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QUANTUM AND CLASSICAL ASPECTS
of
INFORMATION OPTICS

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Optical Integral Transforms for Information Processing

Lecture 2: Basic Properties and Optical Schemes

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Lecture 2: Basic Properties and Optical Schemes

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Outlines

- One dimensional linear Canonical integral Transforms (CTs)
- Fractional Fourier transform
- Two dimensional CTs: image rotation, fractional FT, twisting
  - Properties of the CTs
  - Optical schemes
  - Stable modes
- New transform generation
  - Fractional transforms

Separable canonical integral transform

- Spherical lenses
- Cylindrical lenses parallel or perpendicular to each other

$$M_{x,y} = \begin{bmatrix} A_{x,y} & B_{x,y} \\ C_{x,y} & D_{x,y} \end{bmatrix}$$

$$\begin{aligned} \Psi_o^{M_x,M_y}(\mathbf{r}_o) &= R^{M_x,M_y}\left\{\Psi_i(\mathbf{r}_i)\right\} \\ &= \int \Psi_i(\mathbf{r}_i) K_{M_x}(x_i, x_o) K_{M_y}(y_i, y_o)\, d\mathbf{r}_i \end{aligned}$$

$$K_{M_{x,y}}(x_i, x_o) = \begin{cases} \dfrac{1}{\sqrt{iB_{x,y}}} \exp\left(i\pi \dfrac{A_{x,y} x_i^2 + D_{x,y} x_o^2 - 2x_i x_o}{B_{x,y}} \right), & B_{x,y} \neq 0 \\ \dfrac{1}{\sqrt{\left|A_{x,y}\right|}} \exp\left(i\pi \dfrac{C_{x,y} x_o^2}{A_{x,y}} \right) \delta\left(x_i - x_o / A_{x,y} \right), & B_{x,y} = 0 \end{cases}$$

Dimensionless variables are used here and below

One dimensional CT

- One dimensional CT has 3 free parameters since det **M**=1
- If direct CT is parameterized by ray transformation matrix

$$\mathbf{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

  the inverse CT is parameterized by

$$\mathbf{M}^{-1} = \begin{bmatrix} D & -B \\ -C & A \end{bmatrix}$$

- Transformation of the angular spectrum under the CT

$$(\mathbf{M}^{-1})^t = \begin{bmatrix} D & -C \\ -B & A \end{bmatrix}$$

$$\begin{array}{ccccc} & \Rightarrow & \mathbf{M} & \Rightarrow & \\ \Psi_i(\mathbf{r}_i) & & & & \Psi_o(\mathbf{r}_o) \\ & \Leftarrow & \mathbf{M}^{-1} & \Leftarrow & \\ \Downarrow FT & & & & \Downarrow FT \\ & \Rightarrow & (\mathbf{M}^{-1})^t & \Rightarrow & \\ F_i(\mathbf{u}_i) & & & & F_o(\mathbf{u}_o) \\ & \Leftarrow & \mathbf{M}^t & \Leftarrow & \end{array}$$

Basic uniparametric CTs

- Transforms are additive with respect a parameter *p:*
  $R^{p_1}R^{p_2}=R^{p_1+p_2}$
- Fresnel transform $p=z$ $\qquad \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$
- Lens transform $p=\beta$ $\qquad \begin{pmatrix} 1 & 0 \\ -\beta & 1 \end{pmatrix}$
- Scaling transform $p=a$ $\qquad \begin{pmatrix} \exp(a) & 0 \\ 0 & \exp(-a) \end{pmatrix}$
- Fractional Fourier transform $p=\alpha$ $\qquad \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$
- Fractional hyperbolic transform $p=\alpha$ $\qquad \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}$

Iwasawa decomposition

- CT can be represented as a composition of three basic uniparametric transforms

Kenkichi Iwasawa,

岩澤 健吉 1917-1998

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

$$a^2 = A^2 + B^2$$

$$\beta = -\left(AC + BD\right) a^{-2}$$

$$\alpha = \arctan\left(B / A\right)$$

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  - Optical schemes
  - Stable modes
- New transform generation
  - Fractional transforms

History of the fractional FT

- **H. Kober**, "Wurzeln aus der Hankel, Fourier und aus anderen stetigen transformationen," *Quart. J. Math. Oxford. Ser.* 10, pp. 45-49, **1939**
- **K. B. Wolf**, "Construction and properties of canonical transforms," In *Integral Transforms in Science and Engineering*, Plenum Press, NY, **1979**
- **V. Namias**, "The fractional order Fourier transform and its applications to quantum mechanics," *J. Inst. Math. Appl.* 25, pp. 241-265, **1980**
- **D. Mendlovic, and H. M. Ozaktas**, "Fractional Fourier transform and their optical implementation I," *J. Opt. Soc. Am. A* 10, pp. 1875-1881, **1993**
- **L. Almeida**, "The fractional Fourier transform and time-frequency representations," *IEEE Trans. Signal Process.* 42, pp. 3084-3091, **1994**

Definition of the fractional FT

Fractional Fourier transform of $f(x)$ at angle $\alpha=\pi p/2$

$$F_\alpha(u) = R^\alpha[f(x)](u) = \int K_\alpha(x,u) f(x) dx$$

where
$$K_\alpha(x,u) = \sqrt{1 - i\cot\alpha}\,\exp\left(i\pi \frac{(x^2+u^2)\cos\alpha - 2xu}{\sin\alpha} \right)$$

$\alpha=0$ $\Rightarrow$ identity transform $K_{\alpha=0}(x,u) = \delta(x-u)$

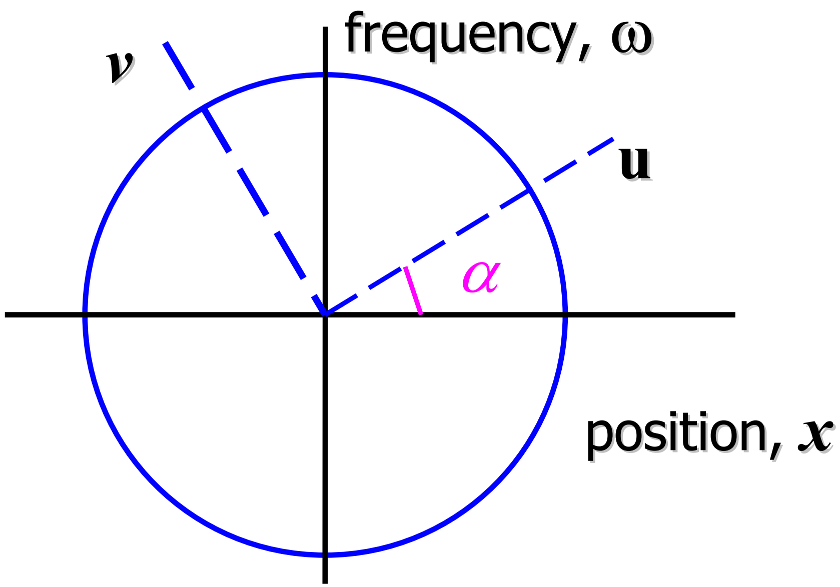
$\alpha=\pi/2$ $\Rightarrow$ Fourier transform $K_{\alpha=\pi/2}(x,u) = \exp(-i\pi 2xu)$

$\alpha=\pi$ $\Rightarrow$ reverse transform $K_{\alpha=\pi}(x,u) = \delta(x+u)$

$\alpha=3\pi/2$ $\Rightarrow$ inverse FT $K_{\alpha=3\pi/2}(x,u) = \exp(i\pi 2xu)$

Interpretations of the fractional FT

Phase plane



$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ \omega \end{pmatrix}$$

- Quantum mechanics: Schrödinger equation for harmonic oscillator

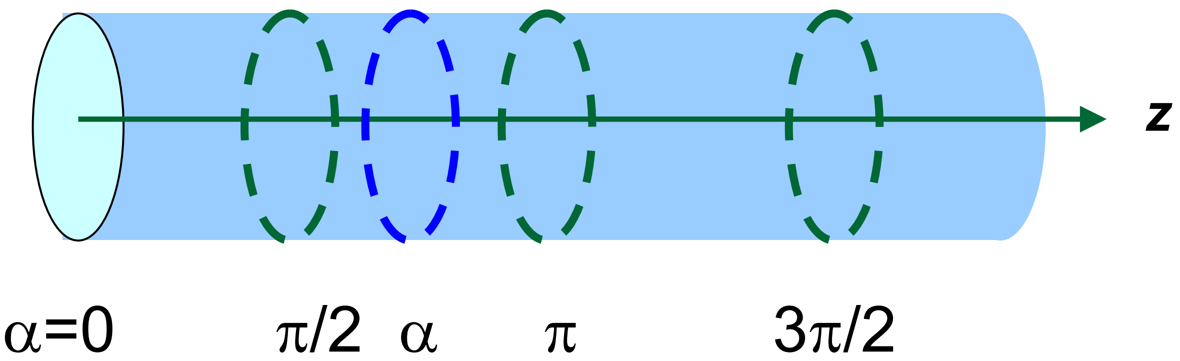
$$\left[\frac{\partial}{\partial\,\alpha} - \frac{i}{2} \left(\frac{\partial^2}{\partial x^2} - x^2 - 1 \right) \right] \Psi(x) = 0$$

- Optics: quadratic refractive index medium (lenses, fibers, spherical mirrors, etc.)
- Signal processing: time-frequency representations

Fractional Fourier transform optical systems

- Optical fibers with quadratic refractive index profile:
  $n^2(r)=n_0^2(1-g^2 r^2)$ ➔ $\alpha=gz$

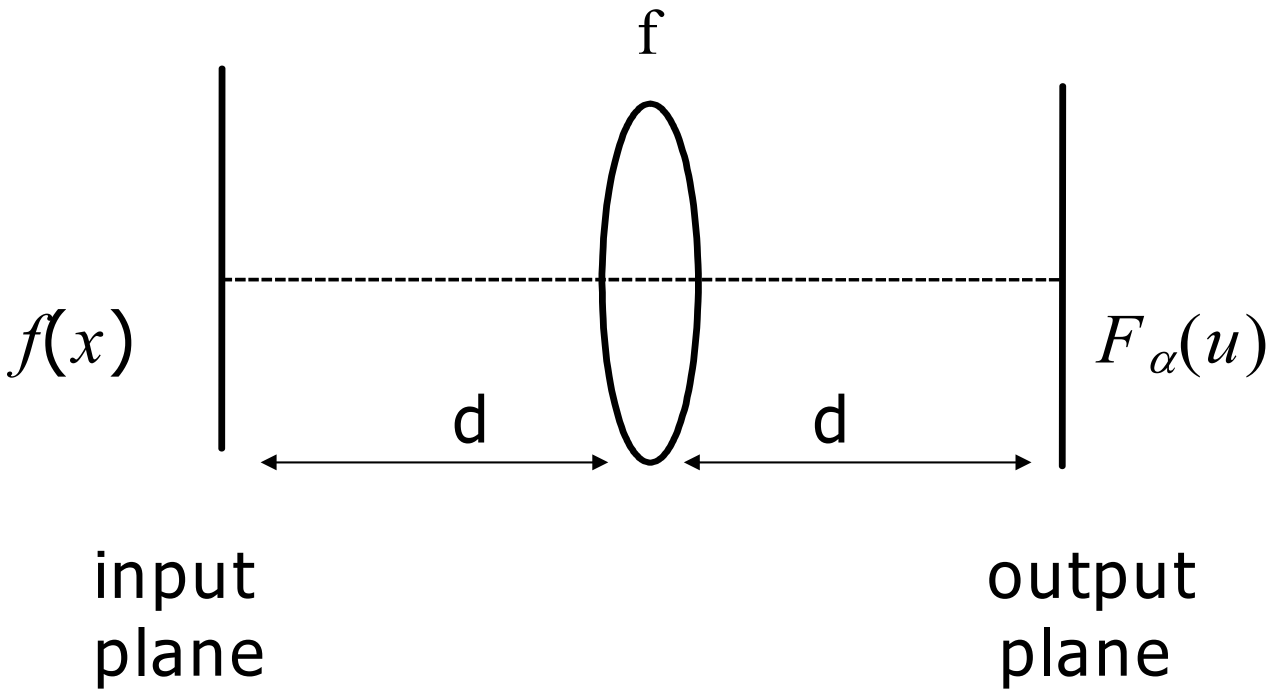
  D. Mendlovic and H. M. Ozaktas, *J. Opt. Soc. Am. A* 10, 1875 (1993)



- Thin lens configurations

  A. Lohmann, *J. Opt. Soc. Am. A* **10**, 2181 (1993)

One lens fractional Fourier transformer



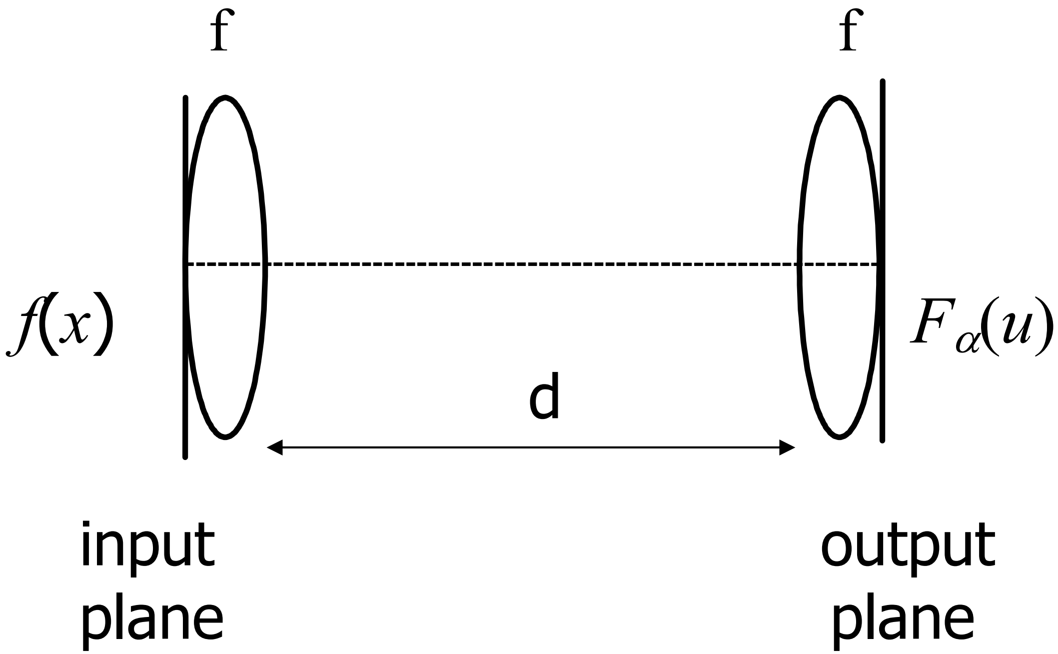
Adolf W. Lohmann

(1926)

$$d = 2f \sin^2(\alpha/2)$$

$$F_\alpha(u) = R^\alpha\left[f(x)\right](u)$$

Two lens fractional Fourier transformer



$$d = 2f \sin^2(\alpha/2)$$

$$F_\alpha(u) = R^\alpha\left[f(x)\right](u)$$

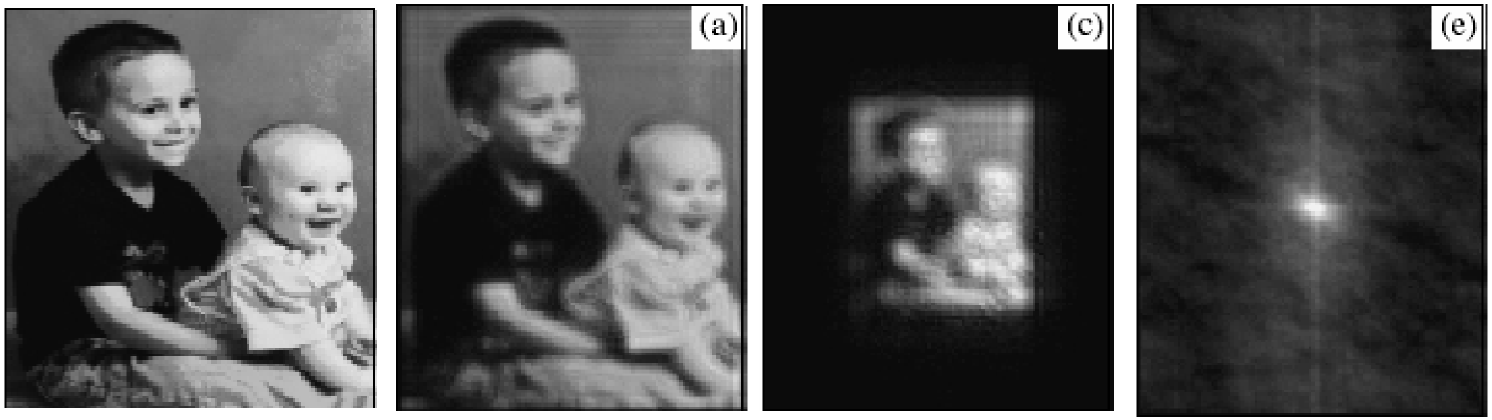
Separable fractional FT: example

- Rotation in x - q_x and y - q_y planes of phase space

$$M_{x,y} = \begin{bmatrix} \cos\alpha_{x,y} & \sin\alpha_{x,y} \\ -\sin\alpha_{x,y} & \cos\alpha_{x,y} \end{bmatrix}$$

- Isotropic fractional FT: $\alpha=\alpha_x=\alpha_y$

$\alpha=0$ $\alpha=\pi/20$ $\alpha=\pi/4$ $\alpha=\pi/2$



Properties of the fractional FT

- Periodicity $$R^{\alpha + n2\pi} = R^{\alpha}$$
- Additivity $$R^{\alpha}R^{\beta} = R^{\alpha+\beta}$$
- Parseval theorem $$\int \left| f(x) \right|^2 dx = \int \left| F_{\alpha}(u) \right|^2 du$$
- Scaling theorem: if $$\cot\alpha = \lambda^2 \cot\beta$$

  then
$$R^{\alpha}[f(x\lambda)](u) = \lambda^{-1}\sqrt{\frac{\sin\beta}{\sin\alpha}}R^{\beta}[f(x)]\left(u\lambda^{-1}\frac{\sin\beta}{\sin\alpha}\right)\exp(i\varphi)$$
$$\varphi = (\alpha-\beta)/2 + \pi\ u^2\cot\alpha\left(1-\frac{\cos^2\beta}{\cos^2\alpha}\right)$$

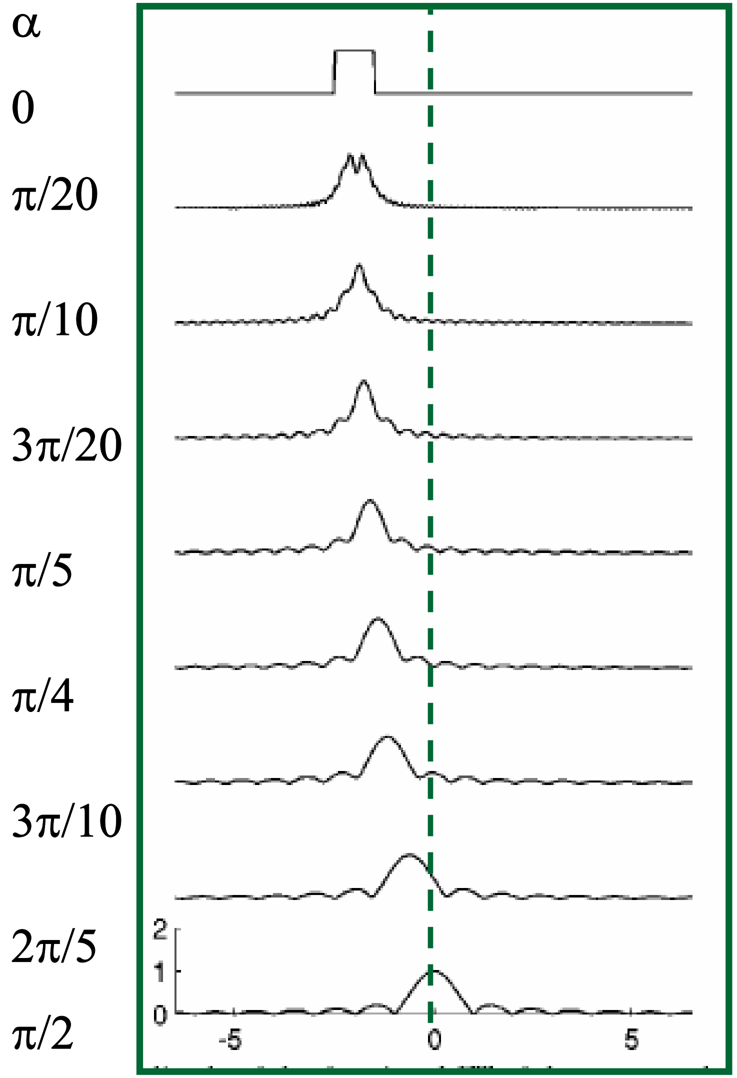
Shift theorem for the fractional FT

- Fractional FT: amplitude and phase are changing

$$R^{\alpha}[f(x-y)](u) = R^{\alpha}[f(x)](u - y\cos\alpha)\exp(i\varphi)$$
$$\varphi = 2\pi(y^2 \sin(2\alpha)/4 - uy\sin\alpha)$$

- FT ($\alpha=\pi/2$) : amplitude doesn't change under signal shift ➔ shift-invariant signal processing

$$R^{\pi/2}[f(x-y)](u) = R^{\pi/2}[f(x)](u)\exp(i\varphi)$$
$$\varphi = -2\pi uy$$



Amplitude of fractional FT

Rotation of Wigner Distribution under the fractional FT

- Wigner distribution function of $f(x)$

$$W_f(x,\omega) = \int f(x + x_0/2) f^*(x - x_0/2) \exp(-i2\pi x_0 \omega) dx_0$$

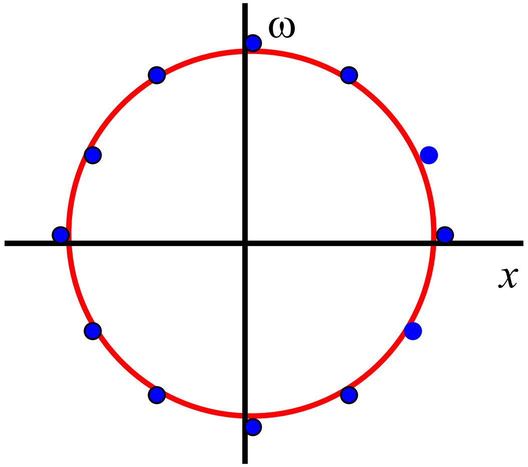
$$\begin{array}{ccc} f(x) & \rightarrow & W_f(x,\omega) \\ \downarrow & & \downarrow \\ F_\alpha(u) & \rightarrow & W_{F_\alpha}(u,v) \end{array}$$

$$W_{F_\alpha}(u,v) = W_f(u\cos\alpha - v\sin\alpha, u\sin\alpha + v\cos\alpha)$$

- Applications: phase space tomography, signal analysis and synthesis, filtering

Eigenfunctions for the fractional FT

- Self-fractional Fourier function (SFFF) $f_\alpha(x)$ for angle α is an eigenfunction of fractional FT operator R^α with eigenvalue *A*: $R^\alpha\left[f_\alpha(x)\right](u) = Af_\alpha(u)$



- From Parseval relation for the fractional FT $\Rightarrow$ $A = \exp(i\varphi)$
- From *periodicity* :

$R^{\alpha+2\pi n}=R^\alpha \Rightarrow$ if $\alpha = 2\pi/M$, then $A^M=1$

$A=\exp(i2\pi L/M)$, where *L* is an integer.

D. Mendlovic, H. M. Ozaktas, and A. W. Lohmann, *Optics Comm.* 105, 36 (1994);
T. Alieva, *J. Phys. A* 29, L377 (1996).

What functions are SFFFs?

- even or odd functions are SFFFs for angle π
- self-Fourier functions are SFFFs for angle $\pi/2$

  comb(x); sech(πx); cos[$\pi(x^2$-1/8)]; $|x|^{-1/2}$;

  $$f(x) + f(-x) + F_{\pi/2}(x) + F_{\pi/2}(-x)$$

- Hermite-Gaussian functions are SFFFs for *any angle* α with eigenvalue A=exp($-i\alpha n$)

$$\Psi_n(\mathrm{u}) = (2^{n-1/2} n!)^{-1/2} \exp(-\pi u^2) H_n(\sqrt{2\pi} u)$$

- A SFFF for angle $\alpha = 2\pi / M$ with eigenvalue

A=exp($-i\, 2\pi L/M$)

$$f_{1/M}{}^{L}(x) = \sum_{m=0}^{\infty} f_{L+Mm} \Psi_{L+Mm}$$

M. J. Caola, *J. Phys. A* 24, L1143 (1991).
G. Cincotti, F. Gori, and M. Santarsiero, *J. Phys. A* 25, L1191 (1992)

Signal decomposition on the SFFFs

- A signal $g(x)$ can be decomposed into the set of orthogonal SFFFs for some angle $2\pi/M$

$$g(x) = \sum_{n=0}^{\infty} g_n \Psi_n(x) = \sum_{L=0}^{M-1} \left(\sum_{m=0}^{\infty} g_{L+Mm} \Psi_{L+Mm}(x) \right) = \sum_{L=0}^{M-1} f_{1/M}^{L}(x)$$

where

$$f_{1/M}^{L}(x) = C \sum_{k=1}^{M} \exp\left(\frac{i2kL\pi}{M} \right) R^{(k-1)/M}[g(u)](x)$$

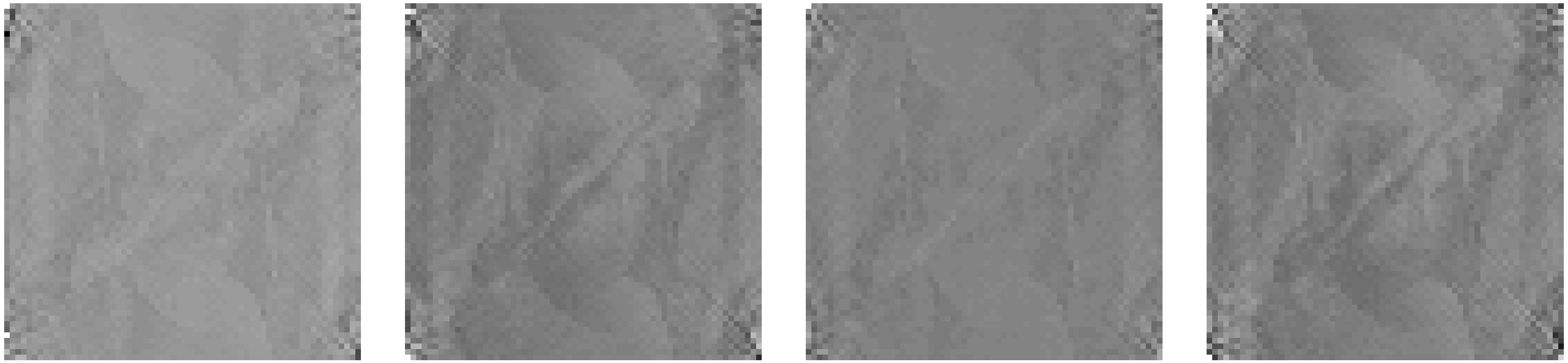
- SFFFs for the same angle $2\pi/M$ but different indexes L are orthogonal to each other.
- The fractional FT of $g(x)$

$$R^{N/M}[g(x)](u) = \sum_{L=0}^{M-1} f_{1/M}^{L}(u) \exp\left(-i \frac{NL2\pi}{M} \right)$$

T. Alieva and A. M. Barbé, *J. Phys. A : Math. Gen*. 30, L211 (1997);
T. Alieva and A. M. Barbé, *J. Mod. Opt.* 46, 83 (1999).

Image decomposition into SFFs

Generator image is decomposed into 4 self-Fourier images. Amplitude of the SFFs are displayed.



L=0 L=1 L=2 L=3

T. Alieva, I. Aizenberg, and A. M. Barbe, *SPIE Proc.* 3480, 457 (1998).

Applications of fractional Fourier transform

- phase retrieval
- beam characterization
- signal analysis (in particular fractal analysis)
- shift-variant filtering
- encryption
- watermarking
- noise reduction
- shift-variant pattern recognition
- neural networks
- motion analysis

H. M. Ozaktas, Z. Zalevsky, and M. A. Kutay, *The fractional Fourier transform with applications in optics and signal processing*, John Willy &Sons, New York (2001);
T. Alieva, M. J. Bastiaans, M. L. Calvo, *EURASIP J. Appl. Sign. Proc.* 2005, 1498 (2005).

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Ray transformation matrix

- Separable system

$$\begin{pmatrix} x_o \\ y_o \\ \theta_{xo} \\ \theta_{yo} \end{pmatrix} = \begin{bmatrix} A_x & 0 & B_x & 0 \\ 0 & A_y & 0 & B_y \\ C_x & 0 & D_x & 0 \\ 0 & C_y & 0 & D_y \end{bmatrix} \begin{pmatrix} x_i \\ y_i \\ \theta_{xi} \\ \theta_{yi} \end{pmatrix}$$

- Add generalized cylindrical lenses ⇒ matrices **A**, **B**, **C**, **D** may not be diagonal

$$\begin{pmatrix} \mathbf{r}_o \\ \mathbf{q}_o \end{pmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{pmatrix} \mathbf{r}_i \\ \mathbf{q}_i \end{pmatrix} = \mathbf{T} \begin{pmatrix} \mathbf{r}_i \\ \mathbf{q}_i \end{pmatrix}$$

Symplectic matrix

- Matrix **T** is symplectic

$$\mathbf{J} = \mathbf{T}^t \mathbf{J} \mathbf{T}, \qquad \mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{-I} & \mathbf{0} \end{pmatrix}$$

$$\mathbf{A}\mathbf{B}^t = \mathbf{B}\mathbf{A}^t \qquad \mathbf{C}\mathbf{D}^t = \mathbf{D}\mathbf{C}^t \qquad \mathbf{A}\mathbf{D}^t - \mathbf{B}\mathbf{C}^t = \mathbf{I}$$

$$\mathbf{A}^t\mathbf{C} = \mathbf{C}^t\mathbf{A} \qquad \mathbf{B}^t\mathbf{D} = \mathbf{D}^t\mathbf{B} \qquad \mathbf{A}^t\mathbf{D} - \mathbf{C}^t\mathbf{B} = \mathbf{I}$$

- det **T**=1
- 16 elements ➡ 10 free parameters

Linear canonical integral transform: general case

- Complex field amplitude evolution during propagation through the first-order optical system (= paraxial approximation)

$$\Psi_o^{\mathbf{T}}(\mathbf{r}_o) = R^{\mathbf{T}}\{\Psi_i(\mathbf{r}_i)\} = \int \Psi_i(\mathbf{r}_i) K_{\mathbf{T}}(\mathbf{r}_i, \mathbf{r}_o)\, d\mathbf{r}_i$$

$$K_{\mathbf{T}}(\mathbf{r}_i, \mathbf{r}_o) = \begin{cases} \dfrac{1}{\sqrt{\det i\mathbf{B}}} \exp\left(i\pi\left[\mathbf{r}_i^t \mathbf{B}^{-1}\mathbf{A}\mathbf{r}_i - 2\mathbf{r}_i^t \mathbf{B}^{-1}\mathbf{r}_o + \mathbf{r}_o^t \mathbf{D}\mathbf{B}^{-1}\mathbf{r}_o\right]\right), & \det \mathbf{B} \neq 0 \\ \dfrac{1}{\sqrt{|\det \mathbf{A}|}} \exp\left(i\pi \mathbf{r}_o^t \mathbf{C}\mathbf{A}^{-1}\mathbf{r}_o\right) \delta\left(\mathbf{r}_i - \mathbf{A}^{-1}\mathbf{r}_o\right), & \mathbf{B} = 0 \end{cases}$$

- Linearity $R^{\mathbf{T}}[f(\mathbf{r}) + g(\mathbf{r})](\mathbf{u}) = R^{\mathbf{T}}[f(\mathbf{r})](\mathbf{u}) + R^{\mathbf{T}}[g(\mathbf{r})](\mathbf{u})$

S. A. Collins, *J. Opt. Soc. Am.* 60,1168 (1970);
M. Moshinsky and C. Quesne, *J. Math. Phys.* 12, 1772 (1971)

Why the transforms are canonical?

- Hamiltonian in paraxial approximation $n(\mathbf{r},z)=n_0-\Delta n(\mathbf{r},z)$

$$h(\mathbf{r},\mathbf{q},z)\approx\frac{\mathbf{q}^2}{2n_0}+\Delta n(\mathbf{r},z),\qquad |\mathbf{q}|\ll q_z,\qquad \Delta n(\mathbf{r},z)<<n_0$$

- Hamiltonian equations

$$\frac{d\mathbf{r}}{dz}=\frac{\partial h(\mathbf{r},\mathbf{q},z)}{\partial\mathbf{q}}\qquad \frac{d\mathbf{q}}{dz}=-\frac{\partial h(\mathbf{r},\mathbf{q},z)}{\partial\mathbf{r}}$$

$$\begin{pmatrix}\dfrac{dr_i}{dz}\\ \dfrac{dq_i}{dz}\end{pmatrix}=\begin{pmatrix}0 & \delta_{i,j}\\ -\delta_{i,j} & 0\end{pmatrix}\begin{pmatrix}\dfrac{\partial}{\partial r_j}\\ \dfrac{\partial}{\partial q_j}\end{pmatrix}h(\mathbf{r},\mathbf{q},z)$$

- Transformations in phase space that preserve the Hamiltonian structure are canonical

K. B. Wolf, “Geometric optics on phase space”, Springer, Berlin (2004)

What is new?

- More parameters

2D separable CT:
6 parameters

⇒

2D general CT:
10 parameters

- New interactions between vector components

2D separable CT:
$x - q_x, y - q_y$

⇒

2D general CT:
$x - q_x, y - q_y, x - y,$
$q_x - q_y, x - q_y, y - q_x$

- det **B**=0 : What is the input-output relation?

Properties of canonical transforms

- Scaling theorem

$$R^{\mathbf{T}}\left[f\left(\mathbf{Wr}\right)\right]\left(\mathbf{u}\right)=(\det\mathbf{W})^{-1/2}R^{\mathbf{TS}}\left[f\left(\mathbf{r}\right)\right]\left(\mathbf{u}\right)$$

$$\mathbf{W}=\begin{bmatrix} w_x & 0 \\ 0 & w_y \end{bmatrix} \qquad \mathbf{TS}=\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}\begin{bmatrix} \mathbf{W}^{-1} & 0 \\ 0 & \mathbf{W} \end{bmatrix}=\begin{bmatrix} \mathbf{AW}^{-1} & \mathbf{BW} \\ \mathbf{CW}^{-1} & \mathbf{DW} \end{bmatrix}$$

- Parseval theorem ➔ Energy conservation law

$$\int f(\mathbf{r})g^{*}(\mathbf{r})d\mathbf{r}=\int F_{\mathbf{T}}(\mathbf{u})G_{\mathbf{T}}^{*}(\mathbf{u})d\mathbf{u}$$

$$\int\left|f\left(\mathbf{r}\right)\right|^{2}d\mathbf{r}=\int\left|F_{\mathbf{T}}\left(\mathbf{u}\right)\right|^{2}d\mathbf{u}$$

Properties of the canonical transforms II

- Shift theorem $R^{\mathbf{T}}\left[f(\mathbf{r}-\mathbf{v})\right](\mathbf{u})=\exp[i\varphi]R^{\mathbf{T}}\left[f(\mathbf{r})\right](\mathbf{u}-\mathbf{A}\mathbf{v})$

$$\varphi=\pi\left[-\mathbf{v}^t\mathbf{C}^t\mathbf{A}\mathbf{v}+2\mathbf{u}^t\mathbf{C}\mathbf{v}\right]$$

- CT of convolution

$$R^{\mathbf{T}}\left[\int f(\mathbf{r}-\mathbf{v})h(\mathbf{v})d\mathbf{v}\right](\mathbf{u})=$$

$$=\int F_{\mathbf{T}}(\mathbf{u}-\mathbf{A}\mathbf{r})h(\mathbf{v})\exp\left[-i\pi\mathbf{v}^t\mathbf{C}^t\mathbf{A}\mathbf{v}\right]\exp\left[i2\pi\mathbf{u}^t\mathbf{C}\mathbf{v}\right]d\mathbf{v}$$

In the case A=0

$$\left|R^{\mathbf{T}}\left[\int f(\mathbf{r}-\mathbf{v})h(\mathbf{v})d\mathbf{v}\right](\mathbf{u})\right|=\left|\sqrt{\det\mathbf{B}}F_{\mathbf{T}}(\mathbf{u})H_{\mathbf{T}}(\mathbf{u})\right|$$

Wigner distribution evolution under the canonical transforms

- Affine transformations of the WD

$$\begin{pmatrix} \mathbf{r}_o \\ \mathbf{q}_o \end{pmatrix} = \mathbf{T}\begin{pmatrix} \mathbf{r}_i \\ \mathbf{q}_i \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}\begin{pmatrix} \mathbf{r}_i \\ \mathbf{q}_i \end{pmatrix}$$

$$\begin{array}{ccc} f(\mathbf{r}_i) & \rightarrow & W_f(\mathbf{r}_i, \mathbf{q}_i) \\ \downarrow & & \downarrow \\ F_{\mathbf{T}}(\mathbf{r}_o) & \rightarrow & W_{F_{\mathbf{T}}}(\mathbf{r}_o, \mathbf{q}_o) \end{array}$$

$$\boxed{W_{F_{\mathbf{T}}}(\mathbf{r}, \mathbf{q}) = W_f(\mathbf{D}^t\mathbf{r} - \mathbf{B}^t\mathbf{q}, -\mathbf{C}^t\mathbf{r} + \mathbf{A}^t\mathbf{q})}$$

Canonical transform calculation: example

- Canonical transform of signal $f(\mathbf{r}) = \exp\left[2\sqrt{2\pi}\mathbf{s}^t\mathbf{K}_i\mathbf{r} - \pi\mathbf{r}^t\mathbf{L}_i\mathbf{r}\right]$

$$R^{\mathbf{T}}\left[f(\mathbf{r})\right](\mathbf{u}) = (\det(\mathbf{A}+i\mathbf{B}\mathbf{L}_i))^{-1/2}\exp\left[-\mathbf{s}^t\mathbf{M}_o\mathbf{s} + 2\sqrt{2\pi}\mathbf{s}^t\mathbf{K}_o\mathbf{u} - \pi\mathbf{u}^t\mathbf{L}_o\mathbf{u}\right]$$

$$\mathbf{K}_o = \mathbf{K}_i(\mathbf{A}+i\mathbf{B}\mathbf{L}_i)^{-1}$$

$$i\mathbf{L}_o = (\mathbf{C}+i\mathbf{D}\mathbf{L}_i)(\mathbf{A}+i\mathbf{B}\mathbf{L}_i)^{-1}$$

$$\mathbf{M}_o = -2i\mathbf{K}_o\mathbf{B}\mathbf{K}_i^t$$

- If $\mathbf{K}_i = 0$

$$R^{\mathbf{T}}\left[f(\mathbf{r})\right](\mathbf{u}) = (\det(\mathbf{A}+i\mathbf{B}\mathbf{L}_i))^{-1/2}\exp\left[-\pi\mathbf{u}^t\mathbf{L}_o\mathbf{u}\right]$$

- If $\mathbf{L}_i = 0$

$$R^{\mathbf{T}}\left[f(\mathbf{r})\right](\mathbf{u}) = (\det\mathbf{A})^{-1/2}\exp\left[2i\mathbf{s}^t\mathbf{K}_i\mathbf{A}^{-1}\mathbf{B}\mathbf{K}_i^t\mathbf{s} + 2\sqrt{2\pi}\mathbf{s}^t\mathbf{K}_i\mathbf{A}^{-1}\mathbf{u} + i\pi\mathbf{u}^t\mathbf{C}\mathbf{A}^{-1}\mathbf{u}\right]$$

M. J. Bastiaans and T. Alieva, *Optics Express* 13, 1107 (2005)

Iwasawa decomposition

- Iwasawa decomposition of transformation matrix

$$\mathbf{T}=\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}=\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ -\mathbf{G} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{S} & \mathbf{0}\\ \mathbf{0} & \mathbf{S}^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{X} & \mathbf{Y}\\ -\mathbf{Y} & \mathbf{X}\end{bmatrix}$$

- Astigmatic lens transform

$$\mathbf{G}=-(\mathbf{CA}^t+\mathbf{DB}^t)(\mathbf{AA}^t+\mathbf{BB}^t)^{-1}=\mathbf{G}^t$$

- Scaling transform

$$\mathbf{S}=(\mathbf{AA}^t+\mathbf{BB}^t)^{1/2}=\mathbf{S}^t$$

- Orthogonal and symplectic matrix T_U described by the unitary matrix U: $U^{-1}=U^{*t}$

$$\mathbf{U}=\mathbf{X}+i\mathbf{Y}=(\mathbf{AA}^t+\mathbf{BB}^t)^{-1/2}(\mathbf{A}+i\mathbf{B})$$

R. Simon and K. B. Wolf, *J. Opt. Soc. Am. A* 17, 2368 (2000)

Three basic operations

- Image rotation (rotation at x-y and q_x-q_y planes)

$$\mathbf{T} = \begin{pmatrix} \mathbf{X} & 0 \\ 0 & \mathbf{X} \end{pmatrix} \qquad \mathbf{X} = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} \qquad \mathbf{Y} = \mathbf{0}$$

- Separable fractional FT – rotation at x-q_x, y-q_y planes of phase space

$$\mathbf{T} = \begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ -\mathbf{Y} & \mathbf{X} \end{pmatrix} \qquad \mathbf{X} = \begin{pmatrix} \cos\gamma_x & 0 \\ 0 & \cos\gamma_y \end{pmatrix} \qquad \mathbf{Y} = \begin{pmatrix} \sin\gamma_x & 0 \\ 0 & \sin\gamma_y \end{pmatrix}$$

- Gyrator (rotation at x-q_y and y-q_x planes)

$$\mathbf{T} = \begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ -\mathbf{Y} & \mathbf{X} \end{pmatrix} \qquad \mathbf{X} = \begin{pmatrix} \cos\alpha & 0 \\ 0 & \cos\alpha \end{pmatrix} \qquad \mathbf{Y} = \begin{pmatrix} 0 & \sin\alpha \\ \sin\alpha & 0 \end{pmatrix}$$

R. Simon and K. B. Wolf, J. *Opt. Soc. Am. A* 17, 342 (2000) R. Simon and K. B. Wolf, *J. Opt. Soc. Am. A* 17, 2368 (2000).

Three basic operations: compact form

- Image rotation (rotation at x-y and q_x-q_y planes)

$$\mathbf{U}_{rot}(\vartheta) = \mathbf{X}_{rot} + i\mathbf{Y}_{rot} = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix}$$

- Separable fractional FT – rotation at x-q_x, y-q_y planes of phase space

$$\mathbf{U}_{frFT}(\gamma_x, \gamma_y) = \mathbf{X}_{frFT} + i\mathbf{Y}_{frFT} = \begin{pmatrix} \exp i\gamma_x & 0 \\ 0 & \exp i\gamma_y \end{pmatrix}$$

- Gyrator (rotation at x-q_y and y-q_x planes)

$$\mathbf{U}_{gyr}(\alpha) = \mathbf{X}_{gyr} + i\mathbf{Y}_{gyr} = \begin{pmatrix} \cos\alpha & i\sin\alpha \\ i\sin\alpha & \cos\alpha \end{pmatrix}$$

R. Simon and K. B. Wolf, *J. Opt. Soc. Am. A* 17, 342 (2000) R. Simon and K. B. Wolf, *J. Opt. Soc. Am. A* 17, 2368 (2000).

Details of Iwasawa decomposition

- Ortho-symplectic matrix T_U : 4 free parameters

$$\mathbf{X}\mathbf{Y}^t = \mathbf{Y}\mathbf{X}^t \qquad \mathbf{X}^t\mathbf{Y} = \mathbf{Y}^t\mathbf{X} \qquad \mathbf{X}\mathbf{X}^t + \mathbf{Y}\mathbf{Y}^t = \mathbf{I}$$

- Ortho-symplectic matrix T_U and corresponding unitary matrix U can be decomposed as

$$\mathbf{T}_\mathbf{U} = \begin{bmatrix} \mathbf{X}_{rot} & \mathbf{Y}_{rot} \\ \mathbf{Y}_{rot} & \mathbf{X}_{rot} \end{bmatrix} \begin{bmatrix} \mathbf{X}_{gyr} & \mathbf{Y}_{gyr} \\ \mathbf{Y}_{gyr} & \mathbf{X}_{gyr} \end{bmatrix} \begin{bmatrix} \mathbf{X}_{frFT} & \mathbf{Y}_{frFT} \\ \mathbf{Y}_{frFT} & \mathbf{X}_{frFT} \end{bmatrix}$$

$$\mathbf{U} = \mathbf{U}_{rot}(\vartheta)\mathbf{U}_{gyr}(\alpha)\mathbf{U}_{frFT}(\gamma_x,\gamma_y)$$

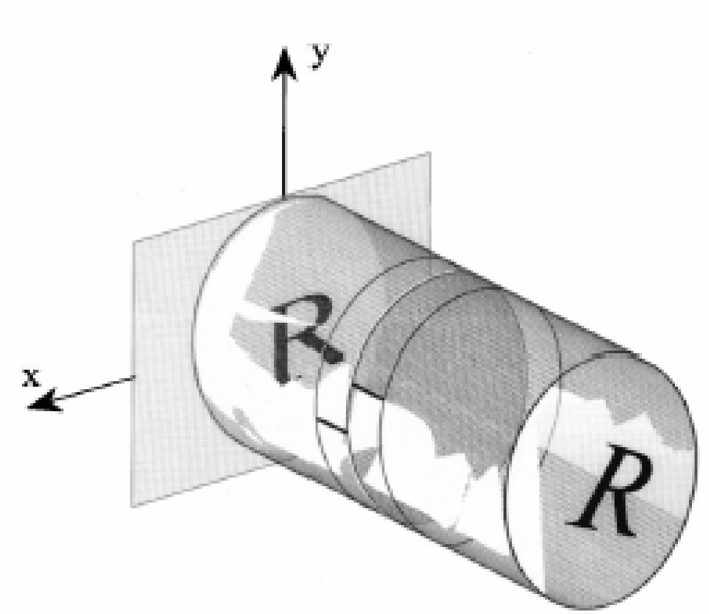
where sub-indices stand for image rotation, gyrator and separable fractional FT operations;

- or as $\mathbf{U} = \mathbf{U}_{rot}(\vartheta)\mathbf{U}_{frFT}(\gamma_x,\gamma_y)\mathbf{U}_{rot}(\theta)$ that leads to the integral expression for the case of singular B

T. Alieva and M. J. Bastiaans, Opt. Lett. 30, 3302 (2005)

Image rotator

- Image is rotated in the x-y planes of phase space
- Its FT is also rotated at the same angle at q_x-q_y plane



$$\mathbf{T} = \begin{pmatrix} \mathbf{X} & 0 \\ 0 & \mathbf{X} \end{pmatrix} \qquad \mathbf{X} = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix}$$

R. Simon and K. B. Wolf , *J. Opt. Soc. Am. A* 17, 342 (2000)

Image reflection and rotation

- Image reflection $f(x,y) \Rightarrow f(\pm x, \mp y)$ is described by matrix

$$\mathbf{T} = \begin{pmatrix} \mathbf{X} & 0 \\ 0 & \mathbf{X} \end{pmatrix} \qquad \mathbf{X} = \begin{pmatrix} \pm 1 & 0 \\ 0 & \mp 1 \end{pmatrix}$$

- Reflector rotated by angle $-\vartheta$ from x-axis

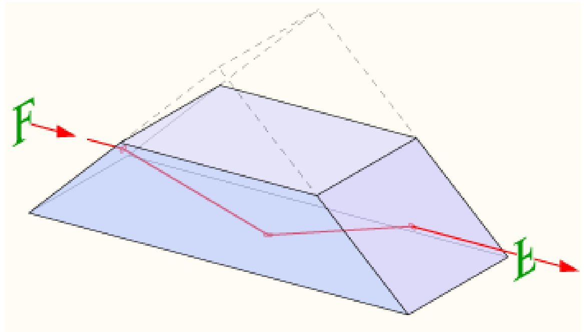
$$\mathbf{X} = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ \sin\vartheta & -\cos\vartheta \end{pmatrix}$$

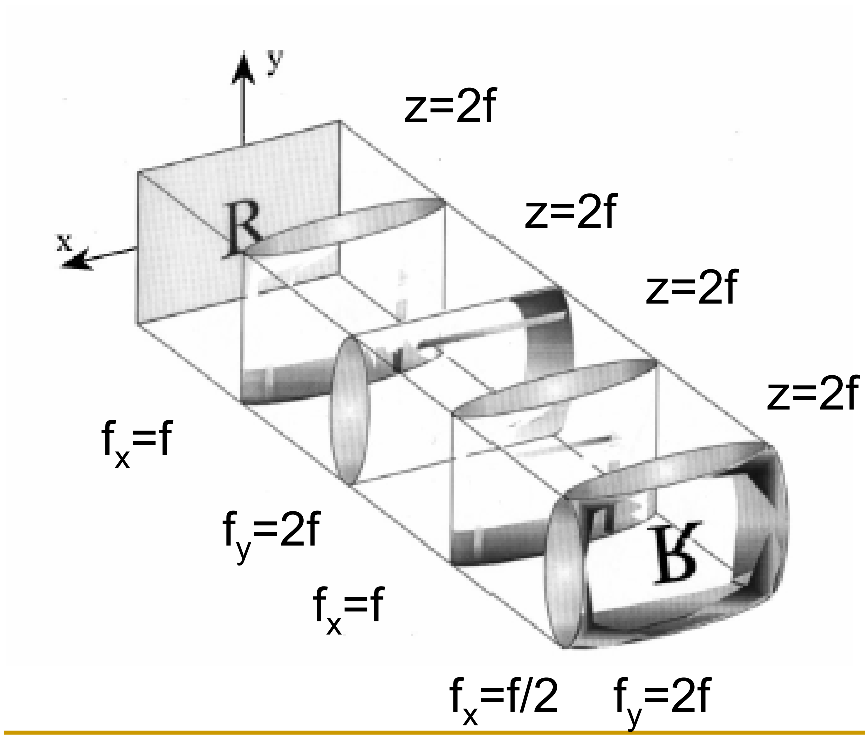
- A cascade of two reflectors - one reflector rotated with respect to the other - yields a rotator

$$\mathbf{X}_{rot} = \mathbf{X}_{ref\,2}\mathbf{X}_{ref\,1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos\vartheta & \sin\vartheta \\ \sin\vartheta & -\cos\vartheta \end{pmatrix} = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix}$$

Optical schemes for image reflection

- Dove prisms
- Lens system



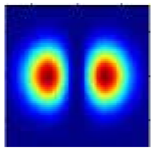


- Minimal lens configuration system for image rotation contains 4 generalized lenses

R. Simon and K. B. Wolf, *J. Opt. Soc. Am. A* 17, 342 (2000); J. A. Rodrigo, T. Alieva and M. L. Calvo *J. Opt. Soc. Am. A* submitted (2006)

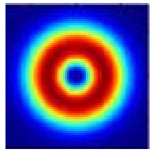
Mode converters

- Hermite-Gaussian $HG_{n,m}$ to Laguerre-Gaussian $LG_{n,m}$ mode converter, Gyrator for $\alpha = \pi/4 + \pi n/2$

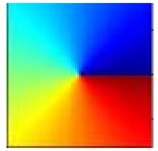


$$\mathbf{X} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{Y} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$$

intensity



phase



- LG modes possess angular orbital momentum (useful for optical particle manipulation)

$$HG_{n,m}(\mathrm{r}) = 2^{1/2}\ \left(2^{n+m} m!n!\right)^{-1/2} \exp(-\pi r^2) H_n(\sqrt{2\pi}x) H_m(\sqrt{2\pi}y)$$

$$LG_{n,m}(\mathrm{r}) = 2^{1/2}\ \left(\frac{\min\{n,m\}!}{\max\{n,m\}!}\right)(\sqrt{2\pi}r)^{|n-m|} \exp\left[i(n-m)\varphi\right]$$

$$\times L_{\min\{n,m\}}^{|n-m|}(2\pi r^2)\exp(-\pi r^2)$$

L. Allen, M. J. Padgett, and M. Babiker, *Progress in Optics* XXXIX, 294 (1999).

Gyrator

- Performs joint rotations in the (x, q_y) and (y, q_x) phase planes

$$\mathbf{T}=\begin{pmatrix}\mathbf{X} & \mathbf{Y}\\ -\mathbf{Y} & \mathbf{X}\end{pmatrix} \qquad \mathbf{X}=\begin{pmatrix}\cos\alpha & 0\\ 0 & \cos\alpha\end{pmatrix} \qquad \mathbf{Y}=\begin{pmatrix}0 & \sin\alpha\\ \sin\alpha & 0\end{pmatrix}$$

- Twisted beam generation:
  - Twisted Gaussian Schell model beams
  - Mode converters

- HG to LG mode converter ($\alpha = \pi/4 + \pi n/2$)

  What mode we obtain for other α?

R. Simon and N. Mukunda, *J. Opt. Soc. Am. A* 10, 95 (1993);
A. T. Friberg, E. Tervonen, and J. Turunen, *J. Opt. Soc. Am. A* 11, 1818 (1994).

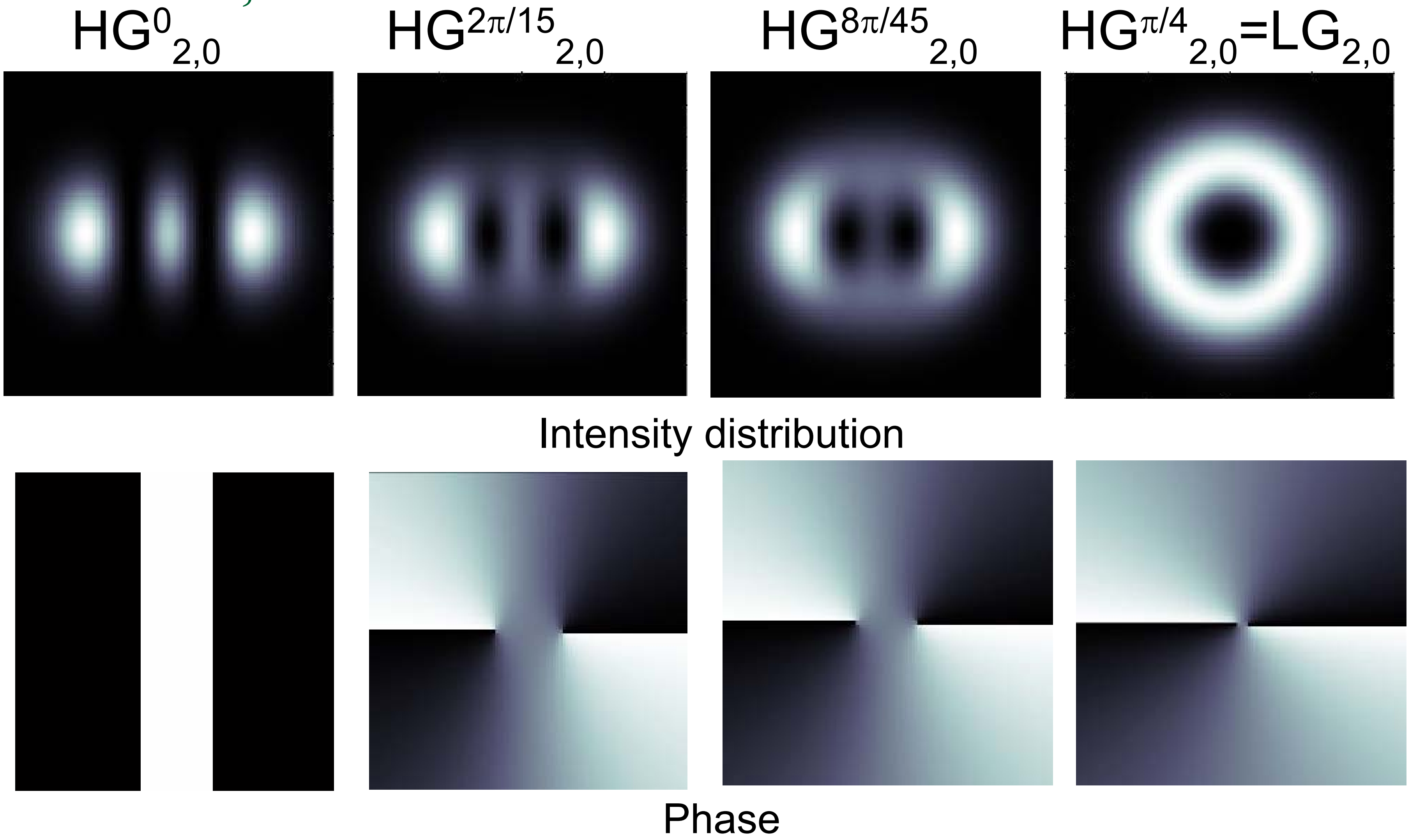
Orthonormal sets of stable modes

- HG and LG functions form the complete orthogonal sets
- Both of these sets are eigenfunctions for the separable fractional FT for angles $\gamma_x = \gamma_y$ $\Rightarrow$ modes are stable
- Matrix corresponding to the isotropic fractional FT commutes with any orthosymplectic matrix, then modes obtained from HGs by gyrator operation at any angle α are also eigenfunctions for the fractional FT and stable
- For any angle these modes form complete orthonormal set

$$R_\alpha^{Gyrator}\left\{HG_{n,m}(\mathbf{r})\right\} = \left\{HG_{n,m}^{\alpha}(\mathbf{r})\right\}$$

M. J. Bastiaans and T. Alieva, *J. Phys. A* 38, L73 (2005);
T. Alieva and M. J. Bastiaans, *Opt. Lett.* 30, 1461 (2005).

$HG^{\alpha}_{n,m}$ modes: examples



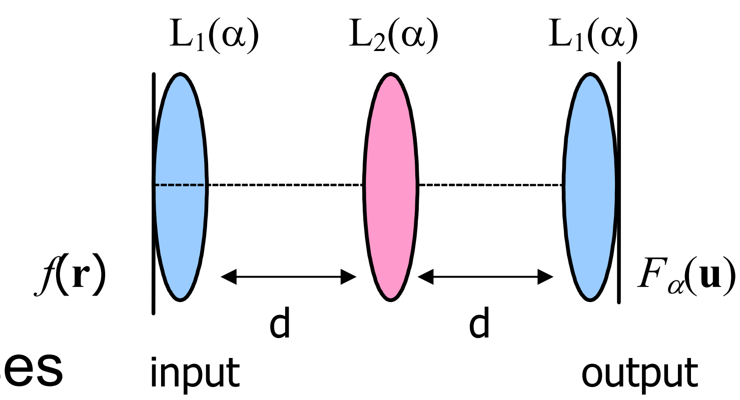
E. G. Abramochkin and V. G. Volostnikov, *J. Opt. A: Pure Appl. Opt.* 6, S157 (2004);
J. A. Rodrigo, T. Alieva, M. L. Calvo, *J. Opt. Soc. A* (submitted)

Optical scheme for gyrator operation

- There are many schemes for HG-LG mode converters due to rotational symmetry of LG mode and the fact that HG and LG modes are eigenfunctions for the fractional FT

$$R^{Rot(\alpha)} R^{frFT(\gamma,\gamma)} R^{Gyr(\pi/4+\pi n/2)} R^{frFT(\gamma_x,\gamma_y)} \left\{ HG_{n,m}(\mathbf{r}) \right\}$$

- Flexible scheme for gyrator: symmetric system with three generalized lenses system
  - *d* is fixed
  - L(α) is changing
- L(α) lens implementation: SLM, composition of rotated cylindrical lenses



M. W. Beijersbergen, L. Allen, et al, *Opt. Commun*. 96, 123 (1993); A.T. O'Neil, J. Courtial, *Optics Commun.* 181, 35 (2000); J.A. Rodrigo, T. Alieva, M.L. Calvo, *J. Opt. Soc. A* (submitted)

Outlines

- One dimensional linear Canonical integral Transforms (CTs)
- Fractional Fourier transform
- Two dimensional CTs: image rotation, fractional FT, twisting
  - Properties of the CTs
  - Optical schemes
  - Stable modes
- **New transform generation**
  - Fractional transforms

New transform generation

- CTs perform affine transformations in phase space: rotation, scaling, skew
- Linear combinations of the CTs produce new transforms
- Linear combination of the canonical fractional FTs leads to
  - Fractional Cosine and Sine transforms
  - Fractional Hartley transform
  - Other (non canonical) fractional Fourier Transforms
- Cascade of the CTs with appropriate masks between them lead to a variety of operations:
  - Convolution, correlation, wavelet transform, etc. (cascade of FTs)
  - Fractional convolution (cascade of fractional FTs)

Why we need fractional transforms

- They arise under the consideration of different problems (fractional FT, fractional Hankel transform in optics and quantum mechanics)

- The fractionalization gives a new degree of freedom (the fractional order) which can be used for more complete object characterization or as encoding parameter

Fractionalization of cyclic transforms

- A linear transform of *f*(*x*) $R\,[f(x)](u) = \int K\,(x,u)\,f(x)\,dx$

is a cyclic one if its *N*-time acting produces the identity transform $R^N = I$

- Examples: Fourier and Hilbert transforms are cyclic with *N=4* ;

Hankel, Hartley, Sine, Cosine transforms: *N*=2

- Desirable properties of a fractional *R*-transform R^p, where *p* is a parameter of the fractionalization:
  - Continuity of R^p for any real value *p*
  - Additivity of R^p with respect to the parameter *p*: $R^{p+q} = R^p R^q$
  - Reproducibility of the ordinary transforms for integer values of *p*.

  In particular $R^0=I$ and $R^1=R$
- Methods of fractional kernel generation:
  - Cyclic kernel harmonic decomposition
  - Cyclic kernel eigenfunctions decompositon

T. Alieva, M. L. Calvo, *J. Opt. Soc. Am. A* 17**,** 2330 (2000); T. Alieva, M. J. Bastiaans, M. L. Calvo *Proc. ICOL 2005, Dehradun, India, Paper IT-OIP-2*, 12-15 December 2005.

Fractional Sine and Cosine transforms

- The Sine and Cosine transforms are defined as

$$R_S[f(x)](u) = 2\int_0^\infty f(x)\sin(2\pi ux)dx \qquad R_C[f(x)](u) = 2\int_0^\infty f(x)\cos(2\pi ux)dx$$

- Fractional Sine and Cosine transforms at angle α are superposition of fractional FTs for angles α and α+π

$$R_S^\alpha[f(x)](u) = i\exp(i\alpha)\left\{R_F^{\alpha+\pi}[f(x)](u) - R_F^\alpha[f(x)](u)\right\}$$

$$R_C^\alpha[f(x)](u) = R_F^{\alpha+\pi}[f(x)](u) + R_F^\alpha[f(x)](u)$$

- Kernels for fractional ST and CT

$$i\exp(-i\alpha)K_S(\alpha,x,u) = 2k_\alpha(x,u)\sin\left[\frac{2\pi ux}{\sin\alpha}\right] \qquad K_C(\alpha,x,u) = 2k_\alpha(x,u)\cos\left[\frac{2\pi ux}{\sin\alpha}\right]$$

where

$$k_\alpha(x,u) = \frac{\exp(i\alpha/2)}{\sqrt{i\sin\alpha}}\exp[i\pi(x^2+u^2)\cot\alpha]$$

T. Alieva, M. J. Bastiaans, *Asian J. Phys.* 12, 25 (2003)

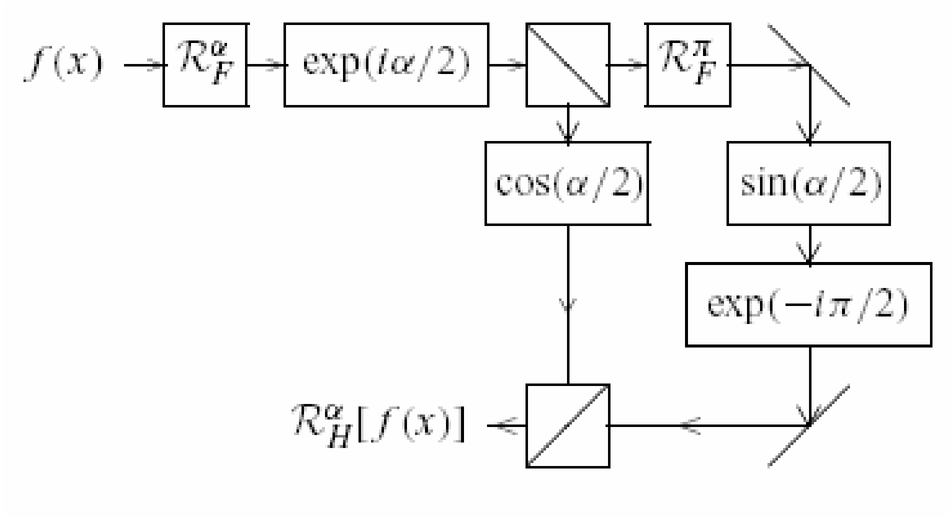
Fractional Hartley transform

- The Hartley transform (HT) of *f*(*x*) is defined as

$$R_H[f(x)](u) = 2\int_{-\infty}^{\infty} f(x)\mathrm{cas}(2\pi ux)dx$$ where $\mathrm{cas}\, x = \cos x + \sin x$

- The fractional HT is a linear superposition of fractional FTs

$$R_H^{\alpha} = \exp(i\alpha/2)[\cos(\alpha/2)R_F^{\alpha} - i\sin(\alpha/2)R_F^{\alpha+\pi}]$$



- Optical set up for fractional Hartley transform

T. Alieva, M. J. Bastiaans, and M. L. Calvo, *EURASIP J. Appl. Sign. Proc.* 2005, 1498 (2005)

Non canonical fractional FT

- Canonical fractional FT is not the only fractional FT
- Other fractional FTs which are periodic and additive with respect to the angle can be constructed as superpositions of the canonical fractional FTs at certain angles
- Kernel of a non canonical fractional FT $K(p,x,u)$ for angle $\pi p/2$ as a linear superposition of the canonical FT kernels $K_F^{n\pi/2l}(x,u)$

$$K(p,x,u) = \frac{1}{M}\sum_{n=0}^{M-1} \exp\left[\frac{i\pi(M-1)(pl-n)}{M}\right]\frac{\sin\left[\pi\left(pl-n\right)\right]}{\sin\left[\pi\left(pl-n\right)/M\right]}K_F^{n\pi/2l}(x,u)$$

S. Liu, J. Jiang, Y. Zhang, J. Zhang, *J. Phys. A* 30, 973 (1997); T. Alieva, M. J. Bastiaans, and M. L. Calvo, *EURASIP J. Appl. Sign. Proc.* 2005, 1498 (2005).

Conclusions

- Canonical integral transforms allow to perform the affine image transformations in phase space (image rotation, scaling, twisting, rotation in position-momentum planes, etc)

- Ray transformation matrix description simplifies
  - interpretation of CT actions in phase space
  - design of optical systems which perform the CT

- Based on the CT other optical operations useful for information processing (convolution, correlation, Wavelet, Hilbert, different classes of fractional transforms etc.) can be performed

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