the convolution $\xi * \mathrm{id}_H : H \to H$ is given by $\lambda\, \mathrm{id}_H$ and hence the Nakayama automorphism $\nu_H = (\xi * \mathrm{id}_H) s^{-2}$ is given by

$$\nu_H(g) = \lambda g^{-1}, \quad \nu_H(x) = x$$

Hence ν_H has order n.

As an algebra $H = H_{n^2}(\lambda)$ is isomorphic to the skew group algebra $A[G]$ where $A = K[x]/(x^n)$, $G = (g)$ of order n, and G acts on A by $g(\bar{x}) = \lambda^{-1}\bar{x}$, $\bar{x}$ = residue class of x, and $g\bar{x}g = g(\bar{x})gg = \lambda^{-1}\bar{x}gg$ implies $\bar{x}g = \lambda g\bar{x}$.

Moreover, $H = H_{n^2}(\lambda)$ is isomorphic to the bound quiver algebra KQ_n/I_n, where Q_n is the cyclic quiver of the form



and I_n is generated by the paths $\alpha_i\alpha_{i+1}\ldots\alpha_{i+n-1}$, $1 \leq i \leq n$. Hence, as an algebra, $H_{n^2}(\lambda)$ is a selfinjective Nakayama algebra.

Selfinjective orbit algebras

A connected K-category R is **locally bounded** if:

- distinct objects of R are nonisomorphic
- $\forall_{x \in obR} R(x,x)$ is a local algebra
- $\forall_{x \in obR} \sum_{y \in obR} (\dim_K R(x,y) + \dim_K R(y,x)) < \infty$

$\Rightarrow R \cong KQ/I$, Q locally finite connected quiver, I admissible ideal of the path category KQ

$\operatorname{mod} R$ category of finitely generated contravariant functors $R \to \operatorname{mod} K$

$\operatorname{mod} R = \operatorname{rep}_K(Q, I)$

R **bounded** (has finitely many objects) $\Rightarrow$
$\bigoplus R = \bigoplus_{x,y \in obR} R(x,y)$ finite dimensional basic connected K-algebra

We will identify a bounded K-category R with the associated finite dimensional algebra $\oplus R$

R locally bounded K-category

G group of K-linear automorphisms of R

G is **admissible** if G acts freely on the objects of R and has finitely many orbits

R/G **orbit (bounded) category**

objects: G-orbits of objects of R

$$(R/G)(a,b) = \left\{ (f_{yx}) \in \prod_{(x,y)\in a\times b} R(x,y) \,\middle|\, g \cdot f_{yx} = f_{g(y),g(x)} \ \forall_{g\in G, x\in a, y\in b} \right\}$$

$F : R \to R/G$ canonical **Galois covering**

$$ob(R) \ni x \mapsto Fx = G\cdot x \in ob(R/G)$$

$\forall_{x\in obR} \ \forall_{a\in ob(R/G)}$ F induces isomorphisms

$$\bigoplus_{Fy=a} R(x,y) \xrightarrow{\sim} (R/G)(Fx,a),$$

$$\bigoplus_{Fy=a} R(y,x) \xrightarrow{\sim} (R/G)(a,Fx)$$

The group G acts also on $\operatorname{mod} R$

$$\operatorname{mod} R \ni M \mapsto gM = Mg^{-1} \in \operatorname{mod} R$$

We have also the **push-down functor** (**Bongartz-Gabriel**)

$$F_\lambda : \operatorname{mod} R \longrightarrow \operatorname{mod} R/G$$

$$M \in \operatorname{mod} R, a \in ob(R/G) \Rightarrow (F_\lambda M)(a) = \bigoplus_{x \in a} M(x)$$

Assume G is torsion-free. Then F_λ induces an injection (**Gabriel**)

$$\left\{ \begin{matrix} G\text{-orbits of} \\ \text{isoclasses of} \\ \text{indecomposable} \\ \text{modules in } \operatorname{mod} R \end{matrix} \right\} \overset{F_\lambda}{\rightarrowtail} \left\{ \begin{matrix} \text{isoclasses of} \\ \text{indecomposable} \\ \text{modules in} \\ \operatorname{mod} R/G \end{matrix} \right\}$$

R is **locally support-finite** if for any $x \in obR$ $\bigcup_{\substack{M \in \operatorname{ind} R \\ M(x) \neq 0}} \operatorname{supp}(M)$ is a bounded category

R locally support-finite $\xRightarrow{\text{Dowbor-Skowroński}}$ F_λ is dense

Then $\Gamma_{R/G} \cong \Gamma_R/G$ (**Gabriel**)

R selfinjective locally bounded K-category

G admissible group of automorphisms of R
$\Rightarrow R/G$ basic connected finite dimensional selfinjective K-algebra

B basic, connected, finite dimensional K-algebra

$1_B = e_1 + \cdots + e_n$
$e_1, \ldots, e_n$ orthogonal primitive idempotents of B

$\widehat{B}$ **repetitive category** of B
(selfinjective locally bounded K-category)

objects: $e_{m,i}, m \in \mathbb{Z}, 1 \leq i \leq n$

$$\widehat{B}(e_{m,i}, e_{r,j}) = \begin{cases} e_j B e_i & , r = m \\ D(e_i B e_j) & , r = m+1 \\ 0 & , \text{otherwise} \end{cases}$$

$e_j B e_i = \mathrm{Hom}_B(e_i B, e_j B)$, $D(e_i B e_j) = e_j D(B) e_i$

$$\bigoplus_{(m,i)\in\mathbb{Z}\times\{1,\ldots,n\}} \widehat{B}(-, e_{r,j})(e_{m,i}) = e_j B \oplus D(B e_j)$$

Therefore, for any admissible group G of automorphisms of $\widehat{B}$, we obtain a basic, connected, finite dimensional selfinjective K-algebra $\widehat{B}/G$.

In particular, consider the **Nakayama automorphism** $\nu_{\widehat{B}}$ of $\widehat{B}$ such that
$\nu_{\widehat{B}}(e_{m,i}) = e_{m+1,i}$ for all $m, i \in \mathbb{Z} \times \{1, \ldots, n\}$.

Then, for each positive integer r, the infinite cyclic group $(\nu_{\widehat{B}}^r)$ is an admissible group of automorphisms of $\widehat{B}$, and we have the selfinjective algebra

$$\mathsf{T}(B)^{(r)} = \widehat{B}/(\nu_{\widehat{B}}^r)$$

$$= \left\{ \begin{array}{c} \begin{bmatrix} b_1 & 0 & 0 & & & & \\ f_2 & b_2 & 0 & & \text{\Large 0} & & \\ 0 & f_3 & b_3 & & & & \\ & & \ddots & \ddots & & & \\ & \text{\Large 0} & & f_{r-1} & b_{r-1} & 0 \\ & & & 0 & f_1 & b_1 \end{bmatrix} \\ b_1, \ldots, b_{r-1} \in B, f_1, \ldots, f_{r-1} \in D(B) \end{array} \right\}$$

r-**fold trivial extension algebra** of B

The Nakayama automorphism of $\mathsf{T}(B)^{(r)}$ has order r.

Observe that $\mathsf{T}(B)^{(1)} \cong \mathsf{T}(B) = B \ltimes D(B)$.

$\Delta \qquad 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \to \cdots \to n-1 \xrightarrow{\alpha_{n-1}} n$

$B = K\Delta$

$\widehat{B} = K\widehat{\Delta}/\widehat{I}_n$

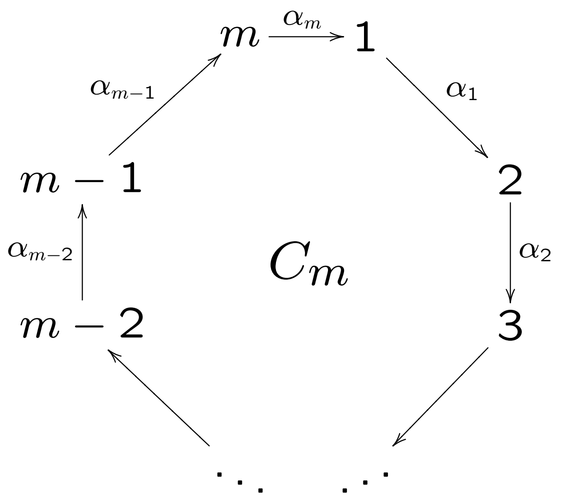
$\widehat{I}_n$ generated by all compositions of $n+1$ consecutive arrows in $\widehat{\Delta}$

$\nu_{\widehat{B}} : \widehat{B} \to \widehat{B}$ Nakayama automorphism, $\nu_{\widehat{B}}(r,i) = (r+1,i)$

$(r,i) \in \mathbb{Z} \times \{1,\dots,n\}$

$\varphi : \widehat{B} \to \widehat{B}$, $\varphi^n = \nu_{\widehat{B}}$

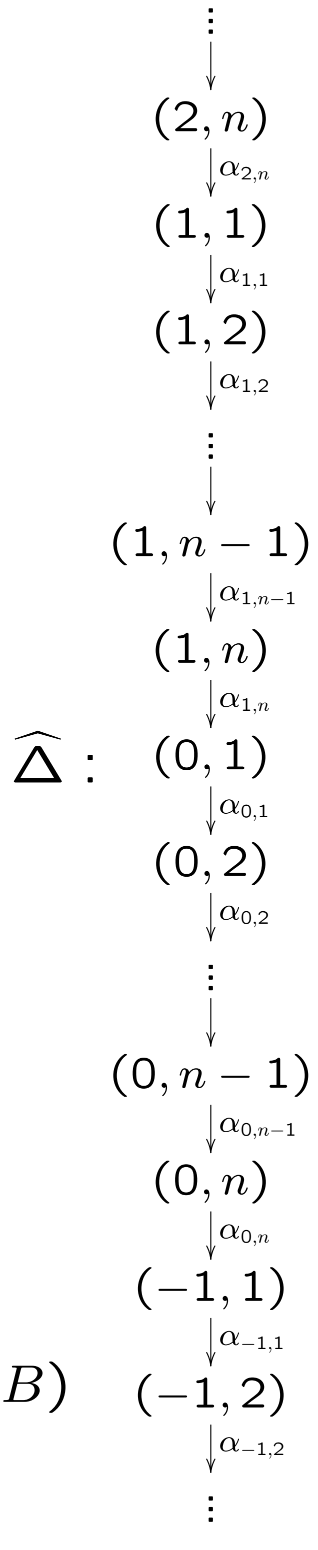
$N_m^n = \widehat{B}/(\varphi^m) = KC_m/J_{m,n}$



$J_{m,n}$ generated by all compositions of $n+1$ consecutive arrows in C_m

N_m^n **Nakayama algebra**, $N_n^n = T(B)$

N_m^n symmetric $\iff m \mid n \iff \varphi^m$ is a root of $\nu_{\widehat{B}}$



A finite dimensional selfinjective K-algebra A is said to be a **Nakayama algebra** if the indecomposable projective A-modules are **uniserial** (the sets of submodules are linearly ordered by inclusion)

Theorem. *Let A be an indecomposable finite dimensional selfinjective K-algebra. The following statements are equivalent:*

(1) *A is a Nakayama algebra.*

(2) *The indecomposable finite dimensional A-modules are uniserial.*

(3) *A is Morita equivalent to N_m^n for some $m, n \geq 1$.*

Assume B is **triangular** (Q_B has no oriented cycles)

Then $\widehat{B}$ is triangular

B is the full bounded subcategory of $\widehat{B}$ given by the objects

$e_{0,i}, 1 \leq i \leq n$

Let i be a sink of Q_B

$B \mapsto S_i^+ B$ **reflection** of B at i

$S_i^+ B$ the full subcategory of $\widehat{B}$ given by the objects

$e_{0,j}$, $1 \le j \le n$, $j \ne i$, and $e_{1,i} = \nu_{\widehat{B}}(e_{0,i})$.

$\sigma_i^+ Q_B = Q_{S_i^+ B}$ reflection of Q_B at i

Observe that $\widehat{B} \cong \widehat{S_i^+ B}$, and hence

$$\mathsf{T}(B)^{(r)} \cong \mathsf{T}(S_i^+ B)^{(r)} \text{ for any } r \ge 1.$$

Reflection sequence of sinks of Q_B: a sequence $i_1, \dots, i_t$ of vertices of Q_B such that i_s is a sink of $\sigma_{i_{s-1}}^+ \dots \sigma_{i_1}^+ Q_B$ for $1 \le s \le t$.

Two triangular algebras B and C are said to be **reflection equivalent** if $C \cong S_{i_t}^+ \dots S_{i_1}^+ B$ for a reflection sequence of sinks $i_1, \dots, i_t$ of Q_B.

B, C reflection equivalent triangular algebras $\Rightarrow \widehat{B} \cong \widehat{C}$, $\mathsf{T}(B)^{(r)} \cong \mathsf{T}(C)^r$ for all $r \ge 1$

II. Periodicity of modules and algebras

Let A be a finite dimensional selfinjective K-algebra. Then A^{op} is also selfinjective and we have the duality between $\operatorname{mod} A$ and A^{op}

$$\operatorname{mod} A \underset{\operatorname{Hom}_{A^{\mathrm{op}}}(-,{}_A A)}{\overset{\operatorname{Hom}_A(-,A_A)}{\rightleftarrows}} \operatorname{mod} A^{\mathrm{op}}$$

Then we have the selfequivalence functor

$$\mathcal{N}_A = D\operatorname{Hom}_A(-,{}_A A) : \operatorname{mod} A \to \operatorname{mod} A$$

called the **Nakayama functor**. Moreover,

$$\mathcal{N}_A^{-1} = \operatorname{Hom}_{A^{\mathrm{op}}}(-,{}_A A)D$$

is the inverse of $\mathcal{N}_A$.

Proposition. *The functors*

$$\mathcal{N}_A, - \otimes_A D(A) : \operatorname{mod} A \to \operatorname{mod} A$$

are equivalent.

Proof. For any module M in $\operatorname{mod} A$, we have a natural isomorphism of right A-modules

$$\phi_M : M \otimes_A D(A) \to D\,\mathrm{Hom}_A(M, A) = \mathcal{N}_A(M)$$

such that $\phi_M(m \otimes f)(g) = f(g(m))$ for $m \in M$, $f \in D(A) = \mathrm{Hom}_K(A, K)$ and $g \in \mathrm{Hom}_A(M, A)$. This induces an equivalence of functors

$$\phi : - \otimes_A D(A) \to \mathcal{N}_A.$$

□

For a K-algebra automorphism σ of A, we denote by

$$(-)_\sigma : \operatorname{mod} A \to \operatorname{mod} A$$

the induced functor such that, for any module M in $\operatorname{mod} A$, M_σ is the module with the twisted right A-module structure

$$m * a = m\sigma(a)$$

for $m \in M$ and $a \in A$.

Proposition. *Let A be a Frobenius algebra and ν_A its Nakayama automorphism. Then the functors*

$$\mathcal{N}_A, (-)_{\nu_A^-} : \operatorname{mod} A \to \operatorname{mod} A$$

are equivalent.

Proof. A required equivalence

$$\psi : (-)_{\nu_A^-} \longrightarrow \mathcal{N}_A$$

is given by the family of isomorphisms of right A-modules

$$\psi_M : M_{\nu_A^-} \longrightarrow \mathcal{N}_A(M) = D\operatorname{Hom}_A(M, A),$$

M modules in $\operatorname{mod} A$, such that

$$\psi_M(m)(g) = (g(m), 1) = (1, g(m))$$

for $m \in M$, $g \in \operatorname{Hom}_A(M, A)$, where $(-,-)$ is the nondegenerate associative K-bilinear form defining the Nakayama automorphism ν_A. □

Hence, if A is a Frobenius algebra, and

$$1_A = \sum_{i=1}^{n_A} \sum_{j=1}^{m_A(i)} e_{ij}$$

is the standard decomposition of 1_A into the sum of pairwise orthogonal primitive idempotents, then we have isomorphisms of right A-modules

$$\mathcal{N}_A(e_{ij}A) \cong (e_{ij}A)_{\nu_A^{-1}} \xrightarrow{\sim} \nu_A(e_{ij})A = \nu_A(e_{ij}A)$$

$$(e_{ij}a) * b = (e_{ij}a)\nu_A^{-1}(b) \longrightarrow \nu_A(e_{ij}a)b$$

for $a, b \in A$. Moreover, $\mathcal{N}_A(e_{ij}A) = D(Ae_{ij})$. Hence we obtain that

$$\mathrm{top}(e_{ij}A) \cong \mathrm{soc}\, \nu_A(e_{ij})A.$$

In particular, the Nakayama automorphism ν_A induces a Nakayama permutation $\nu = \nu_A$ of $\{1, \ldots, n_A\}$.

For a symmetric algebra A, we have $\nu_A = \mathrm{id}_A$ and $\mathcal{N}_A \cong 1_{\mathrm{mod}\, A}$.

In particular, for a symmetric algebra A, we have

$$\operatorname{top} P \cong \operatorname{soc} P$$

for any indecomposable projective A-module P, that is, A is a **weakly symmetric algebra** (the trivial permutation of $\{1, \ldots, n_A\}$ is a Nakayama permutation of A).

Let A be a finite dimensional selfinjective K-algebra

$\underline{\text{mod}} A$ the **stable category** of A

$\underline{\text{mod}} A = \text{mod}\, A / \text{proj}\, A$

The Nakayama functors

$$\mathcal{N}_A, \mathcal{N}_A^{-1} : \text{mod}\, A \to \text{mod}\, A$$

induce the **Nakayama functors**

$$\mathcal{N}_A, \mathcal{N}_A^{-1} : \underline{\text{mod}} A \to \underline{\text{mod}} A$$

because $\mathcal{N}_A(\text{proj}\, A) = \text{inj}\, A = \text{proj}\, A$.

We have also the **Auslander-Reiten functors**

$$\tau_A = D\,\mathrm{Tr}, \tau_A^- = \mathrm{Tr}\,D : \underline{\mathrm{mod}}A \to \underline{\mathrm{mod}}A.$$

Consider also the **(Heller's) syzygy functors**

$$\Omega_A, \Omega_A^{-1} : \underline{\mathrm{mod}}A \to \underline{\mathrm{mod}}A$$

For a module M in $\mathrm{mod}\,A$ without projective direct summands, we have exact sequences

$$0 \to \Omega_A(M) \to P_A(M) \to M \to 0$$

$$0 \to M \to I_A(M) \to \Omega_A^{-1}(M) \to 0$$

where $P_A(M)$ is the projective cover of M and $I_A(M)$ is the injective envelope of M in $\mathrm{mod}\,A$.

Proposition. *Let A be a selfinjective algebra.*

(1) *The functors*

$$D\,\mathrm{Tr}, \Omega_A^2\mathcal{N}_A, \mathcal{N}_A\Omega_A^2 : \underline{\mathrm{mod}}A \to \underline{\mathrm{mod}}A$$

are isomorphic.

(2) *The functors*

$$\mathrm{Tr}\,D, \Omega_A^{-2}\mathcal{N}_A^{-1}, \mathcal{N}_A^{-1}\Omega_A^{-2} : \underline{\mathrm{mod}}A \to \underline{\mathrm{mod}}A$$

are isomorphic.

Proof. For a module M in $\operatorname{mod} A$ without projective direct summands, we have

$$0 \to \Omega_A^2(M) \to P_1(M) \to P_0(M) \to M \to 0$$

minimal projective presentation of M in $\operatorname{mod} A$

$$0 \to \operatorname{Hom}_A(M, A_A) \to \operatorname{Hom}_A(P_0, A_A) \to \operatorname{Hom}_A(P_1(M), A_A) \to \operatorname{Tr} M \to 0$$

$$\begin{array}{ccccccccc} 0 \to D\operatorname{Tr} M \to & D\operatorname{Hom}_A(P_1, A_A) & \to & D\operatorname{Hom}_A(P_0(M), A_A) & \to & D\operatorname{Hom}_A(M, A_A) & \to 0 \\ & \| & & \| & & \| & \\ & \mathcal{N}_A(P_1(M)) & & \mathcal{N}_A(P_0(M)) & & \mathcal{N}_A(M) & \end{array}$$

minimal projective presentation of $\mathcal{N}_A(M)$

Hence, $\Omega_A^2 \mathcal{N}_A(M) \cong D\operatorname{Tr} M \cong \mathcal{N}_A \Omega_A^2(M)$. □

Corollary. *Let A be a symmetric algebra. Then*

(1) *The functors $D\operatorname{Tr}, \Omega_A^2 : \underline{\operatorname{mod}} A \to \underline{\operatorname{mod}} A$ are isomorphic.*

(2) *The functors $\operatorname{Tr} D, \Omega_A^{-2} : \overline{\operatorname{mod}} A \to \overline{\operatorname{mod}} A$ are isomorphic.*

Γ_A **Auslander-Reiten quiver of** A

P indecomposable projective-injective A-module, then we have in $\operatorname{mod} A$ an Auslander-Reiten sequence of the form

$0 \to \operatorname{rad} P \to \operatorname{rad} P/\operatorname{soc} P \oplus P \to P/\operatorname{soc} P \to 0$

For A selfinjective,

Γ_A^s **stable Auslander-Reiten quiver** of A
(obtained from Γ_A by removing the projective-injective vertices and the arrows attached to them)

We may recover Γ_A from Γ_A^s if we know the positions of $\operatorname{rad} P$ (equivalently, $P/\operatorname{soc} P$), P indecomposable projectives, in Γ_A^s.

Two selfinjective algebras A and Λ are said to be **stably equivalent** if the stable module categories $\underline{\operatorname{mod}}A$ and $\underline{\operatorname{mod}}\Lambda$ are equivalent.

Let A be a finite dimensional K-algebra

A module M in $\operatorname{mod} A$ is called Ω_A-**periodic** (shortly, **periodic**) if $\Omega_A^n(M) \cong M$ for some $n \geq 1$.

PROBLEM. Determine the finite dimensional K-algebras A whose all indecomposable nonprojective finite dimensional right A-modules are periodic (say up to Morita equivalence).

We will see later that all such algebras are selfinjective.

Similarly, a module M in $\operatorname{mod} A$ is called $D\operatorname{Tr}$-**periodic** if $(D\operatorname{Tr})^n(M) \cong M$ for some $n \geq 1$.

PROBLEM. Determine the finite dimensional K-algebras A for which all indecomposable nonprojective finite dimensional right A-modules are $D\operatorname{Tr}$-periodic.

It is clear that such algebras are selfinjective, because the $D\,\mathrm{Tr}$-orbit of an indecomposable injective A-module consists of one module, which is an indecomposable projective A-module.

Let A be a selfinjective algebra. Then $D\,\mathrm{Tr} \cong \Omega_A^2 \mathcal{N}_A$ as functors on $\underline{\mathrm{mod}}A$. Hence, the Ω_A-periodicity in $\underline{\mathrm{mod}}A$ coincides with the $D\,\mathrm{Tr}$-periodicity in $\underline{\mathrm{mod}}A$ if the Nakayama functor $\mathcal{N}_A$ on $\underline{\mathrm{mod}}A$ has finite order.

For example, it is the case for all finite dimensional Hopf algebras H, because they are Frobenius algebras with the Nakayama automorphism ν_H of finite order, and $\mathcal{N}_H \cong (-)_{\nu_H^{-1}}$ on $\underline{\mathrm{mod}}H$.

Obviously, it is also the case for all symmetric algebras.

Proposition. *Let A be a finite dimensional selfinjective K-algebra of finite representation type. Then all indecomposable nonprojective finite dimensional A-modules are Ω_A-periodic and $D\,\mathrm{Tr}$-periodic.*

Proof. Let M be an indecomposable nonprojective right A-module. If M is not Ω_A-periodic (respectively, $D\,\mathrm{Tr}$-periodic) then $\Omega_A^n(M)$, $n \geq 0$ (respectively, $(D\,\mathrm{Tr})^n(M)$, $n \geq 0$) is an infinite family of pairwise nonisomorphic indecomposable modules in $\mathrm{mod}\, A$, and hence A is of infinite representation type, a contradiction. □

We will now discuss the Ω_A-periodicity of modules.

Let A be a basic, indecomposable, finite dimensional selfinjective K-algebra.

Lemma. *The following statements for A are equivalent:*

(1) $\Omega_A(S)$ *is simple for any simple A-module S.*

(2) $A \cong N_m^1$ *for some $m \geq 1$.*

Proof. (1) ⇒ (2) For any simple A-module S, we have an exact sequence

$$0 \longrightarrow \Omega_A(S) \longrightarrow P(S) \longrightarrow S \longrightarrow 0.$$

Hence, $\Omega_A(S) \cong \operatorname{soc} P_A(S) = \operatorname{rad} P_A(S)$, and consequently $J(A)^2 = 0$ ($J(A)$ Jacobson radical of A). Then $A \cong N_m^1$ for some $m \geq 1$.

For (2) ⇒ (1) note that $J(N_m^1)^2 = 0$. □

Corollary. *The following statements for A are equivalent:*

(1) $\Omega_A(S) \cong S$ *for any simple A-module S.*

(2) $\Omega_A(M) \cong M$ *for any indecomposable non-projective A-module M.*

(3) $A \cong N_1^1 (\cong K[x]/(x^2) \cong \mathrm{T}(K) = K \ltimes D(K))$.

Theorem. *The following statements for A are equivalent:*

(1) $\Omega^2(S)$ *is simple for any simple A-module S.*

(2) $D\,\mathrm{Tr}(S)$ *is simple for any simple A-module S.*

(3) $A \cong N_m^n$ *for some $m, n \geq 1$.*

(characterization of the Nakayama algebras)

Proof. For (1) $\Leftrightarrow$ (2), observe that for any simple A-module S, $\mathcal{N}_A(S)$ is simple (because the Nakayama functor $\mathcal{N}_A = D\,\mathrm{Hom}_A(-, A)$ is exact) and $D\,\mathrm{Tr}(S) \cong \Omega_A^2 \mathcal{N}_A(S)$.

(1) $\Rightarrow$ (3) Since A is basic and indecomposable, $A \cong KQ/I$ for a connected quiver Q and an admissible ideal I of KQ. For a simple A-module S, we have an exact sequence

$$0 \to \Omega_A^2(S) \to P_A(\mathrm{rad}\, P_A(S)) \to P_A(S) \to S \to 0.$$

$\Omega_A^2(S)$ simple implies $P_A(\mathrm{rad}\, P_A(S))$ indecomposable, and hence $\mathrm{top}(\mathrm{rad}\, P_A(S))$ is simple.

Therefore, every vertex of Q is the starting (respectively, ending) vertex of exactly one arrow.

Then $A \cong N_m^n$ for some $m, n \geq 1$.

(3) $\Rightarrow$ (1) follows from the above exact sequences and the bound quiver presentation of N_m^n. □

Corollary. *The following statements for A are equivalent:*
(1) $\Omega_A^2(S) \cong S$ *for any simple A-module S.*
(2) $\Omega_A^2(M) \cong M$ *for any indecomposable nonprojective finite dimensional A-module M.*
(3) $A \cong N_m^n$, $n+1$ *divisible by m.*

Corollary. *The following statements for A are equivalent:*
(1) *A is symmetric and* $\Omega_A^2(S) \cong S$ *for any simple A-module S.*
(2) $D\,\mathrm{Tr}(S) \cong S$ *for any simple A-module S.*
(3) $D\,\mathrm{Tr}(M) \cong M$ *for any nonprojective indecomposable finite dimensional A-module M.*
(4) $A \cong N_1^n \cong K[x]/(x^{n+1})$ *for some $n \geq 1$.*

Proof. (1) $\Leftrightarrow A \cong N_m^n$ with $m \mid n$, $m \mid n+1$ $\Leftrightarrow A \cong N_1^n$, hence (1) $\Leftrightarrow$ (4).

(4) $\Rightarrow$ (3) $\Rightarrow$ (2) known.

(2) $\Rightarrow$ (4) follows because (2) implies $A \cong N_m^n$. □

Example. Let $H = H_{n^2}(\lambda)$, $n \geq 2$, be the Taft (Hopf) algebra. Then $H \cong N_n^{n-1}$. Hence, for any indecomposable nonprojective finite dimensional H-module M, we have

$$\Omega_H^2(M) \cong M \quad \text{and} \quad \Omega_H^2(M) \ncong M.$$

On the other hand, we have

$$(D\,\mathrm{Tr})^n(M) \cong M \text{ and } (\mathrm{Tr}\,D)^r(M) \ncong M,$$

for $1 \leq r < n$, because

$$D\,\mathrm{Tr}(M) \cong \Omega_H^2 \mathcal{N}_H(M) \cong \mathcal{N}_H(M) \cong M_{\nu_H^{-1}},$$

and the Nakayama automorphism ν_H has order n.

Proposition. *Let H be a finite dimensional Hopf algebra over K. The following statements are equivalent:*

(1) *The trivial H-module K is Ω_H-periodic.*

(2) *All indecomposable nonprojective finite dimensional H-modules are Ω_H-periodic.*

Proof. Let $H = (H, m, \eta, \Delta, \varepsilon, s)$. Then the counit $\varepsilon : H \to K$ induces on K the structure of trivial right H-module

$$\lambda * h = \lambda\varepsilon(h), \text{ for } \lambda \in K, h \in H.$$

Clearly, K is an indecomposable H-module. Moreover, K is projective if and only if H is semisimple. Hence (2) $\Rightarrow$ (1) holds.

For (1) $\Rightarrow$ (2), we first observe that for any projective module P in $\operatorname{mod} H$ and any module M in $\operatorname{mod} H$, $P \otimes_K M$ is a projective-injective module in $\operatorname{mod} H$. The structure of right module on $P \otimes_K M$ is given by

$$(P \otimes_K M) \otimes H \xrightarrow{1\otimes 1\otimes \Delta} P \otimes_K M \otimes_K H \otimes_K H \xrightarrow{1\otimes \tau \otimes 1} (P \otimes_K H) \otimes_K (M \otimes_K H) \xrightarrow{\alpha\otimes\beta} P \otimes_K M$$

where $\tau : M \otimes_K H \to H \otimes_K M$ is the exchanging map, and $\alpha : P \otimes_K H \to P$, $\beta : M \otimes_K H \to M$ are the right H-module structure maps.

Moreover, the following well-known isomorphism of functors on $\operatorname{mod} K$

$$\mathrm{Hom}_K(P \otimes_K M, -) \xrightarrow{\sim} \mathrm{Hom}_K(P, \mathrm{Hom}_K(M, -))$$

induces an isomorphism of functors on $\operatorname{mod} H$

$$\mathrm{Hom}_H(P \otimes_K M, -) \xrightarrow{\sim} \mathrm{Hom}_H(P, \mathrm{Hom}_H(M, -)).$$

Hence the functor $\mathrm{Hom}_H(P \otimes_K M, -) : \operatorname{mod} H \to \operatorname{mod} H$ is exact, and consequently $P \otimes_K M$ is a projective right H-module. Since H is a Frobenius algebra, $P \otimes_K M$ is also injective.

Assume now that $\Omega_H^n(K) \cong K$ for some $n \geq 1$. Then there exists a long exact sequence of the form in $\operatorname{mod} H$

$$0 \to \Omega_H^n(K) \to P_{n-1} \to \cdots \to P_1 \to P_0 \to K \to 0$$

with $P_0, P_1, \ldots, P_{n-1}$ projective modules. Let M be an indecomposable nonprojective module in $\operatorname{mod} H$. Then we obtain a long exact sequence in $\operatorname{mod} H$

$$0 \to \Omega_H^n(K) \otimes_K M \to P_{n-1} \otimes_K M \to \cdots \to P_1 \otimes_K M \to P_0 \otimes_K M \to K \otimes_K M \to 0$$

with $P_0 \otimes_K M, P_1 \otimes_K M, \ldots, P_{n-1} \otimes_K M$ projective H-modules.

We know that $\Omega_H^n(M)$ is an indecomposable nonprojective H-module. Hence

$$\Omega_H^n(K) \otimes_K M \cong \Omega_H^n(M) \oplus P$$

for some projective H-module P. On the other hand, we have

$$\Omega_H^n(K) \otimes_K M \cong K \otimes_K M \cong M.$$

Hence $\Omega_H^n(M) \cong M$, and M is Ω_H-periodic. Therefore, (1) $\Rightarrow$ (2). □

Proposition. *Let A be a selfinjective algebra, M a module in* $\operatorname{mod} A$*, and r a positive integer. Then*

(1) *The functors* $\operatorname{Ext}^r_A(M,-), \underline{\operatorname{Hom}}_A(\Omega^r_A(M),-) : \underline{\operatorname{mod}} A \to \underline{\operatorname{mod}} A$ *are equivalent.*

(2) *The functors* $\operatorname{Ext}^r_A(-,M), \underline{\operatorname{Hom}}_A(-,\Omega^{-r}_A(M)) : \underline{\operatorname{mod}} A \to \underline{\operatorname{mod}} A$ *are equivalent.*

A selfinjective, M module in $\operatorname{mod} A$

$$\begin{aligned} \operatorname{Ext}^*_A(M,M) &= \bigoplus_{r=0}^{\infty} \operatorname{Ext}^r_A(M,M) \\ &\cong \bigoplus_{r=0}^{\infty} \underline{\operatorname{Hom}}_A(\Omega^r_A(M),M) \end{aligned}$$

Ext-algebra of M (graded K-algebra)

$\underline{f} \in \underline{\operatorname{Hom}}_A(\Omega^r_A(M),M)$, $\underline{g} \in \underline{\operatorname{Hom}}_A(\Omega^s_A(M),M)$

$\underline{f} * \underline{g} = \underline{f} \circ \Omega^r_A(\underline{g}), \quad \Omega^{r+s}_A(M) \to \Omega^r_A(M) \to M.$

Observe that, if M is Ω_A-periodic of period d, then

$$\mathrm{Ext}_A^{i+d}(M,N) \cong \mathrm{Ext}_A^{i}(M,N)$$

for all $i \geq 1$ and modules N in $\mathrm{mod}\, A$. Indeed,

$$\begin{aligned} \mathrm{Ext}_A^{i+d}(M,N) &\cong \underline{\mathrm{Hom}}_A(\Omega_A^{i+d}(M),N) \\ &\cong \underline{\mathrm{Hom}}_A(\Omega_A^{i}(\Omega_A^{d}(M)),N) \\ &\cong \underline{\mathrm{Hom}}_A(\Omega_A^{i}(M),N) \\ &\cong \mathrm{Ext}_A^{i}(M,N) \end{aligned}$$

Theorem (Carlson, 1977). *Let A be a finite dimensional selfinjective K-algebra and M be an indecomposable Ω_A-periodic A-module of period d. Moreover, let $\mathcal{N}(M)$ be the ideal of the algebra $\mathrm{Ext}_A^*(M,M)$ generated by all nilpotent homogeneous elements. Then*

$$\mathrm{Ext}_A^*(M,M)/\mathcal{N}(M) \cong K[x]$$

as graded K-algebras, where x is of degree d.

Proof. We identify

$$\underline{\mathrm{Hom}}_A(\Omega^i_A(M), M) = \mathrm{Ext}^i_A(M, M)$$
$$= \underline{\mathrm{Hom}}_A(M, \Omega^{-i}_A(M))$$

for any $i \geq 1$.

Let $\underline{f} \in \underline{\mathrm{Hom}}_A(\Omega^s_A(M), M)$ be a homogeneous nilpotent element of $\mathrm{Ext}^*_A(M, M)$ and $\underline{g} \in \underline{\mathrm{Hom}}_A(\Omega^m_A(M), M)$ an arbitrary homogeneous element of $\mathrm{Ext}^*_A(M, M)$.

We claim that

$$\underline{f} * \underline{g} = \underline{f}\Omega^s_A(\underline{g}) \in \underline{\mathrm{Hom}}_A(\Omega^{m+s}_A(M), M)$$

is again a nilpotent element of $\mathrm{Ext}^*_A(M, M)$.

Choose r such that $r(m + s) = qd$ for some $q \geq 1$, and consider $\underline{h} = (\underline{f}\Omega^s_A(\underline{g}))^r$ in $\mathrm{Ext}^*_A(M, M)$. Then

$$\underline{h} \in \underline{\mathrm{Hom}}_A(\Omega^{qd}_A(M), M) \cong \underline{\mathrm{Hom}}_A(M, M),$$

because $\Omega^{qd}_A(M) \cong M$. Suppose $\underline{h}$ is an isomorphism. Then $f : \Omega^s_A(M) \to M$ is a split epimorphism, and hence an isomorphism, since M and $\Omega^s_A(M)$ are indecomposable. But then

$\underline{f}$ is not nilpotent in $\text{Ext}^*_A(M, M)$, a contradiction. Therefore, $\underline{h}$ belongs to the radical of the local algebra $\underline{\text{End}}_A(M)$, and hence $\underline{h}$ is nilpotent. Then $\Omega^{id}_A(\underline{h}) \in \underline{\text{End}}_A(M)$ are nilpotent for all $i \geq 0$, and hence belong to the radical of $\underline{\text{End}}_A(M)$. Since $(\text{rad}\,\underline{\text{End}}_A(M))^l = 0$ for some $l \geq 1$, we get $\underline{h}^l = 0$. But then $\underline{f} * \underline{g} = \underline{f}\Omega^s_A(\underline{g})$ is a nilpotent element in $\text{Ext}^*_A(M, M)$. Similarly, using

$$\text{Ext}^i_A(M, M) = \underline{\text{Hom}}_A(M, \Omega^{-i}_A(M)), i \geq 1,$$

we prove that $g * f$ is nilpotent in $\text{Ext}^*_A(M, M)$.

Let $s \neq pd$ for all $p \geq 1$. We show that any element $\underline{f} \in \underline{\text{Hom}}_A(\Omega^s_A(M), M)$ is a nilpotent element of $\text{Ext}^*_A(M, M)$.

Choose $r \geq 1$ such that $rs = qd$ for some $q \geq 1$, and take $\underline{h} = f^r$ in $\text{Ext}^*_A(M, M)$. Since d is period of M and s is not divisible by d, we conclude that f is not an isomorphism. Then $\underline{h}$ is not an isomorphism, hence $\underline{h} \in \underline{\text{End}}_A(M)$ is nilpotent. Therefore, h is a nilpotent element in $\text{Ext}^*_A(M, M)$, and so $\underline{f}$ is nilpotent in $\text{Ext}^*_A(M, M)$.

Let $x \in \underline{\mathrm{Hom}}(\Omega_A^d(M), M) \cong \underline{\mathrm{Hom}}_A(M, M)$ corresponds to the residue class of the identity map from M to M. Observe that x is not nilpotent in $\mathrm{Ext}_A^*(M, M)$. We claim that $x^n \notin \mathcal{N}(M)$ for any $n \geq 1$. Suppose that $x^t \in \mathcal{N}(M)$ for some $t \geq 1$. Then $x^t = \sum g_i * f_i * h_i$, where f_i are homogeneous nilpotent elements of $\mathrm{Ext}_A^*(M, M)$ and g_i, h_i are elements of $\mathrm{Ext}_A^*(M, M)$. We may assume that the elements g_i, h_i are also homogeneous.

It follows from the first part of the proof that $g_i * f_i * h_i = (g_i * f_i) * h_i$ are nilpotent elements in $\mathrm{Ext}_A^*(M, M)$, and hence are nilpotent in $\underline{\mathrm{End}}_A(M)$. But then $\sum g_i * f_i * h_i$ are nilpotent in $\underline{\mathrm{End}}(M)$, and hence in $\mathrm{Ext}_A^*(M, M)$. This implies that x^t, and hence x, is nilpotent in $\mathrm{Ext}_A^*(M, M)$, a contradiction. Since $\underline{\mathrm{End}}_A(M)/\operatorname{rad}\underline{\mathrm{End}}_A(M) \cong K$, we conclude that $\mathrm{Ext}_A^*(M, M)/\mathcal{N}(M) \cong K[x]$ as graded K-algebras, with x of degree d. $\square$

Let A be a finite dimensional K-algebra

$1_A = e_1 + e_2 + \cdots + e_m$

$e_1, e_2, \dots, e_m$ pairwise orthogonal primitive idempotents of A

$A^e = A^{\mathrm{op}} \otimes_K A$ **enveloping algebra** of A

$1_{A^e} = \sum_{1 \le i,j \le m} e'_i \otimes e_j$

$$1_{A^e} = \sum_{1 \le i,j \le m} e'_i \otimes e_j$$

$e'_1 = e_1, e'_2 = e_2, \dots, e'_m = e_m$ primitive idempotents of A^{op}

$\operatorname{mod} A^e$ = category of finite dimensional A-A-bimodules

A is a right A^e-module by

$a(x \otimes y) = xay$ for $a \in A$, $x \in A^{\mathrm{op}}$, $y \in A$

$$\begin{aligned} P(i', j) &= (e_i \otimes e_j)A^e = e_i A^{\mathrm{op}} \otimes_K e_j A \\ &= Ae_i \otimes_K e_j A \end{aligned}$$

$$A^e = \bigoplus_{1 \le i,j \le m} P(i', j)$$

$P(i', j)$ indecomposable projective right A^e-modules (projective A-A-bimodules)

${}_A P(i', j) \cong (Ae_i)^{\dim_K e_j A}$ projective left A-module

$P(i', j)_A \cong (e_j A)^{\dim_K Ae_i}$ projective right A-module

Hence every projective right A^e-module is a projective left A-module and a projective right A-module.

Lemma. *Let A be a finite dimensional K-algebra. For each $i \ge 0$, $\Omega^i_{A^e}(A)$ is a projective left A-module and a projective right A-module.*

Proof. Consider a minimal projective resolution of A in $\operatorname{mod} A^e$

$$\cdots \to P_{i+1} \to P_i \to \cdots \to P_1 \to P_0 \to A \to 0$$

For each $i \geq 0$, we have an exact sequence in $\operatorname{mod} A^e$

$$0 \to \Omega^{i+1}_{A^e}(A) \to P_i \to \Omega^i_{A^e}(A) \to 0,$$

which is an exact sequence in $\operatorname{mod} A^{\mathrm{op}}$ and in $\operatorname{mod} A$. Since the projective right A^e-modules are projective left A-modules and projective right A-modules, by induction on i, we conclude that these sequences split in $\operatorname{mod} A^{\mathrm{op}}$ and in $\operatorname{mod} A$, and hence $\Omega^i_{A^e}(A)$ are projective left A-modules and projective right A-modules. $\square$

Lemma. *Let A be a selfinjective algebra and M be a module in* $\operatorname{mod} A$ *without projective direct summands. Then, for each $i \geq 0$, we have*

$$\Omega^i_A(M) \cong M \otimes_A \Omega^i_{A^e}(A) \text{ in } \underline{\operatorname{mod}} A.$$

Proof. We may assume that M is indecomposable. The splitting exact sequences (as in the above lemma)

$$0 \to \Omega^{i+1}_{A^e}(A) \to P_i \to \Omega^i_{A^e}(A) \to 0,$$

for $i \geq 0$, induce the exact sequences

$$0 \to M \otimes_A \Omega^{i+1}_{A^e}(A) \to M \otimes_A P_i \to M \otimes_A \Omega^i_{A^e}(A) \to 0$$

in $\operatorname{mod} A$, and

$$\cdots \to M \otimes_A P_{i+1} \to M \otimes_A P_i \to \cdots \to M \otimes_A P_0 \to M \otimes_A A \to 0$$

is a projective resolution of $M \cong M \otimes_A A$ in $\operatorname{mod} A$. Since, for each $i \geq 0$, $\Omega^i_A(M)$ is an indecomposable nonprojective A-module, we conclude that

$$M \otimes_A \Omega^i_{A^e}(A) \cong \Omega^i_A(M) \oplus P(i)$$

for some projective module $P(i)$ in $\operatorname{mod} A$. Therefore,

$$\Omega^i_A(M) \cong M \otimes_A \Omega^i_{A^e}(A) \text{ in } \underline{\operatorname{mod}} A.$$

$\square$

Lemma (Green-Snashall-Solberg, 2003). *Let A be a finite dimensional K-algebra. Assume there exists a positive integer d and an automorphism σ of A such that $\Omega^d_{A^e}(A) \cong {}_1A_\sigma$ in* $\operatorname{mod} A^e$. *Then A is selfinjective.*

Proof. We have an isomorphism of A-A-bimodules

$$\alpha : D(A) \otimes_A {}_1A_\sigma \longrightarrow D(A)_\sigma$$

such that $\alpha(f \otimes a) = fa$ for $f \in D(A)$ and $a \in {}_1A_\sigma$.

Consider a minimal projective resolution

$$\cdots \to P_{i+1} \to P_i \to \cdots \to P_1 \to P_0 \to A \to 0$$

of A in $\operatorname{mod} A^e$. Hence we obtain an exact sequence

$$0 \to D(A) \otimes_A \Omega^d_{A^e}(A) \to D(A) \otimes_A P_{d-1} \to D(A) \otimes_A \Omega^{d-1}_{A^e}(A) \to 0$$

in $\mathrm{mod}\, A$. Moreover, $D(A) \otimes_A P_{d-1}$ is a projective right A-module. On the other hand, $\Omega^d_{A^e}(A) \cong {}_1A_\sigma$ in $\mathrm{mod}\, A^e$ implies that there is a monomorphism $D(A)_\sigma \to D(A) \otimes_A P_{d-1}$ in $\mathrm{mod}\, A$. Further, the automorphism σ induces an isomorphism ${}_1A_{\sigma^{-1}} \xrightarrow{\sim} {}_\sigma A_1$ of A-A-bimodules, and then the right A-modules $D(A)_\sigma = D({}_\sigma A_1)$ and ${}_{\sigma^{-1}}D(A) = D({}_1A_{\sigma^{-1}})$ are isomorphic. Therefore, the injective cogenerator $D(A)$ in $\mathrm{mod}\, A$ is a direct summand of the projective module $D(A) \otimes_A P_{d-1}$, and so is projective. Clearly then A is selfinjective. $\square$

A finite dimensional K-algebra A is called **periodic** if A is a periodic module in $\operatorname{mod} A^e$, that is, $\Omega^d_{A^e}(A) \cong A$ in $\operatorname{mod} A^e$ for some $d \geq 1$. It follows from the above lemma that then A is selfinjective.

Corollary. *Let A be a finite dimensional periodic K-algebra. Then all indecomposable nonprojective modules in* $\operatorname{mod} A$ *are periodic.*

Proof. Assume $\Omega^d_{A^e}(A) \cong A$ in $\operatorname{mod} A$ for some $d \geq 1$. Let M be an indecomposable nonprojective module in $\operatorname{mod} A$. Since A is selfinjective, invoking the corresponding lemma, we have in $\underline{\operatorname{mod}} A$ isomorphisms

$$\Omega^d_A(M) \cong M \otimes_A \Omega^d_{A^e}(M) \cong M \otimes_A A \cong M.$$

Then $\Omega^d_A(M) \cong M$ in $\operatorname{mod} A$, because $\Omega^d_A(M)$ and M are indecomposable nonprojective modules. □

PROBLEM. Determine the finite dimensional periodic algebras (up to Morita equivalence).

Lemma. *Let A be a finite dimensional K-algebra. Then A is selfinjective if and only if A^e is selfinjective.*

Proof. Since $(A^e)^{\mathsf{b}} \cong (A^{\mathsf{b}})^e$ and the class of selfinjective algebras is closed under Morita equivalences, we may assume that A is basic. Then A^e is basic. Assume A is selfinjective. Then A is a Frobenius algebra and we obtain isomorphisms

$$\begin{aligned} A^e &\cong A^{\mathsf{op}} \otimes_K A \cong D(A^{\mathsf{op}}) \otimes_K D(A) \\ &\cong D(A^{\mathsf{op}} \otimes_A A) \cong D(A^e) \end{aligned}$$

in $\operatorname{mod} A^e$, and hence A^e is selfinjective.

Conversely, if A^e is selfinjective then

$$A^{\mathsf{op}} \otimes_K A \cong D(A^{\mathsf{op}}) \otimes_K D(A)$$

in $\operatorname{mod} A^e$, and hence

$$A^{\mathsf{dim}_K(A^{\mathsf{op}})} \cong D(A)^{\mathsf{dim}_K D(A^{\mathsf{op}})}$$

in $\operatorname{mod} A$. Then A_A is injective, and hence A is selfinjective. $\square$

Theorem (Green-Snashall-Solberg, 2003). *Let A be a finite dimensional indecomposable K-algebra. The following statements are equivalent:*

(1) *All simple right A-modules are Ω_A-periodic.*

(2) *There exists a natural number d and an algebra automorphism σ of A such that $\Omega^d_{A^e}(A) \cong {}_1A_\sigma$ in $\operatorname{mod} A^e$, and $\sigma(e)A \cong eA$ for any primitive idempotent e of A.*

Proof. (1) $\Rightarrow$ (2). Let d be a minimal natural number such that $\Omega^d_A(S) \cong S$ for any simple right A-module S.

Let $B = \Omega^d_{A^e}(A)$. We know that $\Omega^d_{A^e}(A)$ is a projective left A-module. Hence we have the exact functor $- \otimes_A B : \operatorname{mod} A \to \operatorname{mod} A$. Moreover, for any simple right A-module S, we have $S \otimes_A B = S \otimes_A \Omega^d_A(A) \cong \Omega^d_A(S) \cong S$. Then by induction on the length of a module, we conclude that $\ell(M \otimes_A B) \cong \ell(M)$ for any module M in $\operatorname{mod} A$.

We prove now that $P \otimes_A B \cong P$ for any projective module P in $\operatorname{mod} A$. Let P be an indecomposable projective right A-module. Then the exact sequence

$$0 \to PJ(A) \to P \to P/PJ(A) \to 0,$$

where $J(A)$ is the Jacobson radical of A, induces the exact sequence

$$0 \to PJ(A) \otimes_A B \to P \otimes_A B \to (P/PJ(A)) \otimes_A B \to 0.$$

The module $P \otimes_A B$ is a projective right A-module, as a direct summand of the projective right A-module $A \otimes_A B \cong \Omega^d_{A^e}(A)$, and $\ell(P \otimes_A B) = \ell(P)$. Further, $(P/PJ(A)) \otimes B \cong P/PJ(A)$, and hence $P/PJ(A)$ is a direct summand of the top $P \otimes_A B/(P \otimes_A B)J(A)$ of $P \otimes_A B$. Then P is a direct summand of $P \otimes_A B$, and consequently $P \otimes_A B \cong P$, because $\ell(P \otimes_A B) = \ell(P)$. Therefore, there exists an isomorphism $A \otimes_A B \to A$ of right A-modules, and hence B as a right A-module is isomorphic to A_A.

We claim now that B as a left A-module is isomorphic to ${}_AA$. Let T be a simple left A-module. Since B is isomorphic to A_A in $\operatorname{mod} A$, we have $B \otimes_A T \cong A \otimes_A T \cong T$ as K-vector spaces. Further, for any simple right A-module S, we have $S \otimes_A B \otimes_A T \cong S \otimes_A T$ (from the first part of the proof) and $S \otimes_A T \neq 0$ if and only if $S = D(T) = \operatorname{Hom}_K(T, K)$. Then $(A/J(A)) \otimes_A B \otimes_A T \cong T$. On the other hand, we have in $\operatorname{mod} A^{\text{op}} = A$-mod the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & J(A) \otimes_A (B \otimes_A T) & \to & A \otimes_A (B \otimes_A T) & \to & (A/J(A)) \otimes_A (B \otimes_A T) & \to & 0 \\ & & \downarrow \approx & & \downarrow \approx & & \downarrow \approx & & \\ 0 & \to & J(A)(B \otimes_A T) & \to & B \otimes_A T & \to & B \otimes_A T / J(A)(B \otimes_A T) & \to & 0 \end{array}$$

and hence $(B \otimes_A T)/J(A)(B \otimes_A T) \cong T$ in $\operatorname{mod} A^{\text{op}}$.

Since $\dim_K B \otimes_A T = \dim_K T$ we obtain that $B \otimes_A T \cong T$ as left A-modules. Therefore, $B \otimes_A T \cong T$ in A-mod for all simple left A-modules T. Applying now arguments from the first part of the proof we conclude that B as a left A-module is isomorphic to ${}_AA$.

Let $\psi : A \to B$ be an isomorphism of left A-modules, and $b = \psi(1)$. Then $\psi(a) = ab$ for $a \in A$, and $Ab = B$.

Define $\sigma : A \to A$ by $\sigma(a) = \psi^{-1}(ba)$ for $a \in A$. Then, for $a \in A$, we have

$$\begin{aligned} ba &= \psi(\psi^{-1}(ba)) = \psi(\sigma(a)) = \psi(\sigma(a)1) \\ &= \sigma(a)\psi(1) = \sigma(a)b \end{aligned}$$

Next we show that σ is a homomorphism of K-algebras. Obviously, σ is K-linear and $\sigma(1) = \psi^{-1}(b) = 1$.

Moreover, for $a, a' \in A$, we have

$$\begin{aligned}\sigma(aa')b &= b(aa') = (ba)a' = (\sigma(a)b)a' \\ &= \sigma(a)(ba') = \sigma(a)(\sigma(a')b) \\ &= (\sigma(a)\sigma(a'))b.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\psi(\sigma(aa')) &= \psi(\sigma(aa')1) = \sigma(aa')\psi(1) = \sigma(aa')b \\ &= (\sigma(a)\sigma(a'))b = (\sigma(a)\sigma(a'))\psi(1) \\ &= \psi(\sigma(a)\sigma(a'))\end{aligned}$$

and so $\sigma(aa') = \sigma(a)\sigma(a')$.

Therefore, σ is a homomorphism of K-algebras.

We claim that σ is an automorphism. It is enough to show that $\ker \sigma = 0$. Let $a \in \ker \sigma$. Then $0 = \sigma(a)b = ba$ and hence $Ba = (Ab)a = A(ba) = 0$. Since B is isomorphic to A as a right A-module, we obtain $Aa = 0$, and hence $a = 0$. Therefore, indeed $\ker \sigma = 0$.

Finally, observe that the isomorphism $\psi : A \to B$ of left A-modules is an isomorphism $\psi : {}_1A_\sigma \to B$ of A-A-bimodules. Indeed, for $x, a \in A$, we have

$$\begin{aligned}\psi(x\sigma(a)) &= (x\sigma(a))b = x(\sigma(a)b) = x(ba) \\ &= (xb)a = \psi(x)a\end{aligned}$$

Therefore, $\Omega^d_{A^e}(A) \cong {}_1A_\sigma$ in $\operatorname{mod} A^e$.

Let e be a primitive idempotent of A. Then we have isomorphisms of right A-modules

$$\begin{aligned}\sigma(e)A/\sigma(e)J(A) &\xrightarrow{\sim} \Omega^d_A(\sigma(e)A/\sigma(e)J(A)) \\ &\xrightarrow{\sim} (\sigma(e)A/\sigma(e)J(A)) \otimes_A {}_1A_\sigma \\ &\xrightarrow{\sim} (\sigma(e)A/\sigma(e)J(A))_\sigma \\ &\xrightarrow{\sim} eA/eJ(A).\end{aligned}$$

Hence, $\sigma(e)A \xrightarrow{\sim} eA$ in $\operatorname{mod} A$.

(2) $\Rightarrow$ (1) Let $\Omega^d_{A^e}(A) \cong {}_1A_\sigma$ for some $d \geq 1$ and an automorphism σ of A such that $\sigma(e)A \cong eA$ for any primitive idempotent e of A. We know that then A and A^e are selfinjective. Then for any simple right A-module S, the right A-modules $\Omega^d_A(S)$ and $S \otimes \Omega^d_A(A) \cong S \otimes_A A_\sigma \cong S_\sigma$ are isomorphic. Every simple right A-module S is isomorphic to a module of the form $eA/eJ(A)$ for some primitive idempotent e of A. Since $eA \cong \sigma(e)A$ in $\operatorname{mod} A$, the automorphism σ induces isomorphisms of right A-modules $eA \xrightarrow{\sim} (eA)_\sigma$, $eJ(A) \xrightarrow{\sim} (eJ(A))_\sigma$, and hence $S \xrightarrow{\sim} S_\sigma$ in $\operatorname{mod} A$. Therefore, $\Omega^d_A(S) \cong S$ for any simple right A-module S. $\square$

Corollary. *Let A be a finite dimensional K-algebra whose all simple right A-modules are periodic. Then A is a selfinjective algebra.*

A finite dimensional K-algebra

$$HH^*(A) = \mathrm{Ext}^*_{A^e}(A, A) = \bigoplus_{i \geq 0} \mathrm{Ext}^i_{A^e}(A, A)$$

Hochschild cohomology algebra (graded commutative K-algebra with the Yoneda product)

$HH^0(A) = \mathcal{Z}(A)$ the center of A

$HH^1(A) = \mathrm{Der}_K(A, A) / \mathrm{Der}^0_K(A, A)$

$$\mathrm{Der}_K(A, A) = \left\{ \delta \in \mathrm{Hom}_K(A, A) \,\middle|\, \begin{matrix} \delta(ab) = a\delta(b) + \delta(a)b \\ \text{for all } a, b \in A \end{matrix} \right\}$$

(**derivations** of A)

$$\mathrm{Der}^0_K(A, A) = \left\{ \delta_x \in \mathrm{Hom}_K(A, A) \,\middle|\, \begin{matrix} \delta_x(a) = ax - xa \\ x, a \in A \end{matrix} \right\}$$

(**inner derivations** of A)

$HH^1(A)$ the space of outer derivations of A

$HH^n(A)$, $n \geq 2$, control deformations of A

Two algebras A and B are said to be **derived equivalent** if the derived categories $D^{\mathsf{b}}(\operatorname{mod} A)$ and $D^{\mathsf{b}}(\operatorname{mod} B)$ are equivalent as triangulated categories.

For selfinjective algebras we have

$$\begin{array}{c} \text{Morita} \\ \text{equivalence} \end{array} \Rightarrow \begin{array}{c} \text{derived} \\ \text{equivalence} \end{array} \xRightarrow{\textbf{Rickard}} \begin{array}{c} \text{stable} \\ \text{equivalence} \end{array}$$

Theorem (Happel, Rickard, 1989). *Let A and B be two derived equivalent K-algebras. Then $HH^*(A) \cong HH^*(B)$ as graded K-algebras.*

Theorem (Green-Snashall-Solberg, 2003). *Let A be a finite dimensional indecomposable K-algebra. Assume that $\Omega^n_{A^e}(A) \cong {}_1A_\sigma$ for a positive integer n and an algebra automorphism σ of A. Then*

$$HH^*(A)/\mathcal{N}(A) \cong \begin{cases} K, \text{or} \\ K[x] \end{cases}$$

where $\mathcal{N}(A)$ is the ideal of $HH^*(A)$ generated by all nilpotent homogeneous elements.

Moreover, $HH^*(A) \cong K$, if $\Omega^m_{A^e}(A) \not\cong A$ for all $m \geq 1$.

Proof. Since $\Omega^n_{A^e}(A) \cong {}_1A_\sigma$, A is selfinjective. Then A^e is selfinjective, and we may identify

$$HH^i(A) = \mathrm{Ext}^i_{A^e}(A, A) = \underline{\mathrm{Hom}}_{A^e}(\Omega^i_{A^e}(A), A)$$

If $\Omega^m_{A^e}(A) \cong A$ for some $m \geq 1$, then by the Carlson's theorem we have $HH^*(A)/\mathcal{N}(A) \cong K[x]$, where x is of degree $d =$ period of A in mod A^e. In particular, it is the case if σ has finite order.

Assume now that $\Omega^m_{A^e}(A) \ncong A$ in $\operatorname{mod} A^e$ for any $m \geq 1$. Then σ has infinite order. Let $s \geq 1$ and $\underline{\eta} \in \underline{\mathrm{Hom}}(\Omega^s_{A^e}(A), A) = HH^s(A)$. We claim that $\underline{\eta}$ is nilpotent in $HH^*(A)$. Assume first that $s = np$ for some $p \geq 1$. Then, for any $i \geq 1$ we have $\Omega^{inp}_{A^e}(A) \cong {}_1A_{\sigma^{ip}}$ is an indecomposable right A^e-module and

$$\Omega^{(i-1)np}_{A^e}(\underline{\eta}) : \Omega^{inp}_{A^e}(A) \longrightarrow \Omega^{(i-1)np}_{A^e}(A)$$

is not an isomorphism. Further, our assumption $\Omega^n_{A^e}(A) \cong {}_1A_\sigma$ implies that the A^e-modules $\Omega^{inp}_{A^e}(A)$, $i \geq 1$, have bounded length (dimension). Then, applying the Harada-Sai lemma, we conclude that there exists a natural number t such that

$$\underline{\eta}^t = \Omega^{tnp}_{A^e}(\underline{\eta}) \dots \Omega^{2np}_{A^e}(\underline{\eta})\Omega^{np}_{A^e}(\underline{\eta}) = 0$$

in the algebra $HH^*(A)$. Hence, $\underline{\eta}$ is nilpotent. Assume now that $n \not| \, s$. Then there are positive integers r and q such that $rs = nq$. Then $\underline{\eta}^r \in HH^{nq}(A)$, and hence (by the above argument) $\underline{\eta}^r$ is nilpotent, and consequently $\underline{\eta}$ is nilpotent.

We proved that every homogeneous element of $HH^*(A)$ of positive degree is nilpotent. Moreover, A is indecomposable, and then $HH^0(A) \cong \mathcal{Z}(A)$ is a commutative local algebra, $J(\mathcal{Z}(A))$ is nilpotent, and $\mathcal{Z}(A)/J(\mathcal{Z}(A)) \cong K$. Therefore, we conclude that $HH^*(A)/\mathcal{N}(A) \cong K$. □

Corollary. *Let A be a finite dimensional indecomposable selfinjective K-algebra of finite representation type. Then*

$$HH^*(A)/\mathcal{N}(A) \cong \begin{cases} K, \text{ or} \\ K[x] \end{cases}$$

Proof. Since all indecomposable nonprojective (hence simple) modules in $\operatorname{mod} A$ are periodic, applying the two Green-Snashall-Solberg theorems, we get the claim. □

Corollary. *Let A and B be two derived equivalent indecomposable finite dimensional self-injective K-algebras. Then A is periodic if and only if B is periodic.*

Proof. We have (Happel-Rickard theorem) that $HH^*(A)$ and $HH^*(B)$ are isomorphic graded K-algebras. Assume that A is periodic in $\operatorname{mod} A^e$, say of period d. Then, by Carlson's theorem $HH^*(A)/\mathcal{N}(A) \cong K[x]$, where x is of degree d. Hence $HH^*(B)/\mathcal{N}(B) \cong K[x]$. Applying Green-Snashall-Solberg theorem, we then infer that B is periodic in $\operatorname{mod} B^e$ (in fact, we have $\Omega^d_{B^e}(B) \cong B$ in $\operatorname{mod} B^e$). $\square$

III. Periodicity of finite groups

G finite group

$\mathbb{Z}$ ring of integers

$\mathbb{Z}G$ group algebra of G over $\mathbb{Z}$

We may consider the group $\mathbb{Z}$ as the trivial $\mathbb{Z}G$-module ($m * g = m$ for any $m \in \mathbb{Z}$ and $g \in G$)

For $n \geq 0$ and a $\mathbb{Z}G$-module M, let

$$H^n(G, M) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M)$$

n-th cohomology group of G with coefficients in M

In particular, we may consider the cohomology groups of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$

$$H^i(G, \mathbb{Z}) = \mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}),\ i \geq 0.$$

A group G is called **(globally) periodic** if there exists a positive integer d such that

$$H^i(G, \mathbb{Z}) \cong H^{i+d}(G, \mathbb{Z}) \text{ for all } i \geq 1.$$

The minimal such $d =$ **the (cohomological) period of G**

Example. Let $m \geq 2$, and $G = \mathbb{Z}_m$ the cyclic group of order m, say generated by an element g. Then we have the following periodic free $\mathbb{Z}G$-resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$

$$\cdots \xrightarrow{N} \mathbb{Z}G \xrightarrow{g-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{g-1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

where $\varepsilon(g) = 1$ for $g \in G$, $g - 1$ is the left multiplication by $g-1$, and N is the left multiplication by $N = 1 + g + \cdots + g^{m-1}$.

Applying $\text{Hom}_{\mathbb{Z}G}(-, \mathbb{Z})$ we obtain the periodic complex whose i-th cohomology is the group $\text{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}) = H^i(G, \mathbb{Z})$. Then one obtains $H^0(G, \mathbb{Z}) \cong \mathbb{Z}$, $H^{2i}(G, \mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z}$ and $H^{2i-1}(G, \mathbb{Z}) = 0$ for $i \geq 1$. In particular, $G = \mathbb{Z}_m$ is a periodic group of period 2.

In fact, the following is true.

Theorem. *Let G be a finite group. Then G is periodic of period 2 if and only if G is cyclic.*

Moreover, we have also the following theorem.

Theorem. *Let G be a periodic finite group. Then $H^{2i-1}(G, \mathbb{Z}) = 0$ for any $i \geq 1$. Hence the period of G is* **even**.

Zassenhaus considered the following problem, motivated by some topological problems (free group actions on spheres).

PROBLEM. Describe all finite groups G whose all commutative subgroups are cyclic.

Zassenhaus solved this problem in the solvable case. This was completed by Suzuki to the general case.

Theorem (Suzuki-Zassenhaus, 1954-1955). *A complete list of finite groups with all commutative subgroups cyclic is given by the following table*

Family	*Definition*	*Conditions*
I	$\mathbb{Z}/a \times_\alpha \mathbb{Z}/b$	$(a,b) = 1$
II	$\mathbb{Z}/a \times_\beta (\mathbb{Z}/b \times \mathcal{Q}_{2^i})$	$(a,b) = (ab,2) = 1$
III	$\mathbb{Z}/a \times_\gamma (\mathbb{Z}/b \times T_i)$	$(a,b) = (ab,6) = 1$
IV	$\mathbb{Z}/a \times_\tau (\mathbb{Z}/b \times O_i^*)$	$(a,b) = (ab,6) = 1$
V	$(\mathbb{Z}/a \times_\alpha \mathbb{Z}/b) \times \mathrm{SL}_2(\mathbb{F}_p)$	$(a,b) = (ab, p(p^2-1)) = 1$
VI	$\mathbb{Z}/a \times_\mu (\mathbb{Z}/b \times TL_2(\mathbb{F}_p)$	$(a,b) = (ab, p(p^2-1)) = 1$

These 6 *families of groups are given as semidirect products of certain finite groups.*

We will exhibit (now and later) only some natural examples of such groups.

Examples. (1) For $m \geq 1$, consider the **dihedral group**

$$D_{2m} = \left\{x, y \;\middle|\; x^2 = 1 = y^m, yx = xy^{m-1}\right\}$$

of order $2m$.

For $m = 2r$, $\{1, x, y^r, xy^r = y^r x\}$ is a noncyclic commutative subgroup of D_{4r}.

For m odd, all commutative subgroups of D_{2m} are cyclic.

Hence, D_{2m} is periodic if and only if m is odd.

(2) For $m \geq 1$ consider the **generalized quaternion** 2-**group**

$$Q_{2^{m+2}} = \left\{x, y \;\middle|\; x^{2^m} = y^2, xyx = x\right\}$$

of order 2^{m+2}. Then every commutative subgroup of $Q_{2^{m+2}}$ is cyclic.

(3) Let p be a prime and $\mathbb{F}_p$ the field with p elements, and

$$\mathsf{SL}_2(\mathbb{F}_p) = \left\{ M \in M_{2\times 2}(\mathbb{F}_p) \;\middle|\; \det M = 1 \right\}$$

(2×2 **special linear group of** $\mathbb{F}_p$). Then

$$|\mathsf{SL}_2(\mathbb{F}_p)| = p(p-1)(p+1).$$

Moreover, all commutative subgroups of $\mathsf{SL}_2(\mathbb{F}_p)$ are cyclic. We also note that for p odd the groups $\mathsf{SL}_2(\mathbb{F}_p)$ are not solvable.

For a prime number p, $\mathbb{Z}_p^r = \underbrace{\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{r}$ is called the **elementary p-group of rank** r.

For a finite group G and a prime p with $p \,\Big|\, |G|$, denote by $r_p(G)$ the maximal rank of elementary p-subgroup of G.

The following characterizations of periodic groups show that the Suzuki-Zassenhaus theorem provides a complete classification of all periodic finite groups.

Theorem (Artin-Tate, Cartan-Eilenberg, 1956). *Let G be a finite group. The following statements are equivalent:*

(1) *G is periodic.*

(2) *$H^d(G, \mathbb{Z}) \cong \mathbb{Z}/|G|\mathbb{Z}$ for some $d \geq 1$.*

(3) *$H^{i+d}(G, M) \cong H^i(G, M)$ for some $d \geq 1$, all $i \geq 1$ and an arbitrary finitely generated $\mathbb{Z}G$-module M.*

(4) *$H^{i+d}(G, \mathbb{Z}_p) \cong H^i(G, \mathbb{Z}_p)$ for some $d \geq 1$, all $i \geq 1$ and any prime p dividing $|G|$.*

(5) *$r_p(G) \leq 1$ for any prime p dividing $|G|$.*

(6) *For any prime p dividing $|G|$, the p-Sylow subgroups of G are cyclic or generalized quaternion 2-groups.*

(7) *Every commutative subgroup of G is cyclic.*

Therefore, the subgroups of periodic groups are periodic.

For a prime p, we have

$\dim_{\mathbb{Z}_p} H^n(\mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_p) = n + 1$ for any $n \geq 0$,

hence $\mathbb{Z}_p \times \mathbb{Z}_p$ is not periodic (application of the **Künneth formula**).

p-**periodic groups**

Let p be a prime number

$\mathbb{Z}_{(p)} = \left\{ \frac{m}{n} \in \mathbb{Q}, m, n \in \mathbb{Z}, p \not| n \right\}$ localization of $\mathbb{Z}$ at p.

Let G be a finite group such that $p \,\Big|\, |G|$.

For each $i \geq 1$, let

$$H^i(G, \mathbb{Z})_{(p)} \cong H^i(G, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}.$$

A group G is called p-**periodic** if there exists a positive integer d such that

$$H^i(G, \mathbb{Z})_{(p)} \cong H^{i+d}(G, \mathbb{Z})_{(p)} \text{ for all } i \geq 1.$$

The minimal such $d = d_p =$ **the (cohomological)** p-**period of** G.

Theorem. *Let G be a finite group, p a prime number, and $p \,\Big|\, |G|$. The following statements are equivalent:*

(1) *G is p-periodic.*

(2) *$H^{i+d}(G, \mathbb{Z}_p) \cong H^i(G, \mathbb{Z}_p)$ for some $d \geq 1$ and any $i \geq 1$.*

(3) *$\mathrm{Ext}^{i+d}_{\mathbb{Z}_pG}(\mathbb{Z}_p, M) \cong \mathrm{Ext}^{i}_{\mathbb{Z}_pG}(\mathbb{Z}_p, M)$ for some $d \geq 1$, any $i \geq 1$, and arbitrary finite dimensional $\mathbb{Z}_pG$-module M.*

(4) *$\Omega^d_{\mathbb{Z}_pG}(\mathbb{Z}_p) \cong \mathbb{Z}_p$ for some $d \geq 1$.*

(5) *$r_p(G) \leq 1$.*

(6) *Every p-Sylow subgroup of G is either cyclic or generalized quaternion 2-group.*

(7) *Every commutative p-subgroup of G is cyclic.*

(8) *For any algebraically closed field K of characteristic p, $\Omega^d_{KG}(K) \cong K$ for some $d \geq 1$.*

(9) *For any algebraically closed field K of characteristic p, there exists $d \geq 1$ such that $\Omega^d_{KG}(M) \cong M$ for any indecomposable nonprojective finite dimensional KG-module M.*

Observe that a finite group G is periodic if and only if G is p-periodic for any prime p dividing $|G|$.

Example. Let p be an odd prime number, $q = p^n$, $n \geq 2$, $\mathbb{F}_q$ the field with q elements, and $G = SL_2(\mathbb{F}_q)$. Then $|G| = q(q^2 - 1)$. Moreover, we have

- the 2-Sylow subgroups of G are generalized quaternion 2-groups
- for any odd prime $l \neq p$ the l-Sylow subgroups of G are cyclic
- the p-Sylow subgroups of G are **not** cyclic

Then G is not p-periodic, and hence is not periodic. Moreover, G is l-periodic for any prime such that $l \,\Big|\, |G|$ and $l \neq p$.

There is no chance for a classification of all finite p-periodic groups, for any fixed prime p.

Let G be a finite group, p a prime number, $p \,\Big|\, |G|$. Let

$$H^{ev}(G, \mathbb{Z}_p) \cong \bigoplus_{n \geq 0} H^{2n}(G, \mathbb{Z}_p)$$

even cohomology algebra of G at p.

$H^{ev}(G, \mathbb{Z}_p)$ is a graded commutative ring.

Theorem (Evans-Venkov, 1959-1961). $H^{ev}(G, \mathbb{Z}_p)$ *is a noetherian ring.*

$\dim H^{ev}(G, \mathbb{Z}_p) =$ Krull dimension of $H^{ev}(G, \mathbb{Z}_p)$ (length d of the maximal chain of distinct graded prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_d$ of $H^{ev}(G, \mathbb{Z}_p)$).

Theorem (Quillen, 1971). *Let G be a finite group and p a prime number dividing $|G|$. Then*

$$\dim H^{ev}(G, \mathbb{Z}_p) = r_p(G).$$

Hence the Krull dimensions of the rings $H^{ev}(G, \mathbb{Z}_p)$, $p \,\Big|\, |G|$, p prime, measure the complexity of the group G.

Corollary. *Let G be a finite group and p a prime number dividing $|G|$. Then G is p-periodic if and only if* $\dim H^{ev}(G, \mathbb{Z}_p) = 1$.

Representation type of group algebras

Let K be an algebraically closed field of characteristic p. By the well-known **Maschke's theorem** the group algebra KG of a finite group G is semisimple if and only if $p \nmid |G|$.

Theorem (Higman, 1954). *Let G be a finite group and $p \mid |G|$. Then KG is of finite representation type if and only if the p-Sylow subgroups of G are cyclic.*

Theorem (Bondarenko-Drozd, 1975). *Let G be a finite group and $p \,\Big|\, |G|$. Then KG is tame of infinite representation type if and only if $p = 2$ and the 2-Sylow subgroups of G are of one of the following types: dihedral, semidihedral, or generalized quaternion groups.*

Corollary. *Let G be a finite group, p a prime number, $p \,\Big|\, |G|$, and assume that $|G|$ is p-periodic. Then*

(1) *KG is of tame representation type.*
(2) *If p is odd, then KG is of finite representation type.*

Theorem (Erdmann-Holm (1999), Erdman-Skowroński (2005)). *Let G be a finite group, p a prime, $p \,\Big|\, |G|$, and $A = KG$. If G is p-periodic then A is periodic in $\operatorname{mod} A^e$.*

More generally,

char $K = p > 0$

G finite group, $p \Big| |G|$

$KG = B_0 \times B_1 \times \cdots \times B_r$,
$B_0, B_1, \ldots, B_r$ indecomposable algebras (**blocks** of KG)

B_0 block containing the trivial module K

B a block of KG

$B \mapsto D = D_B$ **defect group** of KG
D p-subgroup of G

mod $B \ni X \Rightarrow X$ is a direct summand of $Y \otimes_{KD} KG$, for some $Y \in$ mod KD

$D_{B_0} = p$-Sylow subgroup of G

B of finite type $\iff$ D_B is cyclic

B tame of infinite type $\iff$
$p = 2$ and D_B is dihedral, semidihedral or generalized quaternion

B is periodic in mod B^e $\iff$
D_B is cyclic or generalized quaternion 2-group

Topological sources of periodic groups

Let G be a finite group.

We may consider G as a topological group with the discrete topology.

G **acts on a topological space** X if there is a group homomorphism

$G \to$ Homeo(X) group of homeomorphisms of X

G acts freely on X if $gx \neq x$ for all $x \in X$ and $g \in G \setminus \{e\}$.

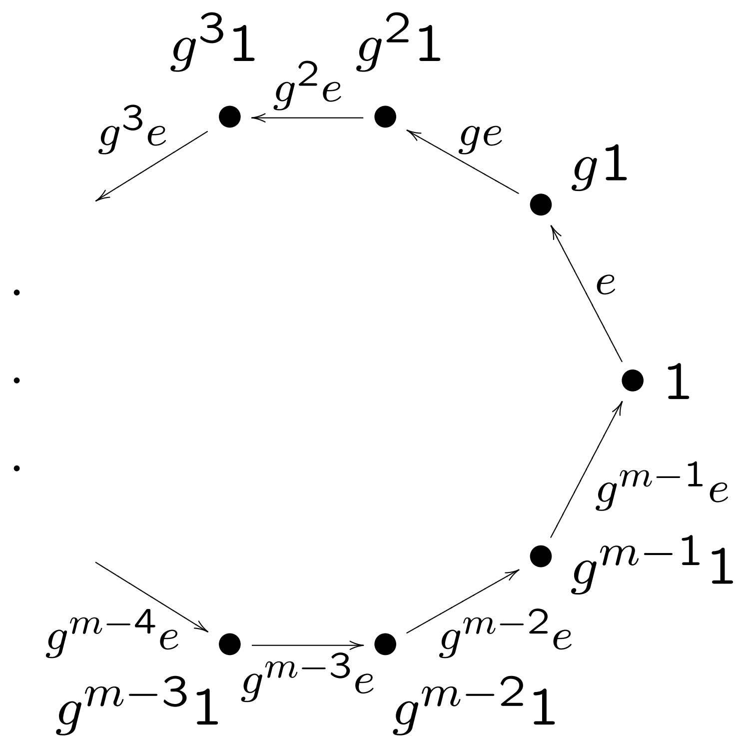
Assume X is a CW-complex (admits a cell decomposition) and G is a finite group of homeomorphisms of X.

We say that G **acts freely** on X if G acts freely on a cell decomposition of X:

$$g(\sigma) \subseteq \bigcup_{\tau \neq \sigma} \tau$$

for all $g \in G \setminus \{1\}$ and all cells σ of X.

Example. For any $m \geq 2$, the cyclic group $G = (g)$ of order m acts freely on the one-dimensional sphere $\mathbb{S}^1$



Spherical space form problem: *Describe the finite groups G acting freely on spheres $\mathbb{S}^m$ and the orbit spaces $\mathbb{S}^m/G$ (spherical spaces).*

Theorem (Smith, 1938-1939). *Let G be a finite group acting freely on a sphere $\mathbb{S}^m$. Then every abelian subgroup of G is cyclic.*

Topological motivation for the Zassenhaus problem.

Theorem. *Let G be a finite group acting freely on a sphere $\mathbb{S}^m$. Then*

(1) *For m even, we have $|G| \leq 2$.*

(2) *For m odd, we have $H^{m+1}(G, \mathbb{Z}) \cong \mathbb{Z}/|G|\mathbb{Z}$. In particular, G is periodic with even period dividing $m+1$.*

Proof. (1) An application of Lefschetz fix point theorem.

(2) Application of cohomological methods (spectral sequence of the fibration $\mathbb{S}^m \to \mathbb{S}^m/G \to BG$). □

Example. (1) Consider the (division) algebra of quaternions

$$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$$

$ij = -ji = k$, $ki = -ik = j$, $jk = -kj = i$, $i^2 = j^2 = k^2 = -1$.

$$\mathbb{S}^3 = \left\{ a + bi + cj + dk \in \mathbb{H} \,\middle|\, a^2 + b^2 + c^2 + d^2 = 1 \right\}$$

$\mathbb{S}^3$ 3-dimensional sphere in $\mathbb{R}^4 = \mathbb{H}$.

There is a group epimorphism $\mathbb{S}^3 \to \mathrm{SO}(3, \mathbb{R})$ (group of rotations of $\mathbb{R}^3$) with the kernel $\{\pm 1\}$.

It is known that every noncyclic finite subgroup of $\mathbb{S}^3$ is conjugate in $\mathbb{S}^3$ (hence isomorphic) to one of the groups

- $\mathbb{D}^*_{2n}$, $n \geq 2$, binary dihedral group
- $\mathcal{T}^*$ binary tetrahedral group
- $\mathcal{O}^*$ binary octahedral group
- $\mathcal{I}^*$ binary icosahedral group

The groups $\mathbb{D}^*_{2n}$, $\mathcal{T}^*$, $\mathcal{O}^*$, $\mathcal{I}^*$ admit a unique normal subgroup $\mathbb{Z}_2 = \{\pm 1\}$ of order 2 such that

- $\mathbb{D}^*_{2n}/\mathbb{Z}_2 = \mathbb{D}_{2n}$ dihedral group
- $\mathcal{T}^*/\mathbb{Z}_2 = \mathcal{T}$ tetrahedral group of rotations of tetrahedron
- $\mathcal{O}^*/\mathbb{Z}_2 = \mathcal{O}$ octahedral group of rotations of octahedron (equivalently, cube)
- $\mathcal{I}^*/\mathbb{Z}_2 = \mathcal{I}$ icosahedral group of rotations of icosahedron (equivalently, dodecahedron)

Then $|\mathbb{D}^*_{2n}| = 4n$, $|\mathcal{T}^*| = 24$, $|\mathcal{O}^*| = 48$, $|\mathcal{I}^*| = 120$.

The groups $\mathbb{D}^*_{2n}$, $\mathcal{T}^*$, $\mathcal{O}^*$, $\mathcal{I}^*$ act freely on the sphere $\mathbb{S}^3$, and hence are periodic groups of period 4 (only cyclic groups may have period 2).

$$Q_{4n} = \mathbb{D}^*_{2n} = \left\langle x, y \;\middle|\; x^n = y^2, xyx = y \right\rangle, n \geq 2$$

is called a **generalized quaternion group**.

For $n = 2^m$, we get the generalized quaternion 2-group $Q_{2^{m+2}}$ considered before.

We have the following embedding of groups

$$Q_{4n} \longrightarrow \mathbb{S}^3 \subseteq \mathbb{H} = \mathbb{R}^4$$

$$x \longrightarrow e^{\pi i/n}$$

$$y \longrightarrow j$$

$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$.

(2) Linear actions on spheres

Let $V = \mathbb{R}^{2n}$, $n \geq 1$,

$(-,-)$ the Euclidean $\mathbb{R}$-bilinear form

$e_1, e_2, \ldots, e_{2n}$ the standard basis of $\mathbb{R}^{2n}$

Let G be a finite group of $\mathbb{R}$-linear automorphisms of V

Assume G acts freely on $V \setminus \{0\}$: the eigenvalues of all $g \in G \setminus \{1\}$ are different from 1.

$(-,-)_G$ G-invariant $\mathbb{R}$-bilinear form induced by $(-,-)$

$$(x,y)_G = \frac{1}{|G|} \sum_{g \in G} (g(x), g(y)) \text{ for } x, y \in V$$

$\mathbb{S} = \{x \in V | (x,x)_G = 1\}$

$\mathbb{S}$ is an $(2n-1)$-dimensional sphere

G acts freely on $\mathbb{S}$. In fact,
G acts freely on a cell decomposition of $\mathbb{S}$.

Indeed, let C be the convex hull of the finite set $\{\pm g(e_i) \mid g \in G, 1 \leq i \leq 2n\}$ in $\mathbb{R}^{2n}$.

Then $\mathbb{S}$ is the border of C and admits the induced cell decomposition.

Since G acts freely on $V \setminus \{0\}$, G acts freely on this cell decomposition of $\mathbb{S}$.

In particular, G is periodic of (even) period dividing $2n$.
(one can construct such groups of period $2n$)

Does every periodic group act freely on a sphere?

No.

Theorem (Milnor, 1957). *Let G be a finite group acting freely on a sphere $\mathbb{S}^m$. Then G admits at most one element of order 2, and such an element is central.*

For example, for m odd, the dihedral group $\mathbb{D}_{2m}$ is periodic, but does not act freely on a sphere.

In particular, this is the case for the symmetric group $S_3 \cong \mathbb{D}_{2\cdot 3} = \mathbb{D}_6$.

The following theorem proved by Swan shows that the periodic groups are finite groups acting freely on CW-complexes homotopically equivalent to spheres.

Theorem (Swan, 1960). *Let G be a finite group. The following statements are equivalent:*

(1) *G is periodic.*

(2) *There exists an odd natural number m, an m-dimensional CW-complex X (Swan complex) homotopically equivalent to $\mathbb{S}^m$ such that G acts freely on X.*

Theorem (Madsen-Thomas-Wall, 1978). *Let G be a finite group. The following statements are equivalent:*

(1) *G acts freely on a sphere.*

(2) *G admits at most one element of order 2, and such element is central.*

(3) *For each prime p, every subgroup G of order p^2 or $2p$ is cyclic.*

Example. For each odd prime p, the group $\mathrm{SL}_2(\mathbb{F}_p)$ acts freely on a sphere. Indeed,

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

is the unique element of order 2 in $\mathrm{SL}_2(\mathbb{F}_p)$, and is central.

We note that $\mathrm{SL}_2(\mathbb{F}_2) \cong S_3 \cong \mathbb{D}_6$, $\mathrm{SL}_2(\mathbb{F}_3) \cong \mathcal{T}^*$, and $\mathrm{SL}_2(\mathbb{F}_5) \cong \mathcal{I}^*$.

The groups $\mathrm{SL}_2(\mathbb{F}_p)$, $p > 5$, do not admit linear free actions on spheres.

Theorem (Wolf, 1967). *A finite group G acts freely and linearly on some sphere if and only if the following conditions are satisfied:*

(1) *For all primes p and q, the subgroups of G of orders pq are cyclic.*

(2) *G has no subgroup isomorphic to $\mathrm{SL}_2(\mathbb{F}_p)$ for a prime $p > 5$.*

IV. Periodicity of tame symmetric algebras

K algebraically closed field

Λ finite dimensional K-algebra

Λ **tame**: $\forall_{d\geq 1}\exists_{M_1,\dots,M_{n_d}}$ $K[x]$-Λ-bimodules such that

- M_i free left $K[x]$-modules of finite rank
- all but finitely many isoclasses of indecomposable right Λ-modules of dimension d are of the form $K[x]/(x-\lambda)\otimes_{K[x]} M_i$, $1\leq i\leq n_d$, $\lambda\in K$

$\mu_\Lambda(d) =$ least number on $K[x]$-Λ-bimodules satisfying the above condition for d

Λ tame $\Longrightarrow$

$$\operatorname{ind}_d \Lambda = \left\{ \begin{array}{c}\textit{finite dis-}\\ \textit{crete set}\end{array}\right\} \bigcup \left\{ \begin{array}{c}\mu_\Lambda(d)\ \textit{one-para-}\\ \textit{meter families}\end{array}\right\}$$

Λ is not tame $\xRightarrow{\textbf{Drozd}}$ Λ is **wild** (representation theory of Λ comprises the representation theories of all finite dimensional K-algebras)

Λ is of finite (representation) type if and only if $\mu_\Lambda(d) = 0$ for all $d \geq 1$ (solution of the **second Brauer-Thrall conjecture**).

THEOREM (Erdmann-Skowroński, 2004). *Let Λ be a nonsimple, basic, indecomposable, finite dimensional algebra over an algebraically closed field K. Then Λ is symmetric, tame, with all indecomposable nonprojective finite dimensional modules periodic if and only if Λ is isomorphic to an algebra of one of the forms:*

- *symmetric algebra of Dynkin type;*
- *symmetric algebra of tubular type;*
- *algebra of quaternion type.*

algebra = basic, indecomposable, finite dimensional K-algebra

Λ algebra $\Rightarrow$ $\Lambda \cong KQ/I$

$Q = Q_\Lambda$ **Gabriel quiver** of Λ, I admissible ideal in the path algebra KQ of Q

$$\operatorname{mod}\Lambda \cong \operatorname{rep}_K(Q, I)$$

tame (basic) selfinjective algebras
- **standard algebras** (admit simply connected Galois coverings)
- **nonstandard algebras**

Representation theory of tame standard selfinjective algebras can be reduced to the representation theory of tame algebras of finite global dimension (tame simply connected algebras with nonnegaive Euler forms)

B basic connected K-algebra

$\mathsf{T}(B) = B \ltimes D(B)$ **trivial extension**

$\mathsf{T}(B) = B \oplus D(B)$ as K-vector spaces

$(a, f) \cdot (b, g) = (ab, fb + ag)$

$\mathsf{T}(B)$ **symmetric algebra**

$((a, f), (b, g)) = f(b) + g(a)$

G finite group of K-algebra automorphisms of $\mathsf{T}(B)$

We may consider the **invariant algebra**

$$\mathsf{T}(B)^G = \left\{ x \in \mathsf{T}(B) \,\middle|\, g(x) = x \text{ for all } g \in G \right\}$$

G **acts freely** on $\mathsf{T}(B)$ if there is a decomposition

$$1_{\mathsf{T}(B)} = e_1 + e_2 + \cdots + e_n$$

where $e_1, e_2, \ldots, e_n$ are orthogonal primitive idempotents of $\mathsf{T}(B)$ such that

(1) $g(e_i) \in \{e_1, \ldots, e_n\}$ for all $g \in G$ and $i \in \{1, \ldots, n\}$.

(2) if $g(e_i) = e_i$ for some $i \in \{1, \ldots, n\}$ then $g = 1$.

It is known that G acts freely on $\mathsf{T}(B)$ if and only if G acts freely on the isoclasses of simple $\mathsf{T}(B)$-modules, for the induced action of G on $\operatorname{mod} \mathsf{T}(B)$.

Proposition. *Assume G acts freely on $\mathsf{T}(B)$. Then $\mathsf{T}(B)^G$ is a weakly symmetric (hence selfinjective) algebra.*

Proof. The invariant algebra $\mathsf{T}(B)^G$ is isomorphic to the orbit algebra $\mathsf{T}(B)/G$ (in the sense of Gabriel). Since $\mathsf{T}(B)$ is symmetric, $\mathsf{T}(B)$ is weakly symmetric, and hence $\mathsf{T}(B)^G \cong \mathsf{T}(B)/G$ is weakly symmetric. □

We note that in general $\mathsf{T}(B)^G$ is not necessarily a symmetric algebra.

Symmetric algebras of Dynkin type

$\Delta \in \{\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$ **Dynkin graph**

$\vec{\Delta}$ a Dynkin quiver with underlying graph Δ

$H = K\vec{\Delta}$ the path algebra of $\vec{\Delta}$

$T \in \operatorname{mod} H$ **tilting** H-module:

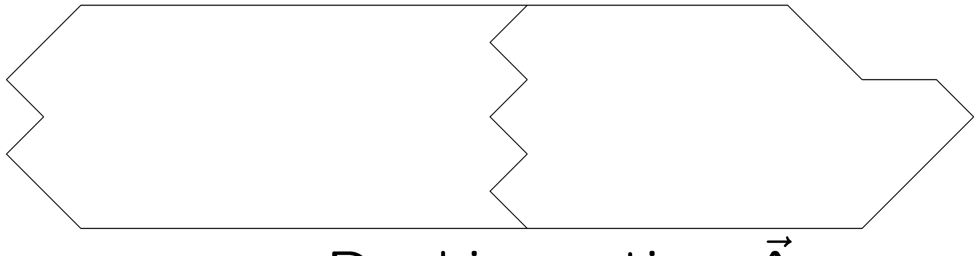
$\operatorname{Ext}^1_H(T, T) = 0$

$T = T_1 \oplus \cdots \oplus T_n,\ n = |\Delta_0|$

$T_1, \ldots, T_n$ indecomposable pairwise nonisomorphic

$B = \operatorname{End}_H(T)$ **tilted algebra** of type $\vec{\Delta}$

- $\operatorname{gl.dim} B \leq 2$
- B is of finite type
- The Auslander-Reiten quiver Γ_B of B is of the form



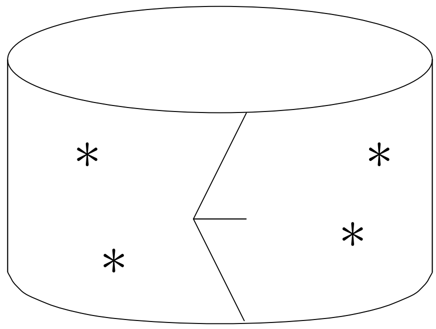
Dynkin section $\vec{\Delta}$

Trivial extensions of finite type

Theorem (Hughes-Waschbüsch,1983). *Let A be an algebra. Then $\mathsf{T}(A)$ is of finite type if and only if $\mathsf{T}(A) \cong \mathsf{T}(B)$ for a tilted algebra B of Dynkin type.*

B tilted of Dynkin type $\vec{\Delta}$

The Auslander-Reiten quiver of $\Gamma_{\mathsf{T}(B)}$



$\vec{\Delta}$

$=$ the stable finite cylinder $\mathbb{Z}\vec{\Delta}/(\tau^{m_\Delta})$ completed by $|\Delta_0|$-projective-injective modules

$m_\Delta = h_\Delta - 1$, h_Δ **Coxeter number** of Δ

$|$isoclases of indecomposable $\mathsf{T}(B)$-modules$|$ $=$ number of roots $|\Delta_0| h_\Delta$ of type Δ

$h_{\mathbb{A}_m} = m+1$, $h_{\mathbb{D}_m} = 2m-2$, $h_{\mathbb{E}_6} = 12$, $h_{\mathbb{E}_7} = 18$, $h_{\mathbb{E}_8} = 30$

B, B' tilted of Dynkin type
$\mathsf{T}(B) \cong \mathsf{T}(B') \iff B' = S^{+}_{i_t} \dots S^{+}_{i_1} B$ (finite number of reflections)

PROBLEM. When a finite group G acts freely on the trivial extension $\mathsf{T}(B)$ of a tilted algebra B of Dynkin type?

By general theory such a group G is cyclic.

Theorem (Bretscher-Läser-Rietdmann, 1981). *Let G be a finite group acting freely on the trivial extension $\mathsf{T}(B)$ of a tilted algebra B of Dynkin type $\vec{\Delta}$, $\Delta \in \{\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$. Then $G = \{1\}$.*

There are respectively 22, 143, 598 isoclasses of the trivial extensions $\mathsf{T}(B)$ of tilted algebras B of types $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$ (**Riedtmann**).

These are all symmetric algebras of Dynkin types $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$.

The tilted algebras B of Dynkin types for which $\mathsf{T}(B)$ admit a free action of a **non-trivial** finite group G are **very exceptional**.

Brauer tree algebras

Brauer tree: a finite connected tree $T = T_S^m$ together with

- a circular ordering of the edges converging at each vertex
- one exceptional vertex S with multiplicity $m \geq 1$

Brauer tree $T \mapsto$ **Brauer quiver** Q_T:

- the vertices of Q_T are the edges of T
- there is an arrow $i \to j$ in $Q_T \iff j$ is the consecutive edge of i in the circular ordering of the edges converging at a vertex of T

Q_T has the following structure:

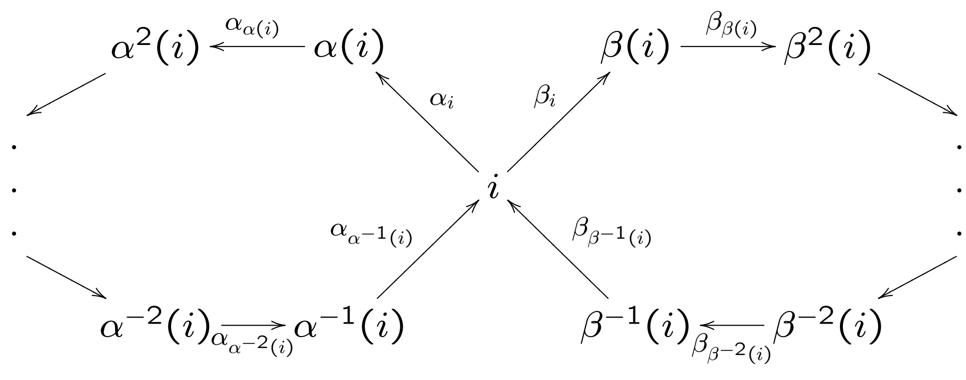
- Q_T is a union of oriented cycles corresponding to the vertices of T
- Every vertex of Q_T belongs to exactly two cycles

The cycles of Q_T are divided into two camps: α-**camps** and β-**camps** such that two cycles of Q_T having nontrivial intersection belong to different camps. We assume that the cycle of Q_T corresponding to the exceptional vertex S of T is an α-cycle.

i vertex of Q_T

$i \xrightarrow{\alpha_i} \alpha(i)$ the arrow in α-camp of Q_T starting at i

$i \xrightarrow{\beta_i} \beta(i)$ the arrow in β-camp of Q_T starting at i



$$A_i = \alpha_i \alpha_{\alpha(i)} \dots \alpha_{\alpha^{-1}(i)} \qquad B_i = \beta_i \beta_{\beta(i)} \dots \beta_{\beta^{-1}(i)}$$

$$T = T_S^m \mapsto A(T) = A(T_S^m) = KQ_{T_S^m}/I_S^m$$

Brauer tree algebra

I_S^m ideal in $KQ_{T_S^m}$ generated by elements :

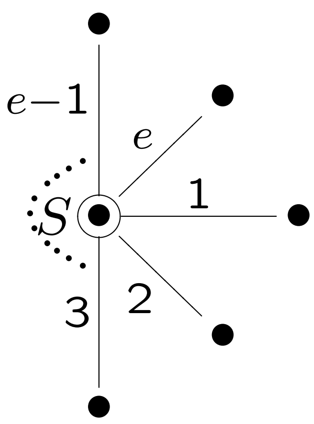
- $\beta_{\beta^{-1}(i)}\alpha_i$ and $\alpha_{\alpha^{-1}(i)}\beta_i$
- $A_i^m - B_i$ if the α-cycle passing through i is exceptional
- $A_i - B_i$ if the α-cycle passing through i is not exceptional

We note that the ideal I_S^m is not an admissible ideal of $KQ_{T_S^m}$.

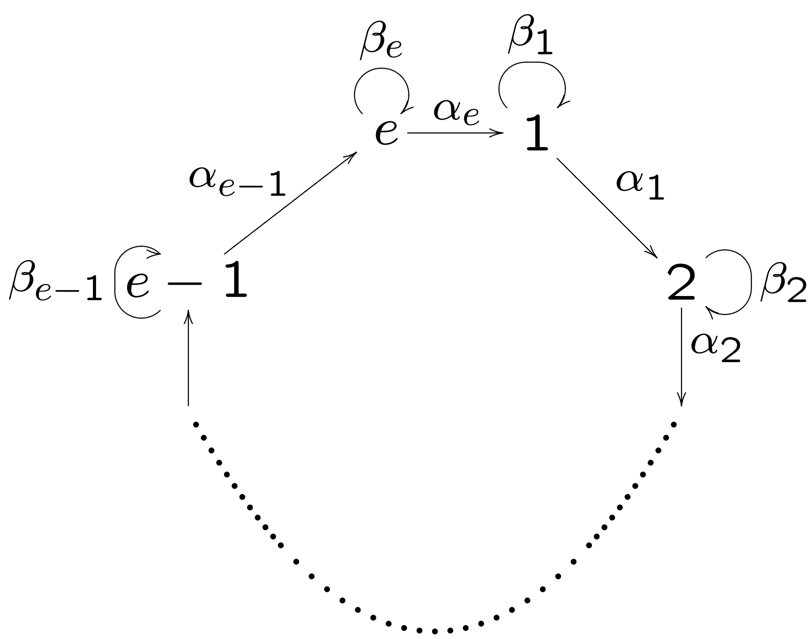
For the multiplicity $m = 1$, the Brauer tree algebras $A(T) = A(T_S^1)$ are exactly the trivial extension algebras $\mathrm{T}(B)$ of the tilted algebras of types $\mathbb{A}_n$.

For the multiplicity $m \geq 2$, we have $A(T_S^m) \cong \mathrm{T}(B)^{\mathbb{Z}_m}$ for an exceptional tilted algebra $B = B(T_S^m)$ of type $\mathbb{A}_n$ and the cyclic group $\mathbb{Z}_m$ acting freely on $\mathrm{T}(B)$.
(here $n = me$, e the number of edges of T_S^m)

Example. Let $T = T_S^m$ be the star



$Q_T = Q_{T_S^m}$ is of the form



$A(T_S^m)$ is a **symmetric Nakayama algebra**

Moreover, $A(T_S^m) = A(T')^{\mathbb{Z}_m}$ for the star T' with me edges and the multiplicity 1, and $A(T') = T(B)$ for the path algebra KQ of the equioriented quiver of type $\mathbb{A}_{me}$

$$1 \longrightarrow 2 \longrightarrow \ldots \longrightarrow me$$

Theorem (Dade-Janusz-Kupisch,1966-1969). *Let B be a block of a group algebra KG with cyclic defect group D_B. Then B is Morita equivalent to a Brauer tree algebra $A(T_S^m)$.*

(Here $me + 1 = p^n$ if $\mid D_B \mid = p^n$ and B has e simple modules)

Remark. *Most of the Brauer tree algebras $A(T_s^m)$* **are not** *Morita equivalent to blocks of group algebras (***Feit, 1984***).*

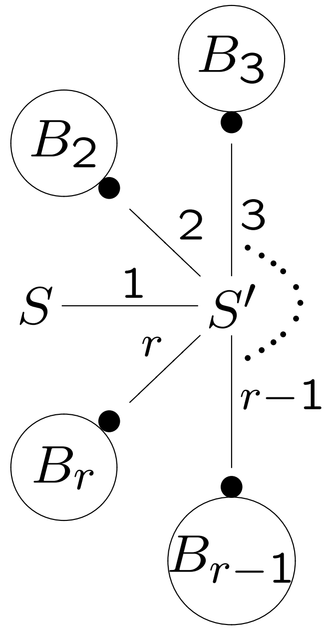
Theorem (Gabriel-Riedtmann (1979), Rickard (1989)). *Let A be a selfinjective algebra. TFAE:*

(1) *A is Morita equivalent to a Brauer tree algebra.*

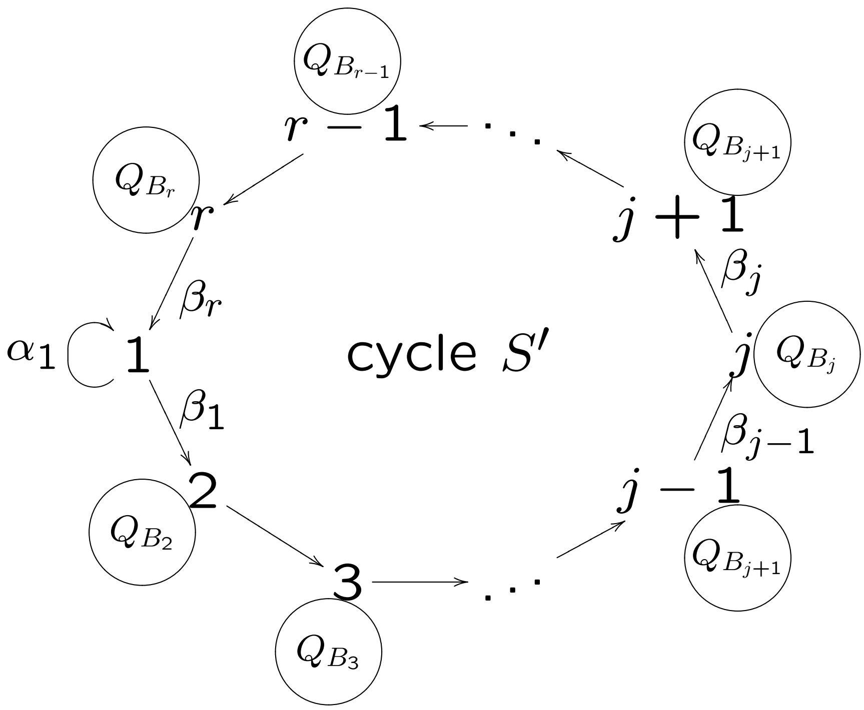
(2) *A is stably equivalent to a symmetric Nakayama algebra.*

(3) *A is derived equivalent to a symmetric Nakayama algebra.*

Let $T = T_S$ be a Brauer tree with at least two edges and an **extreme** exceptional vertex S



Then the Brauer quiver Q_T is of the form



For each edge i of T (vertex i of Q_T) we have the cycles A_i and B_i around i

Define $B'_j = \beta_j \dots \beta_r \alpha_1 \beta_1 \dots \beta_{i-1}$, $j \neq 1$, $j \in S'_0$

For each $\lambda \in K$, define the algebra

$$D(T_S, \lambda) = KQ_T / I(T_S, \lambda)$$

where $I(T_S, \lambda)$ is the ideal of KQ_T generated by

- $\beta_{\beta^{-1}(i)}\alpha_i$ and $\alpha_{\alpha^{-1}(i)}\beta_i$, $i \in (Q_T)_0 \setminus \{1\}$,
- $A_1^2 = B_1$,
- $A_i - B_i$, $i \in (Q_T)_0 \setminus S'_0$,
- $A_j - B'_j$, $j \in S'_0 \setminus \{1\}$,
- $\beta_r \beta_1 - \lambda \beta_r \alpha_1 \beta_1$.

Proposition. (1) *$D(T_S, \lambda)$, $\lambda \in K$, are symmetric algebras of finite type.*

(2) *For $\lambda, \mu \in K \setminus \{0\}$, $D(T_S, \lambda) \cong D(T_S, \mu)$.*

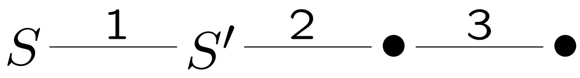
(3) *$D(T_S, 0) \cong D(T_S, 1) \iff \operatorname{char} K \neq 2$.*

(4) *$D(T_S, 0)$ and $D(T_S, 1)$ are socle equivalent.*

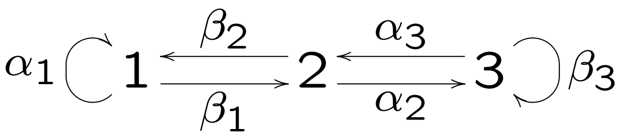
(5) *$D(T_S, 0) = \mathsf{T}(B)^{\mathbb{Z}_3}$, for an exceptional tilted algebra $B = B^*(T_S)$ of Dynkin type $\mathbb{D}_{3m}$ and $\mathbb{Z}_3$ acting freely on $\mathsf{T}(B)$.*

(6) *For $\operatorname{char} K = 2$, $D(T_S, 1)$ is nonstandard and degenerates to $D(T_S, 0)$.*

Example. $T = T_S^2$ of the form



$Q_T = Q_{T_S^2}$ of the form



$D(T_S, 0) = KQ_T/I(T_S, 0)$
$I(T_S, 0)$ generated by

$$\beta_1\alpha_2,\ \alpha_3\beta_2$$
$$\beta_3\alpha_3,\ \alpha_2\beta_3$$
$$\alpha_1^2 - \beta_1\beta_2$$
$$\alpha_2\alpha_3 - \beta_2\alpha_1\beta_1$$
$$\alpha_3\alpha_2 - \beta_3$$
$$\beta_2\beta_1$$

$\prod_1(Q_T, I(T_S, 0)) \cong \mathbb{Z}$

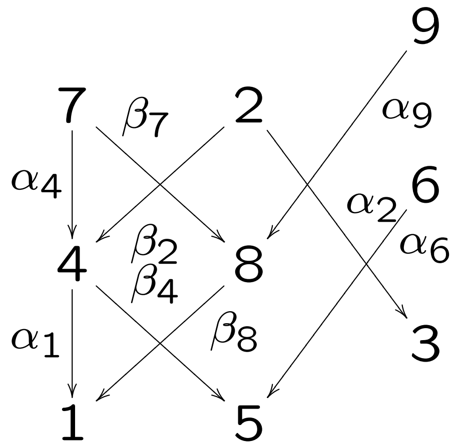
$D(T_S, 1) = KQ_T/I(T_S, 0)$
$I(T_S, 1)$ generated by

$$\beta_1\alpha_2,\ \alpha_3\beta_2$$
$$\beta_3\alpha_3,\ \alpha_2\beta_3$$
$$\alpha_1^2 - \beta_1\beta_2$$
$$\alpha_2\alpha_3 - \beta_2\alpha_1\beta_1$$
$$\alpha_3\alpha_2 - \beta_3$$
$$\beta_2\beta_1 - \beta_2\alpha_1\beta_1$$

$\prod_1(Q_T, I(T_S, 1))$ trivial

Let $B = KQ/I$ where

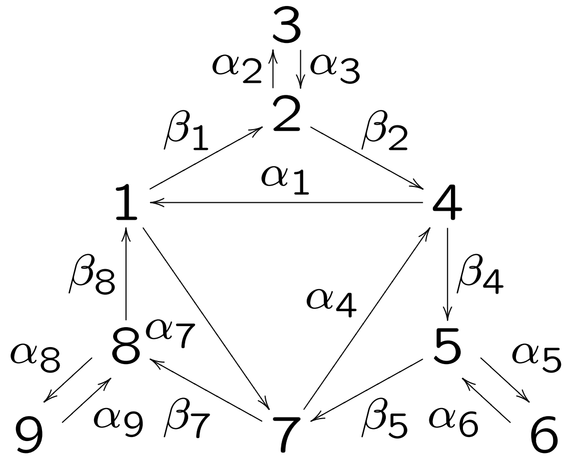
Q :



and I is generated by $\alpha_4\alpha_1 - \beta_7\beta_8$, $\beta_2\beta_4$, $\alpha_9\beta_8$. Then B is a tilted algebra of type $\mathbb{D}_9 = \mathbb{D}_{3\cdot 3}$

Moreover, $\mathsf{T}(B) \cong KQ'/I'$, where

Q' :



and the ideal I' is generated by $\alpha_4\alpha_1 - \beta_7\beta_8$, $\alpha_1\alpha_7 - \beta_4\beta_5$, $\alpha_7\alpha_4 - \beta_1\beta_2$, $\beta_2\beta_4$, $\beta_5\beta_7$, $\beta_8\beta_1$, $\beta_1\alpha_2$, $\alpha_3\beta_2$, $\beta_4\alpha_5$, $\alpha_6\beta_5$, $\beta_7\alpha_8$, $\alpha_9\beta_8$, $\alpha_2\alpha_3 - \beta_2\alpha_1\beta_1$, $\alpha_5\alpha_6 - \beta_5\alpha_4\beta_4$, $\alpha_8\alpha_9 - \beta_8\alpha_7\beta_7$.

Then $\mathbb{Z}_3$ acts freely on $\mathsf{T}(B)$ by the rotation and $\mathsf{T}(B)^{\mathbb{Z}_3} \cong D(T_S, 0)$.

Theorem (Riedtmann, Waschbüsch, ...). *Let Λ be a nonsimple standard selfinjective algebra. TFAE:*

(1) *Λ is symmetric of finite type.*

(2) *Λ is isomorphic to $\mathsf{T}(B)^G$, B tilted algebra of Dynkin type, G finite group acting freely on $\mathsf{T}(B)$.*

(3) *Λ is isomorphic to one of the algebras:*

   (a) *$\mathsf{T}(B)$, B tilted of Dynkin type.*

   (b) *$A(T_S^m)$, T_S^m Brauer tree, S exceptional of multiplicity $m \geq 2$.*

   (c) *$D(T_S, 0)$, T_S Brauer tree, S extreme exceptional.*

Theorem (Riedtmann (1983), Waschbüsch (1981)). *Let Λ be a selfinjective algebra over K. TFAE:*

(1) *Λ is nonstandard of finite type,*

(2) *Λ is nonstandard symmetric of finite type,*

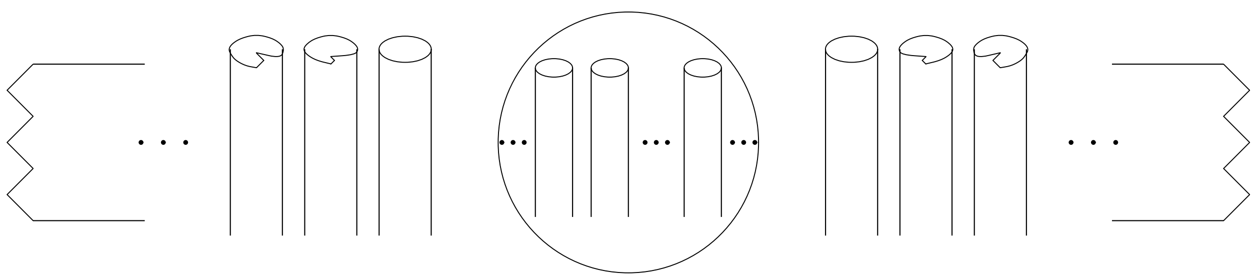
(3) *$\Lambda \cong D(T_S, 1)$, T_S Brauer tree, S extreme exceptional, and $\operatorname{char} K = 2$.*

Symmetric algebras of tubular type

B **tubular algebra** (in the sense of **Ringel**) = tilted algebra $\text{End}_C(T)$ of a canonical tubular algebra C (T tilting module of nonnegative rank) of one of tubular types $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$, or $(2,3,6)$.

B tubular $\Longrightarrow$

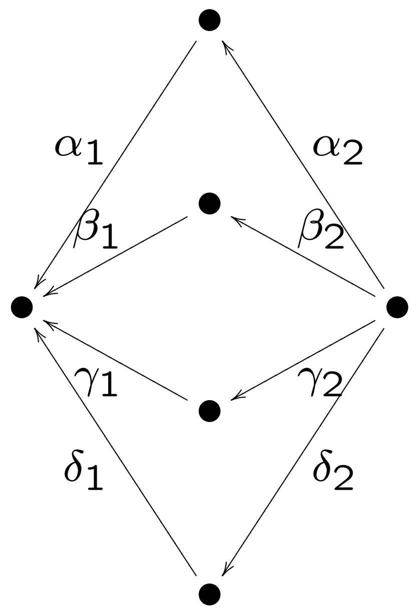
- $\text{gl.dim}\, B = 2$
- $\text{rk}\, K_0(B) = 6$, 8, 9, or 10
- B is of polynomial growth
- The Auslander-Reiten quiver Γ_B of B is of the form



$\mathcal{P}$ $\quad$ $\mathcal{T}_0$ $\quad$ $\bigvee_{q\in\mathbb{Q}^+} \mathcal{T}_q$ $\quad$ $\mathcal{T}_\infty$ $\quad$ $\mathcal{Q}$

Canonical tubular algebras

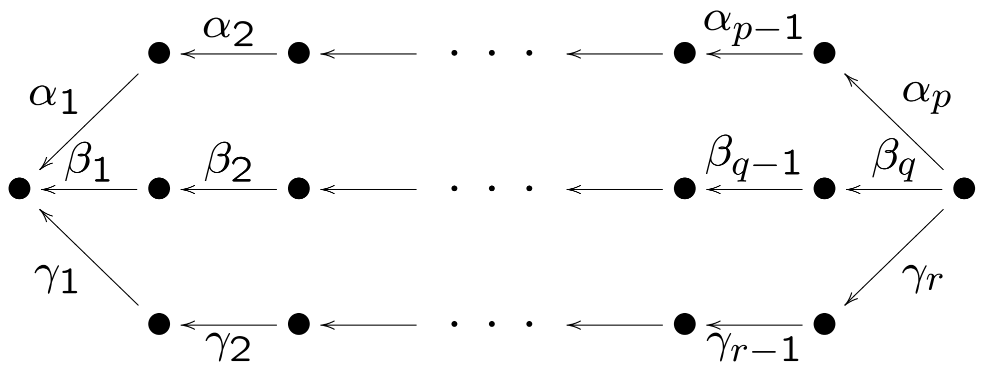
$C_\lambda = C(2,2,2,2,\lambda),\ \lambda \in K \setminus \{0,1\},$



bounded by $\alpha_2\alpha_1 + \beta_2\beta_1 + \gamma_2\gamma_1 = 0,$

$$\alpha_2\alpha_1 + \lambda\beta_2\beta_1 + \delta_2\delta_1 = 0.$$

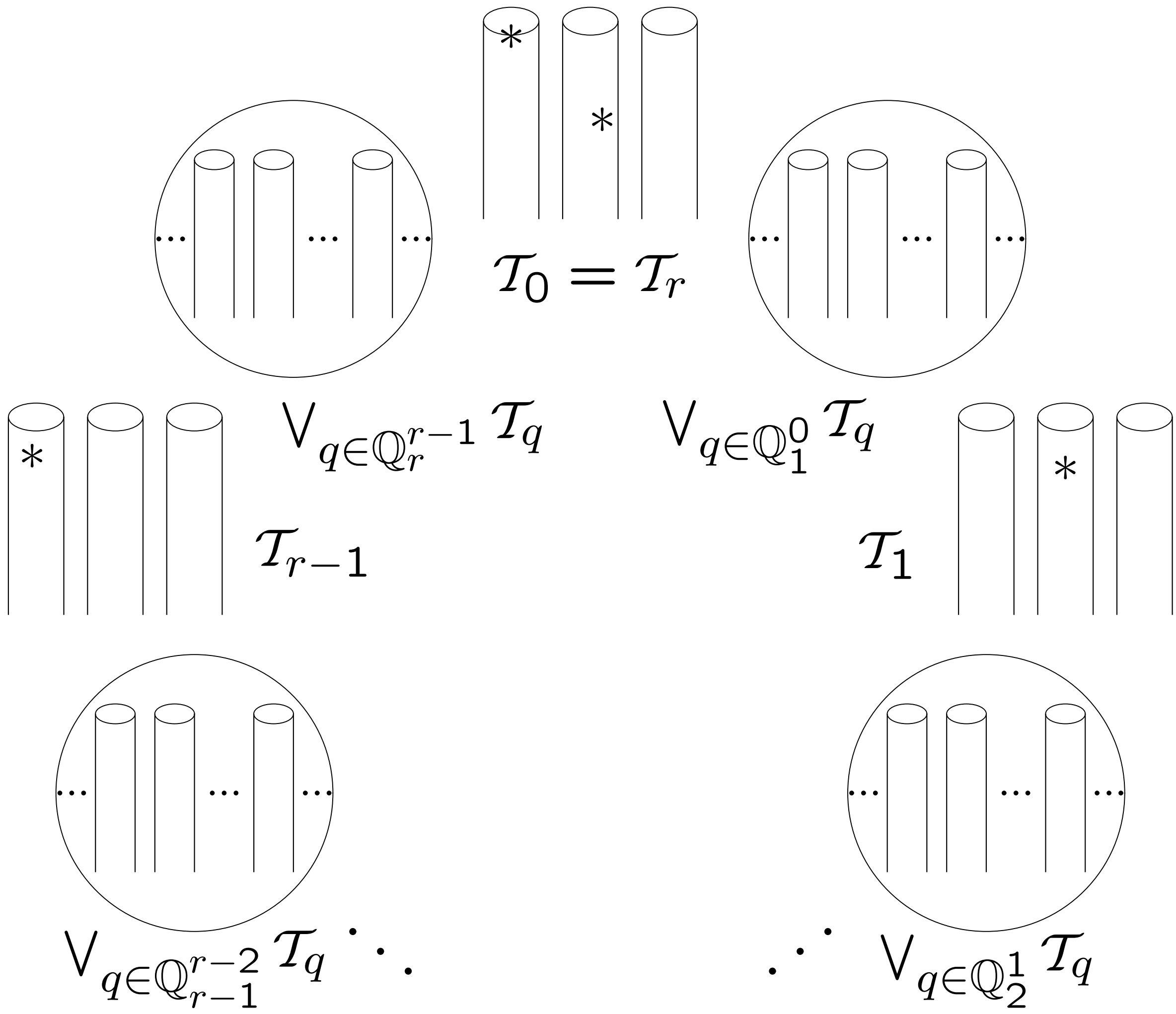
$C(p,q,r),\ (p,q,r) = (3,3,3),(2,4,4),(2,3,6)$



bounded by

$$\alpha_p \dots \alpha_2\alpha_1 + \beta_q \dots \beta_2\beta_1 + \gamma_r \dots \gamma_2\gamma_1 = 0$$

B tubular algebra $\xrightarrow{\textbf{Nehring-Skowroński (1989)}}$ $\mathsf{T}(B)$ symmetric standard tame algebra of polynomial growth and the Auslander-Reiten quiver of $\mathsf{T}(B)$ is of the form



$\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_r$ $\mathbb{P}_1(K)$-families of quasi-tubes (stable tubes with inserted projective-injective vertices $*$)

$\mathcal{T}_q, q \in \mathbb{Q}_i^{i-1} = \mathbb{Q} \cap (i-1, i), 1 \le i \le r$, $\mathbb{P}_1(K)$-families of stable tubes

Theorem (Białkowski-Skowroński, 2003). *Let Λ be a representation-infinite algebra. TFAE:*

(i) Λ *is tame, standard, weakly symmetric, with all indecomposable nonprojective modules periodic and singular Cartan matrix.*

(ii) Λ *is tame, standard, symmetric, with all indecomposable nonprojective modules periodic and singular Cartan matrix.*

(iii) $\Lambda \cong \mathrm{T}(B)$ *for a tubular algebra B.*

B, B' tubular algebras $\xRightarrow{\textbf{Nehring-Skowroński (1989)}}$
$\mathrm{T}(B) \cong \mathrm{T}(B') \iff B' = S_{i_t}^{+} \dots S_{i_1}^{+} B$ (finite number of reflections)

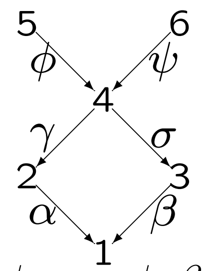
There are 4 families of nonisomorphic trivial extensions of tubular algebras of tubular type $(2,2,2,2)$, and 38, 85, 4953 isoclasses of the trivial extensions of tubular types $(3,3,3)$, $(2,4,4)$, $(2,3,6)$, respectively (**Białkowski**).

PROBLEM. When a finite group acts freely on the trivial extension $\mathsf{T}(B)$ of a tubular algebra B?

By general theory such a group G is cyclic.

Theorem (Lenzing-Skowroński,2000). *Let G be a finite group acting freely on the trivial extension $\mathsf{T}(B)$ of a tubular algebra B of type $(2,3,6)$. Then $G = \{1\}$.*

Theorem (Białkowski-Skowroński,2002). *Let B be a tubular algebra such that a* **non-trivial** *finite group G acts on $\mathsf{T}(B)$. Then $\mathsf{T}(B) \cong \mathsf{T}(B')$ for a tubular algebra B' given by one of the following bound quivers.*

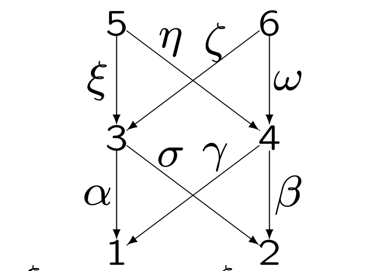


$\phi\gamma\alpha = \phi\sigma\beta$

$\psi\gamma\alpha = \lambda\psi\sigma\beta$

$B_1(\lambda)$

$\lambda \in K \setminus \{0,1\}$

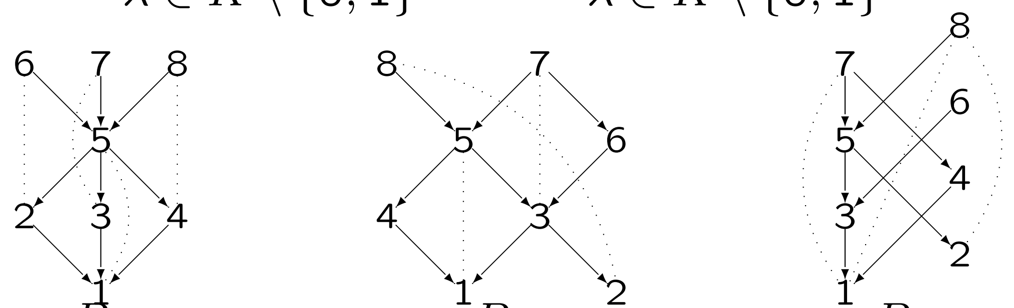


$\xi\alpha = \eta\gamma, \quad \zeta\alpha = \omega\gamma$

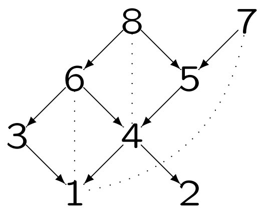
$\xi\sigma = \eta\beta, \quad \zeta\sigma = \lambda\omega\beta$

$B_2(\lambda)$

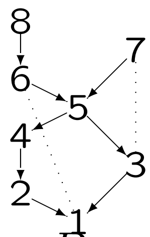
$\lambda \in K \setminus \{0,1\}$



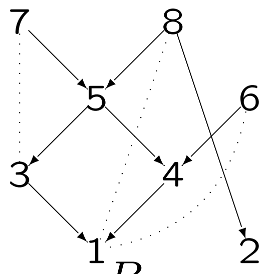
B_3 $\quad$ B_4 $\quad$ B_5



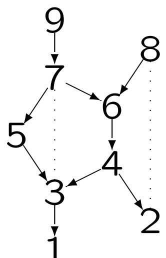
B_6



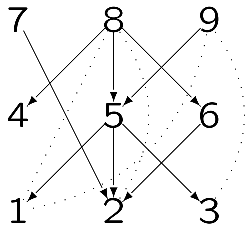
B_7



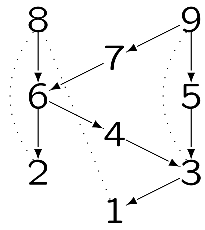
B_8



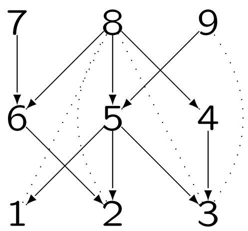
B_9



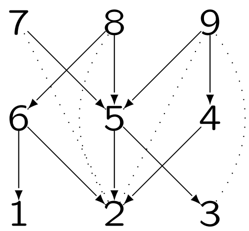
B_{10}



B_{11}



B_{12}



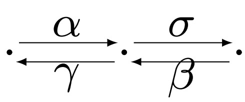
B_{13}

(where a dotted line means that the sum of paths indicated by this line is zero if it indicates exactly three parallel paths, the commutativity of paths if it indicates exactly two parallel paths, and the zero path if it indicates only one path).

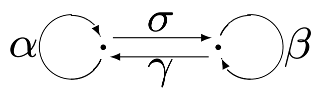
Here, $B_1(\lambda)$, $B_2(\lambda)$ are of type $(2,2,2,2)$, B_3, B_4, B_5, B_6, B_7, B_8 are of type $(3,3,3)$, and B_9, B_{10}, B_{11}, B_{12}, B_{13} are of type $(2,4,4)$.

Theorem (Białkowski-Skowroński, 2003). *Let Λ be a representation-infinite algebra. TFAE:*

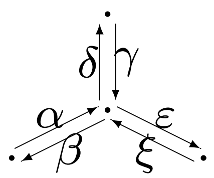
- (i) Λ *tame standard weakly symmetric, with all indecomposable nonprojective finite dimensional modules periodic and non-singular Cartan matrix.*
- (ii) $\Lambda \cong \mathsf{T}(B)^G$ *for a tubular algebra* B *and a* **nontrivial** *finite group* G *acting freely on* $\mathsf{T}(B)$.
- (iii) Λ *is isomorphic to one of the bound quiver algebras.*



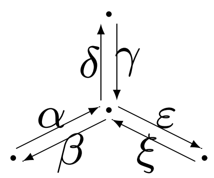
$$\alpha\gamma\alpha = \alpha\sigma\beta$$
$$\beta\gamma\alpha = \lambda\beta\sigma\beta$$
$$\gamma\alpha\gamma = \sigma\beta\gamma$$
$$\gamma\alpha\sigma = \lambda\sigma\beta\sigma$$
$$A_1(\lambda) \cong \mathsf{T}(B_1(\lambda))^{\mathbb{Z}_2}$$
$$\lambda \in K \setminus \{0,1\}$$



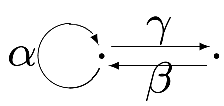
$$\alpha^2 = \sigma\gamma$$
$$\lambda\beta^2 = \gamma\sigma$$
$$\gamma\alpha = \beta\gamma$$
$$\sigma\beta = \alpha\sigma$$
$$A_2(\lambda) \cong \mathsf{T}(B_2(\lambda))^{\mathbb{Z}_3}$$
$$\lambda \in K \setminus \{0,1\}$$



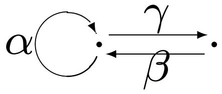
$$\beta\alpha + \delta\gamma + \varepsilon\xi = 0$$
$$\alpha\beta = 0, \quad \xi\varepsilon = 0$$
$$\gamma\delta = 0$$
$$A_3 \cong \mathsf{T}(B_3)^{\mathbb{Z}_2}$$



$$\beta\alpha + \delta\gamma + \varepsilon\xi = 0$$
$$\alpha\beta = 0, \quad \gamma\varepsilon = 0$$
$$\xi\delta = 0$$
$$A_4 \cong \mathsf{T}(B_3)^{\mathbb{Z}_2}$$



$$\alpha^2 = \gamma\beta$$
$$\beta\alpha\gamma = 0$$
$$A_5 \cong \mathsf{T}(B_4)^{\mathbb{Z}_4}$$



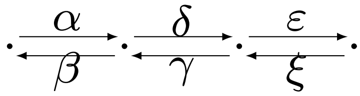
$\alpha^3 = \gamma\beta$

$\beta\gamma = 0$

$\beta\alpha^2 = 0$

$\alpha^2\gamma = 0$

$A_6 \cong \mathsf{T}(B_5)^{\mathbb{Z}_4}$



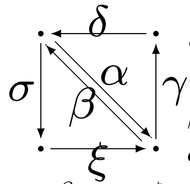
$\beta\alpha = \delta\gamma$

$\gamma\delta = \varepsilon\xi$

$\alpha\delta\varepsilon = 0$

$\xi\gamma\beta = 0$

$A_7 \cong \mathsf{T}(B_4)^{\mathbb{Z}_2}$



$\xi\beta\alpha = 0$

$\delta\alpha\beta = 0$

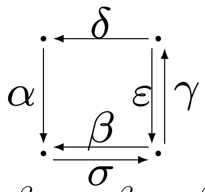
$\beta\alpha\gamma = 0$

$\alpha\beta\sigma = 0$

$\alpha\beta\alpha = \sigma\xi, \quad \xi\gamma = 0$

$\beta\alpha\beta = \gamma\delta, \quad \delta\sigma = 0$

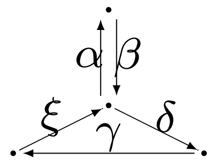
$A_8 \cong \mathsf{T}(B_5)^{\mathbb{Z}_2}$



$\delta\alpha = \varepsilon\beta, \gamma\varepsilon = \beta\sigma, \alpha\sigma\beta = 0$

$\varepsilon\gamma\delta = 0, \sigma\gamma\varepsilon\gamma = 0$

$A_9 \cong \mathsf{T}(B_6)^{\mathbb{Z}_2}$

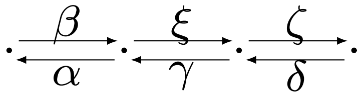


$\xi\alpha\beta = \xi\delta\gamma\xi$

$\alpha\beta\delta = \delta\gamma\xi\delta$

$\beta\alpha = 0, (\gamma\xi\delta)^2\gamma = 0$

$A_{10} \cong \mathsf{T}(B_7)^{\mathbb{Z}_2}$



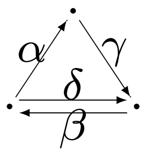
$\gamma\alpha\beta = \gamma\xi\gamma$

$\alpha\beta\xi = \xi\gamma\xi$

$\beta\alpha = 0, \quad \delta\gamma = 0$

$\xi\zeta = 0, \quad (\gamma\xi)^2 = \zeta\delta$

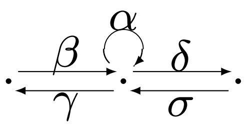
$A_{11} \cong \mathsf{T}(B_8)^{\mathbb{Z}_2}$



$\delta\beta\delta = \alpha\gamma$

$\gamma\beta\alpha = 0, \quad \beta(\delta\beta)^3 = 0$

$A_{12} \cong \mathsf{T}(B_9)^{\mathbb{Z}_3}$

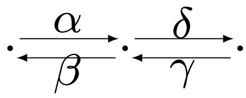


$\alpha^2 = \gamma\beta, \quad \beta\delta = 0, \quad \gamma\beta = 0$

$\sigma\gamma = 0, \quad \alpha\delta = 0, \quad \sigma\alpha = 0$

$\alpha^3 = \delta\sigma$

$A_{13} \cong \mathsf{T}(B_{10})^{\mathbb{Z}_3}$



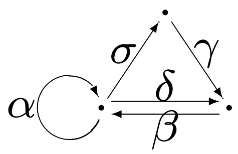
$\beta\alpha = \delta\gamma\delta\gamma$

$\alpha\delta\gamma\delta = 0$

$\gamma\delta\gamma\beta = 0$

$\alpha\beta = 0$

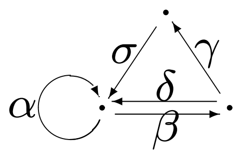
$A_{14} \cong \mathsf{T}(B_{11})^{\mathbb{Z}_3}$



$\gamma\beta\alpha = 0, \quad \alpha^2 = \delta\beta$

$\beta\delta = 0, \alpha\sigma = 0, \alpha\delta = \sigma\gamma$

$A_{15} \cong \mathsf{T}(B_{12})^{\mathbb{Z}_3}$



$\alpha\beta\gamma = 0, \quad \alpha^2 = \beta\delta$

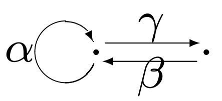
$\delta\beta = 0, \sigma\alpha = 0, \delta\alpha = \gamma\sigma$

$A_{16} \cong \mathsf{T}(B_{13})^{\mathbb{Z}_3}$

(all except A_4 for char $K \neq 2$ are symmetric)

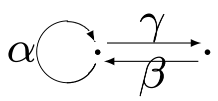
Theorem (Białkowski-Skowroński, 2003). *Let Λ be a nonstandard symmetric algebra over an algebraically closed field K. Then Λ is socle equivalent to a standard representation-infinite tame symmetric algebra A with all indecomposable nonprojective modules periodic if and only if exactly one of the following cases holds:*

(i) *K is of characteristic 3 and Λ is isomorphic to one of the bound quiver algebras*



$$\alpha^2 = \gamma\beta$$
$$\beta\alpha\gamma = \beta\alpha^2\gamma$$
$$\beta\alpha\gamma\beta = 0$$
$$\gamma\beta\alpha\gamma = 0$$

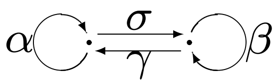
Λ_1



$$\alpha^2\gamma = 0, \quad \beta\alpha^2 = 0$$
$$\gamma\beta\gamma = 0, \quad \beta\gamma\beta = 0$$
$$\beta\gamma = \beta\alpha\gamma$$
$$\alpha^3 = \gamma\beta$$

Λ_2

(ii) *K is of characteristic 2 and Λ is isomorphic to one of the bound quiver algebras*



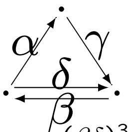
$\alpha^4 = 0, \quad \gamma\alpha^2 = 0, \quad \alpha^2\sigma = 0$

$\alpha^2 = \sigma\gamma + \alpha^3, \quad \lambda\beta^2 = \gamma\sigma$

$\gamma\alpha = \beta\gamma, \quad \sigma\beta = \alpha\sigma$

$\Lambda_3(\lambda)$

$\lambda \in K \setminus \{0,1\}$

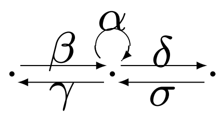


$\delta\beta\delta = \alpha\gamma, \quad (\beta\delta)^3\beta = 0$

$\gamma\beta\alpha\gamma = 0, \quad \alpha\gamma\beta\alpha = 0$

$\gamma\beta\alpha = \gamma\beta\delta\beta\alpha$

Λ_4

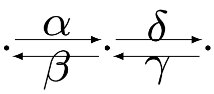


$\alpha^2 = \gamma\beta, \ \alpha^3 = \delta\sigma, \ \beta\delta = 0$

$\sigma\gamma = 0, \ \alpha\delta = 0, \ \sigma\alpha = 0$

$\gamma\beta\gamma = 0, \ \beta\gamma\beta = 0, \ \beta\gamma = \beta\alpha\gamma$

Λ_5



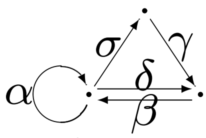
$\alpha\delta\gamma\delta = 0, \quad \gamma\delta\gamma\beta = 0$

$\alpha\beta\alpha = 0, \quad \beta\alpha\beta = 0$

$\alpha\beta = \alpha\delta\gamma\beta$

$\beta\alpha = \delta\gamma\delta\gamma$

Λ_6

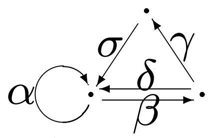


$\beta\delta = \beta\alpha\delta, \alpha\sigma = 0, \alpha\delta = \sigma\gamma$

$\gamma\beta\alpha = 0, \alpha^2 = \delta\beta, \gamma\beta\delta = 0$

$\beta\delta\beta = 0, \quad \delta\beta\delta = 0$

Λ_7

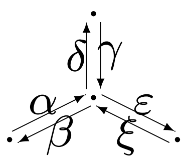


$\delta\beta = \delta\alpha\beta, \sigma\alpha = 0, \delta\alpha = \gamma\sigma$

$\alpha\beta\gamma = 0, \alpha^2 = \beta\delta, \delta\beta\gamma = 0$

$\beta\delta\beta = 0, \quad \delta\beta\delta = 0$

Λ_8

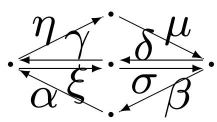


$\beta\alpha + \delta\gamma + \varepsilon\xi = 0$

$\gamma\delta = 0, \ \xi\varepsilon = 0, \ \alpha\beta\alpha = 0$

$\beta\alpha\beta = 0, \ \alpha\beta = \alpha\delta\gamma\beta$

Λ_9



$\mu\beta = 0, \ \alpha\eta = 0, \ \beta\alpha = \delta\gamma$

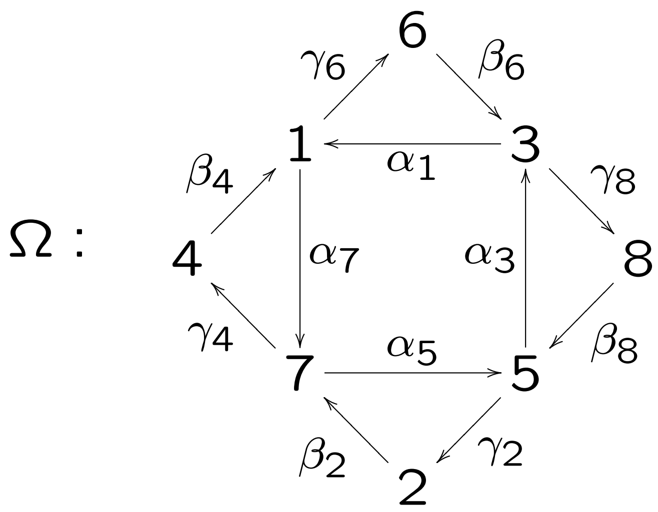
$\xi\sigma = \eta\mu, \ \sigma\delta = \gamma\xi + \sigma\delta\sigma\delta$

$\delta\sigma\delta\sigma = 0, \ \xi\gamma\xi\gamma = 0$

Λ_{10}

Λ nonstandard (above) $\Rightarrow$ Λ degenerates to a standard symmetric algebra $\Lambda' = \mathrm{T}(B)^G$ for an exceptional tubular algebra B and a nontrivial group G acting freely on $\mathrm{T}(B)$.

Example. The trivial extension $\mathsf{T}(B_5)$ of the tubular algebra B_5 of type $(3,3,3)$ is the bound quiver algebra $K\Omega/J$

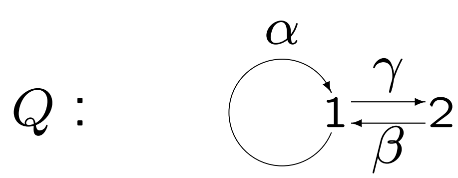


and the ideal J is generated by $\alpha_1\alpha_7\alpha_5 - \gamma_8\beta_8$, $\alpha_3\alpha_1\alpha_7 - \gamma_2\beta_2$, $\alpha_5\alpha_3\alpha_1 - \gamma_4\beta_4$, $\alpha_7\alpha_5\alpha_3 - \gamma_6\beta_6$, $\beta_4\gamma_6$, $\beta_6\gamma_8$, $\beta_8\gamma_2$, $\beta_2\gamma_4$, $\beta_6\alpha_1\alpha_7$, $\beta_8\alpha_3\alpha_1$, $\beta_2\alpha_5\alpha_3$, $\beta_4\alpha_7\alpha_5$, $\alpha_1\alpha_7\gamma_4$, $\alpha_7\alpha_5\gamma_2$, $\alpha_5\alpha_3\gamma_8$, $\alpha_3\alpha_1\gamma_6$.

Then $\mathbb{Z}_4$ acts on $\mathsf{T}(B_5)$ by the rotation and

$$\mathsf{T}(B_5)^{\mathbb{Z}_4} \cong A_6 = KQ/I$$

where



and I is generated by $\alpha^3 - \gamma\beta$, $\beta\gamma$, $\beta\alpha^2$, $\alpha^2\gamma$.

Consider the algebra

$\Lambda_2 = KQ/I^{(1)}$, $I^{(1)} = \left\langle \alpha^3 - \gamma\beta, \beta\gamma - \beta\alpha\gamma, \beta\alpha^2, \alpha^2\gamma \right\rangle$,

A_6, Λ_2 selfinjective algebras of dimension 11

$A_6 \cong \Lambda_2 \iff \operatorname{char} K \neq 3$

$\operatorname{char} K = 3 \;\Rightarrow\; \Lambda_2$ is nonstandard

$A_6/\operatorname{soc} A_6 \cong \Lambda_2/\operatorname{soc} \Lambda_2$

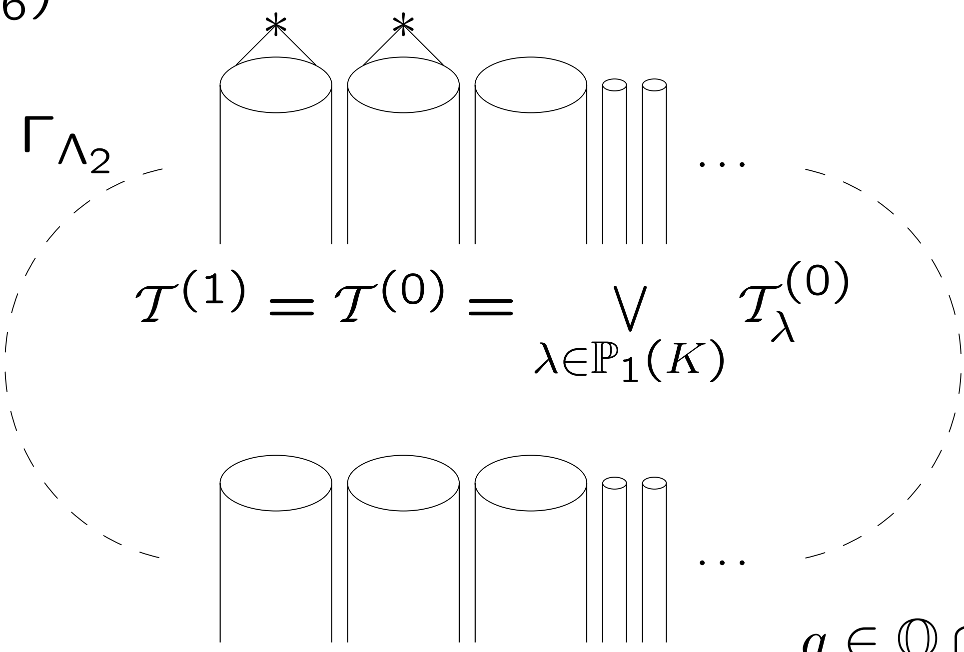
$\Lambda^{(t)} = KQ/I^{(t)}$, $I^{(t)} = \left\langle \alpha^3 - \gamma\beta, \beta\gamma - t\beta\alpha\gamma, \beta\alpha^2, \alpha^2\gamma \right\rangle$, $t \in K$

$\Lambda^{(t)} \cong \Lambda^{(1)} = \Lambda_2$ for $t \in K \setminus \{0\}$

$A_6 = \Lambda^{(0)} = \lim_{t \to 0} \Lambda^{(t)}, \quad A_6 \in \overline{\mathrm{GL}_{11}(K)\Lambda_2}$

A_6 is a **degeneration** of Λ_2 (Λ_2 is a **deformation** of A_6)

$\Gamma_{A_6} = \Gamma_{\Lambda_2}$



$$\mathcal{T}^{(1)} = \mathcal{T}^{(0)} = \bigvee_{\lambda \in \mathbb{P}_1(K)} \mathcal{T}_\lambda^{(0)}$$

$q \in \mathbb{Q} \cap (0,1)$

$$\mathcal{T}^{(q)} = \bigvee_{\lambda \in \mathbb{P}_1(K)} \mathcal{T}_\lambda^{q}$$

Algebras of quaternion type

char $K = p > 0$

G finite group

B block of the group algebra KG

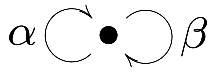
D defect group of B (p-subgroup of G)

B is representation-infinite and all indecomposable non-projective finite dimensional B-modules are Ω_B-periodic	$\Longleftrightarrow$	$p = 2$ and D is a quaternion group

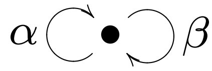
An algebra Λ is of **quaternion type** if

- Λ is symmetric, connected, representation-infinite, tame.
- The indecomposable nonprojective finite dimensional Λ-modules are Ω_Λ-periodic of period dividing 4.
- The Cartan matrix of Λ is nonsingular.

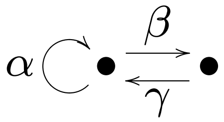
Theorem (Erdmann, 1988). *Let Λ be an algebra of quaternion type. Then Λ is Morita equivalent to one of the bound quiver algebras:*



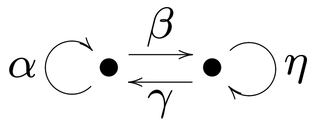
$\alpha^2 = (\beta\alpha)^{k-1}\beta$, $\beta^2 = (\alpha\beta)^{k-1}\alpha$
$(\alpha\beta)^k = (\beta\alpha)^k$, $(\alpha\beta)^k\alpha = 0$
$k \geq 2$



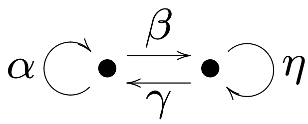
char $K = 2$
$\alpha^2 = (\beta\alpha)^{k-1}\beta + c(\alpha\beta)^k$
$\beta^2 = (\alpha\beta)^{k-1}\alpha + d(\alpha\beta)^k$
$(\alpha\beta)^k = (\beta\alpha)^k$, $(\alpha\beta)^k\alpha = 0$
$(\beta\alpha)^k\beta = 0$
$k \geq 2$, $c, d \in K$, $(c, d) \neq (0, 0)$



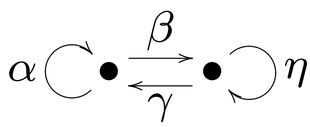
$\gamma\beta\gamma = (\gamma\alpha\beta)^{k-1}\gamma\alpha$
$\beta\gamma\beta = (\alpha\beta\gamma)^{k-1}\alpha\beta$
$\alpha^2 = (\beta\gamma\alpha)^{k-1}\beta\gamma + c(\beta\gamma\alpha)^k$
$\alpha^2\beta = 0$
$k \geq 2$, $c \in K$



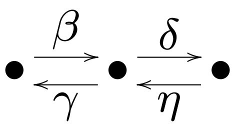
$\gamma\beta = \eta^{s-1}$, $\beta\eta = (\alpha\beta\gamma)^{k-1}\alpha\beta$
$\eta\gamma = (\gamma\alpha\beta)^{k-1}\gamma\alpha$
$\alpha^2 = a(\beta\gamma\alpha)^{k-1}\beta\gamma + c(\beta\gamma\alpha)^k$
$\alpha^2\beta = 0$, $\gamma\alpha^2 = 0$
$k \geq 1$, $s \geq 3$, $a \in K^*$, $c \in K$



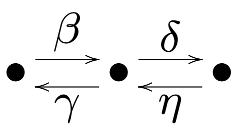
$\alpha\beta = \beta\eta$, $\eta\gamma = \gamma\alpha$, $\beta\gamma = \alpha^2$
$\gamma\beta = \eta^2 + a\eta^{s-1} + c\eta^s$
$\alpha^{s+1} = 0$, $\eta^{s+1} = 0$
$\gamma\alpha^{s-1} = 0$, $\alpha^{s-1}\beta = 0$
$s \geq 4$, $a \in K^*$, $c \in K$



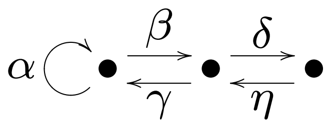
$\alpha\beta = \beta\eta$, $\eta\gamma = \gamma\alpha$, $\beta\gamma = \alpha^2$
$\gamma\beta = a\eta^{t-1} + c\eta^t$
$\alpha^4 = 0$, $\eta^{t+1} = 0$, $\gamma\alpha^2 = 0$
$\alpha^2\beta = 0$
$t \geq 3$, $a \in K^*$, $c \in K$
$(t = 3 \Rightarrow a \neq 1$, $t > 3 \Rightarrow a = 1)$



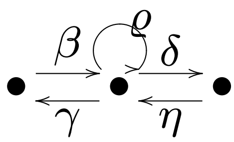
$\beta\delta\eta = (\beta\gamma)^{k-1}\beta$
$\delta\eta\gamma = (\gamma\beta)^{k-1}\gamma$
$\eta\gamma\beta = d(\eta\delta)^{s-1}\eta$
$\gamma\beta\delta = d(\delta\eta)^{s-1}\delta$
$\beta\delta\eta\delta = 0$, $\eta\gamma\beta\gamma = 0$
$k, s \geq 2$, $d \in K^*$
$(k = s = 2 \Rightarrow d \neq 1$, *else* $d = 1)$



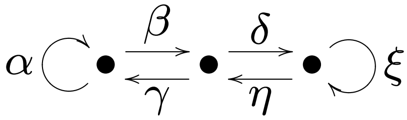
$\beta\gamma\beta = (\beta\delta\eta\gamma)^{k-1}\beta\delta\eta$
$\gamma\beta\gamma = (\delta\eta\gamma\beta)^{k-1}\delta\eta\gamma$
$\eta\delta\eta = (\eta\gamma\beta\delta)^{k-1}\eta\gamma\beta$
$\delta\eta\delta = (\gamma\beta\delta\eta)^{k-1}\gamma\beta\delta$
$\beta\gamma\beta\delta = 0$, $\eta\delta\eta\gamma = 0$
$k \geq 2$



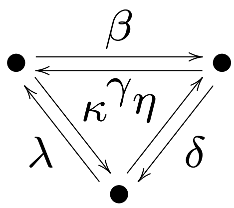
$\beta\gamma = \alpha^{s-1}$
$\alpha\beta = (\beta\delta\eta\gamma)^{k-1}\beta\delta\eta$
$\gamma\alpha = (\delta\eta\gamma\beta)^{k-1}\delta\eta\gamma$
$\eta\delta\eta = (\eta\gamma\beta\delta)^{k-1}\eta\gamma\beta$
$\delta\eta\delta = (\gamma\beta\delta\eta)^{k-1}\gamma\beta\delta$
$\alpha^2\beta = 0$, $\beta\delta\eta\delta = 0$
$k \geq 1$, $s \geq 3$



$\beta\varrho = 0$, $\varrho\gamma = 0$, $\eta\varrho^2 = 0$
$\varrho^2\delta = 0$
$\delta\eta - \gamma\beta = \varrho^{s-1}$, $\eta\varrho = (\eta\delta)^{k-1}\eta$
$\varrho\delta = (\delta\eta)^{k-1}\delta$, $(\beta\gamma)^{k-1}\beta\delta = 0$
$(\eta\delta)^{k-1}\eta\gamma = 0$
$k \geq 2$, $s \geq 3$



$\beta\gamma = \alpha^{s-1}$
$\gamma\alpha = (\delta\eta\gamma\beta)^{k-1}\delta\eta\gamma$
$\alpha\beta = (\beta\delta\eta\gamma)^{k-1}\beta\delta\eta$
$\eta\delta = \xi^{t-1}$
$\delta\xi = (\gamma\beta\delta\eta)^{k-1}\gamma\beta\delta$
$\xi\eta = (\eta\gamma\beta\delta)^{k-1}\eta\gamma\beta$
$\alpha^2\beta = 0$, $\delta\eta\delta = 0$
$k \geq 1$, $s, t \geq 3$



$\beta\delta = (\kappa\lambda)^{a-1}\kappa$
$\eta\gamma = (\lambda\kappa)^{a-1}\lambda$
$\delta\lambda = (\gamma\beta)^{b-1}\gamma$
$\kappa\eta = (\beta\gamma)^{b-1}\beta$
$\lambda\beta = (\eta\delta)^{c-1}\eta$, $\gamma\kappa = (\delta\eta)^{c-1}\delta$
$\gamma\beta\delta = 0$, $\delta\eta\gamma = 0$, $\lambda\kappa\eta = 0$
$a, b, c \geq 1$ (at most one equal 1)

These algebras are of quaternion type:
derived equivalence classification (**Holm, 1999**)
tameness: degeneration argument (**Geiss**)
$\Omega^4_{\Lambda^e}(\Lambda) \cong \Lambda$ (**Erdmann-Skowroński, 2004**)

Corollary (Erdmann-Skowroński, 2004). *Let* Λ *be a basic, connected, finite dimensional symmetric, tame algebra over an algebraically closed field* K*, with all indecomposable nonprojective finite dimensional modules* Ω_Λ*-periodic. Then*

(1) *The Cartan matrix* C_Λ *of* Λ *is singular if and only if* Λ *is isomorphic to the trivial extension* $\mathsf{T}(B)$ *of a tubular algebra* B*.*

(2) *If* Λ *is representation-infinite with nonsingular Cartan matrix* C_Λ *then* Λ *has at most* 4 *simple modules.*

(3) *If* Λ *is representation-infinite then* Λ *has at most* 10 *simple modules.*

$C_{\mathsf{T}(B)} = -(\Phi_B - I_n)C_B$, $n = \operatorname{rk} K_0(B)$,

$\Phi_B = C_B^t C_B^{-1}$ **Coxeter matrix** of B

B tubular algebra $\Rightarrow$ 1 is an eigenvalue of Φ_B
$\Rightarrow \det C_{\mathsf{T}(B)} = 0$.

Calabi-Yau stable module categories

Λ selfinjective algebra

$\operatorname{mod}\Lambda$ category of finite dimensional Λ-modules

$\underline{\operatorname{mod}}\,\Lambda$ **stable category** of $\operatorname{mod}\Lambda$

$\underline{\operatorname{mod}}\,\Lambda$ triangulated category, $T = \Omega_\Lambda^-$ shift functor

$\underline{\operatorname{mod}}\,\Lambda$ has **Serre duality** $\mathcal{S} = \Omega_\Lambda \mathcal{N}_\Lambda$

$\mathcal{N}_\Lambda = D\operatorname{Hom}_\Lambda(-,\Lambda)$ Nakayama functor

$\underline{\operatorname{Hom}}_\Lambda(X,Y) \cong D\underline{\operatorname{Hom}}_\Lambda(Y,\mathcal{S}(X))$

for all $X, Y \in \underline{\operatorname{mod}}\,\Lambda$

Following **Kontsevich** $\underline{\operatorname{mod}}\,\Lambda$ is **Calabi-Yau** if $\mathcal{S} \cong T^n$ (on $\underline{\operatorname{mod}}\,\Lambda$) for some $n \geq 0$
$\iff$ $\Omega_\Lambda^{-n-1} \cong \mathcal{N}_\Lambda$ for some $n \geq 0$.

CYdim($\underline{\operatorname{mod}}\Lambda$) = minimal n with this property

Calabi-Yau dimension of $\underline{\operatorname{mod}}\,\Lambda$

Λ symmetric

$\underline{\text{mod}}\,\Lambda$ is Calabi-Yau $\iff \Omega_\Lambda^{n+1} \cong 1_{\underline{\text{mod}}\,\Lambda}$ for some $n \geq 0$

CYdim($\underline{\text{mod}}\Lambda$) $= n \iff n$ minimal number such that $\Omega_\Lambda^n(M) \cong M$ for all indecomposable nonprojective finite dimensional Λ-modules

Theorem (Erdmann-Skowroński, 2004). *Let Λ be a tame symmetric algebra over an algebraically closed field. Then $\underline{\text{mod}}\,\Lambda$ is Calabi-Yau $\iff$ all indecomposable nonprojective finite dimensional Λ-modules are periodic.*

Theorem (Erdmann-Skowroński, 2004).

(i) *For any natural number n there exists a symmetric algebra Λ of Dynkin type with* CYdim($\underline{\text{mod}}\,\Lambda$) $= n$.

(ii) *Let Λ be a symmetric algebra of tubular type. Then* CYdim($\underline{\text{mod}}\,\Lambda$) $\in \{2, 3, 5, 7, 11\}$.

(iii) *Let Λ be a symmetric algebra of quaternion type. Then* CYdim($\underline{\text{mod}}\,\Lambda$) $\in \{2, 3\}$. *Moreover,* CYdim($\underline{\text{mod}}\,\Lambda$) $= 3$ *if Λ is not of tubular type.*

V. Periodicity and hypersurface singularities

R commutative noetherian local ring

$\mathfrak{m}$ maximal ideal of R

$\dim R$ **Krull dimension** of R (the length of maximal chain of prime ideals of R)

M right R-module

A sequence $x_1, \ldots, x_n \in \mathfrak{m}$ is a **regular sequence** on M if x_i is not a zero-divisor of $M/M(x_1, \ldots, x_{i-1})$, for any $i \in \{1, \ldots, n\}$

depth(M) = the maximal length of regular sequences on M (**depth of** M)

M is a **(maximal) Cohen-Macaulay** R-module if depth(M) = $\dim R$.

R is a **Cohen-Macaulay ring** if R_R is a Cohen-Macaulay R-module.

R is **regular (nonsingular)** if $\mathfrak{m}$ is generated by a regular sequence (equivalently, $\operatorname{gl.dim} R = \dim R$, by the **Auslander-Serre theorem**).

R is an **isolated singularity** if R is nonregular and the localization $R_{\mathfrak{p}}$ is regular (nonsingular) for any prime ideal $\mathfrak{p} \neq \mathfrak{m}$.

K algebraically closed field

$S = K[[x_0, x_1, \ldots, x_n]]$ power series K-algebra

S is a commutative, complete, noetherian, regular, local K-algebra with $\dim S = n + 1$

$\mathfrak{m} = (x_0, x_1, \ldots, x_n)$ unique maximal ideal of S

For $0 \neq f \in \mathfrak{m}^2$,
$R = S/(f)$ is called a **hypersurface singularity**

R is a commutative, complete, noetherian, local K-algebra with $\dim R = n$

$\mathcal{J}(f) = \left(f, \frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$ **Jacobian ideal** of f

R is an **isolated hypersurface singularity** if and only if $\dim_k S/\mathcal{J}(f)$ is finite

Remark. *If $R = S/(f)$ is an isolated hypersurface singularity then $R \cong S/(F)$ for a polynomial $F \in K[x_0, x_1, \ldots, x_n]$ (***Greuel-Kröning***).*

Let R be a hypersurface singularity.

$CM(R)$ category of finitely generated maximal Cohen-Macaulay R-modules.

$CM(R)$ is a Krull-Schmidt category (unique decomposition of objects into direct sums of indecomposable objects).

R is called of **finite Cohen-Macaulay type** (shortly, **finite CM-type**) if $CM(R)$ has only a finite number of pairwise nonisomorphic indecomposable objects.

Theorem (Auslander, 1986). *Let R be a hypersurface singularity of finite CM-type. Then R is an isolated singularity.*

Let R be an isolated hypersurface singularity. Then

- $CM(R)$ is a Frobenius category (projective objects are injective), and R is a unique indecomposable projective object.

- $CM(R)$ admits Auslander-Reiten sequences (**Auslander, 1986**)

$\Gamma_R = \Gamma_R(CM(R))$ **Aulander-Reiten quiver** of R

$\underline{CM(R)}$ stable category of $CM(R)$

$\Gamma^s_R = \Gamma_R(\underline{CM(R)})$ **stable Aulander-Reiten quiver** of R (obtainded from Γ_R by deleting R and the arrows attached to R)

Moreover, we have equivalences of functors from $\underline{CM(R)}$ to $\underline{CM(R)}$:

- $\Omega_R^2 \cong \mathrm{id}_{\underline{CM(R)}}$
- $\tau_R \cong \mathrm{id}_{\underline{CM(R)}}$ if $\dim R$ is even
- $\tau_R \cong \Omega_R$ if $\dim R$ is odd.

$R = S/(f)$ hypersurface singularity

$c(f)$ the set of all proper ideals I of $S = K[[x_0, x_1, \ldots, x_n]]$ such that $f \in I^2$.

R is called a **simple hypersurface singularity**) if $c(f)$ is finite.

Theorem (Arnold, 1972). *Let R be a hypersurface singularity of dimension d over an algebraically closed field K of characteristic 0. Then the following statements are equivalent:*

(1) *R is a simple hypersurface singularity.*

(2) *R is of finite deformation type.*

(3) *$R \cong K[[x_0, x_1, \ldots, x_d]]/(f_{\Delta}^{(d)})$, for a Dynkin graph Δ of type $\mathbb{A}_n (n \geq 1)$, $\mathbb{D}_n (n \geq 4)$, $\mathbb{E}_6$, $\mathbb{E}_7$, or $\mathbb{E}_8$, where*

$$f_{\mathbb{A}_n}^{(d)} = x^2 + y^{n+1} + z_2^2 + \cdots + z_d^2,$$

$$f_{\mathbb{D}_n}^{(d)} = x^2 + y^{n-1} + z_2^2 + \cdots + z_d^2,$$

$$f_{\mathbb{E}_6}^{(d)} = x^3 + y^4 + z_2^2 + \cdots + z_d^2,$$

$$f_{\mathbb{E}_7}^{(d)} = x^3 + xy^3 + z_2^2 + \cdots + z_d^2,$$

$$f_{\mathbb{E}_8}^{(d)} = x^3 + y^5 + z_2^2 + \cdots + z_d^2.$$

Finite deformation type means that R can be deformed only into finitely many other nonisomorphic singularities.

$K[[x_0, x_1, \ldots, x_d]]/(f_\Delta^{(d)})$ is called the **Arnold's simple hypersurface singularity of dimension d and Dynkin type Δ.**

Theorem (Buchweitz-Greuel-Schreyer, Knörrer, 1985-1987). *Let R be a hypersurface singularity of dimension d over an algebraically closed field K of characteristic 0. Then R is of finite Cohen-Macaulay type if and only if $R \cong K[[x_0, x_1, \ldots, x_d]]/(f_\Delta^{(d)})$, for some Dynkin graph Δ.*

Knörrer's periodicity

$S = K[[x_0, x_1, \dots, x_n]]$

$R = S/(f)$ isolated hypersurface singularity

$S^\sharp = S[[u]]$

$R^\sharp = S^\sharp/(f + u^2)$

Theorem (Knörrer, 1987). *Let R be an isolated hypersurface singularity over an algebraically closed field K of characteristic $\neq 2$. Then R is of finite Cohen-Macaulay type if and only if $R^\sharp$ is of finite Cohen-Macaulay type. Moreover, if R is of finite Cohen-Macaulay type, then*

(1) $\underline{CM(R^\sharp)} \cong \underline{CM(R)}[\mathbb{Z}_2]$ *skew group category, and hence $\Gamma^s_{R^\sharp}$ is a twisted quiver of Γ^s_R.*

(2) $\underline{CM((R^\sharp)^\sharp)} \cong \underline{CM(R)}$*, and hence the translation quivers $\Gamma^s_{(R^\sharp)^\sharp}$ and Γ^s_R are isomorphic.*

Solberg's periodicity

$R = S/(f)$ isolated hypersurface singularity

$R^* = S[[u, v]]/(f + uv)$

Theorem (Solberg, 1989). *Let $R = S/(f)$ be an isolated hypersurface singularity over an* **arbitrary** *algebraically closed field K. Then R is of finite Cohen-Macaulay type if and only if R^* is of finite Cohen-Macaulay type. Moreover, if R is of finite Cohen-Macaulay type, then there is an equivalence of categories $\underline{CM(R)} \xrightarrow{\sim} \underline{CM(R^*)}$, which induces an isomorphism of stable Auslander-Reiten quivers $\Gamma^s_R \xrightarrow{\sim} \Gamma^s_{R^*}$.*

For K of characteristic $\neq 2$, the Solberg's periodicity is equivalent to the Knörrer's periodicity.

Kleinian singularities

Let K be an algebraically closed field of characteristic 0.

$$\mathsf{SL}_2(K) = \{A \in M_{2\times 2}(K) \mid \det A = 1\}$$

It is a classical result that every finite subgroup of $\mathsf{SL}_2(K)$ is conjugate in $\mathsf{SL}_2(K)$ to one of the following **Klein groups**

$\mathcal{C}_n^*$	cyclic group of order n, $n \geq 1$
$\mathcal{D}_{2n}^*$	binary dihedral group of order $4n$, $n \geq 2$
$\mathcal{T}^*$	binary tetrahedral group of order 24
$\mathcal{O}^*$	binary octahedral group of order 48
$\mathcal{I}^*$	binary icosahedral group of order 120

Let G be a group of the above form. We associate to G a Dynkin graph $\Delta = \Delta(G)$ as follows:

$$\mathbb{A}_n = \Delta(\mathcal{C}^*_{n+1}), n \geq 1$$

$$\mathbb{D}_n = \Delta(\mathcal{D}^*_{2(n-1)}), n \geq 4$$

$$\mathbb{E}_6 = \Delta(\mathcal{T}^*)$$

$$\mathbb{E}_7 = \Delta(\mathcal{O}^*)$$

$$\mathbb{E}_8 = \Delta(\mathcal{I}^*)$$

G a finite subgroup of $\mathsf{SL}_2(K)$.

Then G acts on the algebra $K[[X,Y]]$: for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}_2(K)$ and $f(X,Y) \in K[[X,Y]]$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(X,Y) = f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} X \\ Y \end{pmatrix}\right)$$
$$= f(dX - bY, -cX + aY).$$

We may consider the **invariant algebra** $K[[X,Y]]^G$.

Theorem (Klein, 1884). *Let K be an algebraically closed field of characteristic* 0, *and G a finite subgroup of* $\mathrm{SL}_2(K)$. *Then*

$$K[[X,Y]]^G \cong K[[x,y,z]]/(f_\Delta)$$

where $\Delta = \Delta(G)$ is the Dynkin graph of G, and

$$f_{\mathbb{A}_n} = x^2 + y^{n+1} + z^2$$
$$f_{\mathbb{D}_n} = x^2 y + y^{n-1} + z^2$$
$$f_{\mathbb{E}_6} = x^3 + y^4 + z^2$$
$$f_{\mathbb{E}_7} = x^3 + xy^3 + z^2$$
$$f_{\mathbb{E}_8} = x^3 + y^5 + z^2$$

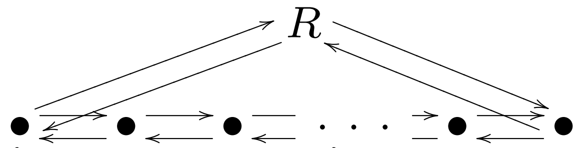
Hence, $f_\Delta = f_\Delta^{(2)}$ with $z = z_2$, and $K[[X,Y]]^G$ are the Arnold's simple hypersurface singularities of dimension 2.

For $K = \mathbb{C}$, the orbit space $\mathbb{C}^2/G$ is a compact Riemann surface with at most 3 singular points, and the Dynkin graph $\Delta(G)$ describes the multiplicities of these singular points.

Theorem (Artin-Verdier, Esnault-Knörrer, 1985). *Let R be a hypersurface singularity of dimension 2 over an algebraically closed field K of characteristic 0. Then R is of finite Cohen-Macaulay type if and only if $R \cong K[[X,Y]]^G$, for a finite subgroup G of $\mathsf{SL}_2(K)$.*

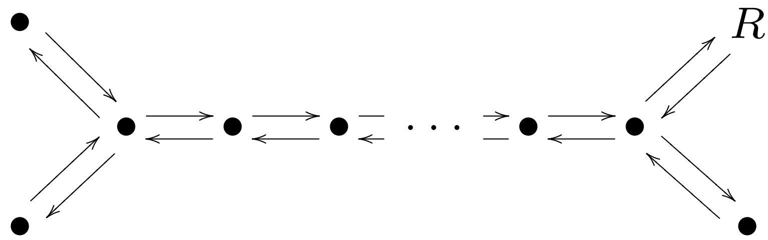
Theorem (Auslander-Reiten, 1986). *Let $R = K[[x,y,z]]/(f_\Delta)$ be an Arnold's simple hypersurface singularity of dimension 2 over an algebraically closed field K of arbitrary characteristic. Then the Auslander-Reiten quiver Γ_R is of the form*

$\Delta = \mathbb{A}_n$
$n \geq 1$



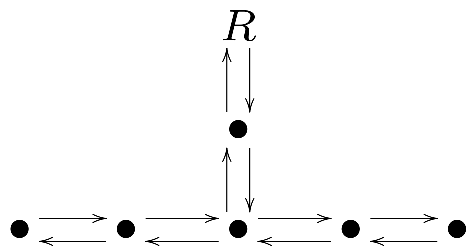
$(n+1$ vertices$)$

$\Delta = \mathbb{D}_n$
$n \geq 4$

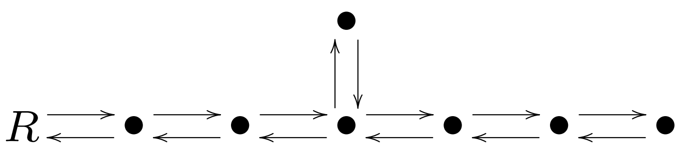


$(n+1$ vertices$)$

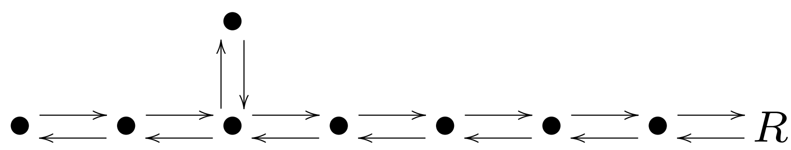
$\Delta = \mathbb{E}_6$



$\Delta = \mathbb{E}_7$



$\Delta = \mathbb{E}_8$



and, in all cases, $\tau_R =$ *identity.*

Simple plane curve singularities

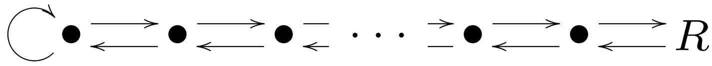
$R = K[[x,y]]/(g_\Delta)$, Δ Dynkin graph

$g_\Delta = f_\Delta^{(1)}$ is of the form

$$g_{\mathbb{A}_n} = x^2 + y^{n+1}$$
$$g_{\mathbb{D}_n} = x^2 y + y^{n-1}$$
$$g_{\mathbb{E}_6} = x^3 + y^4$$
$$g_{\mathbb{E}_7} = x^3 + xy^3$$
$$g_{\mathbb{E}_8} = x^3 + y^5$$

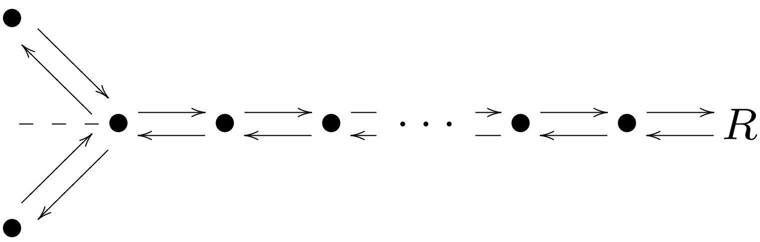
Theorem (Dieterich-Wiedemann (1986), Kiyek-Steineke (1989)). *Let $R = K[[x,y]]/(g_\Delta)$ be a simple plane curve singularity over an algebraically closed field K of arbitrary characteristic. Then the Auslander-Reiten quiver Γ_R is of the form*

$\Delta = \mathbb{A}_{2m}$
$m \geq 1$



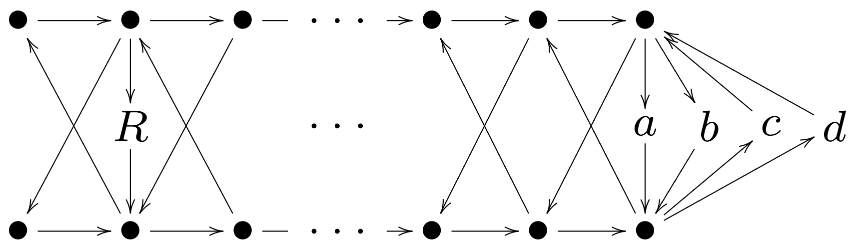
($m+1$ *vertices*, $\tau_R = \Omega_R$ = *identity*)

$\Delta = \mathbb{A}_{2m-1}$
$m \geq 1$



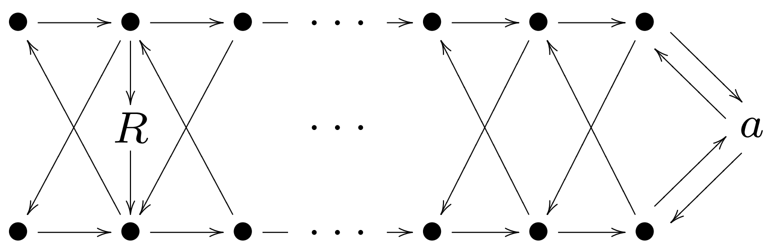
($m+2$ *vertices*, $\tau_R = \Omega_R$ = *reflection at the horizontal line*)

$\Delta = \mathbb{D}_{2m}$
$m \geq 2$



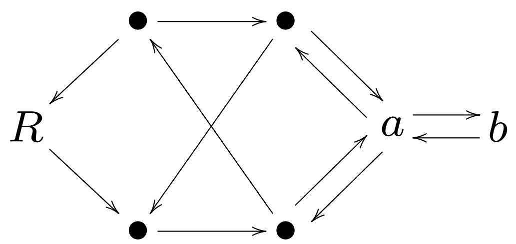
($4m+1$ *vertices*, $\tau_R = \Omega_R$ = *reflection at the horizontal line through* R, a, b, c, d, *with* $\tau_R a = d$, $\tau_R b = c$, $\tau_R^2 = \text{id}$)

$\Delta = \mathbb{D}_{2m+1}$
$m \geq 2$



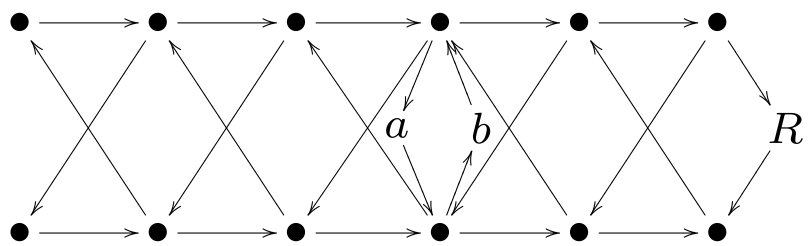
($4m$ *vertices*, $\tau_R = \Omega_R$ = *reflection at the horizontal line through* R *and* a)

$\Delta = \mathbb{E}_6$



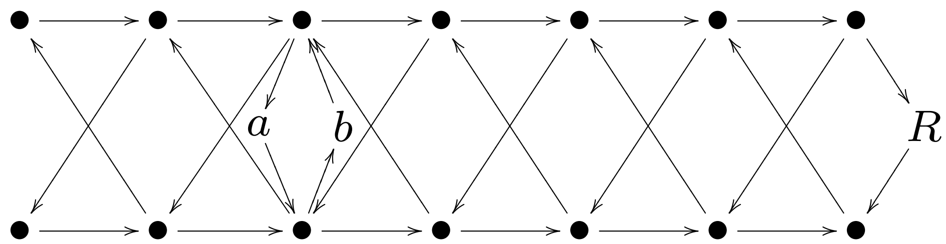
(7 *vertices*, $\tau_R = \Omega_R$ = *reflection at the horizontal line through* R, a *and* b)

$\Delta = \mathbb{E}_7$



(15 *vertices*, $\tau_R = \Omega_R$ = *reflection at the horizontal line through* R, a *and* b, *with* $\tau_R a = b$, $\tau_R b = a$)

$\Delta = \mathbb{E}_8$



(17 *vertices*, $\tau_R = \Omega_R$ = *reflection at the horizontal line through* R, a *and* b, *with* $\tau_R a = b$, $\tau_R b = a$)

Greuel and Kröning introduced the concept of **finite deformation type** of hypersurface singularities for algebraically closed fields of positive characteristic and proved the theorem of the form

Theorem (Greuel-Kröning, 1990). *Let R be a hypersurface singularity. The following statements are equivalent:*

(1) *R is a simple hypersurface singularity.*

(2) *R is of finite deformation type.*

(3) *R is of finite CM-type.*

In characteristic $\neq 2, 3, 5$, the Arnold's simple hypersurface singularities are all simple hypersurface singularities.

The normal forms of simple hypersurface singularities of dimension 1 were classified by **Kiyek and Steineke (1985)**.

The normal forms of simple hypersurface singularities of dimension 2 were classified by **Artin (1977)**.

The normal forms of simple hypersurface singularities of dimensions ≥ 3 can be obtained from the normal forms of dimensions 1 and 2 by Solberg's periodicity theorem (**Solberg (1989), Greuel-Kröning (1990)**).

Theorem (Solberg (1989), Greuel-Kröning (1990)). *Let R be a hypersurface singularity of finite CM-type over an algebraically closed field K of arbitrary characteristic. Then the Auslander-Reiten quiver Γ_R of R is isomorphic to the Auslander-Reiten quiver of an Arnold's simple hypersurface singularity of dimension 1 or 2 (simple plane curve singularity or Kleinian singularity).*

Stable Auslander algebras

Let R be a hypersurface singularity of finite CM-type over an algebraically closed field K of arbitrary characteristic.

$CM(R)$ is a Frobenius category of finite type.

Let $M_1, M_2, \dots, M_n$ be a complete set of pairwise nonisomorphic indecomposable nonprojective objects in $CM(R)$

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$$

Consider the endomorphism algebra

$$\underline{\mathcal{A}}(R) = \mathrm{End}_{\underline{CM(R)}}(\underline{M})$$

of $\underline{M} = M$ in the stable category $\underline{CM(R)}$, and call the **stable Auslander algebra** of R.

For a Dynkin graph Δ, denote

$$P(\Delta) = \underline{\mathcal{A}}(K[[x,y,z]]/(f_\Delta))$$

$$P(\Delta)^* = \underline{\mathcal{A}}(K[[x,y]]/(g_\Delta))$$

Theorem. *Let Δ be a Dynkin graph. The following statements hold:*

(1) *$P(\Delta)$ is a basic finite dimensional selfinjective K-algebra. Moreover, the Nakayama permutation ν of $P(\Delta)$ is the identity for $\Delta = \mathbb{A}_1$, $\mathbb{D}_n$ (n even), $\mathbb{E}_7$, $\mathbb{E}_8$, and of order 2 for $\Delta = \mathbb{A}_n$ ($n \geq 2$), $\mathbb{D}_n$ (n odd), $\mathbb{E}_6$.*

(2) *$P(\Delta)^*$ is a basic finite dimensional, symmetric K-algebra.*

From the above remarks, the stable Auslander algebra $\underline{\mathcal{A}}$ of any hypersurface singularity R of finite CM-type of even dimension (respectively, odd dimension) is isomorphic to $P(\Delta)$ (respectively, $P(\Delta)^*$), for some Dynkin graph Δ.

$P(\Delta)$ is the **preprojective algebra of Dynkin type** Δ (introduced by **Gelfand** and **Ponomarev**)

$P(\Delta)^*$ the **twisted preprojective algebra of Dynkin type** Δ

For K of characteristic $\neq 2$,

$$P(\Delta)^* \cong P(\Delta)[\mathbb{Z}_2] \quad \text{(skew group algebra)}$$

$$P(\Delta) \cong P(\Delta)^*[\mathbb{Z}_2] \quad \text{(skew group algebra)}$$

for the corresponding actions of $\mathbb{Z}_2$ on the algebras $P(\Delta)$ and $P(\Delta)^*$.

We also note that, with few exceptions, the algebras $P(\Delta)$ and $P(\Delta)^*$ are of **wild representation type**.

K algebraically closed field.

Let $\mathcal{B}$ be a K-category of one of the two types:

- $CM(R)$ for an isolated hypersurface singularity R over K.
- $\operatorname{mod}\Lambda$ for a finite dimensional selfinjective K-algebra Λ.

Then $\mathcal{B}$ is a Frobenius category with Auslander-Reiten sequences. Denote $\mathscr{C} = \operatorname{mod}\underline{\mathcal{B}} = (\underline{\mathcal{B}^{\mathrm{op}}}, \mathbb{A}b)$ the category of finitely presented contravariant functors from the stable category $\underline{\mathcal{B}}$ of $\mathcal{B}$ to the category $\mathbb{A}b$ of abelian groups.

Theorem (Auslander-Reiten). *The following statements hold:*

(1) *$\mathscr{C}$ is a Frobenius abelian K-category whose projective objects are the representable functors $\underline{\mathrm{Hom}}_{\mathcal{B}}(-, \underline{B})$, $\underline{B}$ objects of $\underline{\mathcal{B}}$.*

(2) *$\mathscr{C}$ admits Auslander-Reiten sequences.*

Denote by $\mathcal{N}_{\mathcal{B}}$, $\tau_{\mathcal{B}}$, $\Omega_{\mathcal{B}}$ (respectively, $\mathcal{N}_{\mathscr{C}}$, $\tau_{\mathscr{C}}$, $\Omega_{\mathscr{C}}$) the Nakayama, Auslander-Reiten and syzygy functors on $\underline{\mathcal{B}}$ (respectively, on $\underline{\mathscr{C}}$).

Theorem (Auslander-Reiten, 1996). *In the above notation, the following statements hold:*

(1) $\mathcal{N}_{\mathscr{C}}(\underline{\mathrm{Hom}}_{\mathcal{B}}(-,\underline{B})) = \underline{\mathrm{Hom}}_{\mathcal{B}}(-,\Omega_{\mathcal{B}}^{-1}\tau_{\mathcal{B}}(\underline{B}))$ *for any object* $\underline{B}$ *of* $\underline{\mathcal{B}}$.

(2) *The functors* $\tau_{\mathscr{C}}, \Omega_{\mathscr{C}}^2\mathcal{N}_{\mathscr{C}}, \mathcal{N}_{\mathscr{C}}\Omega_{\mathscr{C}}^2 : \underline{\mathscr{C}} \to \underline{\mathscr{C}}$ *are equivalent.*

(3) *If the functor* $\Omega_{\mathcal{B}}^{-1}\tau_{\mathcal{B}} : \underline{\mathcal{B}} \to \underline{\mathcal{B}}$ *has order* s *and the functor* $\Omega_{\mathcal{B}}^2 : \underline{\mathcal{B}} \to \underline{\mathcal{B}}$ *has order* t, *and* $r = lcm(s, 3t)$, *then* $\tau_{\mathscr{C}}^r \xrightarrow{\sim} \mathrm{id}_{\underline{\mathscr{C}}}$.

Theorem (Auslander-Reiten, 1996). *Let* $\mathscr{C} = \operatorname{mod} \underline{CM(R)}$ *for an isolated hypersurface singularity* R *over* K*. The following statements hold:*

(1) *If* R *has even dimension, then each indecomposable object of* $\mathscr{C}$ *is* $\tau_{\mathscr{C}}$*-periodic of period dividing* 6*.*

(2) *If* R *has odd dimension, then each indecomposable object of* $\mathscr{C}$ *is* $\tau_{\mathscr{C}}$*-periodic of period dividing* 3*.*

Proof. We have $\Omega_R^2 \xrightarrow{\sim} \mathrm{id}_{\underline{CM(R)}}$.

(1) If $\dim R$ is even then $\tau_R \xrightarrow{\sim} \mathrm{id}_{\underline{CM(R)}}$. Hence $\Omega_R^{-1}\tau_R = \Omega_R^{-1}$ has order 2, and so $r = lcm(2, 3 \cdot 1) = 6$.

(2) If $\dim R$ is odd then $\tau_R \xrightarrow{\sim} \Omega_R$. Hence $\Omega_R^{-1}\tau_R = \mathrm{id}_{\underline{CM(R)}}$, and so $r = lcm(1, 3 \cdot 1) = 3$. $\square$

Assume R is a hypersurface singularity over K of finite CM-type.

Then $CM(R)$ has only a finite number of indecomposable objects, and hence we have an equivalence

$$\operatorname{mod} \underline{CM(R)} \xrightarrow{\sim} \operatorname{mod} \underline{\mathcal{A}}(R)$$

which commutes with the Auslander-Reiten translations τ_R on $\operatorname{mod} \underline{CM(R)}$ and $\tau_{\underline{\mathcal{A}(R)}} = D\,\mathrm{Tr}$ on $\underline{\mathcal{A}}(R)$. Recall that $\tau_{\underline{\mathcal{A}(R)}} = \Omega^2_{\underline{\mathcal{A}(R)}} \mathcal{N}_{\underline{\mathcal{A}(R)}}$.

We also note that $P(\Delta)$ (respectively, $P(\Delta)^*$) is semisimple if and only if $\Delta = \mathbb{A}_1$.

Therefore we obtain

Theorem (Auslander-Reiten, 1996). *Let* Δ *be a Dynkin graph* $\neq \mathbb{A}_1$. *The following statements hold:*

(1) $\tau^6_{P(\Delta)} \cong 1_{\underline{\mathrm{mod}} P(\Delta)}$, $\Omega^3_{P(\Delta)} \cong \mathcal{N}^{-1}_{P(\Delta)}$ *and* $\Omega^6_{P(\Delta)} \cong 1_{\underline{\mathrm{mod}} P(\Delta)}$ *as functors on* $\underline{\mathrm{mod}} P(\Delta)$.

(2) $\tau^3_{P(\Delta)^*} \cong 1_{\underline{\mathrm{mod}} P(\Delta)^*}$ *and* $\Omega^6_{P(\Delta)^*} \cong 1_{\underline{\mathrm{mod}} P(\Delta)^*}$ *as functors on* $\underline{\mathrm{mod}} P(\Delta)^*$.

In fact, we have the following

Theorem (Schoefield (1990), Erdmann-Snashall (1998)). *Let* Δ *be a Dynkin graph* $\neq \mathbb{A}_1$. *Then* $\Omega^6_{P(\Delta)^e} P(\Delta) \cong P(\Delta)$ *in* $\mathrm{mod}\, P(\Delta)^e$.

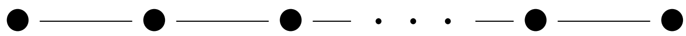
Theorem (Białkowski-Erdmann-Skowroński (2005)). *Let* Δ *be a Dynkin graph* $\neq \mathbb{A}_1$. *Then* $\Omega^6_{(P(\Delta)^*)^e} P(\Delta)^* \cong P(\Delta)^*$ *in* $\mathrm{mod}(P(\Delta)^*)^e$.

Corollary. *Let* Δ *be a Dynkin graph* $\neq \mathbb{A}_1$. *Then* $\underline{\mathrm{mod}} P(\Delta)$ *and* $\underline{\mathrm{mod}} P(\Delta)^*$ *are Calabi-Yau triangulated categories (of Calabi-Yau dimensions* 0, 2, *or* 5*).*

Generalized Dynkin graphs

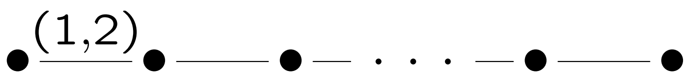
$\mathbb{A}_n$

$n \geq 1$



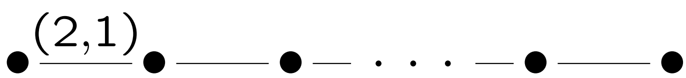
$\mathbb{B}_n$

$n \geq 2$



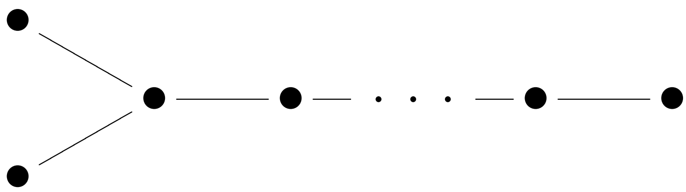
$\mathbb{C}_n$

$n \geq 3$

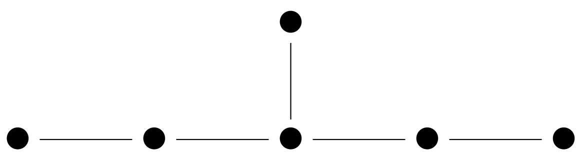


$\mathbb{D}_n$

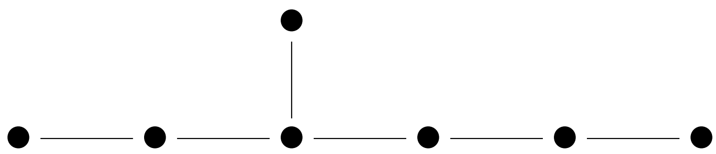
$n \geq 4$



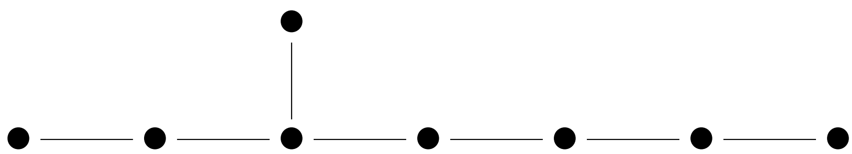
$\mathbb{E}_6$



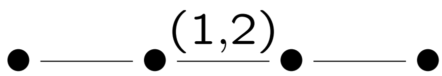
$\mathbb{E}_7$



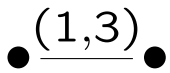
$\mathbb{E}_8$



$\mathbb{F}_4$

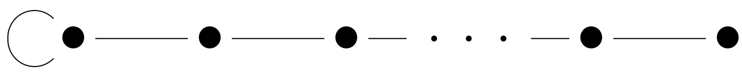


$\mathbb{G}_2$



$\mathbb{L}_n$

$n \geq 1$



Mesh algebras of generalized Dynkin type

(1) The mesh (preprojective) algebras $P(\Delta)$ of types

$$\Delta \in \{\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8, \mathbb{L}_n\},$$

where $P(\mathbb{L}_n) = P(\mathbb{A}_{2n})^*$ for $n \geq 1$.

(2) The mesh (twisted) algebras $\Lambda(\Delta)$ of types

$$\Delta \in \{\mathbb{B}_n, \mathbb{C}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8, \mathbb{F}_4, \mathbb{G}_2\},$$

such that

$\Lambda(\mathbb{B}_2) = P(\mathbb{A}_3)^*$,

$\Lambda(\mathbb{B}_{2m}) = P(\mathbb{D}_{2m+1})^*$ for $m \geq 2$,

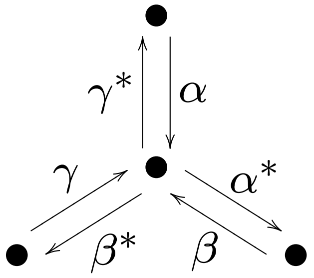
$\Lambda(\mathbb{C}_n) = P(\mathbb{A}_{2n-1})^*$ for $n \geq 3$,

$\Lambda(\mathbb{D}_{2m}) = P(\mathbb{D}_{2m})^*$ for $m \geq 2$,

$\Lambda(\mathbb{E}_7) = P(\mathbb{E}_7)^*$, $\Lambda(\mathbb{E}_8) = P(\mathbb{E}_8)^*$,

$\Lambda(\mathbb{F}_4) = P(\mathbb{E}_6)^*$,

$\Lambda(\mathbb{G}_2)$ is given by the quiver and relations



$$\alpha^*\beta + \beta^*\gamma + \gamma^*\alpha = 0$$

$$\alpha\alpha^* = 0, \ \beta\beta^* = 0, \ \gamma\gamma^* = 0.$$

More generally, one defines the **deformed mesh algebras of generalized Dynkin type**

$$P^f(\Delta),\ \Delta \in \{\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8, \mathbb{L}_n\}$$

$$\Lambda^f(\Delta),\ \Delta \in \{\mathbb{B}_n, \mathbb{C}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8, \mathbb{F}_4, \mathbb{G}_2\}$$

(f elements of products of at most two copies of certain finite dimensional local selfinjective algebras)

which are basic, indecomposable, finite dimensional selfinjective algebras with

$$\dim_K P^f(\Delta) = \dim_K P(\Delta)$$

and

$$\dim_K \Lambda^f(\Delta) = \dim_K \Lambda(\Delta).$$

Theorem (Białkowski-Erdmann-Skowroński, 2005). *Let A be a deformed mesh algebra of a generalized Dynkin type. Then there is a positive integer $m = m_A$ such that $\Omega_{A^e}^{6m}(A) \cong A$ in $\operatorname{mod} A^e$.*

Theorem (Białkowski-Erdmann-Skowroński, 2005). *Let A be a basic, indecomposable, finite dimensional selfinjective but not Nakayama algebra over an algebraically closed field K. The following statements are equivalent:*

(1) $\Omega_A^3(S)$ *is simple for any simple right A-module S.*

(2) A *is isomorphic to a deformed mesh algebra $P^f(\Delta)$ or $\Lambda^f(\Delta)$ of a generalized Dynkin type $\Delta \neq \mathbb{A}_1, \mathbb{A}_2, \mathbb{L}_1$.*

Theorem (Białkowski-Erdmann-Skowroński, 2004). *Let A be a basic, indecomposable, finite dimensional selfinjective algebra over an algebraically closed field K. The following statements are equivalent:*

(1) $\Omega_A^3(S) \cong \mathcal{N}_A(S)$ *for any nonprojective simple right A-module S.*

(2) $\Omega_A^3(S) \cong \mathcal{N}_A^{-1}(S)$ *for any nonprojective simple right A-module S.*

(3) A *is isomorphic to a deformed preprojective algebra $P^f(\Delta)$ of a generalized Dynkin type Δ.*

Theorem (Białkowski-Erdmann-Skowroński, 2005). *Let A be a basic, indecomposable, finite dimensional selfinjective algebra over an algebraically closed field K. The following statements are equivalent:*

(1) $\Omega_A^3(S) \cong S$ *for any simple right A-module S.*

(2) *A is isomorphic to a deformed preprojective algebra $P^f(\Delta)$ of type*

$$\Delta \in \{\mathbb{D}_n(n \text{ even}), \mathbb{E}_7, \mathbb{E}_8, \mathbb{L}_n\}.$$

Let Λ be a finite dimensional selfinjective algebra of finite representation type over an algebraically closed field K.

Let $M_1, M_2, \ldots, M_n$ be a complete set of pairwise nonisomorphic indecomposable nonprojective Λ-modules

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$$

$\underline{\mathcal{A}(\Lambda)} = \underline{\text{End}}_\Lambda(\underline{M})$ **stable Auslander algebra** of Λ.

Observe that the functors τ_Λ, Ω_Λ, $\mathcal{N}_\Lambda = \Omega_\Lambda^{-2}\tau_\Lambda$ have finite orders on $\underline{\text{mod}}\Lambda$.

Applying the Auslander-Reiten theorem to $\mathscr{C} = \text{mod}\,\underline{\text{mod}}\Lambda = \text{mod}\,\underline{\mathcal{A}(\Lambda)}$ we obtain

Theorem. *Let Λ be a finite dimensional self-injective K-algebra of finite representation type and A the stable Auslander algebra of Λ. Moreover, let s be the order of $\Omega_\Lambda^{-1}\tau_\Lambda$ on $\underline{\text{mod}}\Lambda$, t the order of Ω_Λ^2 on $\underline{\text{mod}}\Lambda$ and $r = lcm(s, 3t)$. Then*

(1) *A is a finite dimensional Frobenius K-algebra with the Nakayama automorphism ν_A of order s.*

(2) *$\tau_A^r \xrightarrow{\sim} \text{id}_{\underline{\text{mod}}A}$ and $\Omega_A^{2r} \xrightarrow{\sim} \text{id}_{\underline{\text{mod}}A}$ as functors on $\underline{\text{mod}}A$.*

We note that, if Λ is symmetric, then $\Omega_\Lambda^{-1}\tau_\Lambda = \Omega_\Lambda$ and $r = lcm(s, 3t) = 3s$.

Therefore there are many finite dimensional Frobenius algebras A for which all indecomposable nonprojective modules are τ_A-periodic and Ω_A-periodic.