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WINTER COLLEGE
on
QUANTUM AND CLASSICAL ASPECTS
of
INFORMATION OPTICS

30 January - 10 February 2006

Optical Integral Transforms for Information Processing

Lecture 1: From geometric to wave optics

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Lecture 1: From geometric to wave optics

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Outlines

- Optics and information
- Fourier transform - key for information processing
- Wave optics: Helmholtz equation and spatial frequency spectrum
- Paraxial approximation in wave and geometric optics
- Fresnel and Fraunhofer diffraction
- Phase transformation of thin lens
- Composite system: canonical integral transform and related ray transformation matrix

Optics and information

- acquisition: optical microscopy, optical tomography, speckle imaging, spectroscopy, metrology, optical velocimetry, optical particle manipulation
- processing: image enhancement, feature extraction, texture analysis, classification
- transmission: optical fibres
- archiving: magneto-optical and optical disks, development of holographic memories

Different approaches for optical information processing

- Coherent light
- Incoherent light
- Analogue optical processing
- Digital optical processing

Different approaches for optical information processing

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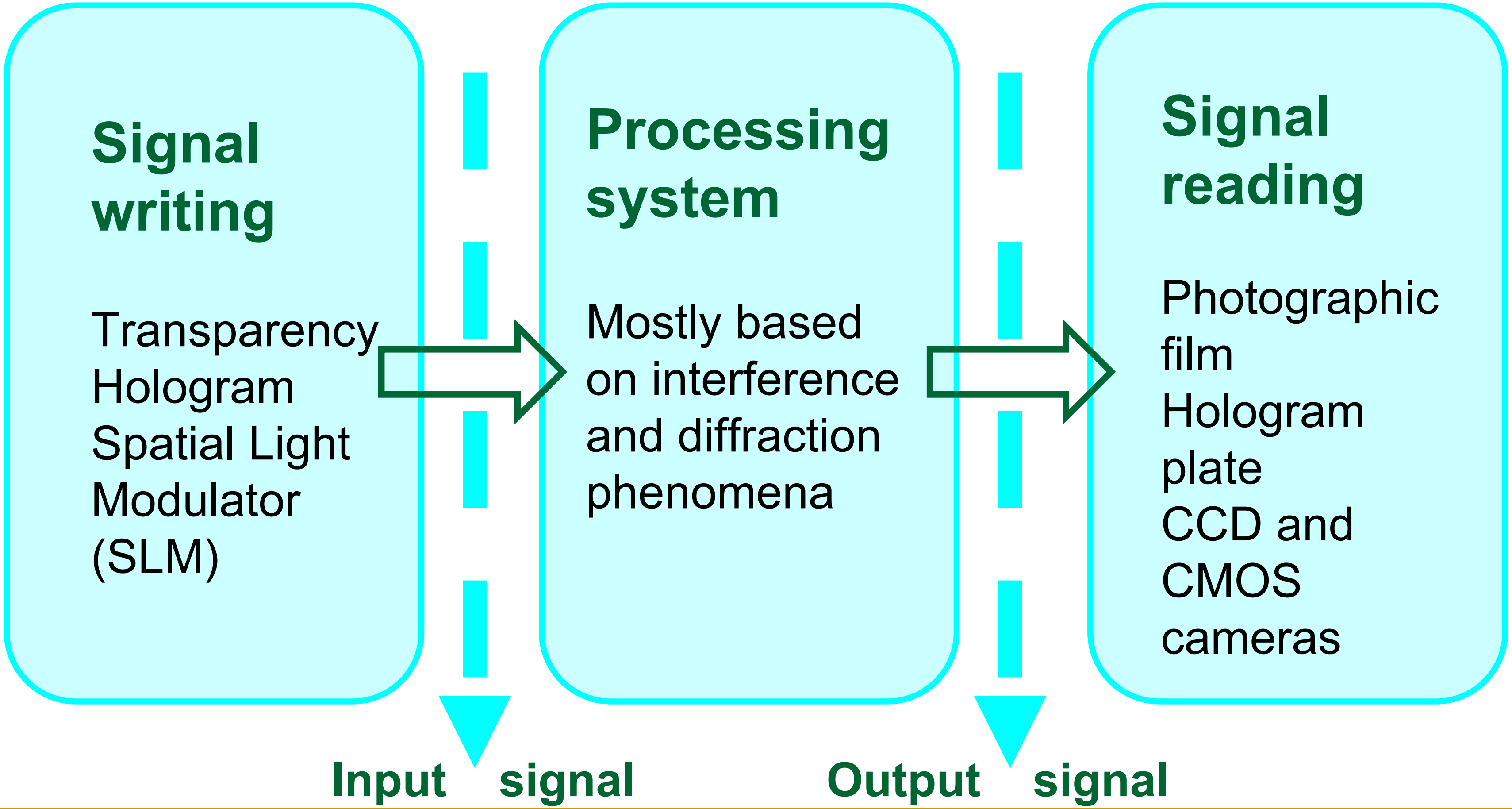
Optical information processing

Optical system transforms the signal in order to

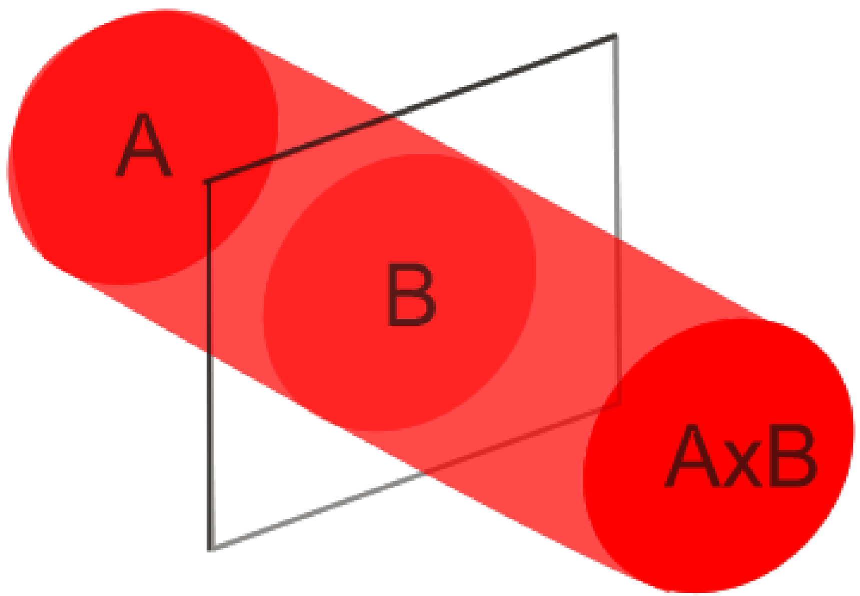
- improve its quality
- perform various operations on it: differentiation, integration, rotation, etc…
- extract its characteristics
- localize a specific pattern
- encrypt/decrypt the information

What we need for optical information processing



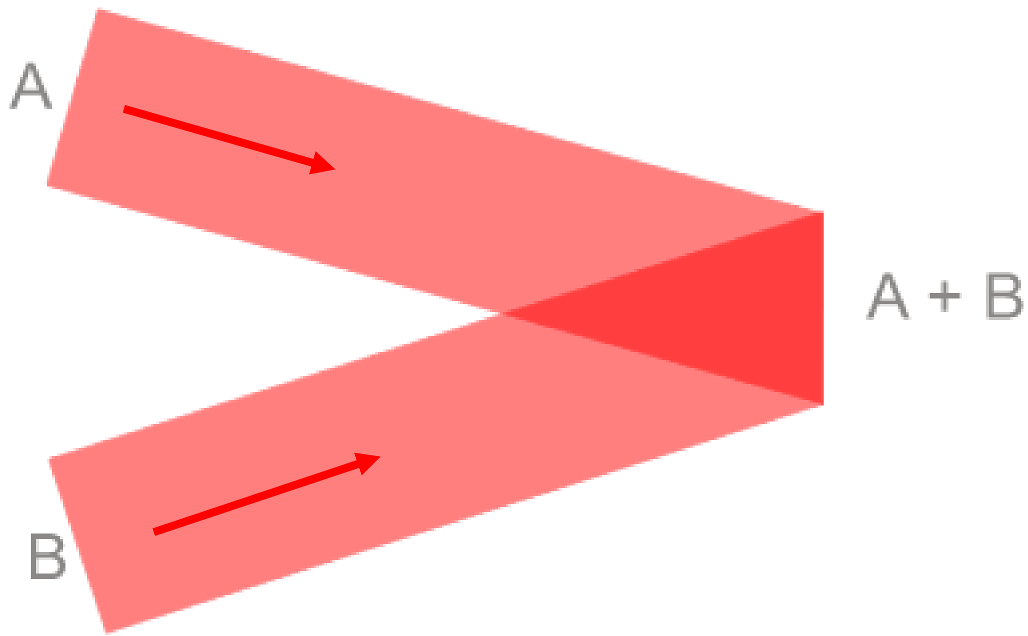
Multiplication operation

- Light propagation through (or reflection from)
  - transparency (with mirror)
  - hologram
  - spatial light modulator



Addition operation

- Superposition principle



Signal description

- Coherent (deterministic) monochromatic signals ⟹ complex field amplitude $f(x,y)$
  - In the coordinate (position) domain $f(x,y)$
  - In the spatial frequency domain $F(u,v)$

- Fourier transform

$$F\left(u,v\right)=FT\left\{f\left(x,y\right)\right\}=\iint f\left(x,y\right)\exp\left[-i2\pi(xu+yv)\right]dxdy$$

- Partially coherent (stochastic) signals ⟹ Mutual intensity distribution, Wigner distribution function

Spatial frequencies

- Low frequencies ⇨ rough image structure
- High frequencies ⇨ fine structure (object edges)

Image

low pass filtering

high pass filtering

Fourier transform properties

- Linearity $FT\left\{\sum_k b_k f_k(x,y)\right\} = \sum_k b_k F_k(u,v)$
- Inverse FT

$$f(x,y) = FT^{-1}\{F(u,v)\} = \iint F(u,v)\exp[i2\pi(xu+yv)]dudv$$

- Shift theorem

$$\iint f(x-a,y-b)\exp\left[-i2\pi(xu+yv)\right]dxdy = \exp\left[-i2\pi(au+bv)\right]F(u,v)$$

- Scaling theorem

$$\iint f(x/a,y/b)\exp\left[-i2\pi(xu+yv)\right]dxdy = |a||b|F(au,bv)$$

- Derivation and FT

$$\frac{\partial^{k+l} f(x,y)}{\partial x^k \partial^l y} = (i2\pi)^{k+l} \iint u^k v^l F(u,v)\exp\left[i2\pi(xu+yv)\right]dudv$$

A. D. Poularikas, ed., *The Transforms and Applications Handbook*, CRC Press, Alabama, (1996)

Fourier transform and convolution

- Convolution of two signals (filtering operation)

$$g(x,y) = f(x,y) * h(x,y) = \iint f(\xi,\eta)\, h(x-\xi, y-\eta)\, d\xi\, d\eta$$

- FT of convolution

$$G(u,v) = FT\{g(x,y)\} = F(u,v)\, H(u,v)$$

$$\boxed{g(x,y) = FT^{-1}\{G(u,v)\} = FT^{-1}\{F(u,v)\, H(u,v)\}}$$

R. N. Bracewell, ed., *The Fourier Transform and Its Applications,* McGraw-Hill, New York, chapter 3 (1978)

Correlation

- Convolution of $f(x,y)$ and $h^*(-x,-y)$ is called correlation

$$c(x,y) = f(x,y) \otimes h(x,y) = f(x,y) * h^*(-x,-y)$$

$$\boxed{c(x,y) = \iint f(\xi,\eta)\, h^*(\xi - x, \eta - y)\, d\xi\, d\eta}$$

- Using FT

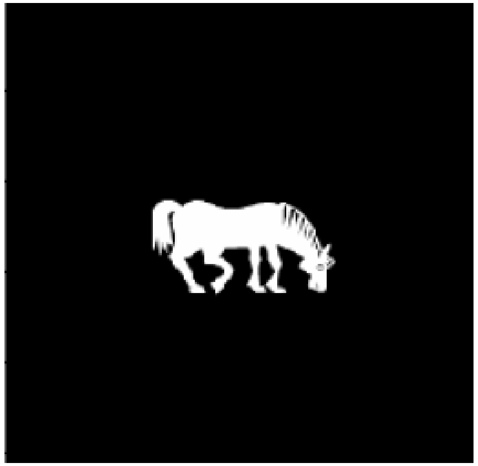
$$\boxed{c(x,y) = FT^{-1}\left\{F(u,v)\, H^*(u,v)\right\}}$$

- Autocorrelation $c_a(x,y) = FT^{-1}\left\{\left|F(u,v)\right|^2\right\}$

  has a maximum in the coordinate origin

- Correlation is a measure of similarity between two signals

Correlation: example



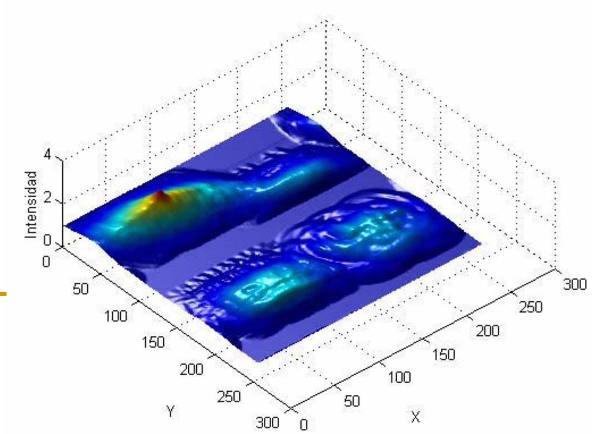
Image

Amplitude of autocorrelation

Horse and other objects

Amplitude of cross correlation with the image of the horse

- Inequality of Schwarz permits to discriminate two signals of equal energy $\left|c_a(0)\right| \geq \left|c(0)\right|$
- Localization of the cross correlation peak indicates the position of the image at the scene



Fourier transform

- Fourier transform is a key for information processing

Optical Fourier transform

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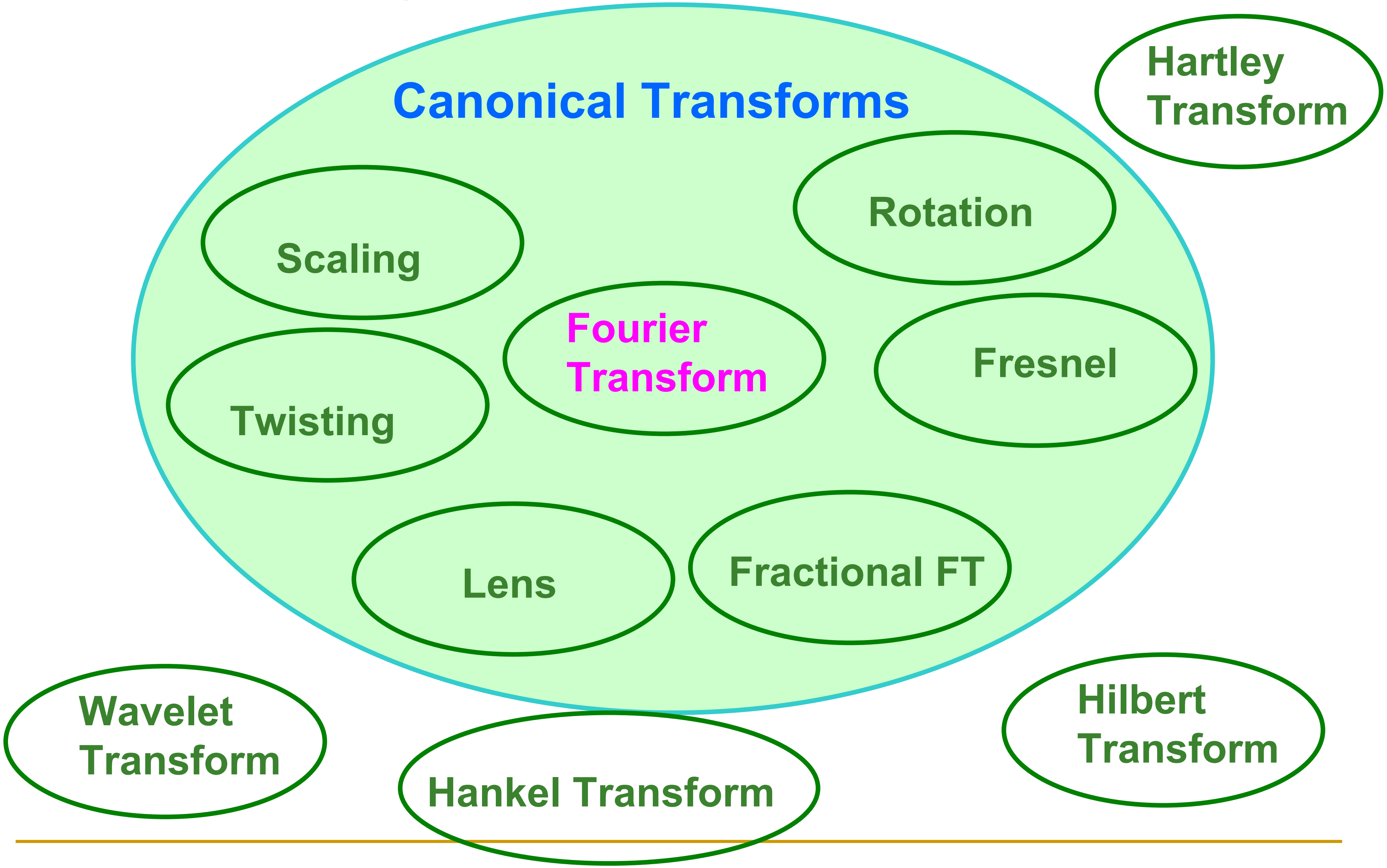
Optical signal multiplication and addition

⇩

Optical computing and information processing (integral transformations, derivation, filtering, recognition, …)

Convolution, Correlation, Wavelet, Hilbert and many other transforms…

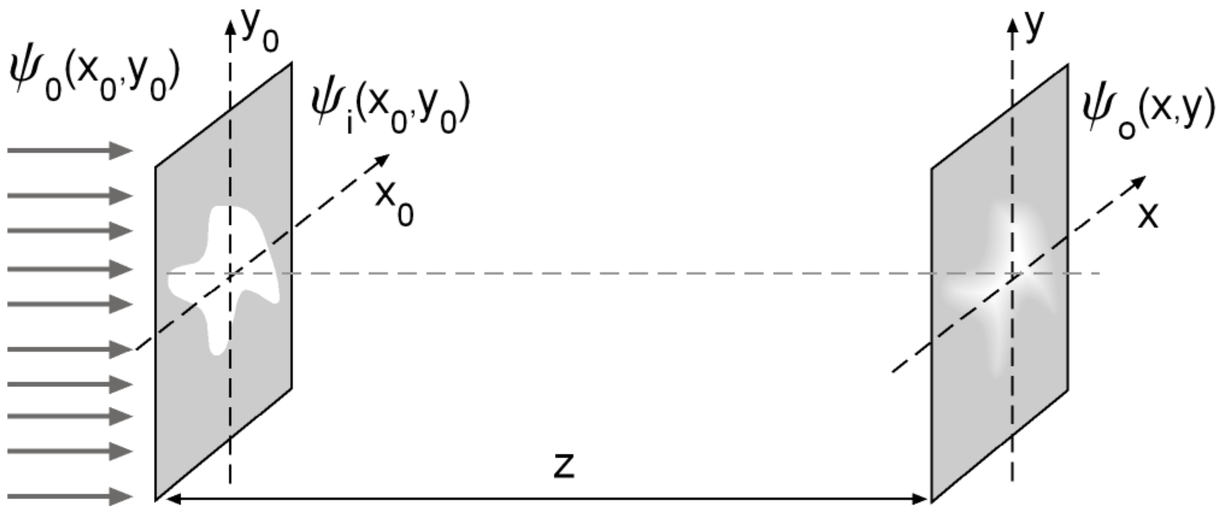
Linear integral transforms



Diffraction by screen

- Consider the diffraction of coherent monochromatic plane wave by the screen characterized by transmittance function $T(x_0,y_0)$

$$\Psi_i(x_0,y_0)=\Psi_0(x_0,y_0)T(x_0,y_0)$$



Helmholtz equation

- In order to calculate the complex field amplitude at the observation (output) plane $\Psi_o(x,y)$ the Helmholtz equation

$$\nabla^2\Psi(x,y,z) + k^2\Psi(x,y,z) = 0$$

which describes the monochromatic wave propagation in free space, has to be resolved with boundary conditions $\Psi(x_0,y_0,0) = \Psi_i(x_0,y_0)$ ($k=2\pi/\lambda$ and λ is the wavelength).

Approximations:

- Monochromatic, linearly polarized waves
- linear, isotropic, homogeneous, no dispersive medium

Angular spectrum

- Methods to resolve the Helmholtz equation:
  - Green function
  - Angular spectrum decomposition
- Angular spectrum

$$F\left(k_x, k_y, z\right) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int \Psi\left(x, y, z\right) \exp\left(-i\left[k_x x + k_y y\right]\right) dxdy$$

where

k_x, k_y, $k_z = (k^2 - k_x^2 - k_y^2)^{1/2}$ are the wave vector components,

$f_x = k_x/2\pi$, $f_y = k_y/2\pi$ are spatial frequencies

Equation for angular spectrum

Introducing

$$\Psi(x,y,z)=\int\int_{-\infty}^{\infty} F(k_x,k_y,z)\,exp\left(i\left[k_x x+k_y y\right]\right)dk_x dk_y$$

in Helmoltz equation
$$\frac{d^2F}{dz^2}+\left(k^2-k_x^2-k_y^2\right)F=0$$

$$F(k_x,k_y,z)=F(k_x,k_y,0)\,exp\left(iz\left[k^2-k_x^2-k_y^2\right]^{1/2}\right)$$

$k^2>k_x^2+k_y^2$ the propagation affects only the phase of the angular spectrum (phase only filter)

$k^2<k_x^2+k_y^2$ evanescent waves

Paraxial approximation

- If the wave vector components $k_{x,}$, and k_y of the angular spectrum of the complex field amplitude $\Psi_i(x_i,y_i)$ satisfy the condition

$$k^2_x \text{ , } k^2_y << k^2$$

then

$$\left[k^2 - k_x^2 - k_y^2\right]^{1/2} \approx k - \frac{k_x^2}{2k} - \frac{k_y^2}{2k}$$

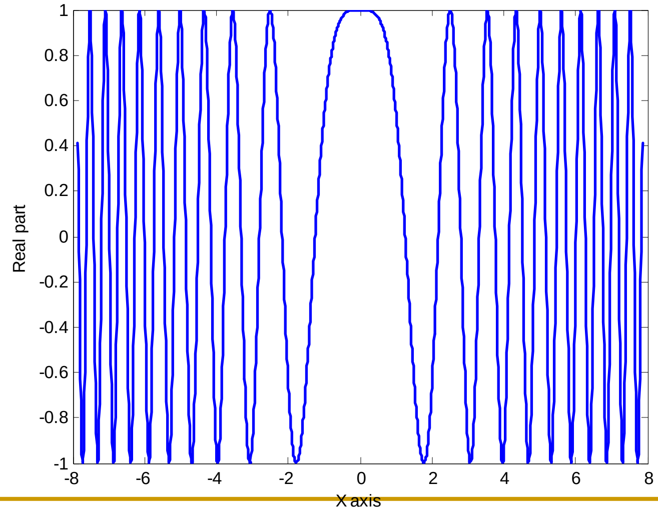
Angular spectrum in paraxial approximation

$$F\left(k_x,k_y,z\right)=F\left(k_x,k_y,0\right)\exp\left(ikz\right)\exp\left(-\frac{iz}{2k}\left[k_x^2+k_y^2\right]\right)$$

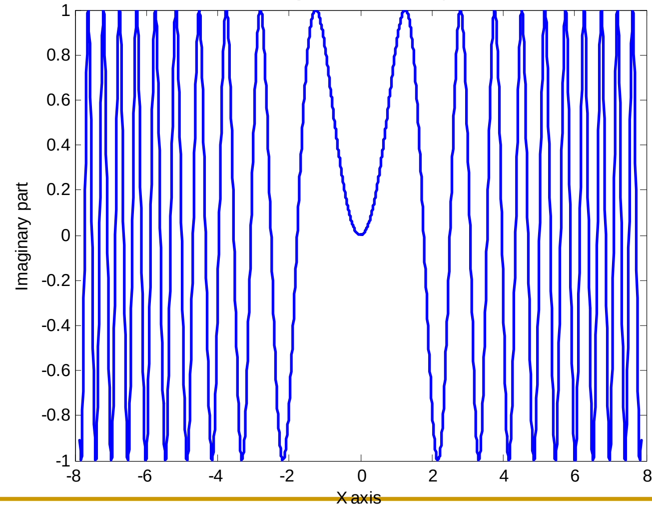
Chirp signals: $\exp(\pm iax^2)$

$\exp(i25x^2/4)$

Real part



Imaginary part



Fresnel diffraction

- Using very important formula for Gaussian optics

$$\int_{-\infty}^{\infty} exp\left(\alpha x^2+\beta x\right)dx=\sqrt{\frac{\pi}{-\alpha}}\,exp\left(-\frac{\beta^2}{4\alpha}\right), \qquad Re(\alpha)\le 0$$

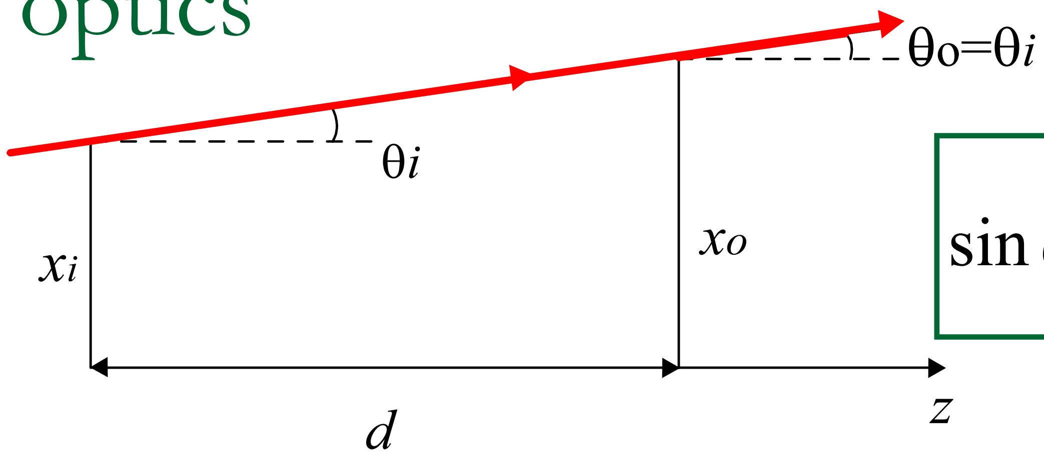
the complex field amplitude at plane z can be represented as a convolution integral (similar to the wavelet) :

$$\Psi_o\left(\mathbf{r}_o\right)=\frac{\exp\left(ikz\right)}{i\lambda z}\int_{-\infty}^{\infty}\Psi_i\left(\mathbf{r}_i\right)\exp\left(\frac{ik}{2z}\left(\mathbf{r}_0-\mathbf{r}_i\right)^2\right)d\mathbf{r}_i$$

$$\Psi_o\left(\mathbf{r}_o\right)=\frac{\exp\left(ikz\right)}{i\lambda z}\left\{\Psi_i\left(\mathbf{r}_o\right)*\exp\left(\frac{ik}{2z}\mathbf{r}_o^2\right)\right\}$$

I. S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products,* Academic Press, NY, (1994).

Paraxial approximation in geometrical optics



$$\sin\theta_{x,y} \approx \theta_{x,y} \qquad \frac{k_{x,y}}{k} << 1$$

- Paraxial rays are described by position $\mathbf{r}=(x,y)$ and direction $\mathbf{q}=(q_x n, q_y n)$ vectors, n – refractive index (further: n=1)

$$\begin{pmatrix} x_o \\ \theta_{xo} \end{pmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_i \\ \theta_{xi} \end{pmatrix}$$

- Rectilinear ray propagation in isotropic homogeneous medium

Ray propagation in free space

- From one dimensional to two dimensional case

$$\begin{pmatrix} x_o \\ \theta_{xo} \end{pmatrix} = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_i \\ \theta_{xi} \end{pmatrix}$$

$$\begin{pmatrix} x_o \\ y_o \\ \theta_{xo} \\ \theta_{yo} \end{pmatrix} = \begin{bmatrix} 1 & 0 & z & 0 \\ 0 & 1 & 0 & z \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_i \\ y_i \\ \theta_{xi} \\ \theta_{yi} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{r}_o \\ \mathbf{q}_o \end{pmatrix} = \begin{bmatrix} \mathbf{I} & z\mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{pmatrix} \mathbf{r}_i \\ \mathbf{q}_i \end{pmatrix}$$

Ray transformation matrix and Fresnel integral

- Another form for the Fresnel integral

$$\Psi_o(\mathbf{r}_o) = \frac{\exp(ikz)}{i\lambda z} \int_{-\infty}^{\infty} \Psi_i(\mathbf{r}_i) \exp\left(\frac{i\pi}{\lambda z}\left[1\cdot\mathbf{r}_i^2 + 1\cdot\mathbf{r}_o^2 - 2\mathbf{r}_i\mathbf{r}_o\right]\right) d\mathbf{r}_i$$

$$\begin{pmatrix} \mathbf{r}_o \\ \mathbf{q}_o \end{pmatrix} = \begin{bmatrix} \mathbf{I} & z\mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{pmatrix} \mathbf{r}_i \\ \mathbf{q}_i \end{pmatrix}$$

Fraunhofer diffraction

- If $z >> k(x_i^2 + y_i^2)_{max}/2$, where x_{imax}, y_{imax} are the maximum horizontal and vertical sizes of diffracted object

  then $\exp\left(\frac{ik}{2z}\mathbf{r}_i^2\right) \approx 1$ ⇨

$$\Psi_o(\mathbf{r}_o) = \frac{\exp(ikz)}{i\lambda z}\exp\left(\frac{ik}{2z}\mathbf{r}_o^2\right)\int_{-\infty}^{\infty}\Psi_i(\mathbf{r}_i)\exp\left(-\frac{ik}{z}\mathbf{r}_i\mathbf{r}_o\right)d\mathbf{r}_i$$

$$= \frac{\exp(ikz)}{i\lambda z}\exp\left(\frac{ik}{2z}\mathbf{r}_o^2\right)FT\left[\Psi_i(\mathbf{r}_i)\right]\left(\frac{k}{z}\mathbf{r}_o\right)$$

- At large distances the complex field amplitude of diffractive field is proportional to the scaled Fourier transform of the input one

Limitations of Fraunhofer system for FT observation

- The distance between the input and output planes is too large

  $$z >> \pi a^2 / \lambda$$

  where a is the size of the object and λ is the wavelength

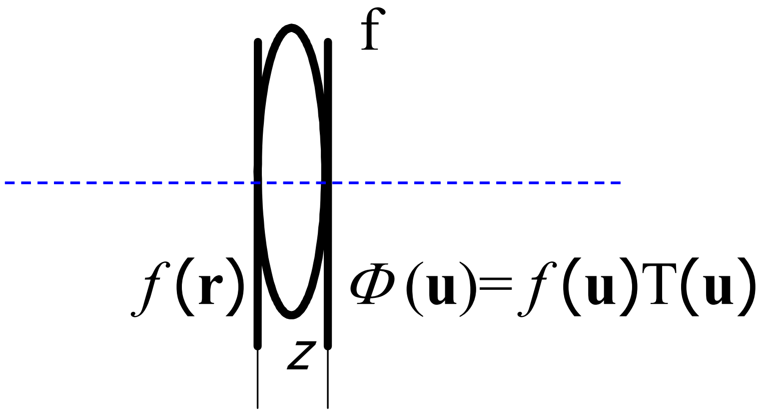
- Scaling of the FT depends on the distance
- Additional quadratic phase factor

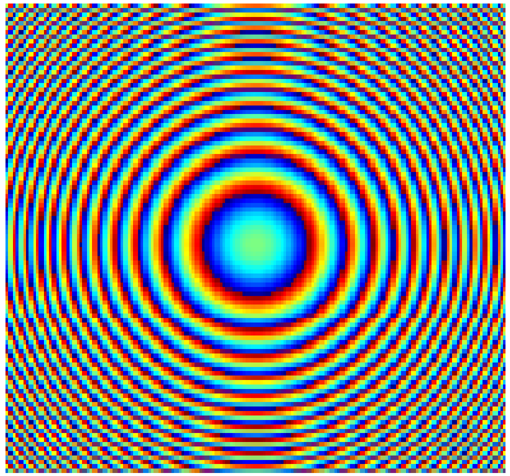
- Solution: use lenses!

Phase transform by thin lens

- Transmittance function of the spherical convergent thin lens (P=exp($izn2\pi/\lambda$) , n is the refractive index of lens material)

$$T(x,y) = P\exp\left[-i\pi\left(x^2+y^2\right)/\lambda f\right]$$

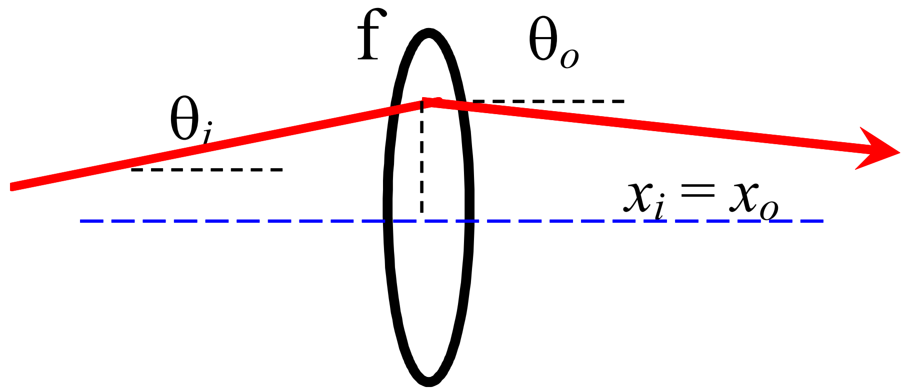




- Action of thin lens: signal multiplication by chirp function (only phase of the signal is changed)

Ray transformation by thin spherical lens

- Position vector doesn't change
- Direction changes
- Transformation matrix



$$\begin{pmatrix} \mathbf{r}_o \\ \mathbf{q}_o \end{pmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -f^{-1}\mathbf{I} & \mathbf{I} \end{bmatrix} \begin{pmatrix} \mathbf{r}_i \\ \mathbf{q}_i \end{pmatrix}$$

2 D

$$\begin{pmatrix} x_o \\ \theta_{xo} \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{pmatrix} x_i \\ \theta_{xi} \end{pmatrix}$$

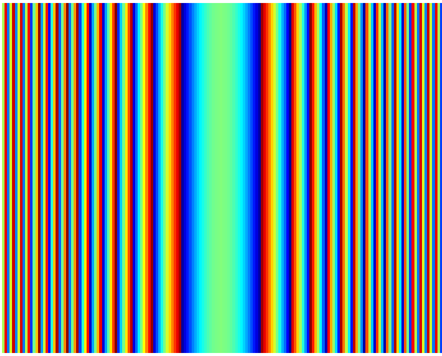
- Integral transformation

$$\Psi_o(\mathbf{r}_o) = P\exp\left(-i\pi\mathbf{r}_o^2/f\right)\int_{-\infty}^{\infty}\Psi_i(\mathbf{r}_i)\,\delta(\mathbf{r}_i-\mathbf{r}_o)\,d\mathbf{r}_i$$

Cylindrical lenses

- Transmittance function of cylindrical lens

$$T(x,y) = P\exp\left[-i\pi x^2/\lambda f\right]$$



- Matrix representation

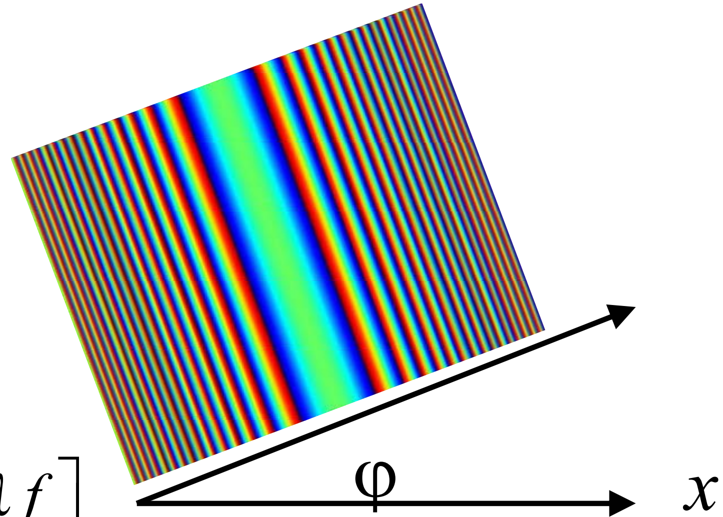
$$\begin{pmatrix} x_o \\ y_o \\ \theta_{xo} \\ \theta_{yo} \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -f^{-1} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_i \\ y_i \\ \theta_{xi} \\ \theta_{yi} \end{pmatrix}$$

- Integral transformation

$$\Psi_o(\mathbf{r}_o) = P\exp\left(-i\pi x_o^2/f\right)\int_{-\infty}^{\infty} \Psi_i(\mathbf{r}_i)\,\delta(\mathbf{r}_i - \mathbf{r}_o)\,d\mathbf{r}_i$$

Cylindrical lens: general form

- Transmittance function of a cylindrical lens rotated at angle φ with respect to the chosen coordinate system



$$T(x,y) = P\exp\left[-i\pi(x\cos\varphi + y\sin\varphi)^2/\lambda f\right]$$

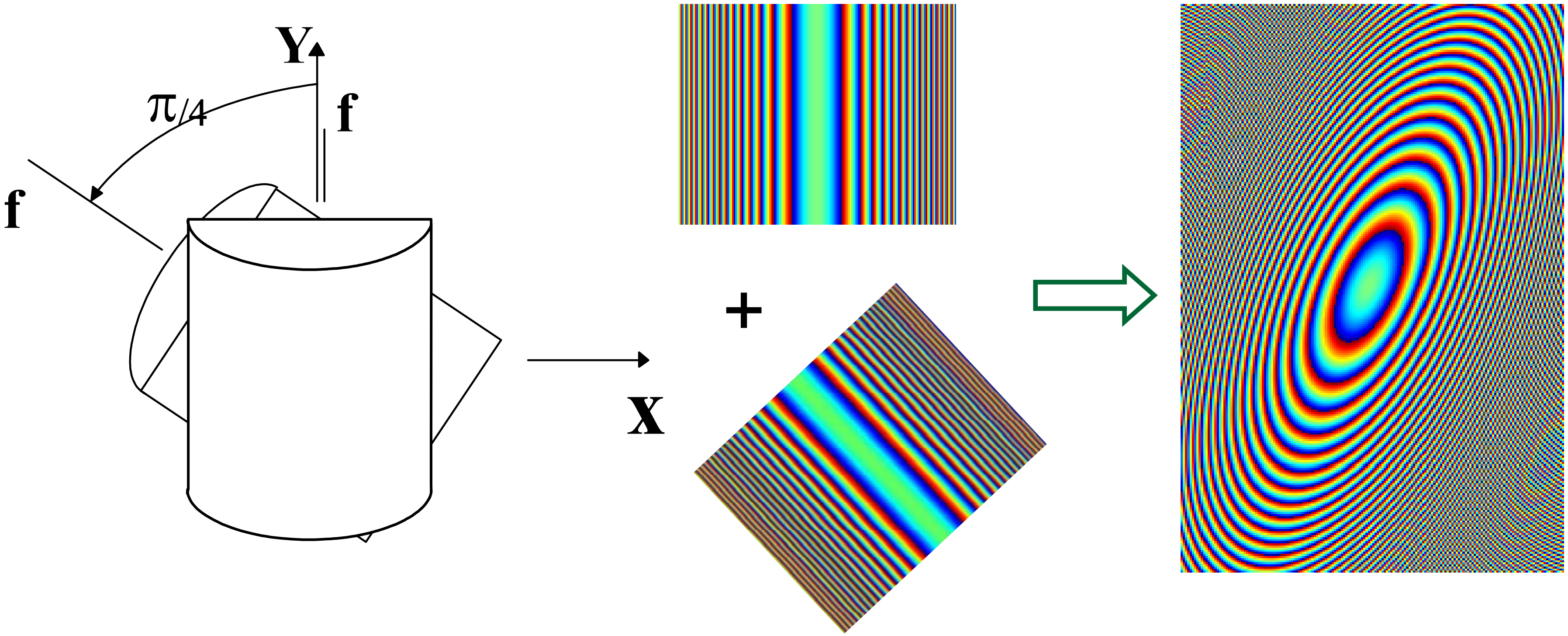
- Transformation matrix
- Sub matrix **G**

$$\begin{pmatrix}\mathbf{r}_o \\ \mathbf{q}_o\end{pmatrix} = \begin{bmatrix}\mathbf{I} & \mathbf{0} \\ -\mathbf{G} & \mathbf{I}\end{bmatrix}\begin{pmatrix}\mathbf{r}_i \\ \mathbf{q}_i\end{pmatrix}$$

$$\mathbf{G} = \frac{1}{2f}\begin{bmatrix}1+\cos 2\varphi & \sin 2\varphi \\ \sin 2\varphi & 1-\cos 2\varphi\end{bmatrix}$$

Composition of cylindrical lens

- Two cylindrical lenses with the same focal distance rotated at angle $\pi/2$ ⇨ spherical lens
- Two cylindrical lenses rotated at angle $\pi/4$



Generalized lens

- Lens matrix for the set of m attached cylindrical lenses of power $p_m = 1/f_m$ rotated at angle φ_m

$$\mathbf{G} = \sum_{i=1}^{m} \mathbf{G}_i = \begin{bmatrix} \sum_{i=1}^{m} p_i \cos^2 \varphi_i & \sum_{i=1}^{m} p_i (\sin 2\varphi_i)/2 \\ \sum_{i=1}^{m} p_i (\sin 2\varphi_i)/2 & \sum_{i=1}^{m} p_i \sin^2 \varphi_i \end{bmatrix}$$

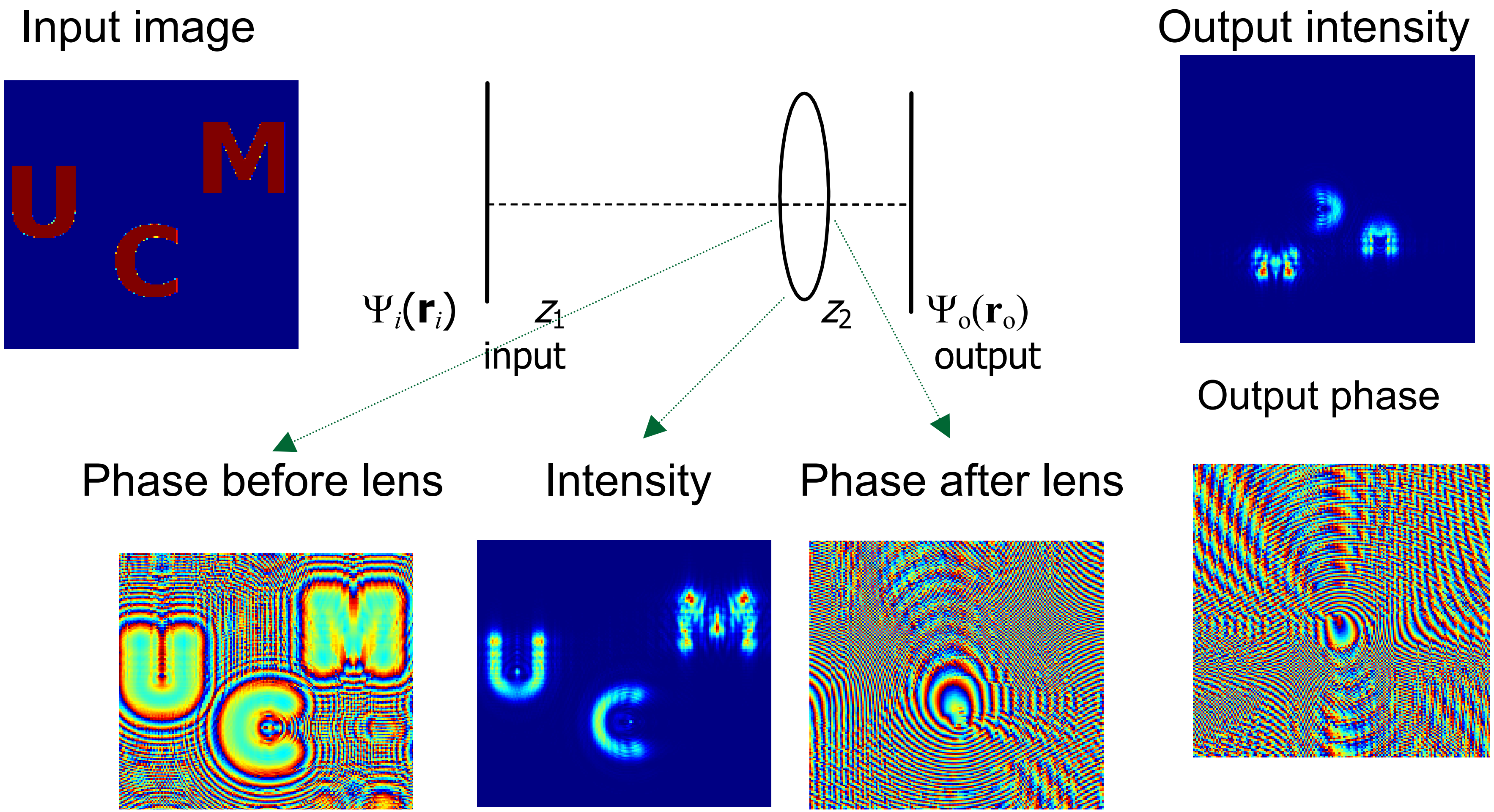
- $\mathbf{G} = \mathbf{G}^T$
- Simple notation of generalized lens

$$\mathbf{G} = \begin{bmatrix} g_{xx} & g_{xy} \\ g_{xy} & g_{yy} \end{bmatrix}$$

G. Nemes and A. E. Siegman, *J. Opt. Soc. Am. A* 11, 2257 (1994)

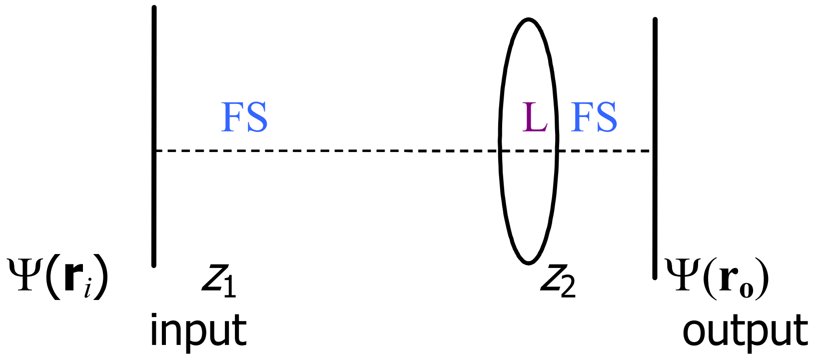
Composite system

- Complex amplitude evolution during propagation



Composite system (analytical description)

- Using the important formula



$$\int_{-\infty}^{\infty} exp\left(\alpha x^2 + \beta x\right)dx = \sqrt{\frac{\pi}{-\alpha}}\, exp\left(-\frac{\beta^2}{4\alpha}\right), \qquad Re(\alpha) \leq 0$$

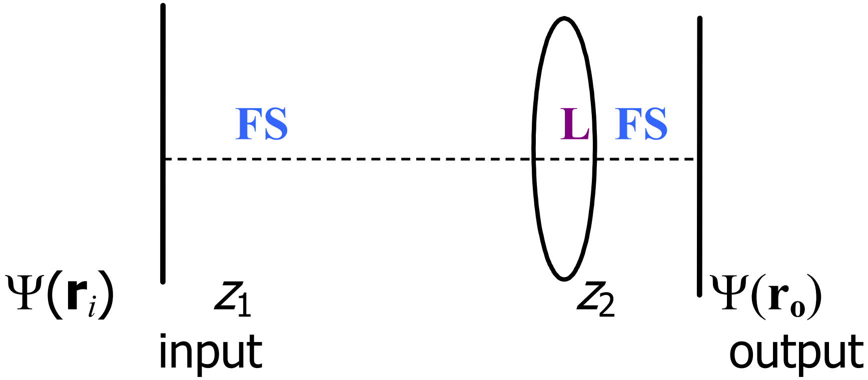
$$\xi = \frac{1}{z_1} + \frac{1}{z_2} - \frac{1}{\mathrm{f}}$$

$$\xi \neq 0$$

$$\Psi_o\left(\mathbf{r}_o\right) = \frac{\exp\left(ik\left(z_1 + z_2\right)\right)}{i\lambda z_1 z_2 \xi} P\int_{-\infty}^{\infty} \Psi_i\left(\mathbf{r}_i\right) \exp\left(\frac{ik\mathbf{r}_i^2}{2}\left[\frac{1}{z_1} - \frac{1}{\xi z_1^2}\right]\right)$$
$$\times \exp\left(\frac{ik\mathbf{r}_o^2}{2}\left[\frac{1}{z_2} - \frac{1}{\xi z_2^2}\right] - \frac{ik\mathbf{r}_o\mathbf{r}_i}{2\xi z_1 z_2}\right) d\mathbf{r}_i$$

Ray transformation in composite system

- Matrix of the composite system is a product of matrices corresponding to its parts in inverse order



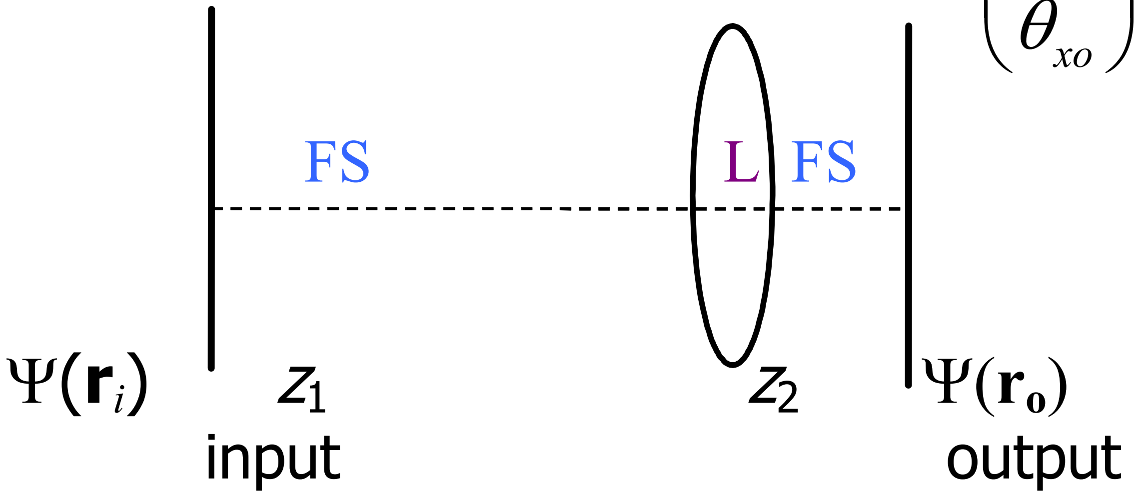
$$\mathbf{M} = \mathbf{M}_n \times \mathbf{M}_{n-1} \ldots \mathbf{M}_1$$

$$\begin{pmatrix} x_o \\ \theta_{xo} \end{pmatrix} = \begin{bmatrix} 1 & z_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & z_1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_i \\ \theta_{xi} \end{pmatrix} = \mathbf{M} \begin{pmatrix} x_i \\ \theta_{xi} \end{pmatrix}$$

Ray transformation matrix vs integral transform

- Ray transformation

$$\begin{pmatrix} x_o \\ \theta_{xo} \end{pmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{pmatrix} x_i \\ \theta_{xi} \end{pmatrix}$$



$$A = 1 - z_2/\mathrm{f}$$
$$B = z_1 + z_2 - z_1 z_2/\mathrm{f}$$
$$C = -1/\mathrm{f}$$
$$D = 1 - z_1/\mathrm{f}$$

- Integral transform for $B \neq 0$

$$\Psi_o(\mathbf{r}_o) = \frac{\exp\left(ik(z_1+z_2)\right)}{i\lambda B} P\int_{-\infty}^{\infty}\int \Psi_i(\mathbf{r}_i)\exp\left(\frac{i\pi}{\lambda B}\left[D\mathbf{r}_o^2 + A\mathbf{r}_i^2 - 2\mathbf{r}_o\mathbf{r}_i\right]\right)d\mathbf{r}_i$$

Imaging condition

- In the case $$\xi = \frac{1}{z_1} + \frac{1}{z_2} - \frac{1}{\mathrm{f}} = 0$$

  the ray transformation matrix can be written as

$$A = -z_2 / z_1$$
$$B = 0$$
$$C = -(z_1 + z_2) / z_2 z_1$$
$$D = -z_1 / z_2$$

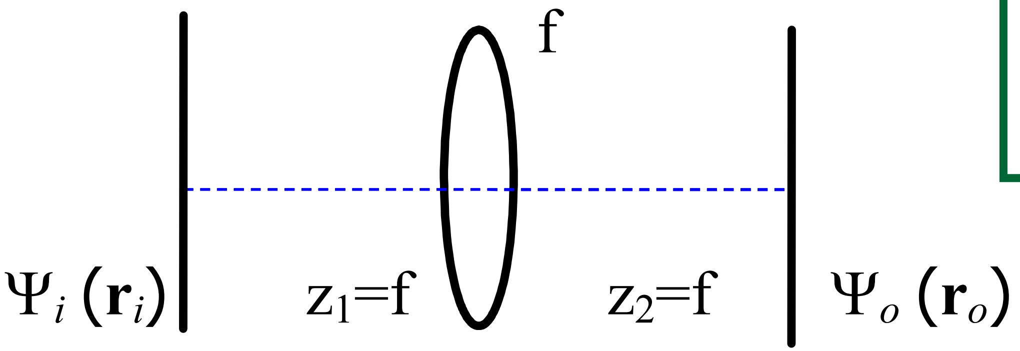
- Output complex amplitude is a product of the scaled input complex amplitude and a quadratic phase factor

$$\Psi_o(\mathbf{r}_o) = \frac{\exp(ik(z_1 + z_2))}{|A|} \exp\left(\frac{i\pi C \mathbf{r}_o^2}{\lambda A}\right) P \int_{-\infty}^{\infty}\int \Psi_i(\mathbf{r}_i)\,\delta(\mathbf{r}_i - \mathbf{r}_o / A)\, d\mathbf{r}_i$$

Optical Fourier transform

- Fourier transform (1D or 2D) : application of cylindrical or spherical lenses

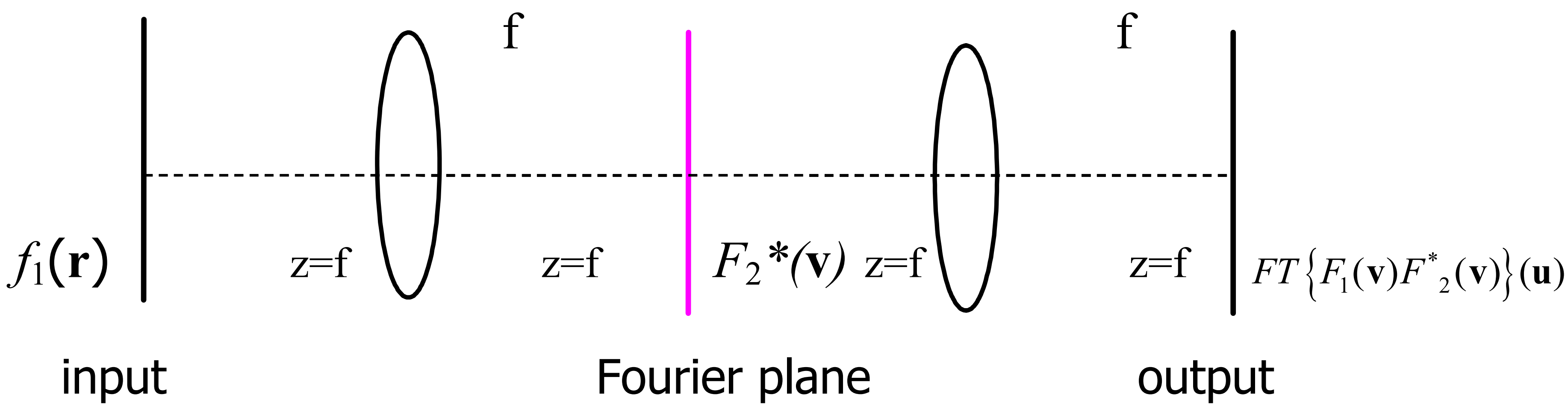
$$\mathbf{M} = \begin{bmatrix} 0 & f \\ -1/f & 0 \end{bmatrix}$$



$$\Psi_o(\mathbf{r}_o) = P\int \Psi_i(\mathbf{r}_i)\exp\left(-i2\pi\frac{\mathbf{r}_i\mathbf{r}_o}{\lambda \mathrm{f}}\right)d\mathbf{r}_i$$

4-f Van der Lugt optical processor

- Correlation or convolution operations ⇨ Fourier plane mask



$$FT\left\{F_1(\mathbf{v})F^*{}_2(\mathbf{v})\right\}(\mathbf{u}) = Correlation(-\mathbf{u})$$

A. Van der Lugt, *IEEE Trans. Inf. Theory* IT-10, 139 (1964); A. Van der Lugt, *Optical Signal Processing*, John Wiley, NY (1992)

Summary: basic operations for coherent optical processing

- Superposition principle $\rightarrow$ sum of optical fields
- Light propagation through (or reflection from) screen $\rightarrow$ multiplication
- Diffraction of Fresnel $\rightarrow$ convolution with chirp
- Diffraction in the far field $\rightarrow$ Fourier transform
- System with thin lens
  - simplifies the observation of the Fourier transform
  - performs canonical transform
- Fourier transform + signal multiplication = convolution, correlation operations
- Cylindrical lenses $\rightarrow$ new operations: rotation, twisting, …