The Abdus Salam
International Centre for Theoretical Physics

United Nations
Educational, Scientific
and Cultural Organization

International Atomic
Energy Agency

SMR1735/12

Advanced School and Conference on Representation Theory and Related Topics

(9 - 27 January 2006)

Hereditary Categories
Lectures 3, 4 and 5

Helmut Lenzing
Institut für Mathematik
Universität Paderborn
Paderborn, Germany

Strada Costiera 11, 34014 Trieste, Italy - Tel. +39 040 2240 111; Fax +39 040 224 163 - sci_info@ictp.it, www.ictp.it

4 Coherent sheaves on a smooth projective curve

We introduce these categories in an axiomatic way, introducing in addition to $(H1)$ – $(H3^*)$ some natural additional axioms. For the whole section we assume that k is an **algebraically closed field**.

4.1 Some further axioms

We continue investigating Hom-finite hereditary abelian k-categories $\mathcal{H}$ with Serre duality. By Gabriel's theorem we understand completely the case when $\mathcal{H}$ is a length category. An object is of finite length if and only if it is together noetherian and artinian. Here, an object X in an abelian category is called **noetherian** if each ascending chain of subobjects $U_0 \subseteq U_1 \subseteq U_2 \subseteq \cdots$ of U becomes stationary, that is, we have $U_n = U_{n+1} = U_{n+2} = \cdots$ for large n. Dually, X is called **artinian** if each descending chain of subobjects becomes stationary.

It is hence natural to add to (H1), (H2) and (H3*) one of the two chain conditions as a further request:

(H4) *$\mathcal{H}$ is noetherian but not every object of $\mathcal{H}$ has finite length.*

Proposition 4.1 *Assume that $\mathcal{H}$ is a noetherian, hereditary, abelian category which is Hom-finite and satisfies Serre duality. Then the following holds:*

(*i*) *The full subcategory $\mathcal{H}_0$ of consisting of all objects of finite length is an exact abelian subcategory of $\mathcal{H}$, in particular, a hereditary abelian length category with Serre duality.*

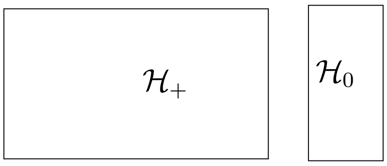
(*ii*) *$\mathcal{H}_0$ is uniserial, and decomposes into a coproduct $\coprod_{x\in C}\mathcal{U}_x$ of connected uniserial subcategories, whose associated quivers are tubes (possibly of infinite τ-period).*

(*iii*) *Each indecomposable object X of $\mathcal{H}$ decomposes into a direct sum $X = X_0 \oplus E$, where X_0 belongs to $\mathcal{H}_0$, and E does not have any simple subobject.*

Proof. Since $\mathcal{H}_0$ is closed in $\mathcal{H}$ under subobjects, quotients and extensions, it is a full abelian subcategory of $\mathcal{H}$ and hence hereditary and abelian. Since moreover τ as an equivalence preserves simple objects and hence finite length, it follows that τ induces an equivalence $\tau_{|\mathcal{H}_0}$ of $\mathcal{H}_0$ such that Serre duality also holds for $\mathcal{H}_0$. Hence Gabriel's theorem applies and (*i*) and (*ii*) follow.

Concerning (*iii*) we choose by noetherianness a maximal subobject X_0 of X having finite length. By Serre duality the sequence $0 \to X_0 \to X \to X_+ \to 0$ splits since X_+ belongs to $\mathcal{H}_+$ and therefore $0 = \mathrm{Hom}(\tau^{-1}X_0, X_+) = \mathrm{D}\,\mathrm{Ext}^1(X_+, X_0)$. □

Let now $\mathcal{H}_+$ denote the full subcategory of $\mathcal{H}$ consisting of all objects E not having a simple subobject, then each indecomposable object of $\mathcal{H}$ is either in $\mathcal{H}_+$ or in $\mathcal{H}_0$; moreover there are no non-zero morphisms from $\mathcal{H}_0$ to $\mathcal{H}_+$. In other words $\mathcal{H} = \mathcal{H}_+ \vee \mathcal{H}_0$. On the other hand there are many morphisms from $\mathcal{H}_+$ to $\mathcal{H}_0$. We depict the situation schematically as follows:



By Gabriel's theorem we basically know the shape of indecomposables of $\mathcal{H}_0$. The classification problem for the indecomposables of $\mathcal{H}$ thus largely reduces to classify the indecomposable objects in $\mathcal{H}_+$.

We next request the existence of a linear form $rk : \mathrm{K}_0(\mathcal{H}) \to \mathbb{Z}$, that is, of an additive function on $\mathcal{H}$ separating the objects of $\mathcal{H}_+$ and $\mathcal{H}_0$:

(H5) *There is an additive function* $\mathrm{rk} : \mathcal{H} \to \mathbb{Z}$*, called* ***rank****, that is* τ*-stable, zero on* $\mathcal{H}_0$ *and* > 0 *on nonzero objects of* $\mathcal{H}_+$*. Moreover* $\mathcal{H}_+$ *admits an (indecomposable) object of rank one.*

We call the objects of $\mathcal{H}_+$ **bundles** and those of rank one **line bundles**.

By noetherianness each non-zero object E has a maximal subobject E', hence a nonzero morphism to a simple object. We now introduce a much stronger request, dealing with the interaction between $\mathcal{H}_+$ and $\mathcal{H}_0$.

(H6) *Each simple object* S *satisfies* $\tau S \cong S$*. Moreover, if* L *is a line bundle and* S *is simple, then* $\mathrm{Hom}(E, S) \cong k$*.*

It follows from **(H6)** *t*hat $\mathcal{H}$ is **connected**.

4.2 Properties of line bundles

Proposition 4.2 *Let L be a line bundle and E be any bundle. Then each nonzero morphism $u : L \to E$ is a monomorphism. In particular* $\mathrm{End}(L) = k$.

Proof. By properties of the rank the image I of u has rank one, hence the kernel K of u has rank zero by additivity of the rank. Being of finite length and a subobject of a bundle K must be zero, proving the first claim. For the second claim note that by the previous argument $\mathrm{End}(L)$ is a finite dimensional algebra without zero divisors, hence a finite dimensional skew field extension of k. Since k is algebraically closed hence $\mathrm{End}(L) = k$. □

The role of the line bundles becomes clear from the next statement which can be proved by induction on the rank.

Proposition 4.3 *Each bundle E of $\mathcal{H}$ has a line bundle filtration, that is, a chain $0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_r = E$ of subobjects of E, where each quotient E_i/E_{i-1} is a line bundle (and hence r is the rank of E).* □

4.3 Categories of coherent sheaves

Theorem 4.4 (Reiten–van den Bergh) *Assume that k is an algebraically closed field. Properties (H1)–(H6) plus Serre duality characterize the categories of coherent sheaves on a smooth projective curve.*

Proof. This can be deduced from [9]. □

In the sequel, therefore, by a **category of coherent sheaves on a smooth projective curve** we just mean a category $\mathcal{H}$ satisfying (H1)–(H6) plus Serre duality (H3*).

4.4 Degree and Riemann-Roch

By its very properties the rank is an important linear form on $\mathrm{K}_0(\mathcal{H})$. It is τ-invariant, vanishes on $\mathcal{H}_0$ and is strictly positive on indecomposables of $\mathcal{H}_+$. We now introduce another linear form, called degree, and having somehow dual properties being > 0 on nonzero objects of $\mathcal{H}_0$ and being τ-stable on $\mathcal{H}_0$.

Fix a line bundle L_0 of $\mathcal{H}$. (We will see later that it is irrelevant which one we choose.) Then the **degree** is the linear form on $\mathrm{K}_0(\mathcal{H})$ given by

$$\deg x = \langle [L_0], x\rangle - \langle [L_0], [L_0]\rangle \mathrm{rk}\, x$$

for each $x \in \mathrm{K}_0(\mathcal{H})$. Put $g = \dim_k \mathrm{Ext}^1(L_0, L_0)$ such that $\langle [L_0], [L_0]\rangle = 1 - g$. We deduce from $(H6)$ that the degree of each simple object equals one, hence **for objects from $\mathcal{H}_0$ degree and length coincide**. We deduce from this:

Remark 4.5 An object X of $\mathcal{H}$ with $\mathrm{rk}\, X = \deg X$ is already 0. In particular, if $[X] = 0$, then $X = 0$, showing that the interaction between $\mathcal{H}$ and $\mathrm{K}_0(\mathcal{H})$ works fine.

The defining equation for the degree can be rewritten as

$$\langle [L_0], x\rangle = (1 - g)\mathrm{rk}\, x + \deg x.$$

Using line bundle filtrations and the symmetry property $\langle x, y\rangle = -\langle y, \tau x\rangle$ this formula generalizes to the **Riemann-Roch formula** (RR)

$$\langle x, y\rangle = (1 - g)\mathrm{rk}\, x\, \mathrm{rk}\, y + \begin{vmatrix} \mathrm{rk}\, x & \mathrm{rk}\, y \\ \deg x & \deg y \end{vmatrix}$$

Note that $g = \dim_k \mathrm{Ext}^1(L_0, L_0)$ initially depends on the choice of the line bundle L_0. It follows however from (RR) that each line bundle L satisfies $\langle L, L\rangle = 1 - g$ such that also $g = \dim_k \mathrm{Ext}^1(L, L)$. Therefore g is an invariant of $\mathcal{H}$, called the **genus** of $\mathcal{H}$.

Remark 4.6 (i) The Riemann-Roch formula is the key for a homological classification of indecomposable bundles (sofar possible).

(ii) The genus $g = g_{\mathcal{H}}$ of $\mathcal{H}$ and the related **Euler characteristic** $\chi_{\mathcal{H}} = 2(1 - g)$, which is sometimes more appropriate to use, are important homological invariants of $\mathcal{H}$. Their value is responsible for the complexity of the classification problem for $\mathcal{H}$.

5 Request of a tilting object: coherent sheaves on the projective line

5.1 Curves of genus zero

Next we are going to show that the assumption of a tilting object for $\mathcal{H}$ is a very severe restriction. We impose this as a further axiom for the whole section in addition to the axioms $(H1)$–(H6).

(H7) *$\mathcal{H}$ has a tilting object T.*

Theorem 5.1 *Assume that $\mathcal{H}$ is a category of coherent sheaves on a smooth projective curve over an algebraically closed field. Then the following assertions are equivalent:*

(i) $\mathcal{H}$ has a tilting object.

(ii) $\mathcal{H}$ has genus zero.

(iii) $\mathcal{H}$ has a tilting object $T = L \oplus \bar{L}$ that is the direct sum of two line bundles L and $\bar{L}$, such that $\mathrm{End}(T)$ is isomorphic to the Kronecker algebra $k[\circ \rightrightarrows \circ]$

The proof follows from the next two lemmas.

Lemma 5.2 *If $\mathcal{H}$ has a tilting object T, then we have $g_{\mathcal{H}} = 0$ and each indecomposable direct summand of T is a line bundle.*

PROOF. Let E be an indecomposable direct summand of a tilting object T. It follows that also E has no self-extensions and so, by a theorem of Happel and Ringel [6] valid for hereditary categories, has trivial endomorphism ring k, that is, E is an **exceptional object** from $\mathcal{H}$. Applying (RR) to $[E]$ hence leads to

$$1 = \langle [E], [E] \rangle = (1 - g)\mathrm{rk}\,(E)^2.$$

It follows that $g = 0$, and E is a line bundle. □

Lemma 5.3 *If $\mathcal{H}$ has genus zero, then each line bundle L is exceptional. Moreover let $\eta : 0 \to L \to \bar{L} \to S \to 0$ be a nonsplit sequence with a simple object S, then $\bar{L}$ is a line bundle and $T = L \oplus \bar{L}$ is a tilting object, whose endomorphism algebra is isomorphic to the Kronecker algebra $k[\circ \rightrightarrows \circ]$.*

PROOF. Applying (RR) to a line bundle L' yields $1 - \dim_k \mathrm{Ext}^1(L', L') = \langle L', L' \rangle = (1 - g) = 1$, so **for genus zero a line bundle has no self-extensions**.

Applying $\mathrm{Hom}(L, -)$ and $\mathrm{Hom}(\bar{L}, -)$ to η shows that

$$\mathrm{Hom}(\bar{L}, L) = 0, \quad \mathrm{Ext}^1(\bar{L}, L) = 0, \quad \mathrm{Ext}^1(L, \bar{L}) = 0, \quad \dim_k \mathrm{Hom}(L, \bar{L}) = 2.$$

Hence $T = L \oplus \bar{L}$ has no self-extensions, and its endomorphism ring is isomorphic to the Kronecker algebra.

It remains to show that T generates $\mathcal{H}$ homologically. Let X be in $\mathcal{H}$ such that $\mathrm{Hom}(T, X) = 0 = \mathrm{Ext}^1(T, X)$. Then $\langle [L], [X] \rangle = 0$ and $\langle [\bar{L}], [X] \rangle = 0$. Since $[\bar{L}] - [L] = [S]$ it follows that $\langle [S], [X] \rangle = 0$ and hence $\mathrm{rk}\, X = 0$, so that $X \in \mathcal{H}_0$.

Invoking (RR) is now follows from $0 = \langle [L], [X] \rangle$ that also $\deg X = 0$, and hence the length of X is zero. □

Corollary 5.4 *Up to equivalence there is only one category of coherent sheaves which has genus zero.*

Any category satisfying (H1)–(H7) will thus be called **category of coherent sheaves on the projective line over** k.

PROOF. We only sketch the argument. By the theorem $\mathrm{D}^b(\mathcal{H})$ is equivalent to the derived category $\mathrm{D}^b(\text{mod-}\Lambda)$, where Λ is the Kronecker algebra. So each other category $\mathcal{H}'$ of coherent sheaves with genus zero has a derived category equivalent to $\mathrm{D}^b(\mathcal{H})$, in particular $\mathcal{H}'$ embeds into $\mathrm{D}^b(\mathcal{H})$. From the bisection of $\mathcal{H}$ into $\mathcal{H}_+$ and $\mathcal{H}_0$ (and the corresponding bisection of $\mathcal{H}'$) it is not difficult to see that under this embedding $\mathcal{H}'$ is one of the shifted copies $\mathcal{H}[n]$ of $\mathcal{H}$, implying the claim. □

5.2 Classification of indecomposables: Grothendieck's theorem

By the definition of degree, each simple sheaf S has degree one. Note that each line bundle L admits nonsplit exact sequences $0 \to L' \to L \to S \to 0$ and $0 \to L \to \bar{L} \to S \to 0$, allowing to construct line bundles of arbitrary degree.

Proposition 5.5 (*i*) *Each line bundle L is determined, up to isomorphism, by its degree d. (We write L_d in this case.)*

(*ii*) *We have* $\mathrm{Hom}(L_d, L_{d'}) \neq 0$ *if and only if* $d \leq d'$.

PROOF by (RR). (*i*) Let L and L' be line bundles with the same degree. By (RR) we have $\langle L, L' \rangle = 1$. It follows the existence of monomorphisms $L \hookrightarrow L'$ and similarly $L' \hookrightarrow L$. The composition $L \hookrightarrow L' \hookrightarrow L$ yields a nonzero member of the endomorphism ring $\mathrm{End}(L) = k$ which is an isomorphism. Hence also $L \hookrightarrow L'$ is an isomorphism.

(ii) If $\mathrm{Hom}(L_d, L_{d'})$ is nonzero we obtain an exact sequence $0 \to L_d \to L_{d'} \to C \to 0$, where C has finite length hence non-negative degree, and $d \leq d'$ follows. Conversely assume that $d \leq d'$. By (RR) $\langle [L_d], [L_{d'}] \rangle = (1-g) + \begin{vmatrix} 1 & 1 \\ d & d' \end{vmatrix} = 1 + (d' - d) > 0$. Hence $\mathrm{Hom}(L_d, L_{d'}) \neq 0$. □

The **key ingredient** for the homological classification of $\mathcal{H}$ is the next result with a surprisingly simple proof.

Proposition 5.6 *Let L be a line bundle. Then* $\deg \tau L = \deg L - 2$.

PROOF. Let $L = L_d$ and $\tau L_d = L_n$. We have to show that $n = d_2$. Now $\langle [L_d], [\tau L_d] \rangle = -\langle [L_d], [L_d] \rangle = -1$. Evaluating by (RR) the same expression yields $\langle [L_d], [L_n] \rangle = 1 + \begin{vmatrix} 1 & 1 \\ d & n \end{vmatrix}$, and $n = d - 2$ follows. □

Note: The number 2 appearing in the above proposition is the Euler characteristic of the projective line (or for $k = \mathbb{C}$ of the Riemann sphere).

Theorem 5.7 (Grothendieck) *Each indecomposable bundle E on the projective line is a line bundle.*

PROOF. Assume the assertion is false and choose a counterexample E from $\mathcal{H}_+$ having **minimal rank**. Next we choose a line bundle L_d contained in L and having **maximal degree**. (For the existence of d we may refer to a line bundle filtration of E.) By the assumptions E/L_d is a bundle and thus, having smaller rank, splits into a direct sum $L_{d_1} \oplus \cdots \oplus Ł_{d_t}$ of line bundles. By indecomposability of E the sequence

$$0 \to L_d \to E \to L_{d_1} \oplus \cdots \oplus L_{d_t} \to 0$$

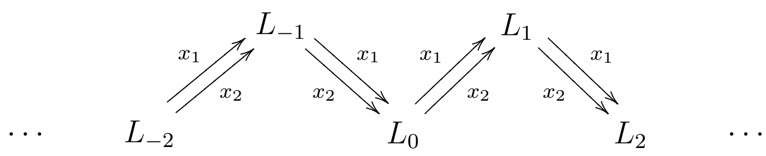
does not split, yielding a non-trivial extension $\mathrm{Ext}^1(L_{d_i}, L_d)$ for some i, hence by Serre duality a non-zero morphism $L_d \to \tau L_{d_i} = L_{d_i - 2}$. This implies $d \leq d_i - 1$.

Now, applying $\mathrm{Hom}(L_{d+1}, -)$ to the above sequence yields

$$0 \to \mathrm{Hom}(L_{d+1}, E) \to \bigoplus_{j=1}^{s} \mathrm{Hom}(L_{d+1}, L_{d_j}) \to \mathrm{Ext}^1(L_{d+1}, L_d).$$

By the previous proposition $\mathrm{Ext}^1(L_{d+1}, L_d) \cong \mathrm{D}\,\mathrm{Hom}(L_d, L_{d-1}) = 0$. Moreover $\mathrm{Hom}(L_{d+1}L_{d_i}, \neq)0$, showing that also $\mathrm{Hom}(L_{d+1}, E) \neq 0$ (use (RR) to see this!) and contradicting the choice of d. □

We thus get a complete picture of the category of indecomposable bundles as follows:



Next, we take another look at $\mathcal{H}_0$:

Proposition 5.8 *The simple objects of $\mathcal{H}$ are naturally parametrized by the points of the projective line $P_1(k) = k \cup \{\infty\}$.*

Proof. Let S be any simple object. In view of (H6) there is a short exact sequence $0 \to L' \to L_0 \to S \to 0$. Since S has degree 1, we conclude that $L' = L_{-1}$, so for each simple we obtain some nonzero $u \in \mathrm{Hom}(L_{-1}, L)$ such that S is isomorphic to the cokernel term of $0 \to L_{-1} \xrightarrow{u} L \to S_u \to 0$. Invoking the degree, each such cokernel has degree one and thus is simple. Moreover, S_u and S_v are isomorphic if and only if $u = \lambda v$ for some non-zero λ from k. Since $\mathrm{Hom}(L_{-1}, L)$ has dimension two, the claim follows. □

5.3 Tilting to Kronecker modules

We know already that $T = L_0 \oplus L_1$ is a tilting object in $\mathcal{H}$ whose endomorphism ring E is isomorphic to the Kronecker algebra. Using the notations of Theorem 3.1 we see that $\mathcal{H}_0$ belongs to $\mathcal{T}$ and, moreover, the indecomposable bundles of $\mathcal{T}$ are exactly the line bundles L_n with $n \geq 0$.

Similarly, it is easily checked that the indecomposables in $\mathcal{F}$ are exactly the line bundles L_m with $m < 0$. Hence, we have $\mathcal{H} = \mathcal{F} \vee \mathcal{T}$ and mod-$E \cong \mathcal{T} \vee \mathcal{F}[1]$. Under this identification reg-E corresponds exactly to $\mathcal{H}_0$, showing that reg-E and $\mathcal{H}_0$ are equivalent categories.

5.4 Various incarnations of $\mathrm{coh}(P_1(k))$

Many theories deal with the same mathematical object, cf. Schiffmann's *general nonsense principle: There are more theories than interesting mathematical objects.* Our case is no exceptions. In the mathematical literature you will find many incarnations of the category of coherent sheaves on the projective line. We just mention a few of them:

1. $\mathcal{H} = \mathrm{coh}(P_1(k))$, algebraic coherent sheaves.

2. $k = \mathbb{C}$, $\mathcal{H} = \mathrm{coh}(\mathbb{S}^2)$, analytic coherent sheaves.

3. $R = k[x_1, x_2]$, graded by total degree, $\mathcal{H} = \mathrm{mod}^{\mathbb{Z}}\text{-}R/\mathrm{mod}_0^{\mathbb{Z}}\text{-}R$, the quotient category of finitely presented $\mathbb{Z}$-graded R-modules modulo its Serre subcategory of finite length graded modules.

4. the pullback
$$\begin{array}{ccc} \mathrm{coh}(P_1(k)) & \longrightarrow & \mathrm{mod}\text{-}k[X] \\ \downarrow & & \downarrow \\ \mathrm{mod}\text{-}k[X^{-1}] & \longrightarrow & \mathrm{mod}\text{-}k[X, X^{-1}] \end{array}$$
in the category of small abelian categories.

5. $\mathrm{mod}\text{-}k[\circ \rightrightarrows \circ]$, if *derived equivalence* replaces equivalence.

There are other incarnations. All of the above, with the exception of the last one, however lead to the **same category** given by axioms (H1)–(H7). The theory developed here is just independent of the actual construction.

6 Weighted projective lines

6.1 Happel's theorem

We call two hereditary categories **derived-equivalent** if their derived categories $D^b(\mathcal{H})$ and $D^b(\mathcal{H}')$ are equivalent (as triangulated categories). An example of derived equivalence is given by the category $\mathrm{coh}(P_1(k))$ of coherent sheaves on the projective line and the category mod-$k[\circ \rightrightarrows \circ]$ of finite dimensional Kronecker modules. One of the two categories is a length category having a generating system of projectives (module-sense), the other is not a length category and has no nonzero projectives. So they are far from being equivalent. Nevertheless, being derived-equivalent they are closely related, in particular the classification problem for one of the categories is equivalent to the classification problem for the other. In particular, one could claim that Kronecker was classifying indecomposable vector bundles on the projective lines already 1884, a long time before Grothendieck proved the theorem carrying his name.

It is therefore natural to aim for a classification of hereditary categories — if wanted with further properties — up to derived equivalence. In general this task is difficult and only partial solutions are known. However there is a surprisingly simple answer if we **request additionally that our categories have a tilting object**. This result is due to Happel [5] and has a difficult proof.

Theorem 6.1 (Happel) *Assume k is an algebraically closed field. Then each connected Ext-finite* ***hereditary category $\mathcal{H}'$ with a tilting object*** *is derived equivalent to a hereditary category $\mathcal{H}$ of one of the following types:*

(i) $\mathcal{H}$ is the category mod-Λ *of* ***finite dimensional modules over a finite dimensional hereditary algebra*** Λ.

(ii) $\mathcal{H}$ is the category $\mathrm{coh}(\mathbb{X})$ *of* ***coherent sheaves on a weighted projective line*** $\mathbb{X}$. □

Weighted projective lines were introduced by Geigle-Lenzing in [4] in an attempt to understand the interaction between preprojective and regular modules for tame hereditary algebras (and also the classification of indecomposable modules over tubular algebras). Roughly speaking, a weighted projective line $\mathbb{X}$ is defined through the attached category $\mathrm{coh}(\mathbb{X})$ of coherent sheaves where $\mathrm{coh}(\mathbb{X})$ is a natural generalization of the category of coherent sheaves on the projective line, obtained by allowing a finite number of simple objects to have τ-period > 1. As in the situation studied previously the classification problem for indecomposables for $\mathcal{H} = \mathrm{coh}(\mathbb{X})$ mainly depends on the Euler characteristic $\chi_{\mathcal{H}}$ of $\mathcal{H}$ (or $\mathbb{X}$).

As in the case of the projective line and the category of modules over the Kronecker algebra, it may happen that a category $\mathrm{coh}(\mathbb{X})$, $\mathbb{X}$ a weighted projective line, and a category mod-Λ, Λ a connected finite dimensional hereditary algebra, are derived equivalent. Actually this is going to happen exactly for the **tame hereditary algebras** Λ and the **weighted projective lines of positive Euler characteristic**. An overview of the situation is given by the following picture. Note that the picture is up to derived equivalence:

	$\mathrm{coh}(\mathbb{X})$ Euler char < 0 wild	
	$\mathrm{coh}(\mathbb{X})$ Euler char 0 tame tubular	
mod-Λ Λ representation-finite	Λ tame hereditary mod-$\Lambda \sim_{der} \mathrm{coh}(\mathbb{X})$ Euler char > 0	mod-Λ Λ wild hereditary

6.2 Coherent sheaves on weighted projective lines

We collect the requirements (H1)–(H7) used to characterize coherent sheaves over the projective line, but now replace (H6) by a more general request:

(H1) *$\mathcal{H}$ is an abelian k-linear category.*

(H2) *$\mathcal{H}$ is (skeletally) small and **Hom-finite**, that is, all morphism spaces* $\mathrm{Hom}(X, Y)$ *from $\mathcal{H}$ are finite dimensional over k.*

(H3*) *(**Serre duality**) We assume the existence of an **equivalence** $\tau : \mathcal{H} \to \mathcal{H}$ and of natural isomorphisms*

$$\mathrm{Ext}^1(X, Y) \xrightarrow{\sim} \mathrm{D}\,\mathrm{Hom}(Y, \tau X)$$

for all objects X, Y from $\mathcal{H}$.

(H4) *$\mathcal{H}$ is noetherian but not every object of $\mathcal{H}$ has finite length.*

(H5) *There is an additive function* rk $: \mathcal{H} \to \mathbb{Z}$, *called* ***rank****, that is* τ*-stable, zero on* $\mathcal{H}_0$ *and* > 0 *on nonzero objects of* $\mathcal{H}_+$*. Moreover* $\mathcal{H}_+$ *admits an (indecomposable) object of rank one.*

(H6$_{new}$) *Each tube of* $\mathcal{H}_0$ *has only finitely many simple objects. Moreover, if* L *is a line bundle and* $\mathcal{U}$ *is a tube,* $\sum_{S \in \mathcal{U} \text{ simple}} \dim_{\mathrm{Hom}(E,S)} = 1$.

(H7) $\mathcal{H}$ *has a tilting object* T.

Since (H7) implies that the Grothendieck group $\mathrm{K}_0(\mathcal{H})$ is finitely generated free (on the classes of a representative system of indecomposable direct summands of T) there are only finitely many tubes having more than one simple object, equivalently having τ-period > 1.

Let $\mathcal{U}_1, \ldots, \mathcal{U}_t$ be the exceptional tubes having more than one simple object. We denote by p_i the number of simple objects from $\mathcal{U}_i$ and call $(p_1, \ldots, p_t)$ the **weight type** of $\mathcal{H}$.

The next assertion is taken from [10], where actually a much stronger version is shown.

Theorem 6.2 *A category is equivalent to a category of coherent sheaves on a weighted projective line if and only if it satisfies the above seven axioms.* □

In the sequel we will therefore say that a category $\mathcal{H}$ satisfying the preceding axioms is a **category of coherent sheaves on a weighted projective line**.

6.3 Rank, degree and Riemann-Roch-formula

As for coherent sheaves on the projective line, we have the bisection $\mathcal{H} = \mathcal{H}_+ \vee \mathcal{H}_0$. Further the rank function is τ-stable and zero exactly on the objects of finite length and is > 0 on nonzero objects of $\mathcal{H}_+$. Line bundles, with the same definition as before, play a key role also here. **We fix a line bundle** L_0 for the rest of the discussion.

Concerning degree, its main property in the previous discussion has been to be strictly positive on all simple objects. Since we allow more than one simple in one tube but only morphisms from L_0 to (exactly) one of them, the linear form $\langle [L_0], - \rangle$ is hence zero on the remaining ones. This is corrected by looking at an average

$$\langle\langle x, y \rangle\rangle = \frac{1}{p} \sum_{j=0}^{p-1} \langle \tau^j x, y \rangle \in \frac{1}{p}\mathbb{Z},$$

$x, y \in \mathrm{K}_0(\mathcal{H})$, of the Euler form. Here $p = \text{l.c.m.}(p_1, \ldots, p_t)$, where $(p_1, \ldots, p_t)$ is the weight type of $\mathcal{H}$.

As before we now introduce the **degree** $\deg : \mathrm{K}_0(\mathcal{H}) \to \mathbb{Z}$ by means of

$$\deg x = \langle\langle [L_0], x \rangle\rangle - \langle\langle [L_0], [L_0] \rangle\rangle \mathrm{rk}\, x.$$

Lemma 6.3 *The degree function* $\deg : \mathrm{K}_0(\mathcal{H}) \to \frac{1}{p}\mathbb{Z}$ *has the following properties:*

(i) $\deg L_0 = 0$.

(ii) *If S is simple of τ-period q, then* $\deg S = 1/q$. *In particular, the degree is τ-stable on $\mathcal{H}_0$ and we have* $\deg X > 0$ *for each nonzero object of* $\mathcal{H}_0$.

The degree, when restricted to $\mathcal{H}_0$, hence plays the role of a weighted length.

With the above definition of degree we have the same **Riemann-Roch formula** (RR)

$$\langle\langle x, y \rangle\rangle = (1 - g)\mathrm{rk}\, x\, \mathrm{rk}\, y + \begin{vmatrix} \mathrm{rk}\, x & \mathrm{rk}\, y \\ \deg x & \deg y \end{vmatrix}$$

as before, introducing the **genus** of $\mathcal{H}$ by $1 - g = \langle\langle [L_0], [L_0] \rangle\rangle$. Finally the **Euler characteristic** is given as

$$\chi_{\mathcal{H}} = 2\langle\langle [L_0], [L_0] \rangle\rangle.$$

Proposition 6.4 *Let $(p_1, \ldots, p_t)$ be the weight type of $\mathcal{H}$. Then the Euler characteristic of $\mathcal{H}$ is given as*

$$\chi_{\mathcal{H}} = 2 - \sum_{i=1}^{t} (1 - 1/p_i).$$

□

Concerning line bundles in general, we have the following:

Proposition 6.5 *Let L be any line bundle in $\mathcal{H}$. Then the following holds*

(i) *L is exceptional, hence* $\mathrm{End}(L) = k$ *and* $\mathrm{Ext}^1(L, L) = 0$.

(ii) *We have* $\deg \tau L = \deg L - \chi_{\mathcal{H}}$. □

Note that a line bundle L is uniquely determined by its class $[L]$ in $\mathrm{K}_0(\mathcal{H})$ but is no longer determined by its degree. However, only finitely many nonisomorphic line bundles can have the same degree.

To summarize: everything here is analogous to the study of coherent sheaves on a projective line, with obvious modification necessary by the inserted weights. In particular, in (RR) $\langle\langle -,-\rangle\rangle$ takes the role of the Euler form $\langle -,-\rangle$. However the control through the averaged Euler form is less strict; so $\langle\langle [X],[Y]\rangle\rangle > 0$ no longer implies that $\mathrm{Hom}(Y,Y) \neq 0$.

Remark 6.6 (1) Note that the number 2 in the for $\chi_{\mathcal{H}}$ is the Euler characteristic for the projective line. Each inserted weight p_i further yields a negative correction term $-(1-1/p_i)$. In the literature this expression for $\chi_{\mathcal{H}}$ is known as *orbifold Euler characteristic*. It appears in many mathematical contexts: function theory, differential geometry, group actions of discrete groups, ...

(2) As for a category of sheaves over a smooth projective curve the Euler characteristic (accordingly the genus) is an important homological invariant of $\mathcal{H}$. It's role is to reflect the representation type of $\mathcal{H}$.

(*i*) If $\chi_{\mathcal{H}} > 0$, equivalently if the star formed by the weights is a **Dynkin diagram** Δ, then the classification problem for $\mathcal{H}$ is very close to the case of $\mathrm{coh}(P_1(k))$. Namely, the indecomposable bundles form a single AR-component whose quiver is of type $\mathbb{Z}\bar{\Delta}$, where $\bar{\Delta}$ is the **extended Dynkin diagram** of Δ. We say in this case that $\mathcal{H}$ is of **domestic type**. To summarize, we are in the domestic type if the weight type of $\mathcal{H}$ is of the form $()$, (p), (p,q) or $(2,2,n)$, $(2,3,3)$, $(2,3,4)$ or $(2,3,5)$.

(*ii*) If $\chi_{\mathcal{H}} = 0$, equivalently if $g_{\mathcal{H}} = 1$, then it is still possible to classify all indecomposables, but this is much more difficult. All AR-components turn out to be tubes we say then that $\mathcal{H}$ has **tubular type**. This happens exactly for the weight types $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$ and $(2,3,6)$.

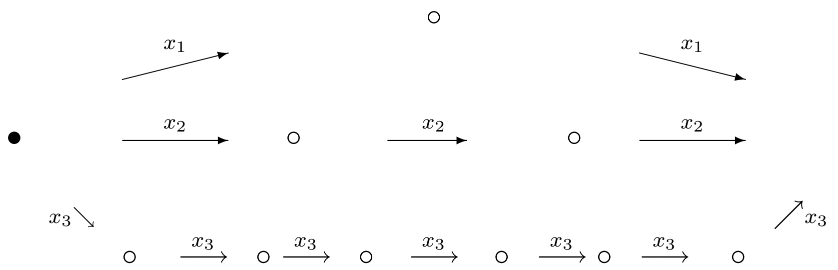
The same classification method works for $\mathcal{H}$ as for coherent sheaves on elliptic curves, a problem first solved by Atiyah [1]. Up to derived equivalence the classification problem for $\mathcal{H}$ was solved for module categories by Ringel [11].

(*iii*) In all the remaining cases where $\chi_{\mathcal{H}} < 0$, the classification problem for $\mathcal{H}$ is wild. For further information we refer to [8].

6.4 Canonical shape of tilting object

It is quite exceptional that a normal form for a tilting module and its endomorphism algebra exists. This is the case for the categories $\mathcal{H}$ of coherent sheaves on a weighted projective line. In the case of trivial weights, the projective line, we have seen that the Kronecker quiver appears as the endomorphism algebra of a tilting object, and it is not difficult to see that this is the only possibility (when dealing with a multiplicity-free tilting module). Recall that there we obtained a tilting object starting with a line bundle L and extending this line bundle by a simple object.

In the weighted projective case one proceeds in a similar fashion, taking a little care. Starting with a line bundle L, we extend L for each tube with all the simples in that tube exactly once. The result is the canonical configuration, here shown for the weight type $(2, 3, 7)$



with relations $x_1^2 + x_2^3 + x_3^7 = 0$.

Theorem 6.7 *Let $\mathcal{H}$ denote a category of coherent sheaves on a weighted projective line. Then $\mathcal{H}$ has a a tilting object T that is a direct sum of line bundles and whose endomorphism ring is a canonical algebra $\Lambda(\mathbf{p}, \underline{\lambda})$.* $\square$

Here, $\mathbf{p} = (p_1, \ldots, p_t)$ denotes the weight typ of $\mathcal{H}$ while $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ denotes pairwise different points of the projective line $P_1(k)$ normalized as follows: $\lambda_1 = \infty$, $\lambda_2 = 0$, $\lambda_3 = 1$.

Remark 6.8 (1) It is best to view a canonical algebra as a generalization of the Kronecker algebra: Look at the full subcategory of the path category consisting of the starting and terminal vertex yielding $k[\circ \rightrightarrows \circ]$.

(2) Λ has global dimension ≤ 2, where equality occurs if and only if there are at most two weights.

(3) The derived categories $\mathrm{D}^b(\mathcal{H})$ and $\mathrm{D}^b(\text{mod-}\Lambda)$ are equivalent (as triangulated categories). If compared, $\mathcal{H}$ has the advantage to be a hereditary category. Also the homological classification machinery, used for $\mathcal{H}$, works less efficient for mod-Λ.

6.5 The main classification tool: stability

For each nonzero object X of $\mathcal{H}$ we have a well-defined **slope** $\mu\, X = \deg X/\mathrm{rk}\, X$ which is in $\mathbb{Q} \cup \{\infty\}$. A nonzero bundle E is called **stable** (resp. **semistable**) if for each proper nonzero subobject E' of E we have $\mu\, E' < \mu\, E$ (resp. $\mu\, E' \leq \mu\, E$). By definition, the zero object is also semistable.

Proposition 6.9 *Let $q \in \mathbb{Q}$, then the full subcategory $\mathcal{H}^{(q)}$ of all semistable objects of slope q is an exact abelian, subcategory which is closed under extensions.*

Moreover, each object of X has finite length in $\mathcal{H}^{(q)}$ bounded by the rank of X. Further the simple objects in $\mathcal{H}^{(q)}$ are exactly the stable objects of slope q.

Proof. (1) Write $q = pd/pr$ with integers d and r. Then $\lambda = p\, r \deg - p\, d \mathrm{rk}$ is an integral linear form on $\mathrm{K}_0(\mathcal{H})$. The full subcategory controlled by λ consists exactly of of the semistable bundles of slope q, hence by Proposition 1.3 is an extension closed exact subcategory and hence a hereditary category.

(2) It remains to show that each object E in $\mathcal{H}^{(q)}$ has finite length bounded by the rank of E. Indeed consider a proper chain of subobjects $0 = E_0 \subsetneqq E_1 \subsetneqq \cdots \subsetneqq E_n = E$ in $\mathcal{H}^{(q)}$. Since all E_n have the same slope q, the corresponding ranks are strictly increasing, showing the claim. □

Accordingly, $\mathcal{H}^{(q)}$ is a hereditary, Hom-finite length category in its own right. Note that this procedure produces many hereditary categories, usually not connected and not having Serre duality. By Schur's lemma the next result immediately follows. □

Corollary 6.10 *Each stable bundle has trivial endomorphism ring k.*

The next proposition is the major basis for classification for non-negative Euler characteristic.

Proposition 6.11 *Assume $\chi_{\mathcal{H}} \geq 0$. Then each indecomposable bundle is* ***semistable****. If, moreover, $\chi_{\mathcal{H}} > 0$, then each indecomposable bundle E is* ***stable and exceptional****.*

PROOF. We are going to sketch the argument. Assume first that $\chi_{\mathcal{H}} \geq 0$ and E is an indecomposable bundle. We then look for subobjects of E having maximal slope (use a line bundle filtration for E to see existence), and among those choose one E_{ss} having maximal rank. Clearly, E_{ss} is semistable and actually uniquely determined, hence called the **maximal semistable subbundle of** E. Invoking $\chi_{\mathcal{H}} \geq 0$ and Proposition 6.5 one next shows that the sequence $0 \to E_{ss} \to E \to E/E_{ss} \to 0$ splits, hence $E = E_{ss}$ is semistable.

Next assume that $\chi_{\mathcal{H}} > 0$. Assume E is an indecomposable bundle of slope q. By the first part E is semistable, hence has finite length in $\mathcal{H}^{(q)}$. It is clear from the definition that the simple objects of $\mathcal{H}^{(q)}$ are exactly the stable ones of the same slope. So there is a stable subobject S of E having the same slope. We claim that $E = S$. Otherwise the sequence $0 \to S \to E \to E/S \to 0$ is nonsplit, yielding a nonzero morphism $\tau^{-1}S \to E/S$, implying $\tau^{-1}S \subseteq E/S$. Invoking $\chi_{\mathcal{H}} > 0$ this yields a subobject of E having slope $> q$, contradicting semistability of E. $\square$

6.6 The domestic case

A category $\mathcal{H}$ of coherent sheaves on a weighted projective line is called **domestic** if $\chi_{\mathcal{H}} > 0$, meaning that

$$2 - \sum_{i=1}^{t}(1 - \frac{1}{p_i}) > 0.$$

This happens only if $t \leq 3$ and the weight type is one of $()$, (p), (p,q), $(2,2,n)$ $(n \geq 2)$, $(2,3,3)$, $(2,3,4)$ or $(2,3,5)$, that is, exactly if the weight type is given by a Dynkin diagram of type A-D-E.

Theorem 6.12 *Assume $\chi_{\mathcal{H}} > 0$ and the weight type of $\mathcal{H}$ is given by the Dynkin diagram Δ. Then the following holds:*

(i) The indecomposable bundles form a single Auslander-Reiten component consisting of exceptional objects. The quiver $\Gamma_{\mathcal{H}_+}$ has the form $\mathbb{Z}\bar{\Delta}$, where $\bar{\Delta}$ is the extended Dynkin diagram corresponding to Δ.

(ii) The direct sum T of a representative system of indecomposable bundles E with slope $0 \leq \mu E < \chi_{\mathcal{H}}$ is a tilting object of $\mathcal{H}$ whose endomorphism ring is isomorphic to the path algebra of $\bar{\Delta}$.

PROOF as for the case $\mathrm{coh}(P_1(k))$ where L_0 and L_1 are the only indecomposable bundles with a slope q in the range $0 \leq q < \chi_{\mathcal{H}} = 2$. □

Corollary 6.13 *Each category $\mathcal{H} = \mathrm{coh}(\mathbb{X})$ with positive Euler characteristic is derived equivalent to the category* mod-Λ*, where Λ is a connected tame hereditary algebra, and conversely.* □

By tilting from $\mathcal{H}$ to mod-Λ we get complete information on mod-Λ. In particular:

Corollary 6.14 *Let Λ be a connected tame hereditary algebra. Then the category* mod-Λ *has a trisection*

$$\text{mod-}\Lambda = \text{prep-}\Lambda \vee \text{reg-}\Lambda \vee \text{prinj-}\Lambda$$

into the preprojective, regular and preinjective Λ modules, where the indecomposables of prep-Λ *and* prinj-Λ*, respectively, form components of $\Gamma_{\mathcal{H}}$, and* reg-Λ *is a category equivalent to $\mathcal{H}_0$.* □

6.7 Euler characteristic zero: the tubular case

We now assume that $\mathcal{H}$ is a category of coherent sheaves on a weighted projective line of Euler characteristic zero. That is, we deal with one of the weight types $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$ or $(2,3,6)$. We have already seen that each indecomposable bundle E is semistable of a rational slope q, such that $E \in \mathcal{H}^{(q)}$. We start with a preliminary result, already showing clearly what is going to be expected here.

Proposition 6.15 *Assume that $\chi_{\mathcal{H}} = 0$. Then each $\mathcal{H}^{(q)}$ is preserved under τ, accordingly decomposes into a coproduct $\coprod_{x \in X} \mathcal{U}_x$ of connected uniserial categories, whose quivers are stable tubes of finite τ-period.*

PROOF. We have already seen that $\mathcal{H}^{(q)}$ is a Hom-finite length category. Since $\mathcal{H}^{(q)}$ is also stable under τ, we additionally have Serre duality. Hence $\mathcal{H}^{(q)}$ decomposes into tubes by Gabriel's theorem. That the τ-period of these tubes is finite, and only a finite number of them have more than one simple object, follows from the fact that the Grothendieck group $\mathrm{K}_0(\Lambda)$ is finitely generated. □

It is convenient to attach slope infinity to the objects from $\mathcal{H}_0$, hence $\mathcal{H}^{(\infty)} = \mathcal{H}_0$ (Convention: the zero object takes any slope).

Theorem 6.16 *Assume that $\mathcal{H}$ is a category of coherent sheaves on a weighted projective line of Euler characteristic zero, equivalently, that the weight type of $\mathcal{H}$ is one of $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$ or $(2,3,6)$. Then the following holds:*

(i) $\mathcal{H} = \bigvee_{q\in\mathbb{Q}\cup\{\infty\}} \mathcal{H}^{(q)}$.

(ii) We have a nonzero morphism from $\mathcal{H}^{(q)}$ to $\mathcal{H}^{(r)}$ if and only if $q \leq r$.

(iii) Each $\mathcal{H}^{(q)}$ decomposes into a one-parameter family of tubes, indexed by the projective line. Moreover, $\mathcal{H}^{(q)}$ is equivalent to $\mathcal{H}_0$.

PROOF. *(i)* is implied by semistability of indecomposables.

(ii) Let E be from $\mathcal{H}^{(q)}$ and F be from $\mathcal{H}^{(r)}$.

(a) If $u : E \to F$ is nonzero, let I be the image of u. Then $q = \mu\, E \leq \mu\, I$ since E is semistable and $\mu\, I \leq \mu\, F = r$ since F is semistable. Hence $q \leq r$.

(b) Assume that $q < r$, then

$$\langle\langle [E], [F] \rangle\rangle = \begin{vmatrix} \mathrm{rk}\, E & \mathrm{rk}\, F \\ \deg E & \deg F \end{vmatrix} = \mathrm{rk}\, E\, \mathrm{rk}\, F(\mu\, F - \mu\, E) > 0.$$

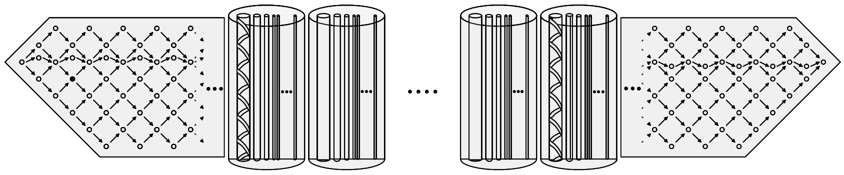
Hence we get a nonzero morphism from the τ-orbit of E to F.

(iii) (a) There exist two automorphisms of $\mathrm{D}^b(\mathcal{H})$ generating an action of the braid group B_3 on $\mathrm{D}^b(\mathcal{H})$ that is transitive on the slopes. It follows that each $\mathcal{H}^{(q)}$ is equivalent to $\mathcal{H}_0$.

(b) Let S be a simple object from $\mathcal{H}$ with $\tau S \cong S$, there is a nonsplit exact sequence $0 \to L \to \bar{L} \to S \to S$, where $\bar{L}$ is also a line bundle and $\mathrm{Hom}(L, \bar{L})$ is two-dimensional over k. One checks that for each nonzero $u : L \to \bar{L}$ the cokernel term of the arising sequence $0 \to L \xrightarrow{u} \bar{L} \to C_u \to 0$ is indecomposable of finite length hence belongs to a tube $\mathcal{T}_u$ of $\mathcal{H}_0$. It is then easily checked that each tube of $\mathcal{H}_0$ arises this way and that u and v yield the same tube if and only if $v \in k^* u$, yielding the one-parameter family over $P_1(k)$ we were looking for. □

Definition 6.17 *A finite dimensional algebra A is called* ***tubular*** *if A is isomorphic to the endomorphism algebra of a tilting object on a category of coherent sheaves $\mathcal{H}$ on a weighted projective of Euler characteristic zero.*

Let A be a tubular algebra. It is straightforward to derive the shape of the module category mod-A from the known structure of $\mathcal{H}$. We are not giving a formal statement here, but instead visualize the shape of the resulting Auslander-Reiten quiver Γ_A:



Therefore, we have a preprojective component (sitting on the left), a preinjective component (sitting on the right) and in between a rational family of 1-parameter families of tubes, where a finite number of them may contain projectives (left hand side) or injectives (right hand sie), and all remaining ones are stable tubes. In particular, this applies to mod-Λ for the module category over a canonical algebra of tubular type.

Remark 6.18 (i) The first classification of indecomposable modules over tubular algebras was achieved by Ringel [11].

(ii) The classification for the coherent sheaves on weighted projective lines of Euler characteristic zero was done by Lenzing-Meltzer [7].

(iii) The classification method for coherent sheaves on weighted projective lines can also be applied to the classification of indecomposable bundles on elliptic curves. This classification — with slightly different methods — was first achieved by Atiyah [1].

References

[1] M. F. Atiyah. Vector bundles over an elliptic curve. *Proc. London Math. Soc.*, **7** (1957), 414–452.

[2] S. Brenner and M.C.R. Butler: Generalizations of the Bernstein-Gel'fand-Ponomarev reflection functors, in: Representation theory II, *Lecture Notes in Math.*, **832**, 103–169, Springer, Berlin-New York, 1980.

[3] P. Gabriel. Indecomposable representations II. *Symposia Mat. Inst. Naz. Alta Mat.* **11**, 1973, 81–104.

[4] W. Geigle and H. Lenzing. A class of weighted projective curves arising in representation theory of finite dimensional algebras. In: Singularities, representations of algebras, and vector bundles, *Lecture Notes Math.* **1273**, Springer 1987, 265–297.

[5] D. Happel. A characterization of hereditary categories with tilting object. *Invent. math.* **144**, 381–398, 2001.

[6] D. Happel and C.M. Ringel: Tilted algebras. *Trans. Amer Math. Soc.* **274** (1982), 399–443.

[7] H. Lenzing and H. Meltzer. Sheaves on a weighted projective line of genus one, and representations of a tubular algebra. *Can. Math. Soc. Conf. Proc.* **14**, 313-337 (1993).

[8] H. Lenzing and J.A. de la Peña. Wild canonical algebras. *Math. Z.* **224** (1997), 403–425.

[9] I. Reiten and M. Van den Bergh. Noetherian hereditary categories satisfying Serre duality, *J. Amer. Math. Soc.* **15** (2002), 295–366.

[10] H. Lenzing. Hereditary noetherian categories with a tilting complex, *Proc. Am. Math. Soc.* **125** (1997), 1893-1901.

[11] C.M. Ringel. *Tame algebras and integral quadratic forms.* Lect. Notes Math. 1099 (1984).