The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization

International Atomic Energy Agency

SMR1735/7

Advanced School and Conference on Representation Theory and Related Topics

(9 - 27 January 2006)

Notes on the Gabriel-Roiter Measure

Henning Krause
Universität Paderborn
Fakultät für Elektrotecnik, Informatik und Mathematik
Institut für Mathematik
Paderborn, Germany

NOTES ON THE GABRIEL-ROITER MEASURE

HENNING KRAUSE

1. Chains and length functions

1.1. **The lexicographic order on finite chains.** Let $(S, \leqslant)$ be a partially ordered set. A subset $X \subseteq S$ is a *chain* if $x_1 \leqslant x_2$ or $x_2 \leqslant x_1$ for each pair $x_1, x_2 \in X$. For a finite chain X, we denote by $\min X$ its minimal and by $\max X$ its maximal element, using the convention

$$\max \emptyset < x < \min \emptyset \quad \text{for all} \quad x \in S.$$

We write $\mathrm{Ch}(S)$ for the set of all finite chains in S and consider on $\mathrm{Ch}(S)$ the *lexicographic order* which is defined by

$$X \leqslant Y \quad :\Longleftrightarrow \quad \min(Y \setminus X) \leqslant \min(X \setminus Y)$$

for $X, Y \in \mathrm{Ch}(S)$.

Remark. (1) We have $X \leqslant Y$ if $X \subseteq Y$.

(2) Suppose that S is totally ordered. Then $\mathrm{Ch}(S)$ is totally ordered. We may think of $X \in \mathrm{Ch}(S) \subseteq \{0,1\}^S$ as a string of 0s and 1s which is indexed by the elements in S. The usual lexicographic order on such strings coincides with the lexicographic order on $\mathrm{Ch}(\mathbb{N})$.

Example. Let $\mathbb{N} = \{1, 2, 3, \cdots\}$ and $\mathbb{Q}$ be the set of rational numbers together with the natural ordering. Then the map

$$\mathrm{Ch}(\mathbb{N}) \longrightarrow \mathbb{Q}, \quad X \mapsto \sum_{x \in X} 2^{-x}$$

is injective and order preserving, taking values in the interval $[2^{-1}, 1]$.

1.2. **Length functions.** Let $(S, \leqslant)$ be a partially ordered set. A *length function* on S is by definition a map $\lambda \colon S \to \mathbb{N}$ such that $x < y$ in S implies $\lambda(x) < \lambda(y)$. A length function $\lambda \colon S \to \mathbb{N}$ induces the following *chain length function*

$$S \longrightarrow \mathrm{Ch}(\mathbb{N}), \quad x \mapsto \lambda^*(x) := \max\{\lambda(X) \mid X \in \mathrm{Ch}(S, x)\},$$

where $\mathrm{Ch}(S, x) = \{X \in \mathrm{Ch}(S) \mid \max X = x\}$.

1.3. **Basic properties.** Let $\lambda \colon S \to \mathbb{N}$ be a length function and $\lambda^* \colon S \to \mathrm{Ch}(\mathbb{N})$ the induced chain length function. We formulate some basic observations and collect a list of properties (C0) – (C5) of λ^*.

Let $x \in S$ and note that $\max \lambda^*(x) = \lambda(x)$. For $X \in \mathrm{Ch}(\mathbb{N})$, we have

$$X \setminus \{\max X\} = \max\{X' \in \mathrm{Ch}(\mathbb{N}) \mid X' < X \text{ and } \max X' < \max X\},$$

and therefore

Preliminary version from January 7, 2006.

(C0) $\lambda^*(x) = \max_{x'<x} \lambda^*(x') \cup \{\lambda(x)\}$.

This shows that the function $\lambda^*\colon S \to \mathrm{Ch}(\mathbb{N})$ can be defined by induction on the length of the elements in S. Next we state some basic properties which suggest to think of λ^* as a refinement of λ.

Proposition. *Let $x, y \in S$.*

(C1) *$x \leqslant y$ implies $\lambda^*(x) \leqslant \lambda^*(y)$.*
(C2) *$\lambda^*(x) = \lambda^*(y)$ implies $\lambda(x) = \lambda(y)$.*
(C3) *$\lambda^*(x') < \lambda^*(y)$ for all $x' < x$ and $\lambda(x) \geqslant \lambda(y)$ imply $\lambda^*(x) \leqslant \lambda^*(y)$.*

Proof. Suppose $x \leqslant y$ and let $X \in \mathrm{Ch}(S, x)$. Then $Y = X \cup \{y\} \in \mathrm{Ch}(S, y)$ and we have $\lambda(X) \leqslant \lambda(Y)$ since $\lambda(X) \subseteq \lambda(Y)$. Thus $\lambda^*(x) \leqslant \lambda^*(y)$. If $\lambda^*(x) = \lambda^*(y)$, then

$$\lambda(x) = \max \lambda^*(x) = \max \lambda^*(y) = \lambda(y).$$

To prove (C3), we use (C0) and apply the following lemma with $X = \lambda^*(x)$ and $Y = \lambda^*(y)$. □

Lemma. *Let $X, Y \in \mathrm{Ch}(\mathbb{N})$. If $X' := X \setminus \{\max X\} < Y$ and $\max X \geqslant \max Y$, then $X \leqslant Y$.*

Proof. The assumption $X' < Y$ implies by definition

$$\min Y \setminus X' < \min X' \setminus Y.$$

We consider two cases. Suppose first that $X' \subseteq Y$. If $X \subseteq Y$, then $X \leqslant Y$. Otherwise,

$$\min Y \setminus X < \max X = \min X \setminus Y$$

and therefore $X < Y$. Now suppose that $X' \not\subseteq Y$. We use again that $\max X \geqslant \max Y$, exclude the case $Y \subseteq X$, and obtain

$$\min Y \setminus X = \min Y \setminus X' < \min X' \setminus Y = \min X \setminus Y.$$

Thus $X \leqslant Y$ and the proof is complete. □

We state some further elementary properties of the map λ^*.

Proposition. *Let $x, y \in S$.*

(C4) *$\lambda^*(x) \leqslant \lambda^*(y)$ or $\lambda^*(x) \geqslant \lambda^*(y)$.*
(C5) *$\{\lambda^*(x) \mid x \in S$ and $\lambda(x) \leqslant n\}$ is finite for all $n \in \mathbb{N}$.*

Proof. (C4) is clear since $\mathrm{Ch}(\mathbb{N})$ is totally ordered. (C5) follows from the fact that $\{X \in \mathrm{Ch}(\mathbb{N}) \mid \max X \leqslant n\}$ is finite for all $n \in \mathbb{N}$. □

1.4. **A recursive definition.** Let $\lambda\colon S \to \mathbb{N}$ be a length function. The function $\lambda^*\colon S \to \mathrm{Ch}(\mathbb{N})$ can be defined by induction on the length of the elements in S because of the formula (C0). This observation suggests the following recursive definition which avoids any reference to $\mathrm{Ch}(\mathbb{N})$. We define a surjective map $\mu\colon S \to S/\lambda^*$ and a partial order on S/λ^*. More precisely, we provide an equivalence relation on S such that S/λ^* denotes the set of equivalence classes and $\mu(x)$ denotes the equivalence class of each $x \in S$. The definition of μ is done by induction, that is, in step $n \geqslant 1$ we define $\mu(x)$ and the relation $\mu(x) \leqslant \mu(y)$ for all $x, y \in S$ of length at most n as follows:

(1) If x or y is minimal, then

$$\mu(x) \leqslant \mu(y) \quad :\Longleftrightarrow \quad \min_{x' \leqslant x} \lambda(x') \geqslant \min_{y' \leqslant y} \lambda(y').$$

(2) If x and y both are not minimal, then

$$\mu(x) = \mu(y) \quad :\Longleftrightarrow \quad \max_{x'<x} \mu(x') = \max_{y'<y} \mu(y') \text{ and } \lambda(x) = \lambda(y);$$

$$\mu(x) < \mu(y) \quad :\Longleftrightarrow \quad \begin{cases} \max_{x'<x} \mu(x') < \max_{y'<y} \mu(y'), \text{ or} \\ \max_{x'<x} \mu(x') = \max_{y'<y} \mu(y') \text{ and } \lambda(x) > \lambda(y). \end{cases}$$

Note that $\max_{x'<x} \mu(x')$ exists for each $x \in S$ because the set $\{\mu(y) \mid y \in S, \lambda(y) \leqslant n\}$ is finite for all n.

Proposition. *Let $\lambda \colon S \to \mathbb{N}$ be a length function. Then there exists an injective map $\nu \colon S/\lambda^* \to \mathrm{Ch}(\mathbb{N})$ making the following diagram commutative.*

$$\begin{array}{ccc} S & \xrightarrow{\lambda} & \mathbb{N} \\ {\scriptstyle\mu}\downarrow & \searrow^{\lambda^*} & \uparrow{\scriptstyle\max} \\ S/\lambda^* & \xrightarrow{\nu} & \mathrm{Ch}(\mathbb{N}) \end{array}$$

Moreover, $\nu(x) \leqslant \nu(y)$ if and only if $x \leqslant y$ for all $x, y \in S/\lambda^*$.

Proof. First observe, using the equation (C0), that λ^* satisfies the defining relations of μ. Now define $\nu(\mu(x)) = \lambda^*(x)$ for $\mu(x) \in S/\lambda^*$. The map ν is well-defined and injective because λ^* and μ satisfy the same relations. □

1.5. **An axiomatic definition.**

Proposition. *Let $\lambda \colon S \to \mathbb{N}$ be a length function. The induced chain length function $\lambda^* \colon S \to \mathrm{Ch}(\mathbb{N})$ is the universal map $\mu \colon S \to P$ into a partially ordered set P satisfying for all $x, y \in S$ the following:*

(P1) *$x \leqslant y$ implies $\mu(x) \leqslant \mu(y)$.*
(P2) *$\mu(x) = \mu(y)$ implies $\lambda(x) = \lambda(y)$.*
(P3) *$\mu(x') < \mu(y)$ for all $x' < x$ and $\lambda(x) \geqslant \lambda(y)$ imply $\mu(x) \leqslant \mu(y)$.*

More precisely, for any such map μ we have

$$\mu(x) \leqslant \mu(y) \quad \Longleftrightarrow \quad \lambda^*(x) \leqslant \lambda^*(y) \quad \textit{for all} \quad x, y \in S.$$

Proof. We have seen in (1.3) that λ^* satisfies (P1) – (P3). So it remains to show that for any map $\mu \colon S \to P$ into a partially ordered set P, the conditions (P1) – (P3) uniquely determine the relation $\mu(x) \leqslant \mu(y)$ for any pair $x, y \in S$. In fact, we claim that (P1) – (P3) imply $\mu(x) \leqslant \mu(y)$ or $\mu(x) \geqslant \mu(y)$. We proceed by induction on the length of the elements in S. For elements of length $n = 1$, the assertion is clear. Now let $n > 1$ and assume the assertion is true for all elements $x \in S$ of length $\lambda(x) < n$. We choose for each $x \in S$ of length $\lambda(x) \leqslant n$ a *Gabriel-Roiter filtration*, that is, a sequence

$$x_1 < x_2 < \ldots < x_{\gamma(x)-1} < x_{\gamma(x)} = x$$

in S such that x_1 is minimal and $\max_{x'<x_i} \mu(x') = \mu(x_{i-1})$ for all $1 < i \leqslant \gamma(x)$. Such a filtration exists because the elements $\mu(x')$ with $x' < x$ are totally ordered. Now fix $x, y \in S$ of length at most n and let $I = \{i \geqslant 1 \mid \mu(x_i) = \mu(y_i)\}$. We consider $r = \max I$ and put $r = 0$ if $I = \emptyset$. There are two possible cases. Suppose first that $r = \gamma(x)$ or $r = \gamma(y)$. If $r = \gamma(x)$, then $\mu(x) = \mu(x_r) = \mu(y_r) \leqslant \mu(y)$ by (P1). Now suppose $\gamma(x) \neq r \neq \gamma(y)$. Then we have $\lambda(x_{r+1}) \neq \lambda(y_{r+1})$ by (P2) and

(P3). If $\lambda(x_{r+1}) > \lambda(y_{r+1})$, then we obtain $\mu(x_{r+1}) < \mu(y_{r+1})$, again using (P2) and (P3). Iterating this argument, we get $\mu(x) = \mu(x_{\gamma(x)}) < \mu(y_{r+1})$. From (P1) we get $\mu(x) < \mu(y_{r+1}) \leqslant \mu(y)$. Thus $\mu(x) \leqslant \mu(y)$ or $\mu(x) \geqslant \mu(y)$ and the proof is complete. □

2. Abelian length categories

2.1. **Additive categories.** A category $\mathcal{A}$ is *additive* if every finite family $X_1, X_2, \ldots, X_n$ of objects has a coproduct

$$X_1 \oplus X_2 \oplus \ldots \oplus X_n,$$

each set $\operatorname{Hom}_{\mathcal{A}}(A, B)$ is an abelian group, and the composition maps

$$\operatorname{Hom}_{\mathcal{A}}(A, B) \times \operatorname{Hom}_{\mathcal{A}}(B, C) \longrightarrow \operatorname{Hom}_{\mathcal{A}}(A, C)$$

are bilinear.

2.2. **Abelian categories.** An additive category $\mathcal{A}$ is *abelian*, if every map $\phi\colon A \to B$ has a kernel and a cokernel, and if the canonical factorization

$$\begin{array}{ccccccc} \operatorname{Ker}\phi & \xrightarrow{\phi'} & A & \xrightarrow{\phi} & B & \xrightarrow{\phi''} & \operatorname{Coker}\phi \\ & & \downarrow & & \uparrow & & \\ & & \operatorname{Coker}\phi' & \xrightarrow{\bar{\phi}} & \operatorname{Ker}\phi'' & & \end{array}$$

of ϕ induces an isomorphism $\bar{\phi}$.

Example. The category $\operatorname{Mod}\Lambda$ of (right) modules over an associative ring Λ is an abelian category.

2.3. **Subobjects.** Let $\mathcal{A}$ be an abelian category. We say that two monomorphisms $X_1 \to X$ and $X_2 \to X$ are *equivalent*, if there exists an isomorphism $X_1 \to X_2$ making the following diagram commutative.

$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

An equivalence class of monomorphisms into X is called a *subobject* of X. Given subobjects $X_1 \to X$ and $X_2 \to X$, we write $X_1 \subseteq X_2$ if there is a morphism $X_1 \to X_2$ making the above diagram commutative. An object $X \neq 0$ is *simple* if $X' \subseteq X$ implies $X' = 0$ or $X' = X$.

2.4. **Length categories.** Let $\mathcal{A}$ be an abelian category. An object X has *finite length* if it has a finite composition series

$$0 = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_{n-1} \subseteq X_n = X$$

(i.e. each X_i/X_{i-1} is simple). In this case the length of a composition series is an invariant of X by the Jordan-Hölder Theorem; it is called the *length* of X and is denoted by $\ell(X)$. Note that X has finite length if and only if X is both artinian (i.e. satisfies the descending chain condition on subobjects) and noetherian (i.e. satisfies the ascending chain condition on subobjects).

Definition. An abelian category is called a *length category* if all objects have finite length and if the isomorphism classes of objects form a set.

An object $X \neq 0$ is called *indecomposable* if $X = X_1 \oplus X_2$ implies $X_1 = 0$ or $X_2 = 0$. A finite length object admits a finite direct sum decomposition into indecomposable objects having local endomorphism rings. Moreover, such a decomposition is unique up to an isomorphism by the Krull-Remak-Schmidt Theorem.

We denote by $\operatorname{ind}\mathcal{A}$ the set of isomorphism classes of indecomposable objects of $\mathcal{A}$.

Example. (1) Let Λ be a right artinian ring. Then the category of finitely generated Λ-modules form a length category which we denote by $\operatorname{mod}\Lambda$.

(2) Let k be a field and Q be any quiver. Then the finite dimensional k-linear representations of Q form a length category.

3. The Gabriel-Roiter measure

3.1. **The definition.** Let $\mathcal{A}$ be an abelian length category. The isomorphism classes of objects of $\mathcal{A}$ are partially ordered as follows:

$$X \leqslant Y \quad :\Longleftrightarrow \quad \text{there exists a monomorphism } X \to Y.$$

We consider the length function $\ell\colon \operatorname{ind}\mathcal{A} \to \mathbb{N}$ and the induced map $\ell^*\colon \operatorname{ind}\mathcal{A} \to \mathrm{Ch}(\mathbb{N})$ is by definition the *Gabriel-Roiter measure* of $\mathcal{A}$. We will not work with this definition but take instead the properties (C1) – (C5) of λ^*. Thus we think of the Gabriel-Roiter measure as a map $\mu\colon \operatorname{ind}\mathcal{A} \to P$ into a partially ordered set P, specifying only the relation $\mu(X) \leqslant \mu(Y)$ for pairs $X, Y \in \operatorname{ind}\mathcal{A}$.

Next we establish further properties (C6) – (C8) of the Gabriel-Roiter measure which depend on the fact that $\mathcal{A}$ is a length category.

3.2. **Gabriel-Roiter filtrations.** A sequence $X_1 < X_2 < \ldots < X_n = X$ in $\operatorname{ind}\mathcal{A}$ is called a *Gabriel-Roiter filtration* of X if X_1 is minimal and $\max_{X' < X_i} \mu(X') = \mu(X_{i-1})$ for all $1 < i \leqslant n$. Clearly, each X admits such a filtration and the values $\mu(X_i)$ are uniquely determined by X. Note that X_1 is a simple object. Moreover, the value $\mu(X_1)$ is minimal among all values $\mu(Y)$.

Proposition. *Let $X, Y \in \operatorname{ind}\mathcal{A}$.*

(C6) *$X \in \operatorname{ind}\mathcal{A}$ is simple if and only if $\mu(X) \leqslant \mu(Y)$ for all $Y \in \operatorname{ind}\mathcal{A}$.*

(C7) *Suppose that $\mu(X) < \mu(Y)$. Then there are $Y' < Y'' \leqslant Y$ in $\operatorname{ind}\mathcal{A}$ such that*

$$\mu(Y') = \max_{U < Y''} \mu(U) \leqslant \mu(X) < \mu(Y'') \quad \text{and} \quad \ell(Y') \leqslant \ell(X).$$

Proof. For (C6), one uses that each indecomposable object has a simple subobject. To prove (C7), fix a Gabriel-Roiter filtration $Y_1 < Y_2 < \ldots < Y_n = Y$ of Y. We have $\mu(Y_1) \leqslant \mu(X)$ because Y_1 is simple and find therefore some i such that $\mu(Y_i) \leqslant \mu(X) < \mu(Y_{i+1})$. Now put $Y' = Y_i$ and $Y'' = Y_{i+1}$. □

3.3. **The main property.**

Proposition (Gabriel)**.** *Let $X, Y_1, \ldots, Y_r \in \operatorname{ind}\mathcal{A}$.*

(C8) *Suppose that $X \subseteq \oplus_{i=1}^r Y_i$. Then $\mu(X) \leqslant \max \mu(Y_i)$ and X is a direct summand if $\mu(X) = \max \mu(Y_i)$.*

Proof. The proof uses only the properties (C1) – (C3) of μ. Fix a monomorphism $\phi\colon X \to Y = \oplus_i Y_i$. We proceed by induction on $n = \ell(X) + \ell(Y)$. First observe that $\mu(X)$ is minimal if and only if X is simple. Thus the case $\ell(X) = 1$ or $n \leqslant 2$ is clear. Now suppose $n > 2$. We can assume that for each i the ith component $\phi_i \colon X \to Y_i$ of ϕ is an epimorphism. Otherwise choose for each i a decomposition $Y_i' = \oplus_j Y_{ij}$ of the image of ϕ_i into indecomposables. Then we use (C1) and have $\mu(X) \leqslant \max \mu(Y_{ij}) \leqslant \max \mu(Y_i)$ because $\ell(X) + \ell(Y') < n$ and $Y_{ij} \leqslant Y_i$ for all j. Now suppose that each ϕ_i is an epimorphism. Thus $\ell(X) \geqslant \ell(Y_i)$ for all i. Let $X' \subset X$ be a proper indecomposable subobject. Then $\mu(X') \leqslant \max \mu(Y_i)$ because $\ell(X') + \ell(Y) < n$, and X' is a direct summand if $\mu(X') = \max \mu(Y_i)$. We can exclude the case that $\mu(X') = \max \mu(Y_i)$ because then X' is a proper direct summand of X, which is impossible. Now we apply (C3) and obtain $\mu(X) \leqslant \max \mu(Y_i)$. Finally, suppose that $\mu(X) = \max \mu(Y_i) = \mu(Y_k)$ for some k. We claim that we can choose k such that ϕ_k is an epimorphism. Otherwise, replace all Y_i with $\mu(X) = \mu(Y_i)$ by the image $Y_i' = \oplus_j Y_{ij}$ of ϕ_i as before. We obtain $\mu(X) \leqslant \max \mu(Y_{ij}) < \mu(Y_k)$ since $Y_{kj} < Y_k$ for all j, using (C1) and (C2). This is a contradiction. Thus ϕ_k is an epimorphism and in fact an isomorphism because $\ell(X) = \ell(Y_k)$ by (C2). In particular, X is a direct summand of $\oplus_i Y_i$. This completes the proof. □

Corollary. *Let $X, Y \in \operatorname{ind} \mathcal{A}$ and suppose that $X \subset Y$ with $\mu(X) = \max_{Y' < Y} \mu(Y')$. If $X \subseteq U \subset Y$ in $\mathcal{A}$, then X is a direct summand of U.*

Proof. Let $U = \oplus_i U_i$ be a decomposition into indecomposables. Now apply (C8). We obtain $\mu(X) \leqslant \max \mu(U_i) \leqslant \mu(Y)$ and our assumption on $X \subset Y$ implies that X is a direct summand of U. □

3.4. **Gabriel-Roiter inclusions.**

Proposition (Ringel). *Let $X, Y \in \operatorname{ind} \mathcal{A}$ and suppose that $X \subset Y$ with $\mu(X) = \max_{Y' < Y} \mu(Y')$. Then Y/X is an indecomposable object.*

Proof. Let $Z = Y/X$ and denote by $\pi \colon Y \to Z$ the canonical map. Now assume that $Z = Z' \oplus Z''$ and $Z'' \neq 0$. Then we have $X \subseteq \pi^{-1}(Z') \subset Y$ and therefore the inclusion $X \to \pi^{-1}(Z')$ splits by Corollary 3.3. Thus the inclusion $Z' \to Z$ factors through π via a map $Z' \to Y$. This map is a split monomorphism and therefore $Z' = 0$. □

Corollary. *Let Y be an indecomposable object in $\mathcal{A}$ which is not simple. Then there exists a short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$ such that X and Z are indecomposable.*

Proof. Take $X \subset Y$ with $\mu(X) = \max_{Y' < Y} \mu(Y')$. □

4. Finiteness results

4.1. **Covariant finiteness.** A subcategory $\mathcal{C}$ of $\mathcal{A}$ is called *covariantly finite* if every object $X \in \mathcal{A}$ admits a *left $\mathcal{C}$-approximation*, that is, a map $X \to Y$ with $Y \in \mathcal{C}$ such that the induced map $\operatorname{Hom}_{\mathcal{A}}(Y, C) \to \operatorname{Hom}_{\mathcal{A}}(X, C)$ is surjective for all $C \in \mathcal{C}$. We have also the dual notion: a subcategory $\mathcal{C}$ is *contravariantly finite* if every object in $\mathcal{A}$ admits a *right $\mathcal{C}$-approximation.*

Lemma. *Let $\mathcal{C}$ be a subcategory of $\mathcal{A}$ which is closed under taking subobjects. Then $\mathcal{C}$ is a covariantly finite subcategory of $\mathcal{A}$.*

Proof. Fix $X \in \mathcal{A}$. Let $X' \subseteq X$ be minimal among the kernels of all maps $X \to Y$ with $Y \in \mathcal{C}$. Then the canonical map $X \to X/X'$ is a left $\mathcal{C}$-approximation. □

Remark. The proof shows that the inclusion functor $\mathcal{C} \to \mathcal{A}$ admits a left adjoint $F\colon \mathcal{A} \to \mathcal{C}$ which takes $X \in \mathcal{A}$ to X/X'. Note that the adjunction map $X \to FX$ is a left $\mathcal{C}$-approximation.

Let M be any set of values $\mu(X)$. Then we define the subcategory

$$\mathcal{A}(M) := \{\oplus_i X_i \in \mathcal{A} \mid \mu(X_i) \in M \text{ for all } i\}.$$

Proposition. *Let M be a set of values $\mu(X)$ which is closed under predecessors, that is, $\mu(X) \leqslant \mu(Y) \in M$ implies $\mu(X) \in M$. Then $\mathcal{A}(M)$ is a covariantly finite subcategory of $\mathcal{A}$.*

Proof. The subcategory $\mathcal{A}(M)$ is closed under taking subobjects by (C8). □

4.2. **Immediate successors.**

Lemma. *Let $X, Y \in \operatorname{ind} \mathcal{A}$ and suppose that $X < Y$ with $\mu(X) = \max_{Y' < Y} \mu(Y')$. If $X \to \bar{X}$ is a left almost split map in $\mathcal{A}$, then Y is a factor object of $\bar{X}$.*

Proof. The monomorphism $X \to Y$ factors through $X \to \bar{X}$ via a map $\phi\colon \bar{X} \to Y$. Let U be the image of ϕ. Applying Corollary 3.3, we find that $U = Y$. □

Proposition. *Suppose that each object in $\operatorname{ind} \mathcal{A}$ admits a left almost split map. Let $X \in \operatorname{ind} \mathcal{A}$ such that $\{Y \in \operatorname{ind} \mathcal{A} \mid \mu(Y) \leqslant \mu(X)\}$ is finite. Then there exists a minimal element in*

$$\{\mu(Y) \mid Y \in \operatorname{ind} \mathcal{A} \text{ and } \mu(X) < \mu(Y)\}$$

provided the set is not empty.

Proof. We fix for each $Y \in \operatorname{ind} \mathcal{A}$ a left almost split map $Y \to \bar{Y}$. Let $n = \ell(\bar{Y})$ be the maximal value such that $\mu(Y) \leqslant \mu(X)$. Now let $\mu(Y) > \mu(X)$. We apply (C7) and find $Y' < Y'' \leqslant Y$ in $\operatorname{ind} \mathcal{A}$ such that $\mu(Y') \leqslant \mu(X) < \mu(Y'') \leqslant \mu(Y)$ and $\mu(Y') = \max_{U < Y''} \mu(U)$. The preceding lemma implies $\ell(Y'') \leqslant n$, and (C5) implies that the number of values $\mu(Y'')$ is finite. Thus there exists a minimal element among those $\mu(Y'')$. □

Remark. The assumption on X that $\{Y \in \operatorname{ind} \mathcal{A} \mid \mu(Y) \leqslant \mu(X)\}$ is finite can be removed provided we have a bound $n \in \mathbb{N}$ such that $\ell(Y) \leqslant \ell(X)$ implies the existence of a left almost split map $Y \to \bar{Y}$ with $\ell(\bar{Y}) \leqslant n$. (Such a bound exists for instance when $\mathcal{A} = \operatorname{mod} \Lambda$ for an artin algebra Λ.)

4.3. **A finiteness criterion.**

Proposition. *Let $\mathcal{A}$ be a length category with left almost split maps and only finitely many isomorphism classes of simple objects. Suppose that $\mathcal{C}$ is an additive subcategory such that*

(1) *$\mathcal{C}$ is covariantly finite in $\mathcal{A}$, and*
(2) *there exists $n \in \mathbb{N}$ such that $\ell(X) \leqslant n$ for all indecomposable $X \in \mathcal{C}$.*

Then there are only finitely many isomorphism classes of indecomposable objects in $\mathcal{C}$.

Proof. We claim that we can construct all indecomposable objects $X \in \mathcal{C}$ in at most 2^n steps from the finitely many simple objects in $\mathcal{A}$ as follows. Choose a non-zero map $S \to X$ from a simple object S and factor this map through the left $\mathcal{C}$-approximation $S \to S'$. Take an indecomposable direct summand X_0 of S' such that the component

$S \to X_0 \to X$ is non-zero. Stop if $X_0 \to X$ is an isomorphism. Otherwise take a left almost split map $X_0 \to Y_0$ and a left $\mathcal{C}$-approximation $Y_0 \to Z_0$. The map $X_0 \to X$ factors through the composition $X_0 \to Y_0 \to Z_0$ and we choose an indecomposable direct summand X_1 of Z_0 such that the component $X_0 \to Y_0 \to X_1 \to X$ is non-zero. Again, we stop if $X_1 \to X$ is an isomorphism. Otherwise, we continue as before and obtain in step r a sequence of non-invertible maps

$$X_0 \to X_1 \to X_2 \to \ldots \to X_r$$

such that the compsition is non-zero. The Harada-Sai Lemma implies that $r < 2^n$ because $\ell(X_i) \leqslant n$ for all i by our assumption. Thus X is isomorphic to X_i for some $i < 2^n$, and we obtain X in at most 2^n steps, having in each step only finitely many choices by taking an indecomposable direct summand. We conclude that $\mathcal{C}$ has only a finite number of indecomposable objects. □

4.4. **The initial segment.**

Theorem (Ringel). *Let $\mathcal{A}$ be a length category such that* $\operatorname{ind}\mathcal{A}$ *is infinite. Suppose also that $\mathcal{A}$ has only finitely many isomorphism classes of simple objects and that every indecomposable object admits a left almost split map. Then there exist infinitely many values $\mu(X_1) < \mu(X_2) < \mu(X_3) < \ldots$ of the Gabriel-Roiter measure of $\mathcal{A}$ having the following properties.*

(1) *If $\mu(X) \neq \mu(X_i)$ for all i, then $\mu(X_i) < \mu(X)$ for all i.*
(2) *The set $\{X \in \operatorname{ind}\mathcal{A} \mid \mu(X) = \mu(X_i)\}$ is finite for all i.*

Proof. We construct the values $\mu(X_i)$ by induction as follows. Take for X_1 any simple object. Observe that $\mu(X_1)$ is minimal among all $\mu(X)$ by (C6) and that only finitely many $X \in \operatorname{ind}\mathcal{A}$ satisfy $\mu(X) = \mu(X_1)$ because $\mathcal{A}$ has only finitely many simple objects. Now suppose that $\mu(X_1) < \ldots < \mu(X_n)$ have been constructed, satisfying the conditions (1) and (2) for all $1 \leqslant i \leqslant n$. We can apply Proposition 4.2 and find an immediate successor $\mu(X_{n+1})$ of $\mu(X_n)$. It remains to show that the set $\{X \in \operatorname{ind}\mathcal{A} \mid \mu(X) = \mu(X_{n+1})\}$ is finite. To this end consider $M = \{\mu(X_1), \ldots, \mu(X_{n+1})\}$. We know from Proposition 4.1 that $\mathcal{A}(M)$ is a covariantly finite subcategory. Clearly, $\ell(X)$ is bounded by $\max\{\ell(X_i) \mid 1 \leqslant i \leqslant n+1\}$ for all indecomposable $X \in \mathcal{A}(M)$. We conclude from Proposition 4.3 that the number of indecomposables in $\mathcal{A}(M)$ is finite. Thus $\{X \in \operatorname{ind}\mathcal{A} \mid \mu(X) = \mu(X_{n+1})\}$ is finite and the proof is complete. □

Corollary (Brauer-Thrall I). *Let $\mathcal{A}$ be a length category satisfying the above conditions. Then for every $n \in \mathbb{N}$ there exists an indecomposable object $X \in \mathcal{A}$ with $\ell(X) > n$.*

Proof. Use that for fixed $n \in \mathbb{N}$, there are only finitely many values $\mu(X)$ with $\ell(X) \leqslant n$, by (C5). □

4.5. **The terminal segment.**

Theorem (Ringel). *Let $\mathcal{A}$ be a length category such that* $\operatorname{ind}\mathcal{A}$ *is infinite. Suppose also that $\mathcal{A}$ has a cogenerator (i.e. an object Q such that each object in $\mathcal{A}$ admits a monomorphism into a direct sum of copies of Q) and that every indecomposable object admits a right almost split map. Then there exist infinitely many values $\mu(X^1) > \mu(X^2) > \mu(X^3) > \ldots$ of the Gabriel-Roiter measure of $\mathcal{A}$ having the following properties.*

(1) *If $\mu(X) \neq \mu(X^i)$ for all i, then $\mu(X^i) > \mu(X)$ for all i.*

(2) *The set* $\{X \in \operatorname{ind} \mathcal{A} \mid \mu(X) = \mu(X^i)\}$ *is finite for all* i.

The proof is based on the following lemma.

Lemma. *Let* $\mathcal{A}$ *be a length category such that every indecomposable object admits a right almost split map. Let* $X \in \mathcal{A}$ *and denote by* $\mathcal{A}_X$ *the subcategory formed by all objects in* $\mathcal{A}$ *having no indecomposable direct summand which is isomorphic to a direct summand of* X. *Then* $\mathcal{A}_X$ *is a contravariantly finite subcategory of* $\mathcal{A}$.

Proof. See Proposition 3.13 in [1]. □

Proof of the theorem. We construct the values $\mu(X^i)$ by induction as follows. Take for X^1 any indecomposable direct summand X of Q such that $\mu(X)$ is maximal. Observe that $\mu(X^1)$ is maximal among all $\mu(X)$ with $X \in \operatorname{ind} \mathcal{A}$ by (C8) and that only finitely many $X \in \operatorname{ind} \mathcal{A}$ satisfy $\mu(X) = \mu(X^1)$ because Q has only finitely many indecomposable direct summands. Now suppose that $\mu(X^1) > \ldots > \mu(X^n)$ have been constructed, satisfying the conditions (1) and (2) for all $1 \leqslant i \leqslant n$. Denote by P the direct sum of all $X \in \operatorname{ind} \mathcal{A}$ with $\mu(X) \geqslant \mu(X^n)$ and let $P' \to P \oplus Q$ be a right $\mathcal{A}_P$-approximation. Now take for X^{n+1} any indecomposable direct summand X of P' such that $\mu(X)$ is maximal. Observe that any indecomposable object $X \in \mathcal{A}_P$ is cogenerated by $P \oplus Q$ and therefore by P'. Thus (C8) implies that $\mu(X)$ is bounded by $\mu(X^{n+1})$. Moreover, if $\mu(X) = \mu(X^{n+1})$, then X is isomorphic to a direct summand of P'. Thus $\{X \in \operatorname{ind} \mathcal{A} \mid \mu(X) = \mu(X^{n+1})\}$ is finite and the proof is complete. □

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Henning Krause, Institut für Mathematik, Universität Paderborn, 33095 Paderborn, Germany.

E-mail address: `hkrause@math.upb.de`