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Representation of signals in a combined domain

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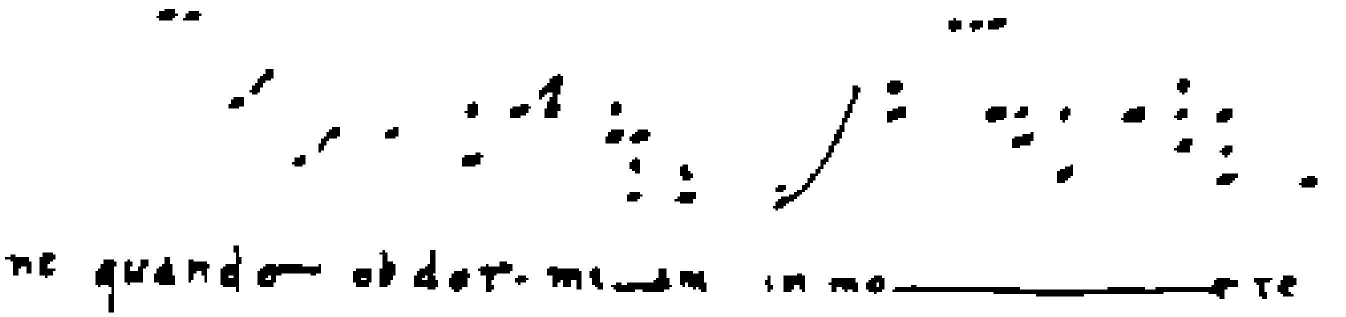
Representation of signals in a combined domain

Examples of mixed signal descriptions (in a combined domain):

music	tone	time	-	frequency
optics	ray	position	-	direction
radar	pulse	time delay	-	Doppler shift
mechanics	particle	position	-	momentum

Mixed signal descriptions have been with us for a long, long time!



Paris, B.N. 776, Gregorian manuscript from Albi, before 1079

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Some basic references

- Franz Hlawatsch and G. Faye Boudreaux-Bartels,
  "Linear and quadratic time-frequency signal representations,"
  IEEE Signal Processing Magazine, vol. 9, nr. 2, pp. 21–67 (1992).

- Leon Cohen,
  "Time-frequency distributions – A review,"
  Proc. IEEE, vol. 77, nr. 7, pp. 941–981 (1989).

- Boualem Boashash,
  "Introduction to the concepts of TFSAP,"
  Part I of Boualem Boashash, ed., Time Frequency Signal Analysis and Processing: A Comprehensive reference, ISBN 0-08-044335-4 (Oxford, Elsevier, 2003).

Linear, shift-invariant system – convolution

selective property of $\delta(t)$ $\qquad \varphi(t) = \int \delta(t-\tau)\,\varphi(\tau)\,\mathrm{d}\tau$

$$\begin{array}{lrcl}
\text{impulse response} & \delta(t) & \longrightarrow & h(t) \\
\text{shift invariance} & \delta(t-\tau) & \longrightarrow & h(t-\tau) \\
\text{linearity} & \delta(t-\tau)\,\varphi_i(\tau)\,\mathrm{d}\tau & \longrightarrow & h(t-\tau)\,\varphi_i(\tau)\,\mathrm{d}\tau \\
\text{linearity} & \int \delta(t-\tau)\,\varphi_i(\tau)\,\mathrm{d}\tau & \longrightarrow & \int h(t-\tau)\,\varphi_i(\tau)\,\mathrm{d}\tau \\
\text{selective property} & \varphi_i(t) & \longrightarrow & \int h(t-\tau)\,\varphi_i(\tau)\,\mathrm{d}\tau = \varphi_o(t)
\end{array}$$

convolution integral

$$\varphi_o(t) = \int h(t-\tau)\,\varphi_i(\tau)\,\mathrm{d}\tau = \int h(\tau)\,\varphi_i(t-\tau)\,\mathrm{d}\tau = h(t) * \varphi_i(t)$$

Auto- and cross-correlation function

Cross-correlation $\Gamma_{\varphi\psi}(\tau) = \int \varphi(t+\tau)\,\psi^*(t)\,\mathrm{d}t = \varphi(\tau) * \psi^*(-\tau)$

Auto-correlation $\Gamma_{\varphi\varphi}(\tau) = \int \varphi(t+\tau)\,\varphi^*(t)\,\mathrm{d}t$

Compare with the convolution $\int \varphi(\tau - t)\,\psi^*(t)\,\mathrm{d}t = \varphi(\tau) * \psi^*(\tau)\,.$

In particular we have $\Gamma_{\varphi\varphi}(0) = \int |\varphi(t)|^2\,\mathrm{d}t$ **= energy.**

For the cross-correlation we have the relation $\Gamma_{\varphi\psi}(\tau) = \Gamma^*_{\psi\varphi}(-\tau)$**, which for the auto-correlation reduces to conjugate (or Hermitian) symmetry:**

$$\Gamma_{\varphi\varphi}(\tau) = \Gamma^*_{\varphi\varphi}(-\tau)\,.$$

Fourier transform – amplitude and power spectrum

The Fourier transform of $\varphi(t)$ reads

$$\varphi(t) \circ\!\!-\!\!\bullet \int \varphi(t)\, e^{-i\omega t}\, \mathrm{d}t = \bar{\varphi}(\omega)\,.$$

For the convolution and the correlation we have

$$\varphi_o(t) = h(t) * \varphi_i(t) = \int h(t-\tau)\, \varphi_i(\tau)\, \mathrm{d}\tau \;\circ\!\!-\!\!\bullet\; \bar{h}(\omega)\, \bar{\varphi}_i(\omega) = \bar{\varphi}_o(\omega)$$

$$\Gamma_{\varphi\psi}(\tau) = \varphi(\tau) * \psi^*(-\tau) = \int \varphi(t+\tau)\, \psi^*(t)\, \mathrm{d}t \;\circ\!\!-\!\!\bullet\; \bar{\varphi}(\omega)\, \bar{\psi}^*(\omega) = \bar{\Gamma}_{\varphi\psi}(\omega)$$

$$\Gamma_{\varphi\varphi}(\tau) = \varphi(\tau) * \varphi^*(-\tau) = \int \varphi(t+\tau)\, \varphi^*(t)\, \mathrm{d}t \;\circ\!\!-\!\!\bullet\; \bar{\varphi}(\omega)\, \bar{\varphi}^*(\omega) = \bar{\Gamma}_{\varphi\varphi}(\omega)$$

While $\bar{\varphi}(\omega)$ is known as the amplitude spectrum (or frequency spectrum, or simply: spectrum), $\bar{\Gamma}_{\varphi\varphi}(\omega)$ is known as the power spectrum.

Outline of the presentation

1. Linear signal dependence
    - Windowed (short-time) Fourier transform
    - Gabor expansion
    - Wavelet transform
2. Quadratic (bilinear) signal dependence
    - Wigner distribution
    - Application to partially coherent light
3. Relatives of the Wigner distribution
    - Ambiguity function
    - Cohen class – kernel design
    - Fractional Fourier transform

Part 1. Linear signal dependence

This part deals with the windowed Fourier transform and its sampled version (also known as the Gabor transform) and the inverse of the latter: Gabor's signal expansion. The relation to filter banks and sub-band coding is observed, as well as the relation to the wavelet transform. Several sampling strategies in the combined (time-frequency) domain are considered.

- M.J. Bastiaans, "Gabor's expansion and the Zak transform for continuous-time and discrete-time signals: critical sampling and rational oversampling," EUT Report, 95-E-295, ISBN 90-6144-295-8; TUE, Eindhoven, 1995, pp. 1–62.
- M.J. Bastiaans, "Gabor's signal expansion based on a non-orthogonal sampling geometry," in Optical Information Processing: A Tribute to Adolf Lohmann, ISBN 0-8194-4498-7, ed. H.J. Caulfield; SPIE - The International Society for Optical Engineering, Bellingham, WA, 2002, pp. 57-82.
- A.J. van Leest, "Non-separable Gabor schemes: their design and implementation," PhD thesis, Technische Universiteit Eindhoven, Eindhoven, Netherlands, 2001.

Outline of part 1

- Windowed Fourier transform
- Gabor's signal expansion
- Determining the Gabor coefficients – Gabor transform
- Transform pair – Bi-orthogonality condition
- Fourier transform and Zak transform
- Product forms
- Interpretation of the Gabor coefficients
- Detour to wavelets and the wavelet transform
- Oversampling
  - rational oversampling
  - integer oversampling
- Optical setup to generate the Gabor coefficients
- Non-orthogonal sampling/tiling geometry
  - Sub-lattices
  - Shearing

Windowed Fourier transform

- Choose a window function $w(t)$.
- Multiply the signal $\varphi(t)$ with the shifted and complex conjugated version $w^*(t-\tau)$ of the window function, to get the product $\varphi(t)\, w^*(t-\tau)$.
- Take the Fourier transform of the product $\varphi(t)\, w^*(t-\tau)$:

$$s(\tau,\omega) = \int \varphi(t)\, w^*(t-\tau)\, e^{-i\omega t}\, \mathrm{d}t\,.$$

The windowed Fourier transform $s(\tau,\omega)$ is a function of two variables, τ and ω, and describes the signal in a time-frequency domain.

Inverse operation

$$s(\tau,\omega) = \int \varphi(t)\, w^*(t-\tau)\, e^{-i\omega t}\, \mathrm{d}t$$

Inverse transformation is possible in many ways, for instance

$$\varphi(t)\, w^*(0) = \frac{1}{2\pi} \int s(t,\omega)\, e^{i\omega t}\, \mathrm{d}\omega$$

or, much nicer,

$$\varphi(t) \int |w(t)|^2 dt = \frac{1}{2\pi} \iint s(\tau,\omega)\, w(t-\tau)\, e^{i\omega t}\, \mathrm{d}\tau \mathrm{d}\omega\,.$$

The function $s(\tau,\omega)$ acts as a distribution function – a local frequency spectrum – and shows how the signal $\varphi(t)$ can be synthesized from the (τ,ω)-parameterized set of shifted and modulated versions of $w(t)$.

Some remarks about the windowed FT

$$s(\tau, \omega) = \int \varphi(t)\, w^*(t - \tau)\, e^{-i\omega t}\, \mathrm{d}t$$

- The windowed FT $s(\tau, \omega)$ is a function of two variables τ and ω, derived from a function $\varphi(t)$ of one variable t.
- Not every function of two variables is a proper windowed Fourier transform.
- A windowed FT has many internal constraints, but which?
- Reconstruction of $\varphi(t)$ from $s(\tau, \omega)$ is possible in many ways, and should be possible if we have only partial knowledge of $s(\tau, \omega)$, but how?

Gabor's signal expansion

In 1946 Gabor introduced the expansion of a time signal $\varphi(t)$ into a discrete set of shifted and modulated versions of a Gaussian-shaped elementary signal $g(t) = 2^{\frac{1}{4}} e^{-\pi(t/\sigma_t)^2}$ [with FT $= 2^{\frac{1}{4}} \sigma_t e^{-\pi(\omega/\sigma_\omega)^2}$],

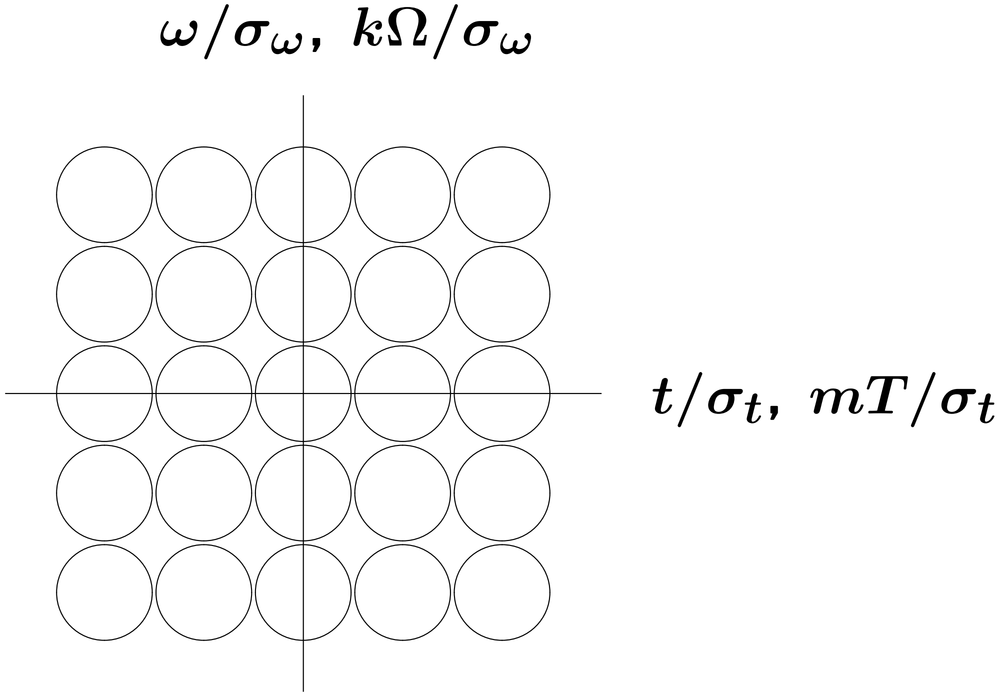
$$\varphi(t) = \sum_{mk} a_{mk}\, g_{mk}(t) = \sum_{mk} a_{mk}\, g(t - mT)\, e^{ik\Omega t},$$

in which the time shift T and the frequency shift Ω satisfy the relationship $\Omega T = 2\pi$, and also $\sigma_t/T = \sigma_\omega/\Omega$ (proportionality condition).

The product of the width of a function and the width of its Fourier transform has a lower bound equal to 2π – Heisenberg's inequality – and this lower bound is only reached for a Gaussian-shaped function.

Information theory – logon – degrees of freedom

Gabor's rectangular sampling/tiling geometry



A rectangular tiling of the time-frequency plane with time-shifted and frequency-shifted (i.e. modulated) Gaussian functions.

$$\int_{-\infty}^{\infty} g(t+\tfrac{1}{2}t')\, g^*(t-\tfrac{1}{2}t')\, e^{-i\omega t'}\,\mathrm{d}t' = 2\,\sigma_t\, e^{-2\pi[(t/\sigma_t)^2 + (\omega/\sigma_\omega)^2]}$$

Gabor's expansion coefficients

$$\varphi(t) = \sum_{mk} a_{mk}\, g_{mk}(t) = \sum_{mk} a_{mk}\, g(t - mT)\, e^{ik\Omega t}$$

The shifted and modulated elementary signals are located on the Gabor lattice $(\tau = mT, \omega = k\Omega)$ in the time-frequency domain.

Question: How can the expansion coefficients a_{mk} be determined? Note that the set of shifted and modulated elementary signals is not orthogonal, and hence

$$a_{mk} \neq \frac{1}{T} \int \varphi(t)\, g^*_{mk}(t)\, \mathrm{d}t = \frac{1}{T} \int \varphi(t)\, g^*(t - mT)\, e^{-ik\Omega t}\, \mathrm{d}t\, .$$

By the way: Gabor's signal expansion need not be limited to a Gaussian-shaped elementary signal, not to the special case $\Omega T = 2\pi$, and not to the rectangular sampling geometry $(\tau = mT, \omega = k\Omega)$.

Comparison of two signal descriptions

Reconstruction from the windowed Fourier transform: continuous

$$\varphi(t) \int |w(t)|^2 \, \mathrm{d}t \;=\; \frac{1}{2\pi} \iint s(\tau, \omega)\, w(t-\tau)\, e^{i\omega t} \, \mathrm{d}\tau \mathrm{d}\omega$$

$$s(\tau, \omega) \;=\; \int \varphi(t)\, w^*(t-\tau)\, e^{-i\omega t} \, \mathrm{d}t$$

Gabor's signal expansion: discrete

$$\varphi(t) \;=\; \sum_{mk} a_{mk}\, g(t - mT)\, e^{ik\Omega t}$$

$$a_{mk} \;=\; ? \;=\; \int \varphi(t)\, w^*(t - mT)\, e^{-ik\Omega t} \, \mathrm{d}t$$

Transform pair – the Gabor transform

Can, for a given elementary signal (or synthesis window) $g(t)$, an analysis window $w(t)$ be found, such that Gabor's signal expansion,

$$\varphi(t) = \sum_{mk} a_{mk}\, g_{mk}(t) = \sum_{mk} a_{mk}\, g(t - mT)\, e^{ik\Omega t},$$

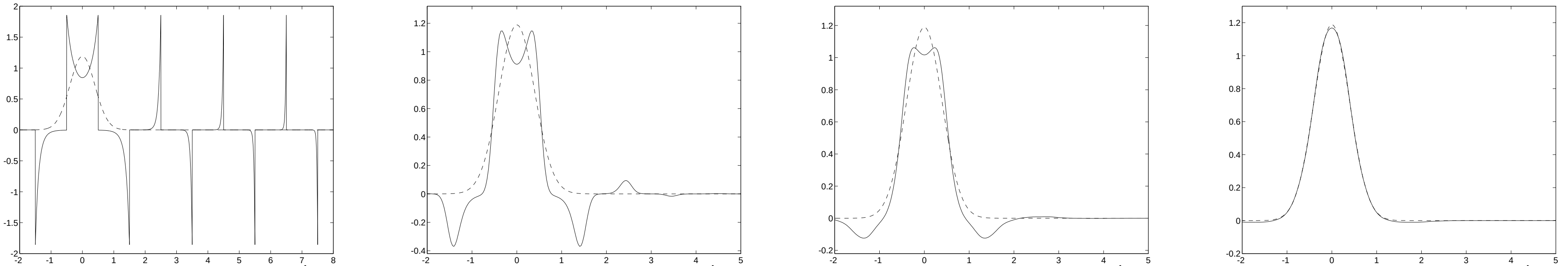
and the sampling values of the windowed FT at the Gabor lattice,

$$a_{mk} = \int \varphi(t)\, w^*_{mk}(t)\, \mathrm{d}t = \int \varphi(t)\, w^*(t - mT)\, e^{-ik\Omega t}\, \mathrm{d}t = s(mT, k\Omega)\,,$$

form a transform pair?

Yes! $$\sum_{mk} g_{mk}(t_1)\, w^*_{mk}(t_2) = \delta(t_1 - t_2)$$ Bi-orthogonality condition

Synthesis and corresponding analysis windows



(a) t/σ_t (b) t/σ_t (c) t/σ_t (d) t/σ_t

A Gaussian synthesis window $g(t) = 2^{1/4}\,\exp[-\pi(t/\sigma_t)^2]$ (dashed line) and its corresponding optimum analysis window $(T/q)\,w(t)$ (solid line), for different values of rational oversampling $2\pi/\Omega T = p/q$, while maintaining the proportionality condition $\sigma_t/T = \sigma_\omega/\Omega = \sqrt{2\pi/\Omega T}$: (a) no oversampling $2\pi/\Omega T = 1$, (b) $2\pi/\Omega T = 7/6$, (c) $2\pi/\Omega T = 3/2$, and (d) $2\pi/\Omega T = 3$

Fourier transform of Gabor's expansion coefficients

$$a_{mk} = \int \varphi(t)\, w_{mk}^*(t)\,\mathrm{d}t = \int \varphi(t)\, w^*(t - mT)\, e^{-ik\Omega t}\,\mathrm{d}t$$

Take the **Fourier transform** $\bar{a}(\xi, \eta)$ of the **two-dimensional array** a_{mk}:

$$\bar{a}(\xi, \eta) = \sum_{mk} a_{mk}\, e^{-i2\pi(m\eta - k\xi)}.$$

This yields the **product relation** (if $\Omega T = 2\pi$)

$$\bar{a}(\xi, \eta) = T \left(\sum_n \varphi(\xi T + nT)\, e^{-i2\pi\eta n} \right) \left(\sum_n w(\xi T + nT)\, e^{-i2\pi\eta n} \right)^*.$$

Zak transform and Gabor's expansion coefficients

The Fourier transform, together with the Zak transform

$$\tilde{\varphi}_\tau(t,\omega) = \sum_n \varphi(t+n\tau)\, e^{-in\tau\omega},$$

is able to bring the two-dimensional array of expansion coefficients into product form (if $\Omega T = 2\pi$):

$$a_{mk} = \int \varphi(t)\, w^*_{mk}(t)\,\mathrm{d}t \qquad\qquad \bar{a}(\xi,\eta) = T\,\tilde{\varphi}_T\left(\xi T, \eta\frac{2\pi}{T}\right)\,\tilde{w}^*_T\left(\xi T, \eta\frac{2\pi}{T}\right).$$

Compare this property with the property of the Fourier transform, which is able to bring a convolution into product form.

Zak transform and Gabor's signal expansion

The Zak transform can also bring Gabor's signal expansion and the bi-orthogonality condition into product form (if $\Omega T = 2\pi$):

$$\varphi(t) = \sum_{mk} a_{mk}\, g_{mk}(t) \qquad \tilde{\varphi}_T\left(\xi T, \eta\frac{2\pi}{T}\right) = \bar{a}(\xi,\eta)\, \tilde{g}_T\left(\xi T, \eta\frac{2\pi}{T}\right)$$

$$a_{mk} = \int \varphi(t)\, w^*_{mk}(t)\, \mathrm{d}t \qquad \bar{a}(\xi,\eta) = T\, \tilde{\varphi}_T\left(\xi T, \eta\frac{2\pi}{T}\right)\, \tilde{w}^*_T\left(\xi T, \eta\frac{2\pi}{T}\right)$$

$$\sum_{mk} g_{mk}(t_1)\, w^*_{mk}(t_2) = \delta(t_1 - t_2) \qquad T\, \tilde{g}_T\left(\xi T, \eta\frac{2\pi}{T}\right)\, \tilde{w}^*_T\left(\xi T, \eta\frac{2\pi}{T}\right) = 1$$

and the analysis window $w(t)$ that corresponds to the synthesis window $g(t)$ can be found from the latter equation.

All questions answered!

- For any synthesis window $g(t)$, an analysis window $w(t)$ can be found, for instance by using the Zak transform:

$$T\,\tilde{w}_T^*\left(\xi T, \eta\frac{2\pi}{T}\right)\,\tilde{g}_T\left(\xi T, \eta\frac{2\pi}{T}\right) = 1\,.$$

- Gabor's expansion coefficients follow as the sampling values of the windowed Fourier transform at the Gabor lattice:

$$a_{mk} = s(mT, k\Omega) = \int \varphi(t)\,w^*(t - mT)\,e^{-ik\Omega t}\,\mathrm{d}t\,.$$

- Knowledge of the values of the windowed Fourier transform $s(\tau, \omega)$ at the Gabor lattice suffices to reconstruct the signal using Gabor's signal expansion:

$$\varphi(t) = \sum_{mk} a_{mk}\,g(t - mT)\,e^{ik\Omega t}\,.$$

Fast algorithms

The Fourier transform, together with the Zak transform

$$\tilde{\varphi}_T\left(\xi T, \eta\frac{2\pi}{T}\right) = \sum_m \varphi(t+mT)\, e^{-im\omega T},$$

is able to bring the Gabor coefficients (or Gabor transform)

$$a_{mk} = s(mT, k\Omega) = \int \varphi(t)\, w^*(t-mT)\, e^{-ik\Omega t}\,\mathrm{d}t$$

into product form:

$$\bar{a}(\xi,\eta) = T\,\tilde{\varphi}_T\left(\xi T, \eta\frac{2\pi}{T}\right)\,\tilde{w}_T^*\left(\xi T, \eta\frac{2\pi}{T}\right).$$

The Zak transform is, in essence, a Fourier transform. Hence, if we translate everything to discrete-time signals, fast algorithms can be used like the FFT, resulting in a fast Zak transform and a fast Gabor transform. Compare this with the fast convolution, well-known in digital signal processing.

Interpretation of the Gabor coefficients

The array of Gabor coefficients a_{mk} (or Gabor transform) can be interpreted as

- the sampled Fourier transform of the windowed signal $\varphi(t)\, w^*(t-mT)$ [= Fourier series expansion of $\varphi(t)\, w^*(t-mT)$]:

$$a_{mk} = s(mT, k\Omega) = \int \varphi(t)\, w^*(t-mT)\, e^{-ik\Omega t}\, \mathrm{d}t\,.$$

- the inner product of $\varphi(t)$ and the elements $w_{mk}(t)$ of an (m,k)-parameterized set of basis functions $w_{mk}(t) = w(t-mT)\, e^{ik\Omega t}$:

$$a_{mk} = \int \varphi(t)\, w^*_{mk}(t)\, \mathrm{d}t\,.$$

- the sampled output signals of a k-parameterized filter bank with impulse responses $w'_k(t) = w^*(-t)\, e^{ik\Omega t}$ and input signal $\varphi(t)$:

$$a_{mk} = e^{-imk\Omega T} \int \varphi(t)\, w'_k(mT-t)\, \mathrm{d}t\,.$$

Interpretation of Gabor's signal expansion

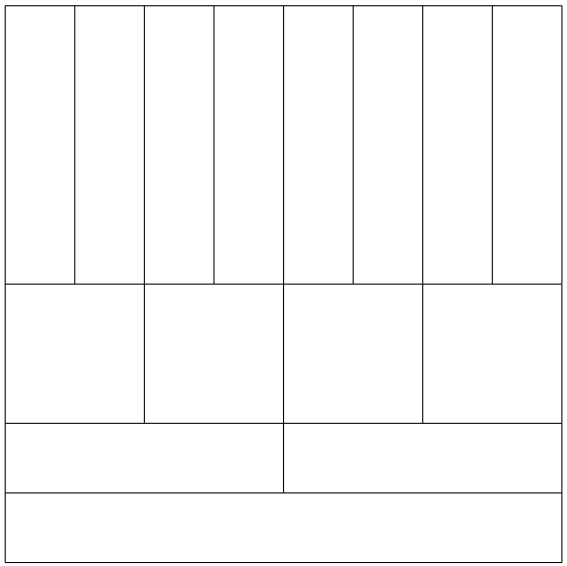
Gabor's signal expansion $$\varphi(t) = \sum_{mk} a_{mk}\, g(t - mT)\, e^{ik\Omega t}$$

can be interpreted as

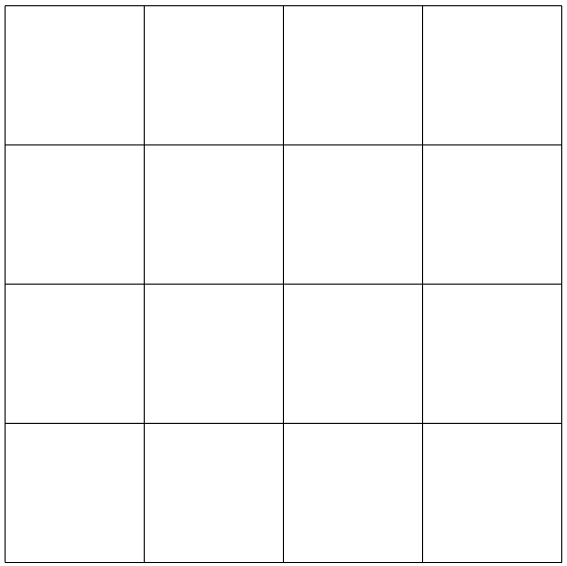
- a superposition of the elements $g_{mk}(t)$ of an (m,k)-parameterized set of basis functions $g_{mk}(t) = g(t - mT)\, e^{ik\Omega t}$ with weights a_{mk}:
$$\varphi(t) = \sum_m \sum_k a_{mk}\, g_{mk}(t)\,.$$
- a superposition of the outputs of a k-parameterized filter bank with impulse responses $g'_k(t) = g(t)\, e^{ik\Omega t}$ and the k-parameterized input sequences a_{mk}:
$$\varphi(t) = \sum_k \left(\sum_m a_{mk}\, g'_k(t - mT) \right)\,.$$

Detour to wavelets and the wavelet transform:

Just a different tiling of the time-frequency plane, resulting in a time-scale representation (wavelet tiling) rather than a time-frequency representation (Gabor tiling) of the signal.



Wavelet tiling



Gabor tiling

Sampling theorem

The well-known sampling theorem for a band-limited signal reads

$$\varphi(t) = \sum_m \varphi(mT)\ \mathsf{sinc}\left(\frac{t}{T} - m\right),$$

where the sinc-function is, in fact, the impulse response of an ideal low-pass filter with cut-off frequency π/T.

If $\varphi_h(t)$ denotes the high-frequency part of the band-limited signal $\varphi(t)$, then $\varphi_h(t)$ can be represented as

$$\varphi_h(t) = \sum_m c_m.\, h(t - mT)\,,$$

in which $h(t)$ is the impulse response of an ideal band-pass filter.

Towards the wavelet transform

$$\varphi_h(t) = \sum_m c_m.\, h(t - mT)$$

If we continue splitting the low-frequency part of the signal in a high-frequency and a low-frequency part, we eventually get the expression

$$\varphi(t) = \sum_{k=0}^{\infty} \left(\sum_m c_{mk}\, h\left(\frac{t}{2^k} - mT\right)\right) .$$

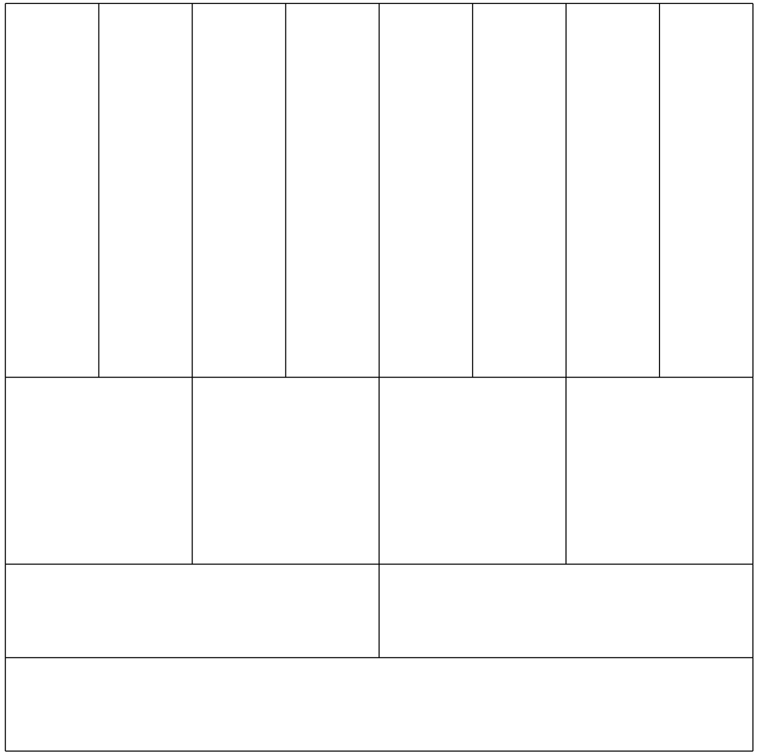
Compare this expression with Gabor's signal expansion

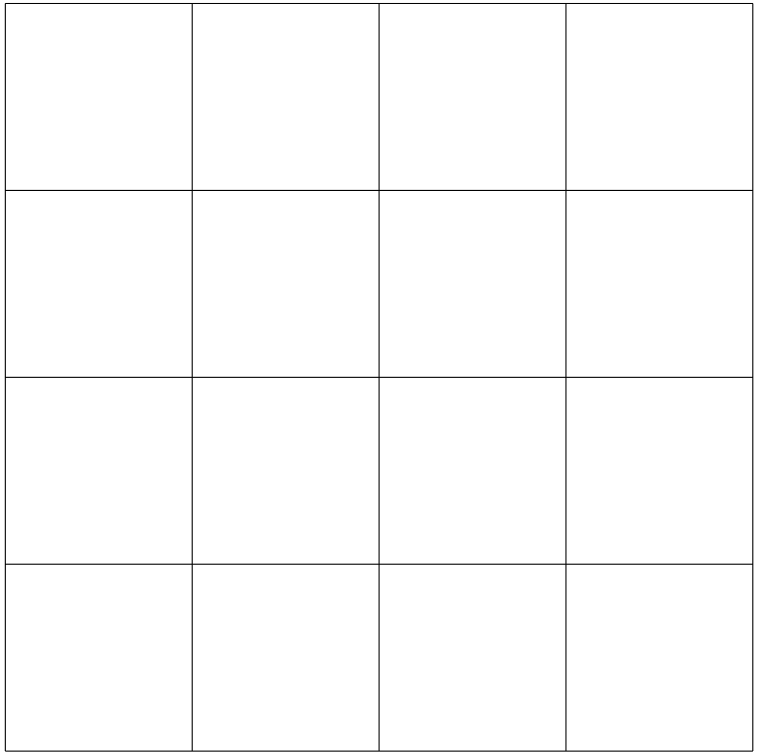
$$\varphi(t) = \sum_{k=-\infty}^{\infty} \left(\sum_m a_{mk}\, g(t - mT)\, e^{ik\Omega t}\right) .$$

Time-scale vs. time-frequency signal representation.

Wavelet transform – subband-coding – multirate filter banks

Wavelet and Gabor tiling





Wavelet tiling

Constant-Q filter bank

multirate

Gabor tiling

Constant-bandwidth filter bank

constant rate

Wavelet detour ends; the Gabor tour continues

- **Oversampling $\Omega T < 2\pi$**
  - **Rational oversampling $2\pi/\Omega T = p/q \quad (p > q \geq 1)$**
  - **Integer oversampling $2\pi/\Omega T = p \quad (p > 1)$**
- **Optical setup to generate the Gabor coefficients**
- **Non-orthogonal sampling/tiling geometry**
  - **Sub-lattices**
  - **Shearing**

Product forms in the case of rational oversampling

In the case of rational oversampling, $2\pi/\Omega T = p/q$, Gabor's signal expansion, the Gabor transform, and the bi-orthogonality condition can be represented as matrix-matrix and matrix-vector multiplications,

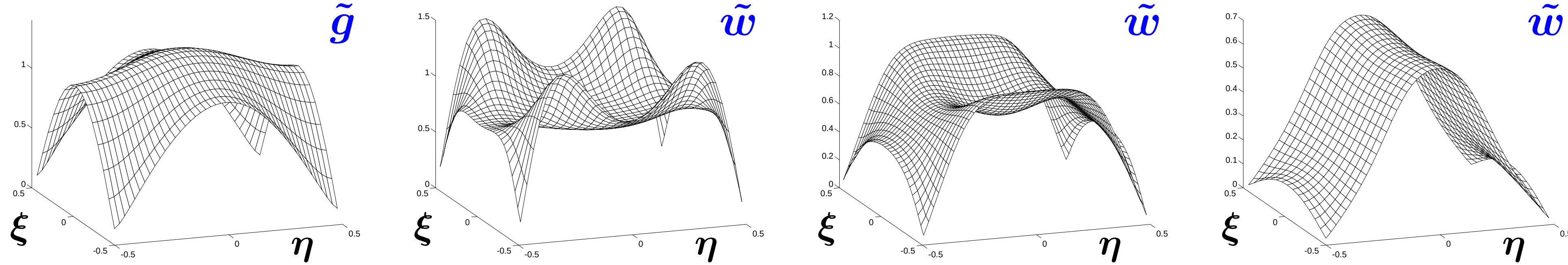
$$
\begin{aligned}
\varphi(t) &= \sum_{mk} a_{mk}\, g_{mk}(t) & \qquad \phi(\xi,\eta) &= \frac{1}{p} G(\xi,\eta)\, a(\xi,\eta) \\
a_{mk} &= \int \varphi(t)\, w^*_{mk}(t)\, \mathrm{d}t & \qquad a(\xi,\eta) &= \frac{pT}{q} W^*(\xi,\eta)\, \phi(\xi,\eta) \\
\delta(t_1 - t_2) &= \sum_{mk} g_{mk}(t_1)\, w^*_{mk}(t_2) & \qquad I_q &= \frac{T}{q} G(\xi,\eta)\, W^*(\xi,\eta)
\end{aligned}
$$

where ϕ is a q-dimensional column vector, a is a p-dimensional column vector, G and W are $q \times p$-dimensional matrices, and I_q is the q-dimensional identity matrix. Moreover: $0 \leq \xi < 1$ and $0 \leq \eta < 1/p$.

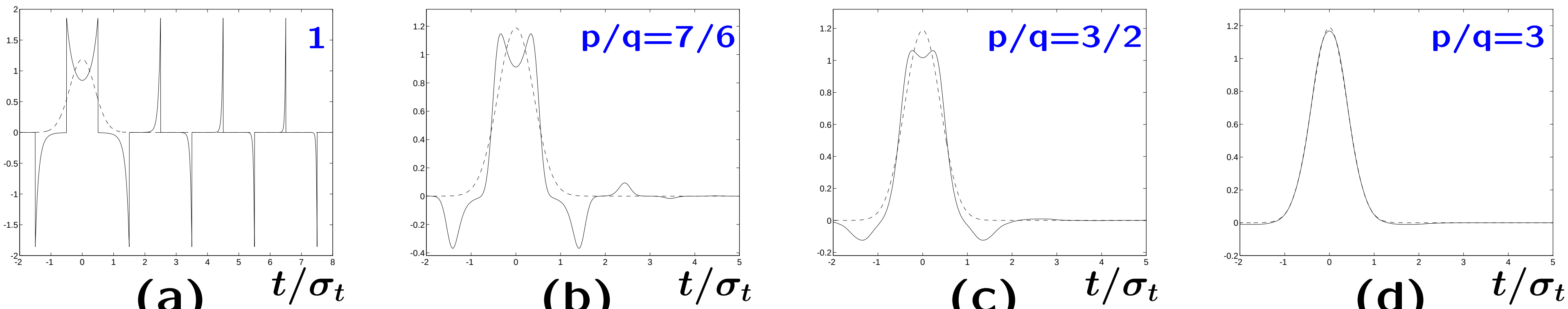
Zak transforms and corresponding window functions

The optimal analysis window follows from the inverse $G^{\dagger}$:

$$W^{*}_{opt}(\xi,\eta) = (q/T)\, G^{\dagger}(\xi,\eta) = (q/T)\, G^{*}(\xi,\eta)\, [G(\xi,\eta)\, G^{*}(\xi,\eta)]^{-1}.$$



(a) (b) (c) (d)



(a) (b) (c) (d)

The special case of integer oversampling

In the special case of integer oversampling, $2\pi/\Omega T = p/q$ with $q = 1$, ϕ is a scalar, a is a p-dimensional column vector, and G and W are p-dimensional row vectors, and the Gabor transform

$$a(\xi, \eta) = pT\, W^*(\xi, \eta)\, \phi(\xi, \eta)$$

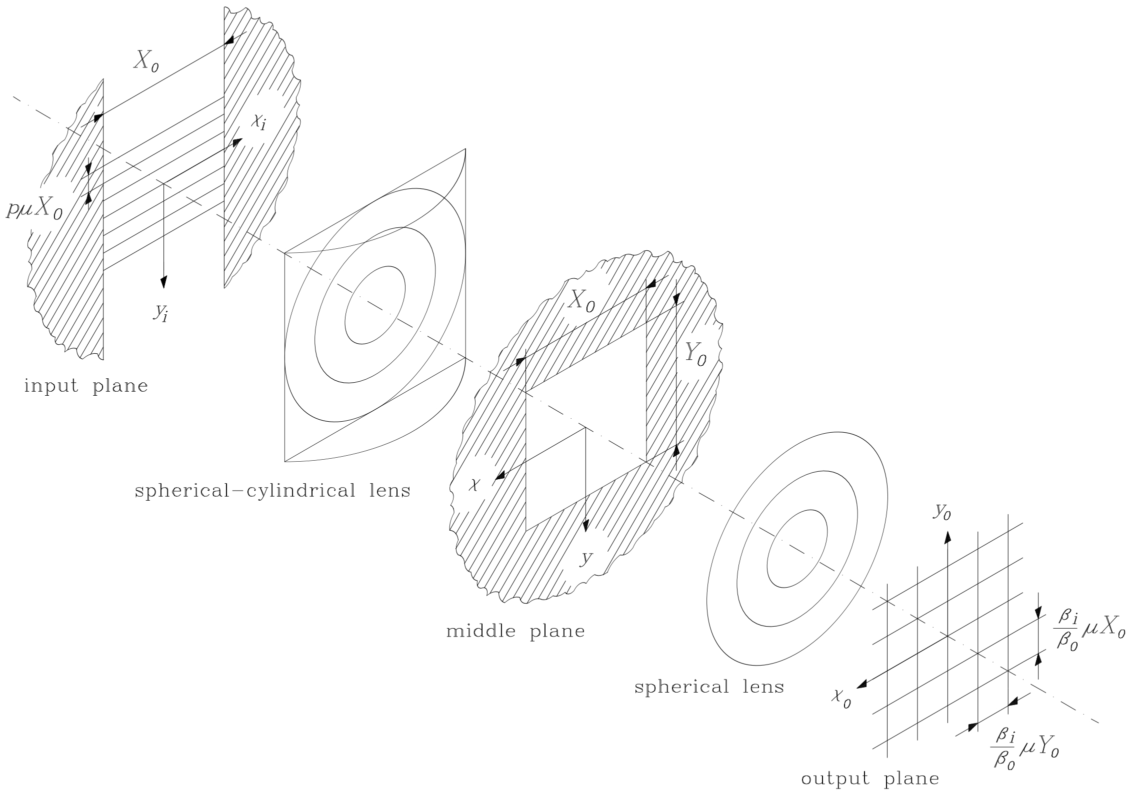
takes the normal product form again:

$$\bar{a}(\xi, \eta) = pT\, \tilde{\varphi}_{pT}\left(\xi pT, \eta\frac{2\pi}{T}\right)\, \tilde{w}_T^*\left(\xi pT, \eta\frac{2\pi}{T}\right) .$$

Note that one period of the periodic Fourier transform $\bar{a}(\xi, \eta)$ contains p horizontal periods (in the ξ-direction) of the (periodic) Zak transform $\tilde{\varphi}_{pT}(\xi pT, \eta 2\pi/T)$, and p vertical quasi-periods (in the η-direction) of the (quasi-periodic) Zak transform $\tilde{w}_T^*(\xi pT, \eta 2\pi/T)$.

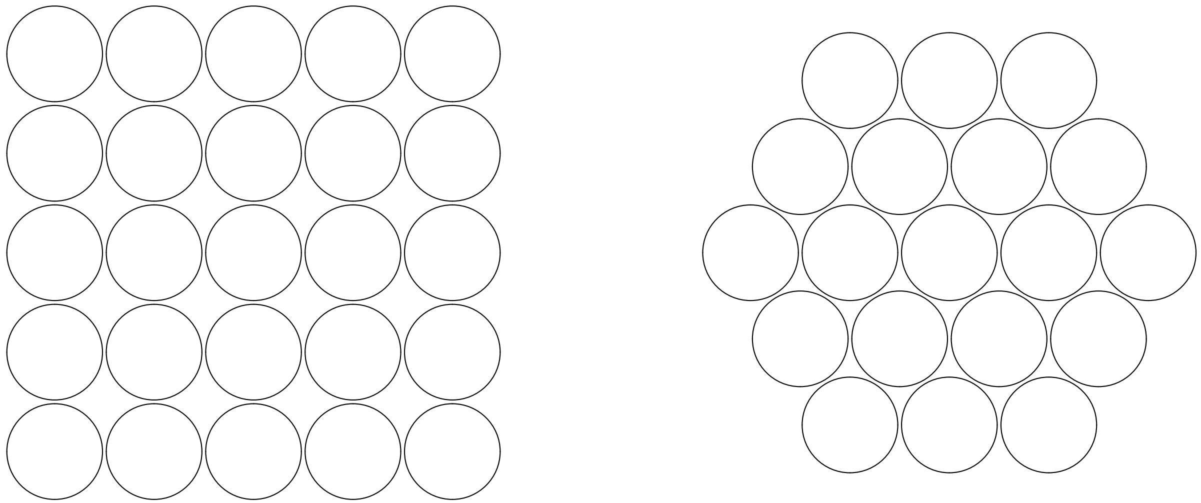
This relationship is the basis for a coherent-optical realization.

Optical generation of the Gabor coefficients



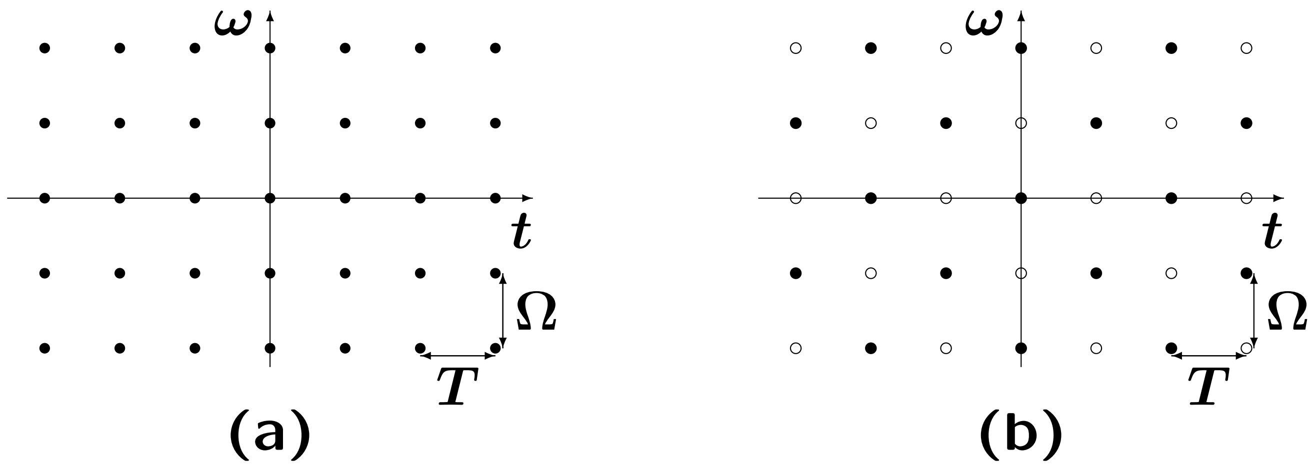
Non-orthogonal sampling/tiling geometry

It is well known that circles are better packed on a hexagonal lattice then on a rectangular one.



A rectangular and a hexagonal packing of circles, with filling factor $\pi/4 = 0.7854$ and $\pi/2\sqrt{3} = 0.9069$, respectively.

Non-orthogonal sampling geometry



(a) A rectangular lattice with lattice vectors $[T, 0]^t$ and $[0, \Omega]^t$, and thus $R = 0$ and $D = 1$; and (b) a hexagonal lattice with lattice vectors $[T, \Omega]^t$ and $[0, 2\Omega]^t$, and thus $R = 1$ and $D = 2$.

$$\begin{bmatrix} T & 0 \\ R\Omega & D\Omega \end{bmatrix} = \begin{bmatrix} T & 0 \\ 0 & \Omega \end{bmatrix} \begin{bmatrix} 1 & 0 \\ R & D \end{bmatrix} \qquad \lambda_{mk} = \sum_n \delta_{k-mR-nD}$$

Gabor's signal expansion on a non-orthogonal lattice

Gabor's expansion (on a non-orthogonal lattice) can be expressed as

$$\varphi(t) = \sum_{mk} \lambda_{mk}\, a_{mk}\, g_{mk}(t) = \sum_{mk} a^s_{mk}\, g_{mk}(t)\,,$$

with the Gabor transform in the usual form [but with a different analysis window $w(t)$, though!],

$$a^s_{mk} = \lambda_{mk}\, a_{mk} = \lambda_{mk} \int \varphi(t)\, w^*_{mk}(t)\, \mathrm{d}t\,,$$

and the bi-orthogonality condition reading

$$\sum_{mk} \lambda_{mk}\, g_{mk}(t_1)\, w^*_{mk}(t_2) = \delta(t_1 - t_2)\,.$$

The (non-orthogonal) bi-orthogonality condition

With $\lambda_{mk} = \sum_n \delta_{k-mR-nD}$, the bi-orthogonality condition

$$\sum_{mk} \lambda_{mk}\, g_{mk}(t_1)\, w^*_{mk}(t_2) = \delta(t_1 - t_2)\,,$$

in the case of rational oversampling, $2\pi/T = p/q$, expressed in terms of the Zak transforms of the window functions, reads

$$\frac{T}{Dq} \sum_{r=0}^{p-1} \tilde{g}_T \left([\xi + s_1] \frac{pT}{q}, \left[\eta + \frac{r}{p}\right] \frac{2\pi}{T} \right)$$

$$\times\ \tilde{w}^*_T \left(\left[\xi + s_2 - \frac{n}{D}\right] \frac{pT}{q}, \left[\eta + \frac{r}{p} - \frac{nR}{D}\right] \frac{2\pi}{T} \right) = \delta_n \delta_{s_1 - s_2}\,,$$

with $s_1, s_2 = 0, 1, \ldots, q-1$, $n = 0, 1, \ldots, D-1$, $0 \leq \xi < 1$, $0 \leq \eta < 1/p$.

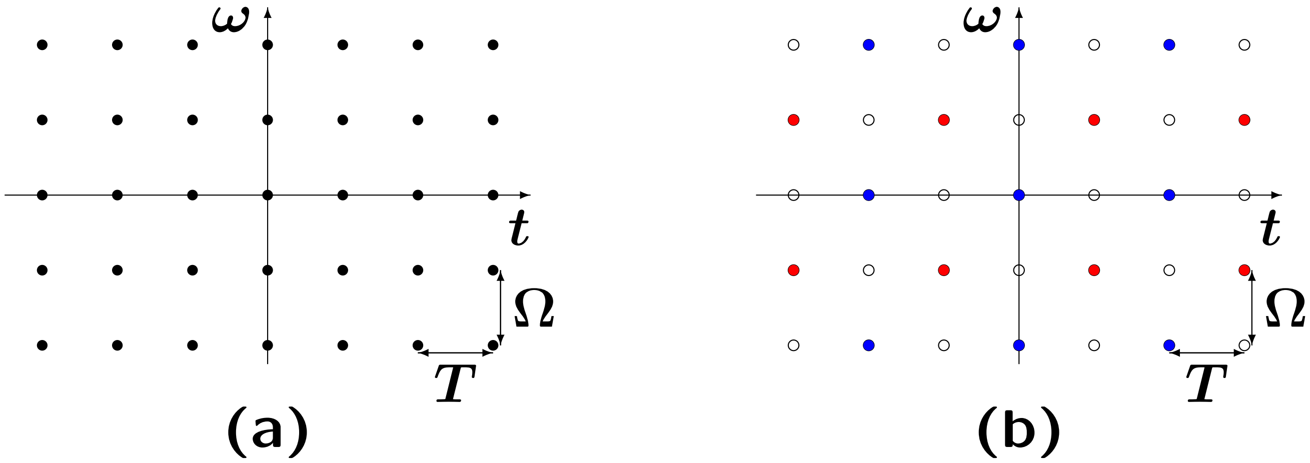
The hexagonal bi-orthogonality condition

For the integer ($p/2$-times) oversampled hexagonal case [$R = 1$, $D = 2$, $n = 0, 1$, $q = 1$, p even, $0 \leq \xi < 1$ and $0 \leq \eta < 1/p$], we have the following bi-orthogonal condition

$$\frac{T}{2} \sum_{r=0}^{p-1} \tilde{g}_T \left(\xi p T, \left[\eta + \frac{r}{p} \right] \frac{2\pi}{T} \right) \tilde{w}_T^* \left(\left[\xi - \frac{n}{2} \right] p T, \left[\eta + \frac{r}{p} - \frac{nR}{2} \right] \frac{2\pi}{T} \right) = \delta_n :$$

$D = 2$ equations ($n = 0, 1, \ldots D-1$) for p variables ($r = 0, 1, \ldots, p-1$) for every ξ and η ($0 \leq \xi < 1$ and $0 \leq \eta < 1/p$), from which the Zak transform $\tilde{w}_T(t, \omega)$ and hence the window function $w(t)$ can easily be determined.

The optical setup to generate the Gabor coefficients can still be used, but now with a hexagonal sampling in the output plane!



(a) A rectangular lattice with lattice vectors $[T, 0]^t$ and $[0, \Omega]^t$; and (b) a hexagonal lattice with lattice vectors $[T, \Omega]^t$ and $[0, 2\Omega]^t$

The hexagonal lattice can be considered as a combination of two rectangular lattices, which can be treated in the normal way! In a filter bank realization, we then switch back and forth between one bank and the other.

The hexagonal lattice can also be considered as arising from a rectangular lattice by shearing.

Non-orthogonal to rectangular sampling via shearing

$$\varphi(t) = \sum_{mk} \lambda_{mk}\, a_{mk}\, g(t - mT)\, e^{jk\Omega t}$$

If we eliminate the array $\lambda_{mk} = \sum_n \delta_{k-mR-nD}$, we can directly write

$$\varphi(t) = \sum_m \sum_n a_{m,mR+nD}\, g(t - mT)\, e^{j(mR+nD)\Omega t},$$

which can be represented in the rectangular form

$$\varphi'(t) = \sum_m \sum_n a'_{mn}\, g'_{mn}(t)\,,$$

with the 'primed' variables being sheared versions

$$f'(t) = f(t)\, e^{-jR\Omega t^2/2T} \quad \text{and} \quad a'_{mn} = a_{m,mR+nD}\, e^{jRm^2\Omega T/2}.$$

Outline of the presentation

1. Linear signal dependence
   - Windowed (short-time) Fourier transform
   - Gabor expansion
   - Wavelet transform
2. Quadratic (bilinear) signal dependence
   - Wigner distribution
   - Application to partially coherent light
3. Relatives of the Wigner distribution
   - Ambiguity function
   - Cohen class – kernel design
   - Fractional Fourier transform

Part 2. Bilinear signal dependence

This part deals with the Wigner distribution and its application to (partially coherent) light. It is shown that the Wigner distribution puts many phenomena of Fourier optics, geometrical optics, matrix optics, radiometry, etc., in one uniform perspective. Among other things, we will consider the propagation of light through first-order optical systems and find transport equations for the Wigner distribution for (weakly) inhomogeneous media.

- M.J. Bastiaans, "Application of the Wigner distribution function to partially coherent light beams," in Optics and Optoelectronics, Theory, Devices and Applications, Proc. ICOL'98, the International Conference on Optics and Optoelectronics, Dehradun, India, 9–12 December 1998, ISBN 81-7319-285-5, ed. O.P. Nijhawan; A.K. Gupta; A.K. Musla; Kehar Singh; Narosa Publishing House, New Delhi, India, 1998, pp. 101–115.

Wigner distribution

In 1932 Wigner introduced the distribution function

$$W(t,\omega) = \int \varphi(t+\tfrac{1}{2}t')\,\varphi^*(t-\tfrac{1}{2}t')\,e^{-i\omega t'}\,\mathrm{d}t'.$$

The Wigner distribution is a kind of windowed Fourier transform

$$W(t,\omega) = e^{i2\omega t}\int \varphi(\tfrac{1}{2}t')\,w^*(\tfrac{1}{2}t'-2t)\,e^{-i\omega t'}\,\mathrm{d}t' = 2\,s(2t,2\omega)\,e^{i2\omega t},$$

where the window function $w(t)$ equals the time-reversed signal $\varphi(-t)$.

$$W(t,\omega) = \int \varphi(t+\tfrac{1}{2}t')\,\varphi^*(t-\tfrac{1}{2}t')\,e^{-i\omega t'}\,\mathrm{d}t'$$

- Real, quadratic signal representation, very well suited for energy considerations.
- Signal intensity = integral over all frequencies:

$$|\varphi(t)|^2 = \frac{1}{2\pi}\int W(t,\omega)\,\mathrm{d}\omega\,.$$

- Spectral intensity = integral over all time moments:

$$\left|\int \varphi(t)\,e^{-i\omega t}\,\mathrm{d}t\right|^2 = \int W(t,\omega)\,\mathrm{d}t\,.$$

$$W(t,\omega) = \int \varphi(t + \tfrac{1}{2}t')\, \varphi^*(t - \tfrac{1}{2}t')\, e^{-i\omega t'}\, \mathrm{d}t'$$

- Time-shift and frequency-shift covariant:

$$\varphi(t) \;\longrightarrow\; W(t,\omega)$$

$$\varphi(t - t_o)\, e^{j\omega_o t} \;\longrightarrow\; W(t - t_o, \omega - \omega_o)$$

- Not necessarily positive; however (Moyal's formula):

$$\frac{1}{2\pi} \iint W_1(t,\omega)\, W_2(t,\omega)\, \mathrm{d}t\mathrm{d}\omega = \left| \int \varphi_1(t)\, \varphi_2^*(t)\, \mathrm{d}t \right|^2 \geq 0$$

- Many other, nice properties!

$$W(t,\omega) = \int \varphi(t+\tfrac{1}{2}t')\,\varphi^*(t-\tfrac{1}{2}t')\,e^{-i\omega t'}\,\mathrm{d}t'$$

- Due to its quadratic signal dependence, the Wigner distribution is especially suited for non-stationary noise, if defined as

$$W(t,\omega) = \int \mathsf{E}\{\varphi(t+\tfrac{1}{2}t')\,\varphi^*(t-\tfrac{1}{2}t')\}\,e^{-i\omega t'}\,\mathrm{d}t'.$$

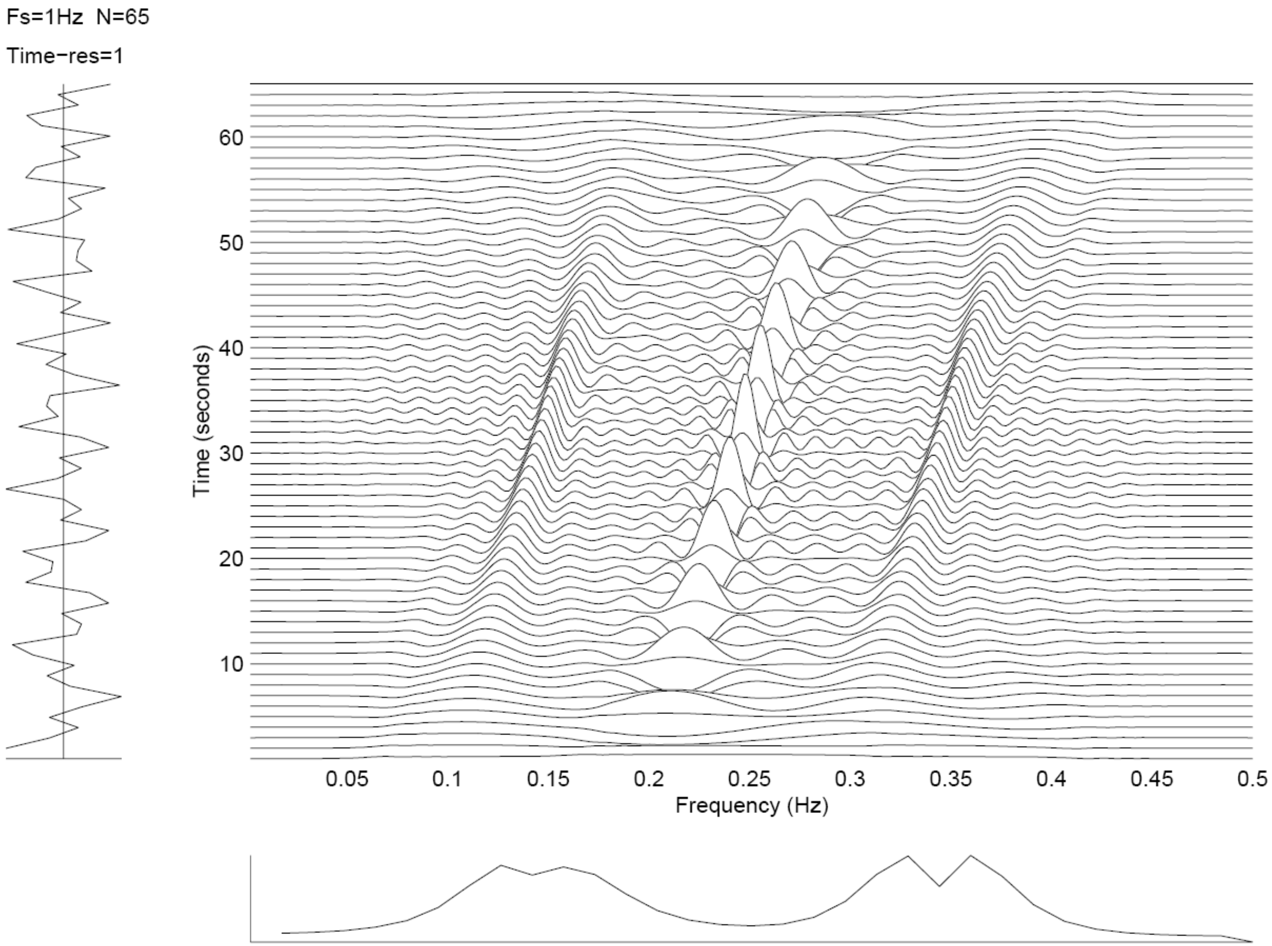
- Interpretation problems arise for multicomponent deterministic signals:

$$\begin{aligned} \varphi(t) &= \varphi_1(t) + \varphi_2(t) \\ W(t,\omega) &= W_1(t,\omega) + W_2(t,\omega) + \text{cross-terms.} \end{aligned}$$

A solution may be found in averaging, for instance (Cohen):

$$\frac{1}{2\pi}\iint W(t',\omega')\,K(t-t',\omega-\omega')\,\mathrm{d}t'\mathrm{d}\omega'.$$

Wigner distribution of a multicomponent signal



Wigner distribution of two chirp signals; note the oscillating cross-term in between the two auto-terms.

$$W(t,\omega) = \int \varphi(t+\tfrac{1}{2}t')\,\varphi^*(t-\tfrac{1}{2}t')\,e^{-i\omega t'}\,\mathrm{d}t'$$

- The Wigner distribution is well adapted to signals with a quadratic phase dependence (i.e., chirp signals)

$$\varphi(t) = e^{i\frac{1}{2}\alpha t^2};$$

  the Wigner distribution then takes the form

$$W(t,\omega) = 2\pi\,\delta(\omega - \alpha t)\,.$$

- Optics, where many signals and systems are described by quadratic-phase functions, is a perfect area of application of the Wigner distribution, in particular if light is considered on a stochastic basis (partially coherent light, cf. spatially non-stationary noise).

We will apply the Wigner distribution to partially coherent light.

Outline of part 2

- Description of partially coherent light:
  - Mutual coherence function $\tilde{\Gamma}(r_1, r_2, \tau)$
  - Positional power spectrum $\Gamma(r_1, r_2, \omega)$
  - Directional power spectrum $\bar{\Gamma}(q_1, q_2, \omega)$
- Wigner distribution:
  - Definition of the Wigner distribution $W(r, q)$
  - Properties of the Wigner distribution
  - Examples of Wigner distributions
- Modal expansions: Inequalities for the Wigner distribution
- Propagation of the Wigner distribution
  - Ray-spread function $K(r_o, q_o, r_i, q_i)$
  - Transport equations for the Wigner distribution
- Miscellaneous topics
  - Second-order moments of the Wigner distribution
  - Invariants for the second-order moments
  - Second-order moments of a (twisted) Gaussian light beam

Description of partially coherent light

Temporally stationary stochastic process $\tilde{\varphi}(r,t) \qquad r = (x, y)^t$

Mutual coherence function

$$\mathrm{E}\{\tilde{\varphi}\,(r_1, t_1)\ \tilde{\varphi}^*\,(r_2, t_2)\} = \tilde{\Gamma}\,(r_1, r_2, t_1 - t_2)$$

where $\mathrm{E}\{\cdot\}$ denotes ensemble averaging

Mutual power spectrum (or cross-spectral density)

$$\Gamma\,(r_1, r_2, \omega) = \int \tilde{\Gamma}\,(r_1, r_2, \tau)\ e^{i\omega\tau}\,\mathrm{d}\tau$$

Basic property: nonnegative definite Hermitian

$$\Gamma(r_1, r_2, \omega) = \int \tilde{\Gamma}(r_1, r_2, \tau)\, e^{i\omega\tau}\, \mathrm{d}\tau$$

$$\Gamma(r_1, r_2, \omega) = \Gamma^*(r_2, r_1, \omega)$$

$$\iint g(r_1, \omega)\, \Gamma(r_1, r_2, \omega)\, g^*(r_2, \omega)\, \mathrm{d}r_1 \mathrm{d}r_2 \geq 0$$

Positional power spectrum $\Gamma(r_1, r_2, \omega)$

Directional power spectrum $\bar{\Gamma}(q_1, q_2, \omega)$

$$\bar{\Gamma}(q_1, q_2, \omega) = \iint \Gamma(r_1, r_2, \omega)\, e^{-i(q_1^t r_1 - q_2^t r_2)}\, \mathrm{d}r_1 \mathrm{d}r_2$$

We will omit the temporal-frequency variable ω.

Wigner distribution

The Wigner distribution $W(r, q)$ is defined as

$$W(r, q) = \int \Gamma\left(r + \tfrac{1}{2}r', r - \tfrac{1}{2}r'\right) e^{-iq^t r'} \, \mathrm{d}r' = W^*(r, q)\,.$$

With r considered as a parameter, the integral represents a Fourier transformation (with conjugate variables r' and q) of the positional power spectrum $\Gamma(r + \frac{1}{2}r', r - \frac{1}{2}r')$.

We also have (in terms of the directional power spectrum)

$$W(r, q) = \int \bar{\Gamma}\left(q + \tfrac{1}{2}q', q - \tfrac{1}{2}q'\right) e^{ir^t q'} \, \mathrm{d}\frac{q'}{2\pi}\,.$$

Completely coherent light

$$W(r,q) = \int \Gamma\left(r + \tfrac{1}{2}r', r - \tfrac{1}{2}r'\right) e^{-iq^t r'}\, \mathrm{d}r' = W^*(r,q)$$

In the case of completely coherent light, the positional power spectrum has the form of a product

$$\Gamma(r_1, r_2) = \varphi(r_1)\,\varphi^*(r_2)\,.$$

Wigner distribution

$$w(r,q) = \int \varphi\left(r + \tfrac{1}{2}r'\right) \varphi^*\left(r - \tfrac{1}{2}r'\right) e^{-iq^t r'}\, \mathrm{d}r'$$

Similar expressions in terms of the directional power spectrum.

Point source and plane wave (harmonic signal)

A point source, $\varphi(r) = \delta(r - r_\circ)$, i.e., an impulse at the position $r_\circ$, has an impulse as Wigner distribution:

$$w(r, q) = \delta(r - r_\circ)\,;$$

note that it does not depend on the frequency variable q.

A plane wave, $\varphi(r) = e^{iq_\circ^t r}$, i.e., a harmonic signal with (spatial) frequency $q_\circ$, has an impulse as Wigner distribution:

$$w(r, q) = \delta\left(\frac{q - q_\circ}{2\pi}\right);$$

note that it does not depend on the space variable r.

Note that these two signals are dual to each other and that their Wigner distributions are related by a rotation through 90°.

Quadratic-phase signal – spherical wave

$$W(r,q) = \int \Gamma\left(r + \tfrac{1}{2}r', r - \tfrac{1}{2}r'\right) e^{-iq^t r'}\, \mathrm{d}r' = W^*(r,q)$$

Spherical wave $\quad \varphi(r) = e^{\frac{1}{2} i r^t H r}$ chirp signal

The curvature of the spherical wave is described by the real symmetric 2×2 matrix H.

The Wigner distribution takes the form $\quad w(r,q) = \delta\left(\dfrac{q - Hr}{2\pi}\right).$

At any point r, only one spatial-frequency q manifests itself: $q = Hr$.

$$W(\boldsymbol{r},\boldsymbol{q}) = \int \Gamma\left(\boldsymbol{r}+\tfrac{1}{2}\boldsymbol{r}',\boldsymbol{r}-\tfrac{1}{2}\boldsymbol{r}'\right) e^{-i\boldsymbol{q}^t\boldsymbol{r}'}\,\mathrm{d}\boldsymbol{r}' = W^*(\boldsymbol{r},\boldsymbol{q})$$

- The positional and directional power spectra of a partially coherent light beam can be reconstructed from $W(\boldsymbol{r},\boldsymbol{q})$ by an inverse Fourier transformation.
- $W(\boldsymbol{r},\boldsymbol{q})$ is real, but not necessarily nonnegative.
- If the light beam is limited to a certain space (or spatial-frequency) interval, $W(\boldsymbol{r},\boldsymbol{q})$ is limited to the same interval.
- A space (or spatial-frequency) shift of the light beam yields the same shift for $W(\boldsymbol{r},\boldsymbol{q})$:

$$\Gamma(\boldsymbol{r}_1-\boldsymbol{r}_\circ,\boldsymbol{r}_2-\boldsymbol{r}_\circ) \;\longrightarrow\; W(\boldsymbol{r}-\boldsymbol{r}_\circ,\boldsymbol{q})$$

$$\bar{\Gamma}(\boldsymbol{q}_1-\boldsymbol{q}_\circ,\boldsymbol{q}_2-\boldsymbol{q}_\circ) \;\longrightarrow\; W(\boldsymbol{r},\boldsymbol{q}-\boldsymbol{q}_\circ)\,.$$

$$W(\boldsymbol{r},\boldsymbol{q}) = \int \Gamma\left(\boldsymbol{r}+\tfrac{1}{2}\boldsymbol{r}', \boldsymbol{r}-\tfrac{1}{2}\boldsymbol{r}'\right) e^{-i\boldsymbol{q}^t\boldsymbol{r}'}\,\mathrm{d}\boldsymbol{r}' = W^*(\boldsymbol{r},\boldsymbol{q})$$

- The integral over the spatial-frequency variable is proportional to the positional intensity of the light beam

$$\int W(\boldsymbol{r},\boldsymbol{q})\,\mathrm{d}\frac{\boldsymbol{q}}{2\pi} = \Gamma(\boldsymbol{r},\boldsymbol{r})\,.$$

- The integral over the space variable is proportional to the directional (cf. radiant) intensity of the light beam

$$\int W(\boldsymbol{r},\boldsymbol{q})\,\mathrm{d}\boldsymbol{r} = \bar{\Gamma}(\boldsymbol{q},\boldsymbol{q})\,.$$

- The total energy follows as

$$\iint W(\boldsymbol{r},\boldsymbol{q})\,\mathrm{d}\boldsymbol{r}\mathrm{d}\frac{\boldsymbol{q}}{2\pi} = \int \Gamma(\boldsymbol{r},\boldsymbol{r})\,\mathrm{d}\boldsymbol{r} = \int \bar{\Gamma}(\boldsymbol{q},\boldsymbol{q})\,\mathrm{d}\frac{\boldsymbol{q}}{2\pi}\,.$$

$$W(\boldsymbol{r},\boldsymbol{q}) = \int \Gamma\left(\boldsymbol{r}+\tfrac{1}{2}\boldsymbol{r}', \boldsymbol{r}-\tfrac{1}{2}\boldsymbol{r}'\right) e^{-i\boldsymbol{q}^t\boldsymbol{r}'}\,\mathrm{d}\boldsymbol{r}' = W^*(\boldsymbol{r},\boldsymbol{q})$$

- The radiant emittance of the light beam is proportional to the integral

$$j_z(\boldsymbol{r}) = \int \frac{\sqrt{k^2-\boldsymbol{q}^t\boldsymbol{q}}}{k}\,W(\boldsymbol{r},\boldsymbol{q})\,\mathrm{d}\frac{\boldsymbol{q}}{2\pi}\,;$$

when we combine this integral with the integral

$$\boldsymbol{j}_r(\boldsymbol{r}) = \int \frac{\boldsymbol{q}}{k}\,W(\boldsymbol{r},\boldsymbol{q})\,\mathrm{d}\frac{\boldsymbol{q}}{2\pi}\,,$$

we can construct the vector $\boldsymbol{j} = \left[\boldsymbol{j}_r^t, j_z\right]^t$, which is proportional to the geometrical vector flux.

Normalized second-order moments

The real, positive definite symmetric 4×4 matrix M

$$M = \iint \begin{bmatrix} rr^t & rq^t \\ qr^t & qq^t \end{bmatrix} W(r,q)\,\mathrm{d}r\mathrm{d}\frac{q}{2\pi} \bigg/ \iint W(r,q)\,\mathrm{d}r\mathrm{d}\frac{q}{2\pi}$$

yields such quantities as the effective width d_x in the x-direction

$$m_{xx} = \frac{\displaystyle\iint x^2\, W(r,q)\,\mathrm{d}r\mathrm{d}\frac{q}{2\pi}}{\displaystyle\iint W(r,q)\,\mathrm{d}r\mathrm{d}\frac{q}{2\pi}} = \frac{\displaystyle\int x^2\,\Gamma(r,r)\,\mathrm{d}r}{\displaystyle\int \Gamma(r,r)\,\mathrm{d}r} = d_x^2$$

and the effective width d_u in the u-direction

$$m_{uu} = \frac{\displaystyle\iint u^2\, W(r,q)\,\mathrm{d}r\mathrm{d}\frac{q}{2\pi}}{\displaystyle\iint W(r,q)\,\mathrm{d}r\mathrm{d}\frac{q}{2\pi}} = \frac{\displaystyle\int u^2\,\bar{\Gamma}(q,q)\,\mathrm{d}\frac{q}{2\pi}}{\displaystyle\int \bar{\Gamma}(q,q)\,\mathrm{d}\frac{q}{2\pi}} = d_u^2\,.$$

Moyal's relationship

$$W(r,q) = \int \Gamma\left(r + \tfrac{1}{2}r', r - \tfrac{1}{2}r'\right) e^{-iq^t r'}\,\mathrm{d}r' = W^*(r,q)$$

$$\iint W_1(r,q)\,W_2(r,q)\,\mathrm{d}r\mathrm{d}\frac{q}{2\pi}$$

$$= \iint \Gamma_1(r_1,r_2)\,\Gamma_2^*(r_1,r_2)\,\mathrm{d}r_1\mathrm{d}r_2$$

$$= \iint \bar{\Gamma}_1(q_1,q_2)\,\bar{\Gamma}_2^*(q_1,q_2)\,\mathrm{d}\frac{q_1}{2\pi}\mathrm{d}\frac{q_2}{2\pi}$$

It can be shown that these integrals are nonnegative; hence, averaging of one Wigner distribution with another one yields a nonnegative result.

Spatially incoherent light

$$W(\boldsymbol{r},\boldsymbol{q}) = \int \Gamma\left(\boldsymbol{r}+\tfrac{1}{2}\boldsymbol{r}', \boldsymbol{r}-\tfrac{1}{2}\boldsymbol{r}'\right) e^{-i\boldsymbol{q}^t\boldsymbol{r}'}\,\mathrm{d}\boldsymbol{r}' = W^*(\boldsymbol{r},\boldsymbol{q})$$

In the case of spatially incoherent light, the positional power spectrum [with intensity $p(\boldsymbol{r}) \geq 0$] reads

$$\Gamma\left(\boldsymbol{r}+\tfrac{1}{2}\boldsymbol{r}', \boldsymbol{r}-\tfrac{1}{2}\boldsymbol{r}'\right) = p(\boldsymbol{r})\,\delta(\boldsymbol{r}')$$

and the Wigner distribution takes the form $W(\boldsymbol{r},\boldsymbol{q}) = p(\boldsymbol{r})$.

The Wigner distribution depends only on the space variable $\boldsymbol{r}$ and not on the spatial-frequency (i.e., direction) variable $\boldsymbol{q}$.

Spatially stationary light

$$W(r,q) = \int \bar{\Gamma}\left(q+\tfrac{1}{2}q', q-\tfrac{1}{2}q'\right) e^{ir^t q'}\, \mathrm{d}\frac{q'}{2\pi} = W^*(r,q)$$

In the case of spatially stationary light, we have:

positional power spectrum $\Gamma\left(r+\tfrac{1}{2}r', r-\tfrac{1}{2}r'\right) = s(r')$

directional power spectrum [intensity $\bar{s}(q) = \int s(r')\, e^{-iq^t r'}\, \mathrm{d}r' \geq 0$]

$$\bar{\Gamma}\left(q+\tfrac{1}{2}q', q-\tfrac{1}{2}q'\right) = \bar{s}(q)\, \delta\left(\frac{q'}{2\pi}\right)$$

and the Wigner distribution takes the form $W(r,q) = \bar{s}(q)$.

Note the similarity between the spatial-frequency behaviour of spatially stationary light and the space behaviour of spatially incoherent light (Van Cittert-Zernike).

Quasi-homogeneous light

$$W(r,q) = \int \Gamma\left(r + \tfrac{1}{2}r', r - \tfrac{1}{2}r'\right) e^{-iq^t r'}\,\mathrm{d}r' = W^*(r,q)$$

Quasi-homogeneous light is in fact spatially stationary light with a slowly varying intensity

$$\Gamma\left(r + \tfrac{1}{2}r', r - \tfrac{1}{2}r'\right) \simeq p(r)\, s(r')\,.$$

The Wigner distribution takes the form $W(r,q) \simeq p(r)\, \bar{s}(q)\,.$

Note that for quasi-homogenous light $W(r,q) \geq 0\,.$

Spatially incoherent for $\bar{s}(q) = 1$.
Spatially stationary for $p(r) = 1$.

The positional power spectrum of the most general partially coherent Gaussian light can be written in the form

$$\Gamma(\mathbf{r}_1,\mathbf{r}_2) = \frac{1}{\pi}\sqrt{\det G_1}\,\exp\left(-\frac{1}{4}\begin{bmatrix}\mathbf{r}_1+\mathbf{r}_2\\ \mathbf{r}_1-\mathbf{r}_2\end{bmatrix}^t \begin{bmatrix}G_1 & -iH\\ -iH^t & G_2\end{bmatrix}\begin{bmatrix}\mathbf{r}_1+\mathbf{r}_2\\ \mathbf{r}_1-\mathbf{r}_2\end{bmatrix}\right),$$

with H a real 2×2 matrix, and G_1 and G_2 (as well as G_2-G_1) real, positive definite symmetric 2×2 matrices.

Note: 10 degrees of freedom

$$W(r,q) = \int \Gamma\left(r + \tfrac{1}{2}r', r - \tfrac{1}{2}r'\right) e^{-iq^t r'} \, \mathrm{d}r' = W^*(r,q)$$

$$\Gamma(r_1, r_2) = \frac{1}{\pi}\sqrt{\det G_1}\, \exp\left(-\frac{1}{4}\begin{bmatrix} r_1 + r_2 \\ r_1 - r_2 \end{bmatrix}^t \begin{bmatrix} G_1 & -iH \\ -iH^t & G_2 \end{bmatrix} \begin{bmatrix} r_1 + r_2 \\ r_1 - r_2 \end{bmatrix}\right)$$

The Wigner distribution of Gaussian light takes the form

$$W(r,q) = 4\sqrt{\frac{\det G_1}{\det G_2}}\, \exp\left(-\begin{bmatrix} r \\ q \end{bmatrix}^t \begin{bmatrix} G_1 + HG_2^{-1}H^t & -HG_2^{-1} \\ -G_2^{-1}H^t & G_2^{-1} \end{bmatrix} \begin{bmatrix} r \\ q \end{bmatrix}\right)$$

and is Gaussian both in r and q.

$$\Gamma(r_1, r_2) = \frac{1}{\pi}\sqrt{\det G_1}\, \exp\left(-\frac{1}{4}\begin{bmatrix} r_1 + r_2 \\ r_1 - r_2 \end{bmatrix}^t \begin{bmatrix} G_1 & -iH \\ -iH^t & G_2 \end{bmatrix} \begin{bmatrix} r_1 + r_2 \\ r_1 - r_2 \end{bmatrix}\right)$$

In a more common way, the positional power spectrum of Gaussian light can be expressed in the form

$$\Gamma(r_1, r_2) = \frac{1}{\pi}\sqrt{\det G_1}\; e^{-\frac{1}{4}(r_1 - r_2)^t G_0 (r_1 - r_2)}\; e^{-\frac{1}{2} r_1^t i \left(H - H^t\right) r_2}$$

$$\times\; e^{-\frac{1}{2} r_1^t \left[G_1 - i\frac{1}{2}\left(H + H^t\right)\right] r_1}\; e^{-\frac{1}{2} r_2^t \left[G_1 + i\frac{1}{2}\left(H + H^t\right)\right] r_2}.$$

$G_0 = G_2 - G_1$ is a real, positive definite symmetric 2×2 matrix.

Note that the asymmetry of the matrix H is a measure for the twist.

Zero-twist Gaussian Schell-model light: $H = H^t$

$$\Gamma\left(r_1, r_2\right) = \frac{1}{\pi}\sqrt{\det G_1}\ \exp\left(-\frac{1}{4}\begin{bmatrix} r_1 + r_2 \\ r_1 - r_2 \end{bmatrix}^t \begin{bmatrix} G_1 & -iH \\ -iH & G_2 \end{bmatrix} \begin{bmatrix} r_1 + r_2 \\ r_1 - r_2 \end{bmatrix}\right)$$

$$= \frac{1}{\pi}\sqrt{\det G_1}\ e^{-\frac{1}{4}\left(r_1 - r_2\right)^t G_0 \left(r_1 - r_2\right)} \times\ e^{-\frac{1}{2} r_1^t \left[G_1 - iH\right] r_1}\ e^{-\frac{1}{2} r_2^t \left[G_1 + iH\right] r_2}$$

The zero-twist Schell-model case applies, for instance, in

- the completely coherent case ($G_1 = G_2$; $G_0 = G_2 - G_1 = 0$);
- the partially coherent one-dimensional case;
- the partially coherent rotationally symmetric case.

Note: 9 degrees of freedom

Symplectic Gaussian light: $H = H^t$ and $G_1 = \sigma^2 G_2$

With $H = H^t$ and $G_1 = \sigma G = \sigma^2 G_2$ $(0 < \sigma \leq 1)$, we get

$$\Gamma(r_1, r_2) = \frac{\sigma}{\pi}\sqrt{\det G}\,\exp\left(-\frac{1}{4}\begin{bmatrix} r_1 + r_2 \\ r_1 - r_2 \end{bmatrix}^t \begin{bmatrix} \sigma G & -iH \\ -iH & G/\sigma \end{bmatrix} \begin{bmatrix} r_1 + r_2 \\ r_1 - r_2 \end{bmatrix}\right)$$

$$W(r, q) = 4\,\sigma^2 \exp\left(-\sigma \begin{bmatrix} r \\ q \end{bmatrix}^t \begin{bmatrix} G + HG^{-1}H & -HG^{-1} \\ -G^{-1}H & G^{-1} \end{bmatrix} \begin{bmatrix} r \\ q \end{bmatrix}\right).$$

The symplectic case applies, again, in

- the completely coherent case ($\sigma = 1$; $G_1 = G_2 = G$);
- the partially coherent one-dimensional case;
- the partially coherent rotationally symmetric case.

Note: 7 degrees of freedom

Modal expansions of the positional power spectrum

$$\Gamma\left(r_1, r_2\right) = \frac{1}{\rho} \sum_{m=0}^{\infty} \mu_m \, \varphi_m \left(\frac{r_1}{\rho}\right) \varphi_m^* \left(\frac{r_2}{\rho}\right) = \Gamma^*\left(r_2, r_1\right)$$

A similar expansion holds for the directional power spectrum.

Integral equation: $\int \Gamma(r_1, r_2)\, \varphi_m \left(\frac{r_2}{\rho}\right) \mathrm{d}r_2 = \mu_m \, \varphi_m \left(\frac{r_1}{\rho}\right)$

Eigenfunctions orthonormal: $\int \varphi_m(\xi)\, \varphi_n^*(\xi)\, \mathrm{d}\xi = \delta_{m-n}$

Eigenvalues nonnegative: $\mu_0 \geq \mu_1 \geq \ldots \geq \mu_m \geq \ldots \geq 0$

Modal expansions of the Wigner distribution

$$\Gamma\left(r_1, r_2\right) = \frac{1}{\rho} \sum_{m=0}^{\infty} \mu_m \, \varphi_m \left(\frac{r_1}{\rho}\right) \varphi_m^* \left(\frac{r_2}{\rho}\right) = \Gamma^*\left(r_2, r_1\right)$$

$$W(r,q) = \int \Gamma\left(r + \tfrac{1}{2}r', r - \tfrac{1}{2}r'\right) e^{-iq^t r'} \, \mathrm{d}r' = \sum_{m=0}^{\infty} \mu_m \, w_m \left(\frac{r}{\rho}, \rho q\right),$$

with $\quad w_m(\xi, \eta) = \int \varphi_m \left(\xi + \tfrac{1}{2}\xi'\right) \varphi_m^* \left(\xi - \tfrac{1}{2}\xi'\right) e^{-i\eta^t \xi'} \, \mathrm{d}\xi'.$

Orthonormality relation for the Wigner distributions $f_m(\xi, \eta)$:

$$\iint w_m(\xi,\eta) \, w_n(\xi,\eta) \, \mathrm{d}\xi \mathrm{d}\frac{\eta}{2\pi} = \left| \int \varphi_m(\xi) \, \varphi_n^*(\xi) \, \mathrm{d}\xi \right|^2 = \delta_{m-n}$$

Inequalities for the Wigner distribution

- $2\,d_x\,d_u \geq 1\,, \qquad 2\,d_y\,d_v \geq 1\,, \qquad (d_x^2 + d_y^2)\left(d_u^2 + d_v^2\right) \geq 1$
- $0 \leq \iint W_1(r,q)\,W_2(r,q)\,\mathrm{d}r\mathrm{d}\frac{q}{2\pi}$
- $$\iint W_1(r,q)\,W_2(r,q)\,\mathrm{d}r\mathrm{d}\frac{q}{2\pi} \leq \sqrt{\iint W_1^2(r,q)\,\mathrm{d}r\mathrm{d}\frac{q}{2\pi}}\,\sqrt{\iint W_2^2(r,q)\,\mathrm{d}r\mathrm{d}\frac{q}{2\pi}}$$
- $\iint W^2(r,q)\,\mathrm{d}r\mathrm{d}\frac{q}{2\pi} \leq \left(\iint W(r,q)\,\mathrm{d}r\mathrm{d}\frac{q}{2\pi}\right)^2$

Linear optical systems

Input-output relationships for completely coherent light:

$$\varphi_o(r_o) = \int h_{rr}(r_o, r_i)\, \varphi_i(r_i)\, \mathrm{d}r_i$$ point-spread function

$$\bar{\varphi}_o(q_o) = \int h_{qr}(q_o, r_i)\, \varphi_i(r_i)\, \mathrm{d}r_i$$ hybrid spread function

$$\varphi_o(r_o) = \int h_{rq}(r_o, q_i)\, \bar{\varphi}_i(q_i)\, \mathrm{d}\frac{q_i}{2\pi}$$ hybrid spread function

$$\bar{\varphi}_o(q_o) = \int h_{qq}(q_o, q_i)\, \bar{\varphi}_i(q_i)\, \mathrm{d}\frac{q_i}{2\pi}$$ wave-spread function

Input-output relationship in terms of positional power spectra:

$$\Gamma_o(r_1, r_2) = \iint h_{rr}(r_1, \rho_1)\, \Gamma_i(\rho_1, \rho_2)\, h^*_{rr}(r_2, \rho_2)\, \mathrm{d}\rho_1 \mathrm{d}\rho_2$$

Ray-spread function of a linear optical system

Input-output relationship in terms of Wigner distributions:

$$W_o(r_o, q_o) = \iint K(r_o, q_o, r_i, q_i)\, W(r_i, q_i)\, \mathrm{d}r_i \mathrm{d}\frac{q_i}{2\pi}$$

The function $K(r, q, r_i, q_i)$ is the response of the optical system in the space-frequency domain to the input signal

$$W_i(r, q) = \delta(r - r_i)\, \delta\left(\frac{q - q_i}{2\pi}\right) \quad (\simeq \text{single ray!})\,.$$

Hence, the function $K(r_o, q_o, r_i, q_i)$ might be called the ray-spread function of the optical system.

Thin lens

Point-spread function of a thin lens:

$$h_{rr}(r_o, r_i) = e^{-\frac{1}{2} i r_o^t C r_o} \; \delta(r_o - r_i) \, ,$$

with $C = C^t$ a real symmetric 2×2 matrix.

Ray-spread function:

$$K(r_o, q_o, r_i, q_i) = \delta(r_i - r_o) \, \delta \left(\frac{q_i - C r_o - q_o}{2\pi} \right)$$

Input-output relationship:

$$W_o(r, q) = W_i(r, Cr + q)$$

Free space (in the Fresnel approximation)

Wave-spread function of free space:

$$h_{qq}(q_o, q_i) = e^{\frac{1}{2} i q_o^t B q_o} \, \delta\left(\frac{q_o - q_i}{2\pi}\right),$$

with $B = B^t$ a real symmetric 2×2 matrix.

Ray-spread function:

$$K(r_o, q_o, r_i, q_i) = \delta(r_i - r_o - Bq_o) \, \delta\left(\frac{q_i - q_o}{2\pi}\right)$$

Input-output relationship:

$$W_o(r, q) = W_i(r + Bq, q)$$

Luneburg's first-order optical system

Ray-spread function of a first-order optical system:

$$K(r_o, q_o, r_i, q_i) = \delta(r_i - Ar_o - Bq_o)\,\delta\left(\frac{q_i - Cr_o - Dq_o}{2\pi}\right),$$

with A, B, C, and D real 2×2 matrices.

Input-output relationship:

$$W_o(r, q) = W_i(Ar + Bq, Cr + Dq)$$

The $ABCD$-matrix is known as the ray transformation matrix:

$$\begin{bmatrix} r_i \\ q_i \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r_o \\ q_o \end{bmatrix}.$$

Symplecticity of the ray transformation matrix

Symplecticity of the ray transformation matrix $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$

With the matrix $J = i \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} = J^{-1} = J^{\dagger} = -J^{t}$, symplecticity can be expressed as $T^{-1} = JT^{t}J$, $\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} D^{t} & -B^{t} \\ -C^{t} & A^{t} \end{bmatrix}$, and hence

$$AB^{t} = BA^{t}, \quad B^{t}D = D^{t}B, \quad DC^{t} = CD^{t}, \quad C^{t}A = A^{t}C,$$

and $\quad AD^{t} - BC^{t} = I = A^{t}D - C^{t}B.$

In the one-dimensional case symplecticity reduces to $AD - BC = 1$.

Lens: $W_o(x,u) = W_i\left(x, u + \frac{k}{f}x\right)$ $\begin{bmatrix} 1 & 0 \\ k/f & 1 \end{bmatrix}$

Free space: $W_o(x,u) = W_i\left(x - \frac{z}{k}u, u\right)$ $\begin{bmatrix} 1 & -z/k \\ 0 & 1 \end{bmatrix}$

Magnifier: $W_o(x,u) = W_i\left(mx, \frac{u}{m}\right)$ $\begin{bmatrix} m & 0 \\ 0 & m^{-1} \end{bmatrix}$

Fourier transformer: $W_o(x,u) = W_i\left(\beta u, -\frac{x}{\beta}\right)$ $\begin{bmatrix} 0 & \beta \\ -\beta^{-1} & 0 \end{bmatrix}$

Fractional Fourier transformer: $\begin{bmatrix} m\cos\alpha & \beta\sin\alpha \\ -\beta^{-1}\sin\alpha & m^{-1}\cos\alpha \end{bmatrix}$

Spherical wave in a first-order optical system

$$w_i(r,q) = \delta\left(\frac{q - H_i r}{2\pi}\right), \text{ with } H_i \text{ the input curvature}$$

$$\begin{aligned} w_o(r,q) &= \delta\left(\frac{(Cr + Dq) - H_i(Ar + Bq)}{2\pi}\right) \\ &= \frac{1}{\det(D - H_i B)}\,\delta\left(\frac{q + (D - H_i B)^{-1}(C - H_i A)r}{2\pi}\right) \\ &= \frac{1}{\det(D - H_i B)}\,\delta\left(\frac{q - H_o r}{2\pi}\right), \text{ with } H_o \text{ the output curvature} \end{aligned}$$

Bilinear relationship: $H_i = (C + DH_o)(A + BH_o)^{-1}$

Free space in the Fresnel approximation is governed by the partial differential equation

$$-i\frac{\partial\varphi}{\partial z} = \left(k + \frac{1}{2k}\frac{\partial^2}{\partial r^2}\right)\varphi\,;$$

partially coherent light satisfies the partial differential equation

$$-i\frac{\partial\Gamma}{\partial z} = \left[\left(k + \frac{1}{2k}\frac{\partial^2}{\partial r_1^2}\right) - \left(k + \frac{1}{2k}\frac{\partial^2}{\partial r_2^2}\right)\right]\Gamma\,.$$

The corresponding transport equation reads

$$\frac{q^t}{k}\frac{\partial W}{\partial r}+\frac{\partial W}{\partial z} = 0, \quad \text{with the solution} \quad W(r,q;z) = W\left(r - \frac{q}{k}z, q; 0\right)\,.$$

Free space is governed by the partial differential equation

$$-i\frac{\partial\Gamma}{\partial z} = \left[\sqrt{k^2 + \frac{\partial^2}{\partial r_1^2}} - \sqrt{k^2 + \frac{\partial^2}{\partial r_2^2}}\right]\Gamma\,.$$

In the Liouville (or geometric-optical) approximation, the transport equation reads

$$\frac{q^t}{k}\frac{\partial W}{\partial r} + \frac{\sqrt{k^2 - q^t q}}{k}\frac{\partial W}{\partial z} = 0\,,$$

with the solution

$$W(r, q; z) = W\left(r - \frac{q}{\sqrt{k^2 - q^t q}}z, q; 0\right).$$

For a weakly inhomogeneous medium we have

$$\frac{q^t}{k}\frac{\partial W}{\partial r} + \frac{\sqrt{k^2 - q^t q}}{k}\frac{\partial W}{\partial z} + \left(\frac{\partial k}{\partial r}\right)^t \frac{\partial W}{\partial q} = 0\,.$$

Along the path $r = r(s)$, $z = z(s)$, $q = q(s)$, defined by

$$\frac{dr}{ds} = \frac{q}{k}, \qquad \frac{dz}{ds} = \frac{\sqrt{k^2 - q^t q}}{k}, \qquad \frac{dq}{ds} = \frac{\partial k}{\partial r},$$

the transport equation reads $\dfrac{dW}{ds} = 0$, and the Wigner distribution has a constant value.

For a rotationally symmetric medium (a fibre, for instance) we have

$$\frac{u}{k}\frac{\partial W}{\partial x} + \frac{v}{k}\frac{\partial W}{\partial y} + \frac{\sqrt{k^2-(u^2+v^2)}}{k}\frac{\partial W}{\partial z} + \left(\frac{\partial k}{\partial x}\right)^t \frac{\partial W}{\partial u} + \left(\frac{\partial k}{\partial y}\right)^t \frac{\partial W}{\partial v} = 0\,,$$

with $k = k\left(\sqrt{x^2+y^2}\right)$. After the transformation of variables

$$x = \rho\cos\theta\,,\quad y = \rho\sin\theta\,,\quad h = vx - uy\,,\quad k^2 = u^2+v^2+w^2,$$

h and w are invariant along a ray, and the transport equation reads

$$\sqrt{k^2 - w^2 - \frac{h^2}{\rho^2}}\frac{\partial W}{\partial \rho} + \frac{h}{\rho^2}\frac{\partial W}{\partial \theta} + w\frac{\partial W}{\partial z} = 0\,.$$

Matrix of normalized second-order moments

The real, positive definite symmetric moment matrix

$$M = \iint \begin{bmatrix} rr^t & rq^t \\ qr^t & qq^t \end{bmatrix} W(r,q)\, \mathrm{d}r \mathrm{d}\frac{q}{2\pi} \Big/ \iint W(r,q)\, \mathrm{d}r \mathrm{d}\frac{q}{2\pi}$$

propagates through a first-order optical system with the symplectic ray transformation matrix

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad \text{and thus} \quad \begin{bmatrix} r_i \\ q_i \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r_o \\ q_o \end{bmatrix} = T \begin{bmatrix} r_o \\ q_o \end{bmatrix},$$

according to the propagation law

$$M_i = T M_o T^t.$$

Similarity transformation

$$\begin{bmatrix} r_i \\ q_i \end{bmatrix} = T \begin{bmatrix} r_o \\ q_o \end{bmatrix}$$

Propagation law: $M_i = T M_o T^t$, with $T^{-1} = J T^t J$

Similarity transformation: $M_i J = T (M_o J) T^{-1}$

With $MJ = S \Lambda S^{-1}$, we have the relationships:

$\Lambda_i = \Lambda_o$. The eigenvalues of MJ remain invariant.

$S_i = T S_o$. The eigenvectors propagate like the rays do.

- If λ is an eigenvalue of MJ, then $-\lambda$ is an eigenvalue, too; this implies that the characteristic polynomial $\det(MJ - \lambda I)$ is a polynomial of λ^2.
- The eigenvalues of MJ are real.
- If M is proportional to a symplectic matrix, then it can be expressed in the form

$$M = m \begin{bmatrix} G^{-1} & G^{-1}H \\ HG^{-1} & G + HG^{-1}H \end{bmatrix},$$

  with m a positive scalar, G and H real symmetric 2×2 matrices, and G positive definite.
- If M is proportional to a symplectic matrix with a (positive) proportionality factor m ($m^4 = \det M$), then the two positive eigenvalues are equal to $+m$ and the two negative eigenvalues equal to $-m$.

- Symplecticity is preserved – with the same proportionality factor m – in (symplectic) first-order optical systems.
- If M is proportional to a symplectic matrix,

$$M = m \begin{bmatrix} G^{-1} & G^{-1}H \\ HG^{-1} & G + HG^{-1}H \end{bmatrix},$$

then the propagation law can be written in the bilinear form

$$H_i \pm iG_i = [C + D(H_o \pm iG_o)][A + B(H_o \pm iG_o)]^{-1},$$

which resembles the bilinear relationship that we already found for a spherical wave ($G = 0$: $H_i = [C + DH_o][A + BH_o]^{-1}$) .

Invariants − one-dimensional case

Moment matrix $M = \begin{bmatrix} m_{xx} & m_{xu} \\ m_{xu} & m_{uu} \end{bmatrix}$

$$0 = \det(MJ - \lambda I) = \lambda^2 - (m_{xx}m_{uu} - m_{xu}^2) = \lambda^2 - \det M$$

The two eigenvalues of MJ take the form $\lambda^{\pm} = \pm\sqrt{\det M}$, and we conclude that $\det M = m_{xx}m_{uu} - m_{xu}^2$ is an invariant under propagation through ABCD-systems.

Invariants – two-dimensional case

Moment matrix $M = \begin{bmatrix} m_{xx} & m_{xy} & m_{xu} & m_{xv} \\ m_{xy} & m_{yy} & m_{yu} & m_{yv} \\ m_{xu} & m_{yu} & m_{uu} & m_{uv} \\ m_{xv} & m_{yv} & m_{uv} & m_{vv} \end{bmatrix}$

$$\begin{aligned} 0 &= \det(MJ - \lambda I) \\ &= \lambda^4 - [(m_{xx}m_{uu} - m_{xu}^2) + (m_{yy}m_{vv} - m_{yv}^2) \\ &\qquad +2(m_{xy}m_{uv} - m_{xv}m_{yu})]\lambda^2 + \det M \end{aligned}$$

The factor in front of λ^2 is the sum of four 2×2 minors of M,

$$\begin{vmatrix} m_{xx} & m_{xu} \\ m_{xu} & m_{uu} \end{vmatrix} + \begin{vmatrix} m_{yy} & m_{yv} \\ m_{yv} & m_{vv} \end{vmatrix} + \begin{vmatrix} m_{xy} & m_{xv} \\ m_{yu} & m_{uv} \end{vmatrix} + \begin{vmatrix} m_{xy} & m_{yu} \\ m_{xv} & m_{uv} \end{vmatrix},$$

and, like $\det M$, remains invariant under propagation.

Second-order moments of Gaussian light

$$W(r,q) = 4\sqrt{\frac{\det G_1}{\det G_2}}\,\exp\left(-\begin{bmatrix} r \\ q \end{bmatrix}^t \begin{bmatrix} G_1 + HG_2^{-1}H^t & -HG_2^{-1} \\ -G_2^{-1}H^t & G_2^{-1} \end{bmatrix} \begin{bmatrix} r \\ q \end{bmatrix}\right)$$

The matrix of second-order moments takes the form

$$M = \begin{bmatrix} R & P \\ P^t & Q \end{bmatrix} = \frac{1}{2}\begin{bmatrix} G_1^{-1} & G_1^{-1}H \\ H^tG_1^{-1} & G_2 + H^tG_1^{-1}H \end{bmatrix},$$

and the matrices G_1, G_2, and H follow directly from the submatrices P, Q, and R of the moment matrix M:

$$G_1 = \tfrac{1}{2}R^{-1} = G_1^t\,, \quad G_2 = 2\,(Q - P^tR^{-1}P) = G_2^t\,, \quad H = R^{-1}P\,.$$

Expression for the twist of Gaussian light

$$M = \begin{bmatrix} R & P \\ P^t & Q \end{bmatrix} = \frac{1}{2} \begin{bmatrix} G_1^{-1} & G_1^{-1} H \\ H^t G_1^{-1} & G_2 + H^t G_1^{-1} H \end{bmatrix}$$

The twist (the asymmetry of the matrix H) can be expressed in terms of the moment matrices R and P:

$$X = PR - RP^t = R(H - H^t)R = \tfrac{1}{4} G_1^{-1}(H - H^t)G_1^{-1} = \tfrac{1}{4}(H - H^t)\det G_1^{-1}.$$

The twist X_o in the output plane of an $ABCD$ system reads

$$\begin{aligned} 4\,X_o &= (D - HB)^t\, G_1^{-1}\,(H - H^t)\, G_1^{-1}\,(D - HB) \\ &+ (D - HB)^t\, G_1^{-1}\, G_2\, B - B^t\, G_2\, G_1^{-1}\,(D - HB)\,, \end{aligned}$$

where G_1, G_2, and H represent input-plane matrices.

Special cases with respect to the twist

$$4\,X = G_1^{-1}(H - H^t)G_1^{-1} = (H - H^t)\det G_1^{-1}$$

$$\begin{aligned} 4\,X_o &= (D - HB)^t\, G_1^{-1}\,(H - H^t)\, G_1^{-1}\,(D - HB) \\ &+ (D - HB)^t\, G_1^{-1}\, G_2\, B - B^t\, G_2\, G_1^{-1}\,(D - HB) \end{aligned}$$

- propagation between conjugate planes: $B = 0$
  $X_o = X_i \det D$
- signal adaptation: $H_i = DB^{-1} = H_i^t$ and $H_o = -B^{-1}A = H_o^t$
  $X_o = X_i = 0$
- $G_1 = \sigma^2\, G_2$
  $4\,X_o = (D - HB)^t\, G_1^{-1}\,(H - H^t)\, G_1^{-1}\,(D - HB) + B^t\,(H - H^t)\,B/\sigma^2$
- symplecticity: $G_1 = \sigma^2\, G_2$ and $H = H^t$
  $X_o = X_i = 0$

Outline of the presentation

1. Linear signal dependence
   - Windowed (short-time) Fourier transform
   - Gabor expansion
   - Wavelet transform
2. Quadratic (bilinear) signal dependence
   - Wigner distribution
   - Application to partially coherent light
3. Relatives of the Wigner distribution
   - Ambiguity function
   - Cohen class – kernel design
   - Fractional Fourier transform

Part 3. Relatives of the Wigner distribution

This part deal with weighted versions of the Wigner distribution, together forming the so-called Cohen class of bilinear signal representations, and with functions that result from elementary operations on the Wigner distribution, like projecting this two-dimensional function in different directions. We will study such things as the Radon transform (well-known from computer tomography), the fractional Fourier transform, non-iterative methods to reconstruct the phase of the signal by only measuring two intensity profiles, etc., which subjects have direct applications in optics.

- M.J. Bastiaans, T. Alieva, L. Stankovic, “On rotated time-frequency kernels,” IEEE Signal Process. Lett., vol. 9, nr. 11, 2002, pp. 378-381.
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Outline of part 3

- Wigner distribution and ambiguity function
- Cohen class
  - Wigner, pseudo Wigner, Page, Rihaczek, and Levin
  - Spectrogram
  - Kernel design; some other kernels
- Fractional Fourier transform
  - Fractional FT and rotation in phase space
  - Optimal kernel alignment using fractional FT moments
  - Signal reconstruction from fractional FT intensity
- Modified pseudo Wigner distribution
  - Pure time / pure frequency / mixed time-frequency averaging
  - Cohen class kernel for the modified pseudo WD

$$\Gamma(t+\tfrac{1}{2}\tau,t-\tfrac{1}{2}\tau) \sim \varphi(t+\tfrac{1}{2}\tau)\,\varphi^*(t-\tfrac{1}{2}\tau)$$

Wigner distribution $W(t,f)$ $A(\tau,\nu)$ ambiguity function

$$\bar{\Gamma}(f+\tfrac{1}{2}\nu,f-\tfrac{1}{2}\nu) \sim \bar{\varphi}(f+\tfrac{1}{2}\nu)\,\bar{\varphi}^*(f-\tfrac{1}{2}\nu)$$

$$\bar{\Gamma}(f_1,f_2) = \iint \Gamma(t_1,t_2)\,e^{-i2\pi(f_1t_t - f_2t_2)}\,\mathrm{d}t_1\mathrm{d}t_2$$

$$\bar{\Gamma}(f+\tfrac{1}{2}\nu,f-\tfrac{1}{2}\nu) = \iint \Gamma(t+\tfrac{1}{2}\tau,t-\tfrac{1}{2}\tau)\,e^{-i2\pi(f\tau+\nu t)}\,\mathrm{d}t\mathrm{d}\tau$$

$$W(t,f) = \int \Gamma(t+\tfrac{1}{2}\tau,t-\tfrac{1}{2}\tau)\,e^{-i2\pi f\tau}\,\mathrm{d}\tau = \int \bar{\Gamma}(f+\tfrac{1}{2}\nu,f-\tfrac{1}{2}\nu)\,e^{i2\pi\nu t}\,\mathrm{d}\nu$$

$$A(\tau,\nu) = \int \Gamma(t+\tfrac{1}{2}\tau,t-\tfrac{1}{2}\tau)\,e^{-i2\pi\nu t}\,\mathrm{d}t = \int \bar{\Gamma}(f+\tfrac{1}{2}\nu,f-\tfrac{1}{2}\nu)\,e^{i2\pi f\tau}\,\mathrm{d}f$$

The ambiguity function is popular in radar signal processing.

t: time f: frequency τ: time lag ν: Doppler shift

The Wigner distribution $W(t,f)$ and the ambiguity function $A(\tau,\nu)$ form a Fourier / inverse Fourier transform pair:

$$W(t,f) = \iint A(\tau,\nu)\, e^{i2\pi(\nu t - f\tau)}\, \mathrm{d}\tau \mathrm{d}\nu$$

$$A(\tau,\nu) = \iint W(t,f)\, e^{i2\pi(f\tau - \nu t)}\, \mathrm{d}t \mathrm{d}f$$

Note that we have the important property: convolution in Wigner space corresponds to multiplication in ambiguity space.

Wigner, pseudo Wigner, Page, Rihaczek, and Levin

Wigner distribution $$W_\varphi(t,f)=\int_{-\infty}^{\infty}\varphi(t+\tfrac{1}{2}\tau)\,\varphi^*(t-\tfrac{1}{2}\tau)\,e^{-i2\pi f\tau}\,\mathrm{d}\tau$$

Pseudo Wigner distribution, with an additional window w:

$$P_\varphi(t,f;w)=\int_{-\infty}^{\infty}\varphi(t+\tfrac{1}{2}\tau)\,w(\tfrac{1}{2}\tau)\,w^*(-\tfrac{1}{2}\tau)\,\varphi^*(t-\tfrac{1}{2}\tau)\,e^{-i2\pi f\tau}\,\mathrm{d}\tau$$

Page: $$\frac{\mathrm{d}}{\mathrm{d}t}\left|\int_{-\infty}^{t}\varphi(t')e^{-i2\pi ft'}\mathrm{d}t'\right|^2=2\,\mathrm{Re}\left\{\int_0^{\infty}\varphi(t)\varphi^*(t-\tau)e^{-i2\pi f\tau}\mathrm{d}\tau\right\}$$

Rihaczek: $$\varphi(t)\,\bar{\varphi}^*(f)\,e^{-i2\pi ft}=\int\varphi(t)\,\varphi^*(t-\tau)\,e^{-i2\pi f\tau}\mathrm{d}\tau$$

Levin: $$\mathrm{Re}\left\{\varphi(t)\,\bar{\varphi}^*(f)\,e^{-i2\pi ft}\right\}=\mathrm{Re}\left\{\int\varphi(t)\,\varphi^*(t-\tau)\,e^{-i2\pi f\tau}\mathrm{d}\tau\right\}$$

Wigner distribution and spectrogram

Spectrogram: $$\left|S_\varphi(t,f)\right|^2 = \left|\int \varphi(t_\circ)\, w^*(t_\circ - t)\, e^{-i2\pi f t_\circ}\, \mathrm{d}t_\circ\right|^2$$

$$\left|S_\varphi(t,f)\right|^2 = \iint \varphi(t_1)\, \varphi^*(t_2)\, w^*(t_1 - t)\, w(t_2 - t)\, e^{-i2\pi f(t_1 - t_2)}\, \mathrm{d}t_1\mathrm{d}t_2$$

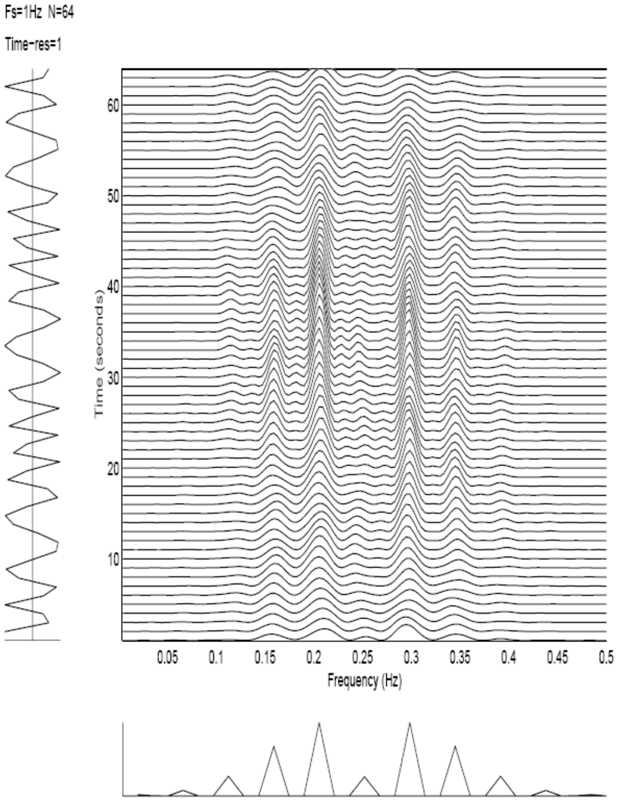
$$= \iint \left[\int W_\varphi(\tfrac{1}{2}(t_1 + t_2), f_\circ)\, e^{i2\pi f_\circ (t_1 - t_2)}\, \mathrm{d}f_\circ\right] \times w^*(t_1 - t)\, w(t_2 - t)\, e^{-i2\pi f(t_1 - t_2)}\, \mathrm{d}t_1\mathrm{d}t_2$$

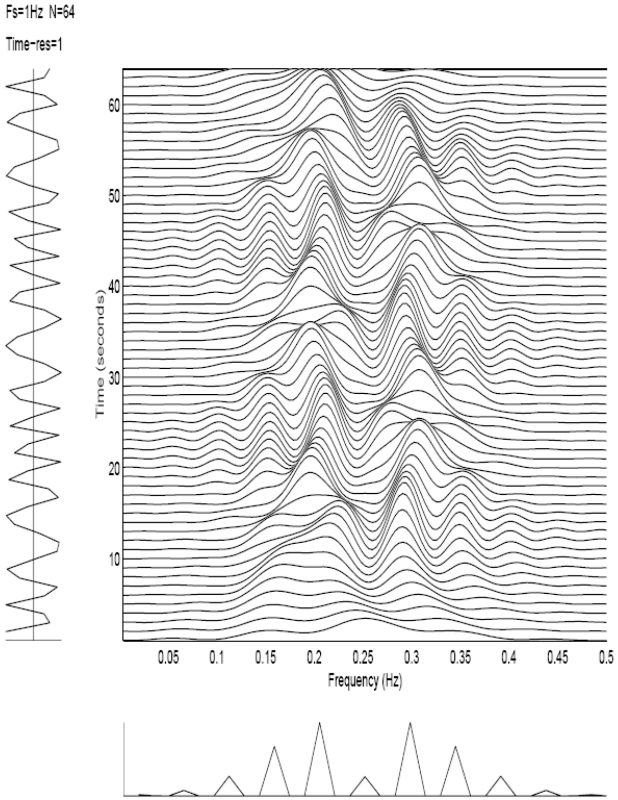
$$= \iint W_\varphi(t_\circ, f_\circ) \times \left[\int w(t_\circ - t + \tfrac{1}{2}t')\, w^*(t_\circ - t - \tfrac{1}{2}t')\, e^{-i2\pi(f_\circ - f)t'}\, \mathrm{d}t'\right]^* \mathrm{d}t_\circ\mathrm{d}f_\circ$$

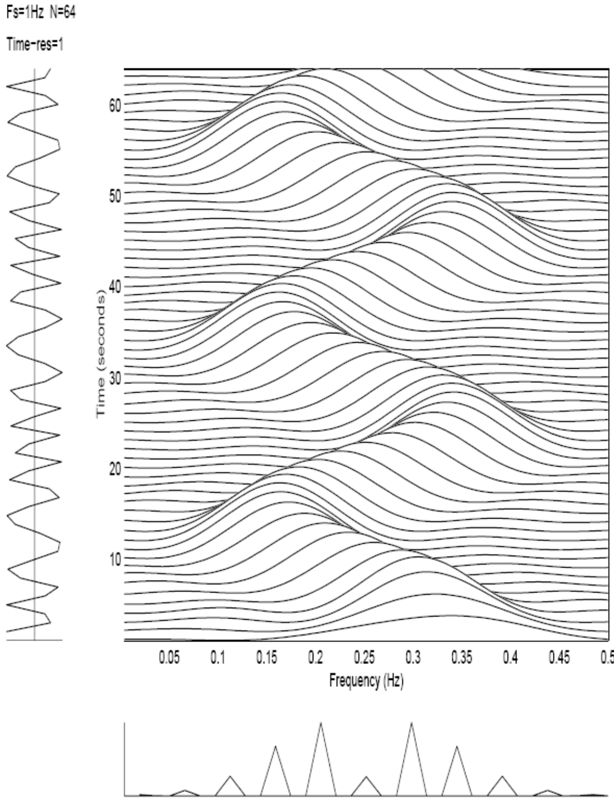
$$= \iint W_\varphi(t_\circ, f_\circ)\, W_w^*(t_\circ - t, f_\circ - f)\, \mathrm{d}t_\circ\mathrm{d}f_\circ = W_w^*(-t, -\omega) * W_\varphi(t, \omega)$$

Pure frequency description for $w = 1$; time description for $w = \delta(t)$.

Spectrogram of a sinusoidal FM signal







Spectrogram of a sinusoidal FM signal, with – from left to right – a broad, medium and narrow analysis window $w(t)$.

Wigner distribution: good localization, but suffers from cross-terms.
Spectrogram: less cross-terms problems, but suffers from smoothing.

Cohen class: convolution of $W_\varphi(t, f)$ with a kernel $K(t, f)$

$$C_\varphi(t, f) = \iint K(t - t_\circ, f - f_\circ)\, W_\varphi(t_\circ, f_\circ)\, \mathrm{d}t_\circ \mathrm{d}f_\circ = K(t, f) * W_\varphi(t, f)$$

Convolution keeps the time and frequency shift covariance:

$$\varphi(t - t_\circ)\, e^{i2\pi f_\circ t} \rightarrow W_\varphi(t - t_\circ, f - f_\circ)\,.$$

In ambiguity space we have the product $\bar{C}_\varphi(\tau, \nu) = \bar{K}(\tau, \nu)\, A_\varphi(\tau, \nu)\,.$

Kernel design (looking for kernels that lead to a good localization together with suppression of the cross-terms) can best be done in ambiguity space.

Depending on the actual form of the kernel $K(t,f)$, properties of the Wigner distribution $W_\varphi(t,f)$ may also hold for $C_\varphi(t,f)$.

$C_\varphi(t,f) = C_\varphi^*(t,f)$ $\qquad$ $\bar{K}(\tau,\nu) = \bar{K}^*(-\tau,-\nu)$ $\qquad$ real

$\int C_\varphi(t,f)\,\mathrm{d}f = |\varphi(t)|^2$ $\qquad$ $\bar{K}(0,\nu) = 1$ $\qquad$ time marginal

$\int C_\varphi(t,f)\,\mathrm{d}t = |\bar{\varphi}(f)|^2$ $\qquad$ $\bar{K}(\tau,0) = 1$ $\qquad$ frequency marginal

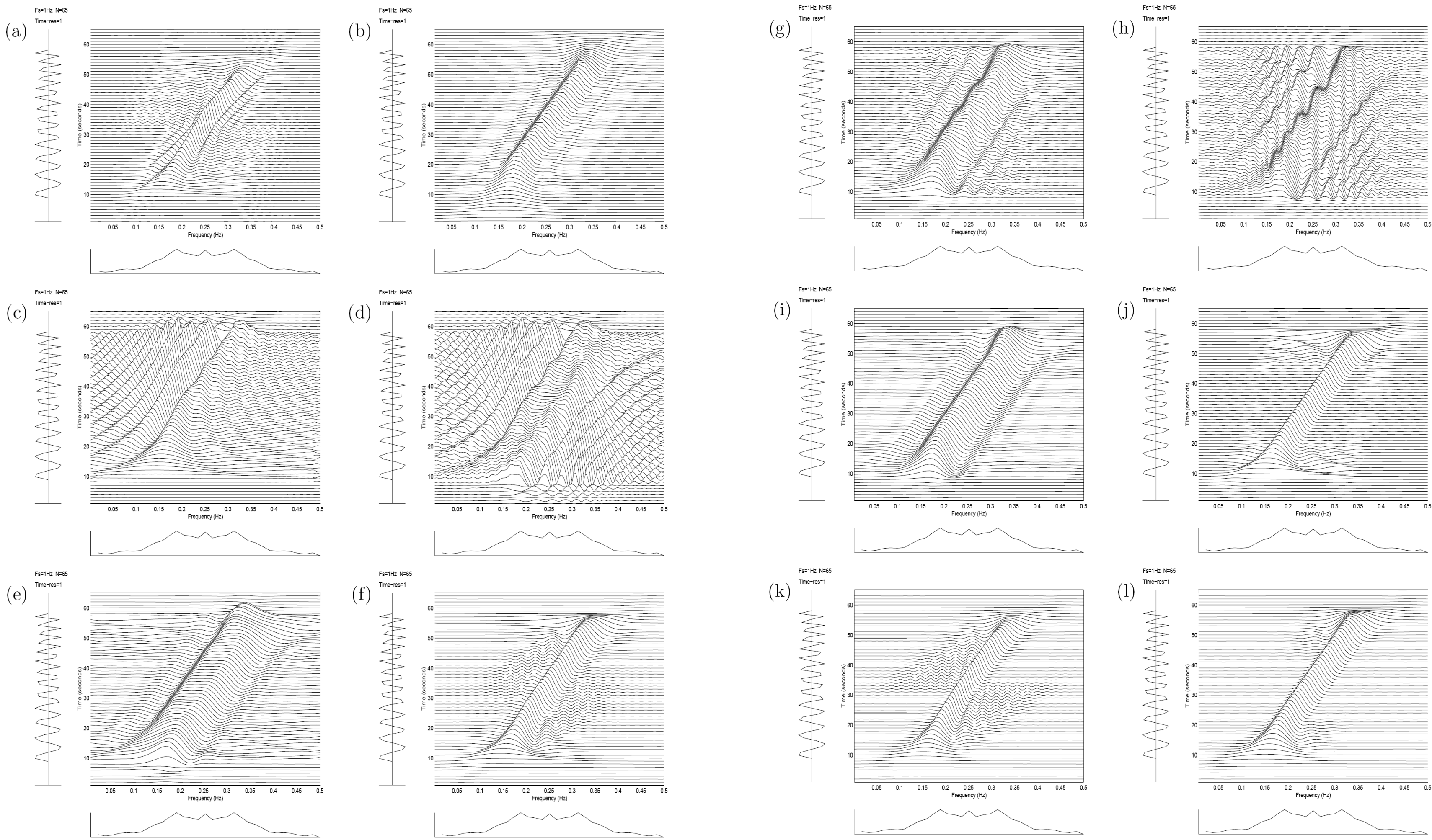
$$\frac{\iint f C_\varphi(t,f)\mathrm{d}t\mathrm{d}f}{\iint C_\varphi(t,f)\mathrm{d}t\mathrm{d}f} = \frac{1}{2\pi}\frac{\mathrm{d}\arg\varphi(t)}{\mathrm{d}t} \qquad \bar{K}(0,\nu) = \text{constant} \qquad \left.\frac{\partial \bar{K}}{\partial \tau}\right|_{\tau=0} = 0$$

instantaneous frequency

Some well-known Cohen-class kernels $\bar{K}(\tau,\nu)$

- Wigner / Pseudo Wigner $\quad 1 \quad / \quad w(\tfrac{1}{2}\tau)\, w^*(-\tfrac{1}{2}\tau)$
- Page/Rihaczek/Levin $\quad e^{-i\pi\nu|\tau|} \quad / \quad e^{-i\pi\nu\tau} \quad / \quad \cos(\pi\nu\tau)$
- Born-Jordan $\quad \dfrac{\sin(\pi\nu\tau)}{\pi\nu\tau}$
- Zhao-Atlas-Marks (cone) $\quad g(\tau)|\tau|\,\dfrac{\sin(\pi\nu\tau)}{\pi\nu\tau}$
- Choi-Williams (exponential) $\quad e^{-(2\pi\tau\nu)^2/\sigma}$
- generalized exponential $\quad e^{-(\nu/\nu_o)^{2N}(\tau/\tau_o)^{2M}}$

Cohen-class time-frequency distributions of a chirp



Wigner, spectrogram, Page, Levin, w-Levin, Choi-Williams; Born-Jordan, Zhao-Atlas-Mark ($2\times$), DI, LI, separable

Fractional Fourier transform

$$\int W_\varphi(t,f)\,\mathrm{d}f = |\varphi(t)|^2 \qquad \text{projections} \qquad \int W_\varphi(t,f)\,\mathrm{d}t = |\bar{\varphi}(f)|^2$$

projection angle α: $$\iint W_\varphi(t,f)\,\delta(t\cos\alpha + f\sin\alpha - u)\,\mathrm{d}t\mathrm{d}f = |\bar{\varphi}_\alpha(u)|^2$$
$$= \int A_\varphi(r\sin\alpha, -r\cos\alpha)\,e^{-i2\pi ur}\,\mathrm{d}r$$

fractional Fourier transform:

$$\bar{\varphi}_\alpha(u) = \begin{cases} \dfrac{\exp(i\alpha/2)}{\sqrt{i\sin\alpha}} \displaystyle\int_{-\infty}^{\infty} \exp\left[i\pi\frac{(u^2+t^2)\cos\alpha - 2\,ut}{\sin\alpha}\right] \varphi(t)\,\mathrm{d}t & (\alpha \neq 0) \\ \varphi(u) & (\alpha = 0) \end{cases}$$

We remark the relation with the Radon transform (computer-aided tomography).

Fractional Fourier transform and rotation in phase space

$\varphi(t) \rightarrow W_\varphi(t, f)$ **Wigner distribution**

$\bar{\varphi}_\alpha(t) \rightarrow W_{\bar{\varphi}_\alpha}(t, f) = W_\varphi(t \cos\alpha - f \sin\alpha, t \sin\alpha + f \cos\alpha)$ **rotation**

$\varphi(t) \rightarrow A_\varphi(\tau, \nu)$ **Ambiguity function**

$\bar{\varphi}_\alpha(t) \rightarrow A_{\bar{\varphi}_\alpha}(\tau, \nu) = A_\varphi(\tau \cos\alpha - \nu \sin\alpha, \tau \sin\alpha + \nu \cos\alpha)$ **rotation**

$C_\varphi(t, f) = K(t, f) * W_\varphi(t, f)$ **Cohen class**

rotated kernel: $C_\varphi(t, f) = K(t \cos\alpha + f \sin\alpha, -t \sin\alpha + f \cos\alpha) * W_\varphi(t, f)$

We can also write: $C_{\bar{\varphi}_\alpha}(t, f) = K(t, f) * W_{\bar{\varphi}_\alpha}(t, f)$

Optimum kernel alignment using fractional FT moments

First-order moment of $\bar{\varphi}_\alpha(t)$: $m_\alpha = \int t \, |\bar{\varphi}_\alpha(t)|^2 \, \mathrm{d}t$

Second-order moment of $\bar{\varphi}_\alpha(t)$: $w_\alpha = \int t^2 \, |\bar{\varphi}_\alpha(t)|^2 \, \mathrm{d}t$

Second-order central moment of $\bar{\varphi}_\alpha(t)$: $p_\alpha = w_\alpha - m_\alpha^2$

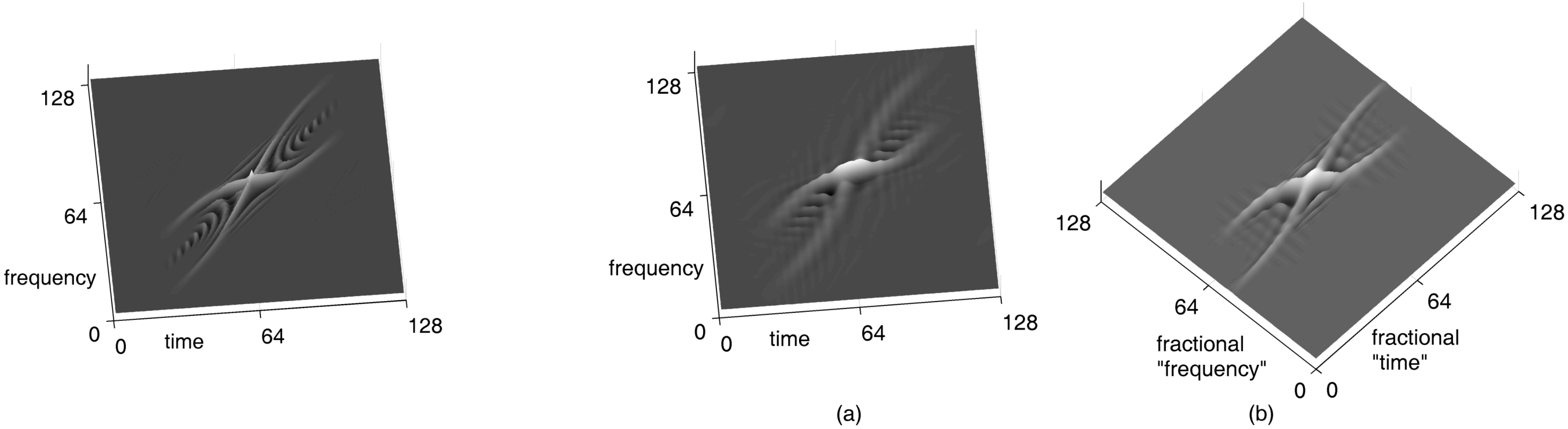
$$p_\alpha = p_0 \cos^2\alpha + p_{\pi/2} \sin^2\alpha + [w_{\pi/4} - m_0 m_{\pi/2} - \tfrac{1}{2}(w_0 + w_{\pi/2})] \sin 2\alpha$$

Extremum widths for $\tan 2\alpha_e = \dfrac{2(w_{\pi/4} - m_0 m_{\pi/2}) - (w_0 + w_{\pi/2})}{p_0 - p_{\pi/2}}$

Kernels can be aligned for optimum performance by rotating them to α_e; the angle α_e follows from measuring only three fractional FTs.

Optimum kernel alignment 2

Extremum widths for $\tan 2\alpha_e = \dfrac{2(w_{\pi/4} - m_0 m_{\pi/2}) - (w_0 + w_{\pi/2})}{p_0 - p_{\pi/2}}$

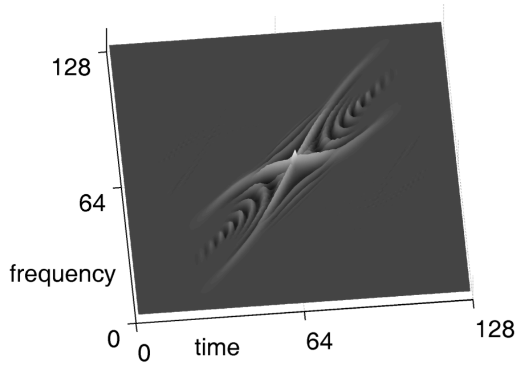


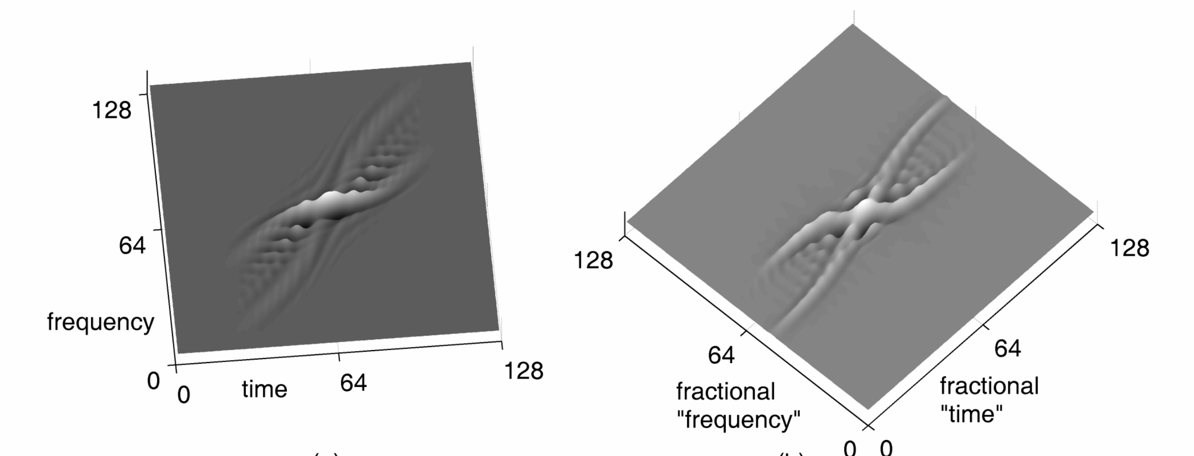
Pseudo WD, Generalized exponential distribution ($M = 1$, $N = 3$) without alignment, and with alignment ($\alpha_e = 41^\circ$) of the kernel

$$\exp[-(3t)^8]\{\exp[j(192\pi t^2 - 8\cos(4\pi t)/\pi)] + \exp[j(64\pi t^2 + 8\cos(4\pi t)/\pi)]\}$$

Optimum kernel alignment 3

Extremum widths for $\tan 2\alpha_e = \dfrac{2(w_{\pi/4} - m_0 m_{\pi/2}) - (w_0 + w_{\pi/2})}{p_0 - p_{\pi/2}}$





(a) (b)

Pseudo WD, Zhao-Atlas-Marks (cone kernel) distribution $[g(\tau) = \cos^4(\pi\tau)]$ without, and with alignment $(\alpha_e = 41^\circ)$ of the kernel

$\exp[-(3t)^8]\{\exp[j(192\pi t^2 - 8\cos(4\pi t)/\pi)] + \exp[j(64\pi t^2 + 8\cos(4\pi t)/\pi)]\}$

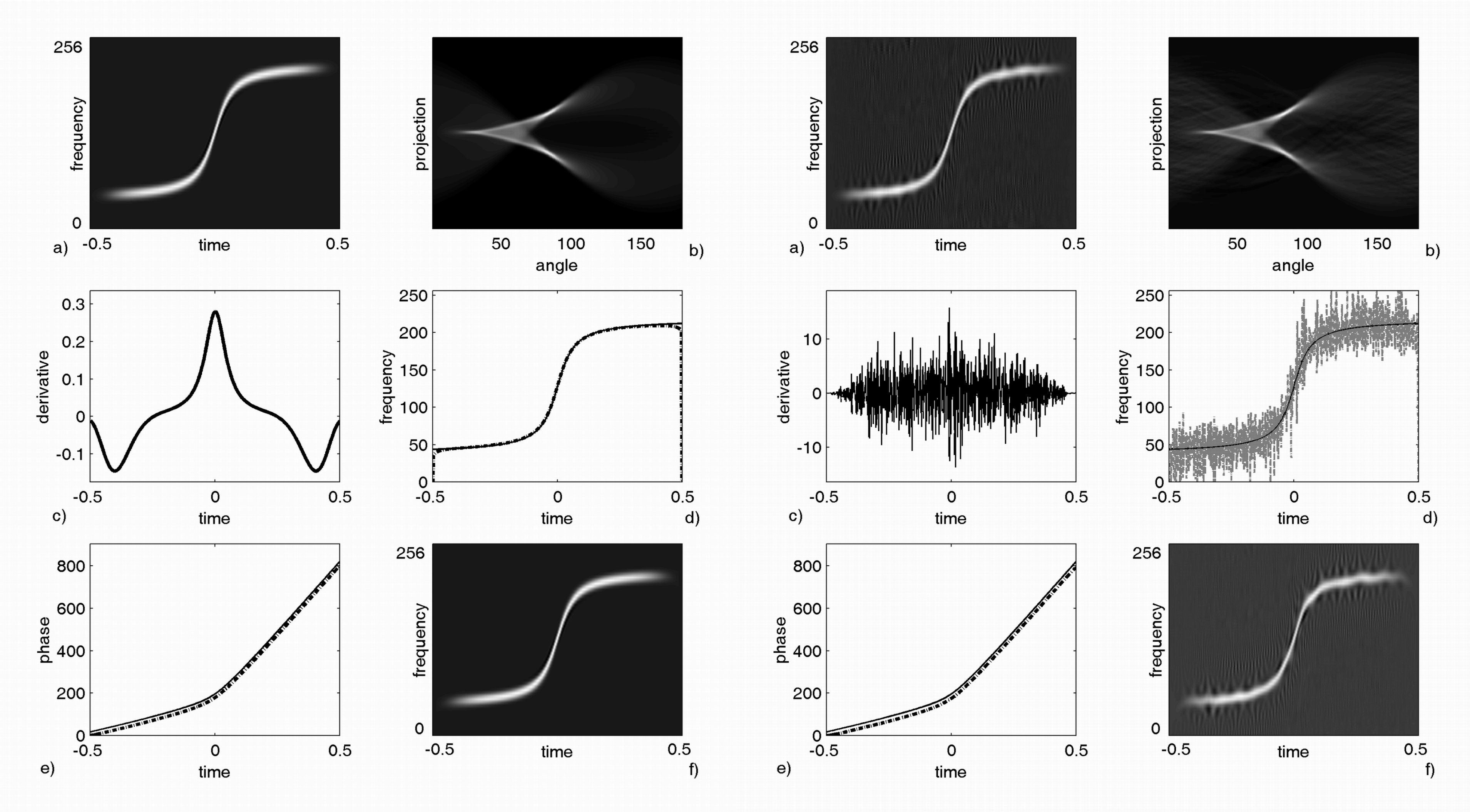
$$\frac{1}{2\pi}\frac{\mathrm{d}\arg\varphi(t)}{\mathrm{d}t} = \frac{\iint fW_\varphi(t,f)\mathrm{d}t\mathrm{d}f}{\iint W_\varphi(t,f)\mathrm{d}t\mathrm{d}f} = \frac{1}{2\pi i}\frac{1}{|\varphi(t)|^2}\int \left.\frac{\partial A_\varphi(\tau,\nu)}{\partial\tau}\right|_{\tau=0} e^{i2\pi t\nu}\mathrm{d}\nu$$

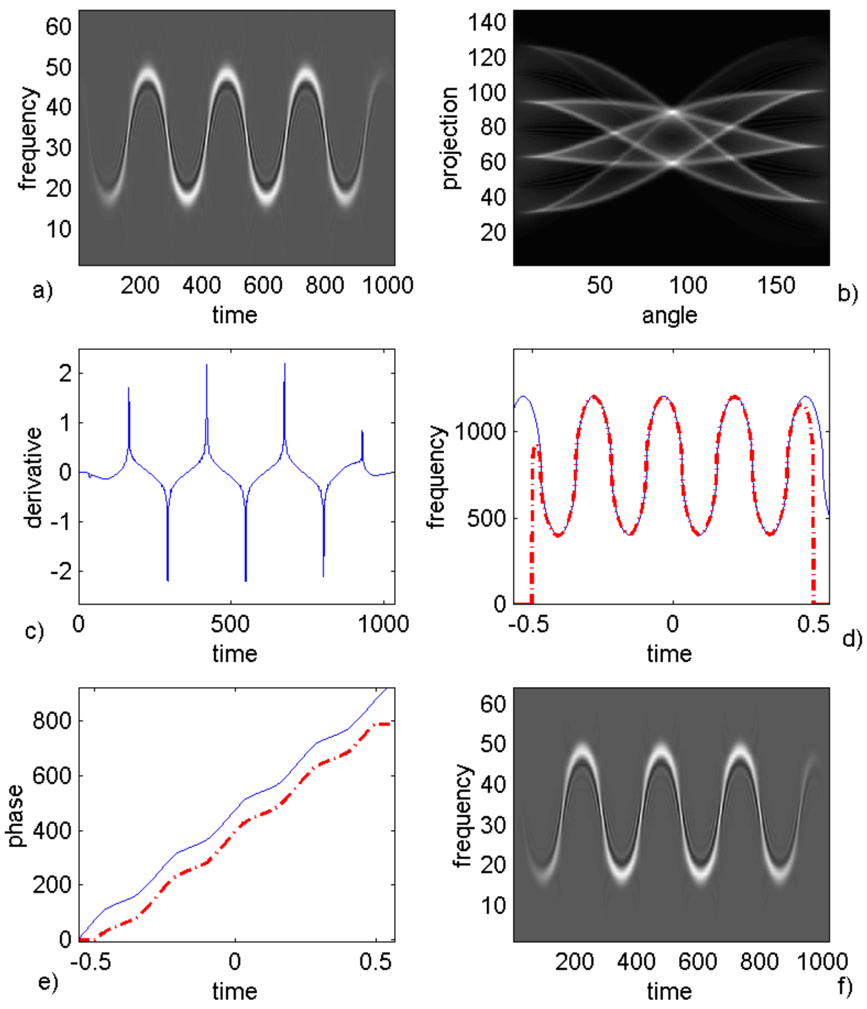
With $\left.\dfrac{\partial A_\varphi(\tau,\nu)}{\partial\tau}\right|_{\tau=0} = -\dfrac{1}{\nu}\displaystyle\int \left.\dfrac{\partial|\bar{\varphi}_\alpha(u)|^2}{\partial\alpha}\right|_{\alpha=0} e^{-i2\pi\nu u}\mathrm{d}u\,,$ we get

$$\frac{1}{2\pi}\frac{\mathrm{d}\arg\varphi(t)}{\mathrm{d}t} = \frac{-1}{2\,|\bar{\varphi}_0(t)|^2}\int \left.\frac{\partial|\bar{\varphi}_\alpha(u)|^2}{\partial\alpha}\right|_{\alpha=0} \mathrm{sgn}(t-u)\,\mathrm{d}u$$

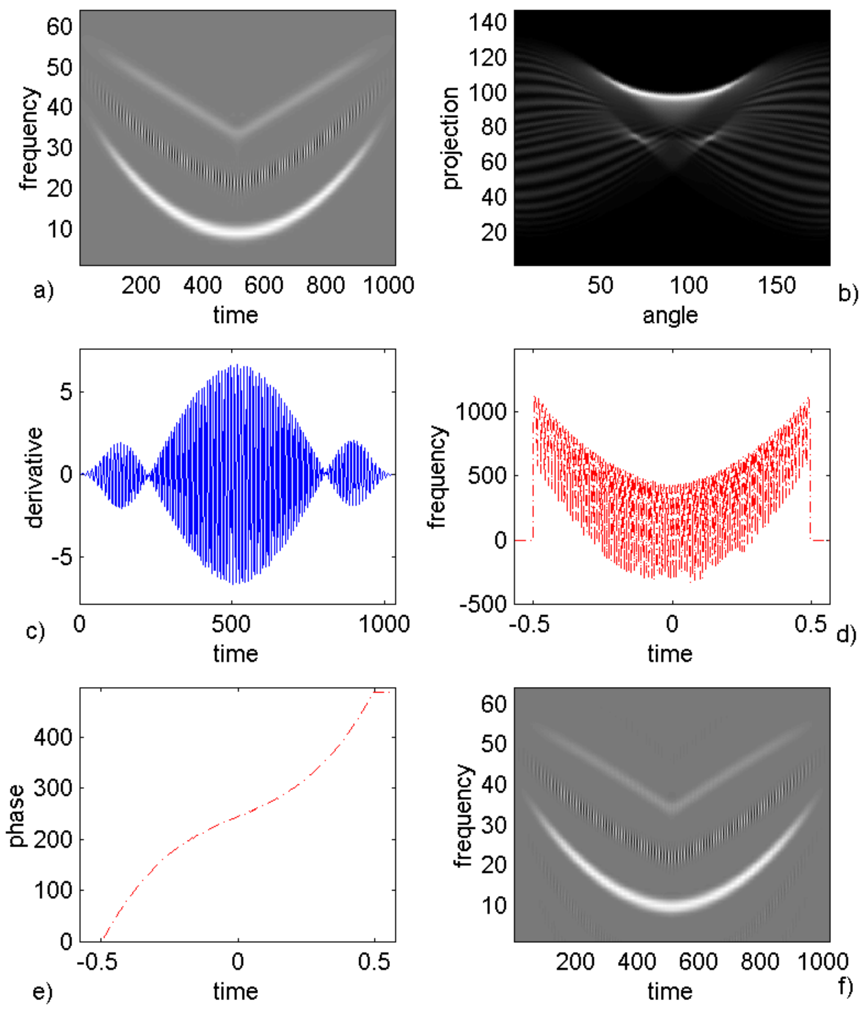
and we conclude that the instantaneous frequency, and hence the phase of $\varphi(t)$, can be reconstructed from the derivative of the fractional Fourier transform intensity distribution $|\bar{\varphi}_\alpha(t)|^2$ with respect to the fractional angle α.

Signal reconstruction from fractional FT intensity 2





a)
frequency
time
b)
projection
angle
c)
derivative
time
d)
frequency
time
e)
phase
time
f)
frequency
time



a)
frequency
time
b)
projection
angle
c)
derivative
time
d)
frequency
time
e)
phase
time
f)
frequency
time

$$W_\varphi(t,f) = \int \varphi(t+\tfrac{1}{2}\tau)\,\varphi^*(t-\tfrac{1}{2}\tau)\,e^{-i2\pi f\tau}\,\mathrm{d}\tau$$ **Wigner distribution**

Pseudo Wigner distribution, with an additional window w:

$$P_\varphi(t,f;w) = \int \varphi(t+\tfrac{1}{2}\tau)\,w(\tfrac{1}{2}\tau)\,w^*(-\tfrac{1}{2}\tau)\,\varphi^*(t-\tfrac{1}{2}\tau)\,e^{-i2\pi f\tau}\,\mathrm{d}\tau$$

In terms of the windowed Fourier transform

$$S_\varphi(t,f;w) = \int \varphi(t+t_\circ)\,w^*(t_\circ)\,e^{-i2\pi f t_\circ}\,\mathrm{d}t_\circ$$

we can also write $P_\varphi(t,f;w) = \int S_\varphi(t,f+\tfrac{1}{2}\theta;w)\,S^*_\varphi(t,f-\tfrac{1}{2}\theta;w)\,\mathrm{d}\theta\,.$

$$P_\varphi(t,f;w) = \int S_\varphi(t,f+\tfrac{1}{2}\theta;w)\,S^*_\varphi(t,f-\tfrac{1}{2}\theta;w)\,\mathrm{d}\theta$$

Frequency-modified version, with an additional window $z(\theta)$:

$$P_\varphi(t,f;w,z) = \int S_\varphi(t,f+\tfrac{1}{2}\theta;w)\,z(\theta)\,S^*_\varphi(t,f-\tfrac{1}{2}\theta;w)\,\mathrm{d}\theta$$

Pseudo Wigner distribution for $z(\theta)=1$; spectrogram for $z(\theta)=\delta(\theta)$

Time-frequency-modified version, with a rotation angle α:

$$P_\varphi^\alpha(t,f;w,z) = \int S_\varphi(t+\tfrac{1}{2}\theta\sin\alpha, f+\tfrac{1}{2}\theta\cos\alpha;w)\,z(\theta)\,e^{-i2\pi f\theta\sin\alpha} \times S^*_\varphi(t-\tfrac{1}{2}\theta\sin\alpha, f-\tfrac{1}{2}\theta\cos\alpha;w)\,\mathrm{d}\theta$$

Pure frequency modifying for $\alpha=0$; pure time modifying for $\alpha=\tfrac{1}{2}\pi$

Also: $$P_{\varphi}^{\alpha}(t,f;w,z)=\int S_{\bar{\varphi}_\alpha}(u,v+\tfrac{1}{2}\theta;w)\,z(\theta)\,S^*_{\bar{\varphi}_\alpha}(u,v-\tfrac{1}{2}\theta;w)\,\mathrm{d}\theta$$

with $$\begin{aligned} S_{\bar{\varphi}_\alpha}(u,v;w) &= \int \bar{\varphi}_\alpha(u+u_\circ)\,w^*(u_\circ)\,e^{-i2\pi v u_\circ}\,\mathrm{d}u_\circ \\ &= e^{i\pi(uv-tf)}\int \varphi(t+t_\circ)\,\bar{w}^*_{-\alpha}(t_\circ)\,e^{-i2\pi f t_\circ}\,\mathrm{d}t_\circ \\ &= e^{i\pi(uv-tf)}\,S_\varphi(t,f;\bar{w}_{-\alpha}) \end{aligned}$$

$$\begin{bmatrix} t \\ f \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

The windowed Fourier transform $S_{\bar{\varphi}_\alpha}(u,v;w)$ of the fractional Fourier transform $\bar{\varphi}_\alpha(u)$ of a signal $\varphi(t)$ corresponds to the windowed Fourier transform $S_\varphi(t,f;\bar{w}_{-\alpha})$ of the signal itself, with the window $\bar{w}_{-\alpha}(t)$ being the fractional FT of the initial one $w(t)$.

Cohen kernels for the modified pseudo Wigner distribution

$C_\varphi(t,f) = K(t,f) * W_\varphi(t,f)$ Cohen class in Wigner space

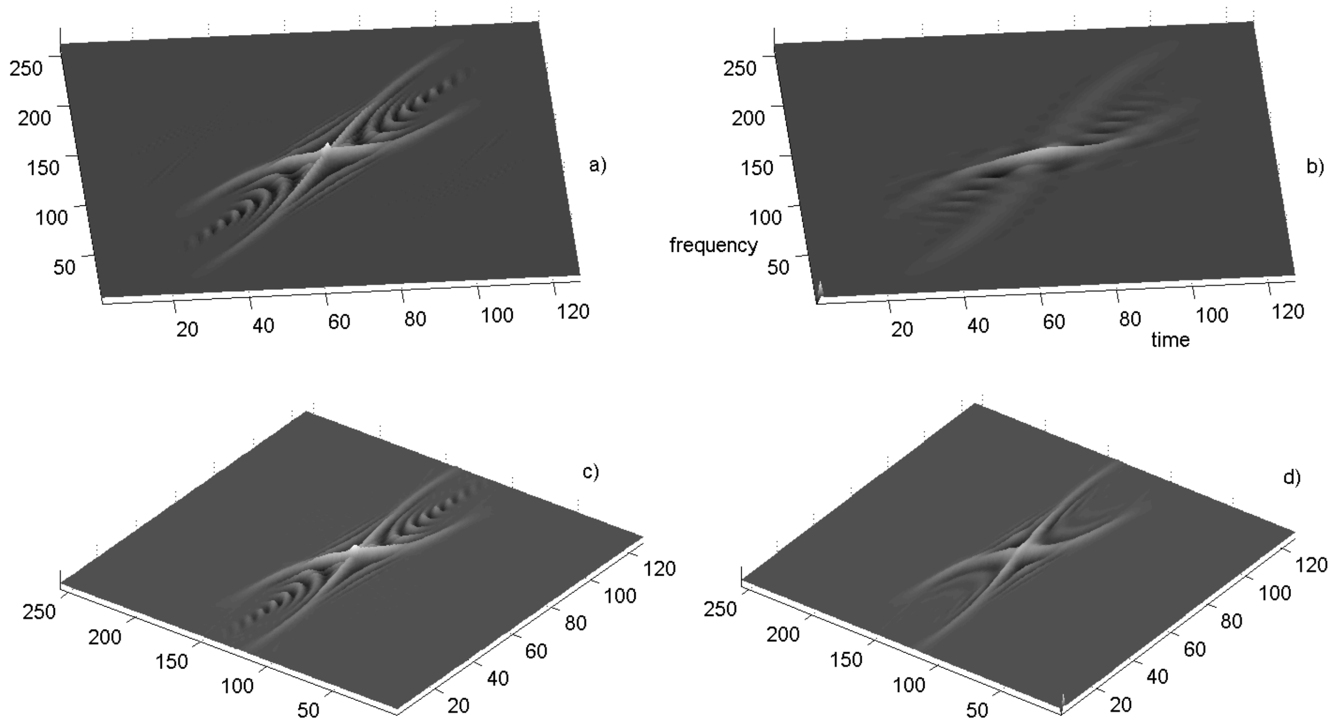
$P_\varphi^\alpha(t,f;w,z) \quad : \quad K(t,f) = W_w(-t,-f)\,\bar{z}(-t\cos\alpha + f\sin\alpha)$

$\bar{C}_\varphi(\tau,\nu) = \bar{K}(\tau,\nu)\,A_\varphi(\tau,\nu)$ Cohen class in ambiguity space

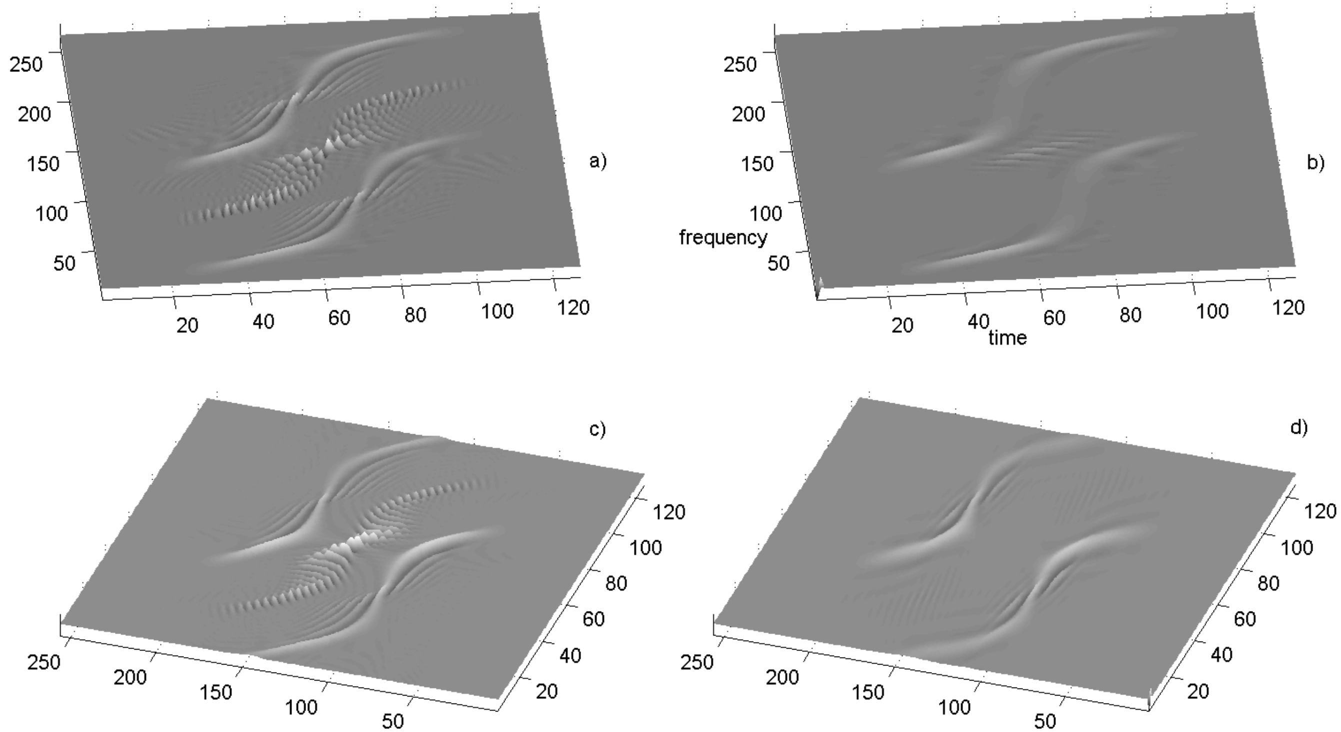
$P_\varphi^\alpha(t,f;w,z) \quad : \quad \bar{K}(t,f) = \int A_w(-\tau + \theta\sin\alpha, -\nu + \theta\cos\alpha)\,z(\theta)\,\mathrm{d}\theta$

Find the optimum fractional angle α_e from fractional FT moments:

$$\tan 2\alpha_e = \frac{2(w_{\pi/4} - m_0 m_{\pi/2}) - (w_0 + w_{\pi/2})}{p_0 - p_{\pi/2}}.$$



a) Pseudo WD, b) Frequency-modified, c) Rotated WD (WD of the fractional FT), d) Modified in the optimum fractional domain



a) Pseudo WD, b) Frequency-modified, c) Rotated WD (WD of the fractional FT), d) Modified in the optimum fractional domain