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WINTER COLLEGE

on

QUANTUM AND CLASSICAL ASPECTS

of

INFORMATION OPTICS

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Lecture 1: Ince-Gaussian Modes of the Paraxial Wave Equation and Laser Resonators

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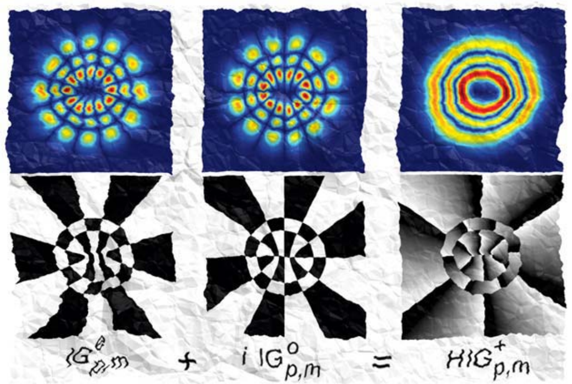
Lecture 1: Ince-Gaussian Modes of the Paraxial Wave Equation and Laser Resonators



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Fundamental Gaussian beams

Hermite-Gaussian beams

Laguerre-Gaussian beams

2. INCE-GAUSSIAN BEAMS

Paraxial wave equation in elliptic coordinates

Form of the Ince-Gaussian beams

Physical properties of the IG beams

Relation with the Hermite-Gauss and Laguerre-Gauss beams

Helical Ince-Gaussian beams

IG beams in quadratic index media

3. ELEGANT (Complex-Argument) INCE-GAUSSIAN BEAMS

Elegant Hermite-Gaussian and Laguerre-Gaussian beams

Elegant Ince-Gaussian beams

Adjoint operator and adjoint solution (biorthogonality)

Introduction

Basics on Gaussian beams

Fundamental Gaussian Beam

For a paraxial field traveling in positive *z* direction, we write

$$U(\mathbf{r}) = \Psi(\mathbf{r}_t, z)\exp(ikz),$$

where the slowly varying complex amplitud Ψ satisfies the Paraxial wave equation (PWE).

$$\left[\nabla_t^2 + 2ik\frac{\partial}{\partial z}\right]\Psi(\mathbf{r}) = 0$$

A fundamental solution of the PWE is the Gaussian Beam (GB)

$$\mathrm{GB}(\mathbf{r}) = \frac{C}{\mu(z)}\exp\left[-\frac{1}{\mu(z)}\frac{r^2}{w_0^2}\right]$$

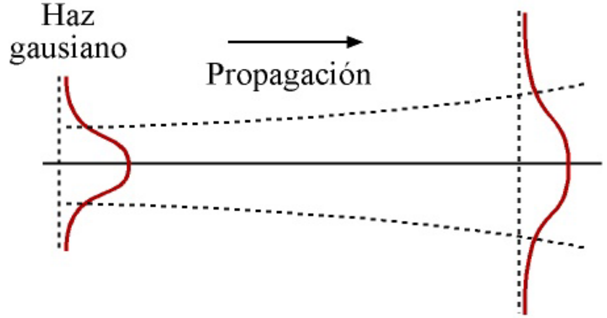
where $\mu = \mu(z) = 1 + iz/z_R$ $\qquad z_R = kw_0^2/2$

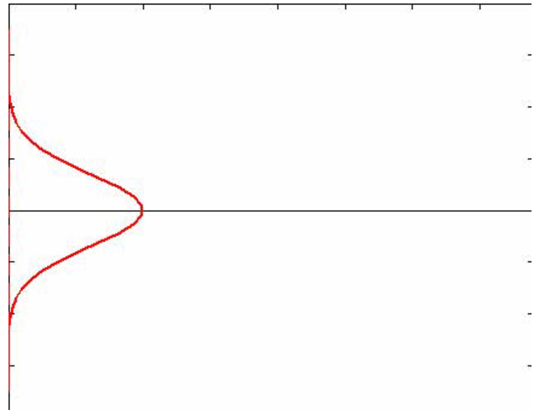
and *C* is the normalization constant $C = \dfrac{1}{w_0}\left(\dfrac{2}{\pi}\right)^{1/2}$

Amplitude and phase propapagation of a Gaussian Beam

Amplitude

$$\frac{C}{w(z)}\exp\left[\frac{-r^2}{w^2(z)}\right]$$

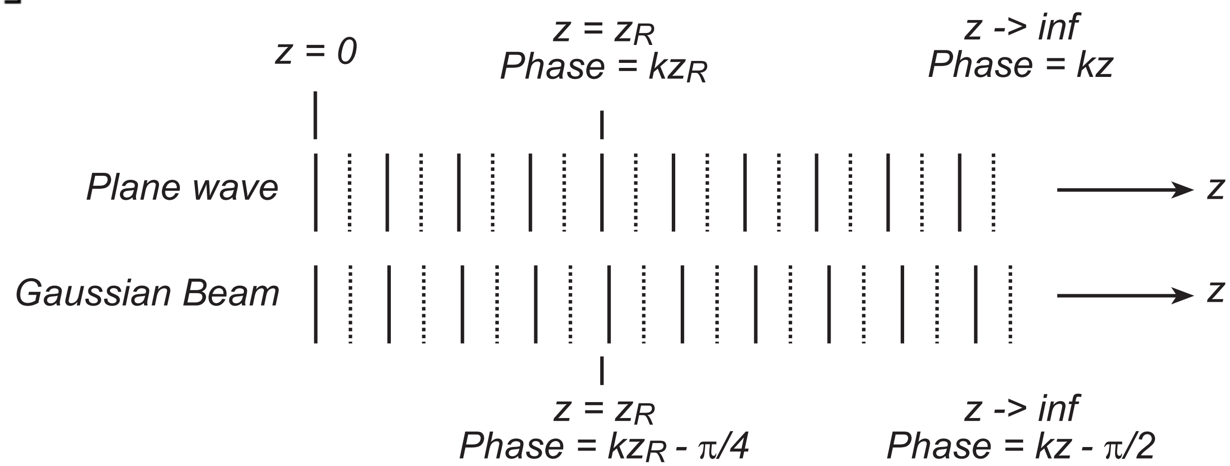




Longitudinal phase

$$\exp\left[ikz - i\underbrace{\arctan\left(\frac{z}{z_R}\right)}\right]$$

Gouy shift



High-order beams: Hermite-Gaussian beams

High-order solutions of the PWE in Cartesian coordinates are given by the Hermite-Gaussian beams

$$\mathrm{HG}_{n_x,n_y}(\mathbf{r}) = C_{n_x,n_y}\left[\frac{\mu^*}{\mu}\right]^{(n_x+n_y)/2} \underbrace{\frac{1}{\mu}\exp\left(-\frac{r^2}{\mu w_0^2}\right)}_{\mathrm{GB}(\mathbf{r})} \mathrm{H}_{n_x}\left(\frac{\sqrt{2}x}{w(z)}\right)\mathrm{H}_{n_y}\left(\frac{\sqrt{2}y}{w(z)}\right)$$

$$\mu = \mu(z) = 1 + iz/z_R \qquad w(z) = w_0|\mu| = w_0\sqrt{1 + z^2/z_R^2}$$

$$C_{n_x,n_y} = \frac{1}{w_0}\left(2^{n_x+n_y-1}\pi n_x! n_y!\right)^{-1/2}$$

HGBs satisfy important mathematical and physical properties

1) Complete families of exact and orthogonal solutions of the PWE
2) They are eigenmodes of stable resonators
3) Their transverse shapes do not change under propapation (structurally stable)

Historical papers by Kogelnik and Li

Laser Beams and Resonators

H. KOGELNIK AND T. LI

Abstract—This paper is a review of the theory of laser beams and resonators. It is meant to be tutorial in nature and useful in scope. No attempt is made to be exhaustive in the treatment. Rather, emphasis is placed on formulations and derivations which lead to basic understanding and on results which bear practical significance.

1. Introduction

THE COHERENT radiation generated by lasers or masers operating in the optical or infrared wavelength regions usually appears as a beam whose transverse extent is large compared to the wavelength. The resonant properties of such a beam in the resonator structure, its propagation characteristics in free space, and its interaction behavior with various optical elements and devices have been studied extensively in recent years. This paper is a review of the theory of laser beams and resonators. Emphasis is placed on formulations and derivations which lead to basic understanding and on results which are of practical value.

Historically, the subject of laser resonators had its origin when Dicke [1], Prokhorov [2], and Schawlow and Townes [3] independently proposed to use the Fabry-Perot interferometer as a laser resonator. The modes in such a structure, as determined by diffraction effects, were first calculated by Fox and Li [4]. Boyd and Gordon [5], and Boyd and Kogelnik [6] developed a theory for resonators with spherical mirrors and approximated the modes by wave beams. The concept of electromagnetic wave beams was also introduced by Goubau and Schwering [7], who investigated the properties of sequences of lenses for the guided transmission of electromagnetic waves. Another treatment of wave beams was given by Pierce [8]. The behavior of Gaussian laser beams as they interact with various optical structures has been analyzed by Goubau [9], Kogelnik [10], [11], and others.

The present paper summarizes the various theories and is divided into three parts. The first part treats the passage of paraxial rays through optical structures and is based on geometrical optics. The second part is an analysis of laser beams and resonators, taking into account the wave nature of the beams but ignoring diffraction effects due to the finite size of the apertures. The third part treats the resonator modes, taking into account aperture diffraction effects. Whenever applicable, useful results are presented in the forms of formulas, tables, charts, and graphs.

Manuscript received July 12, 1966.
H. Kogelnik is with Bell Telephone Laboratories, Inc., Murray Hill, N. J.
T. Li is with Bell Telephone Laboratories, Inc., Holmdel, N. J.

2. Paraxial Ray Analysis

A study of the passage of paraxial rays through optical resonators, transmission lines, and similar structures can reveal many important properties of these systems. One such "geometrical" property is the stability of the structure [6], another is the loss of unstable resonators [12]. The propagation of paraxial rays through various optical structures can be described by ray transfer matrices. Knowledge of these matrices is particularly useful as they also describe the propagation of Gaussian beams through these structures; this will be discussed in Section 3. The present section describes briefly some ray concepts which are useful in understanding laser beams and resonators, and lists the ray matrices of several optical systems of interest. A more detailed treatment of ray propagation can be found in textbooks [13] and in the literature on laser resonators [14].

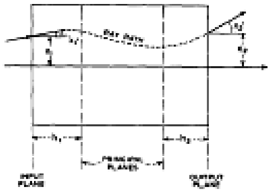


Fig. 1. Reference planes of an optical system. A typical ray path is indicated.

2.1 *Ray Transfer Matrix*

A paraxial ray in a given cross section (z = const) of an optical system is characterized by its distance x from the optic (z) axis and by its angle or slope x' with respect to that axis. A typical ray path through an optical structure is shown in Fig. 1. The slope x' of paraxial rays is assumed to be small. The ray path through a given structure depends on the optical properties of the structure and on the input conditions, i.e., the position x_1 and the slope x_1' of the ray in the input plane of the system. For paraxial rays the corresponding output quantities x_2 and x_2' are linearly dependent on the input quantities. This is conveniently written in the matrix form

$$\begin{vmatrix} x_2 \\ x_2' \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix} \begin{vmatrix} x_1 \\ x_1' \end{vmatrix} \quad (1)$$

1966 KOGELNIK AND LI: LASER B

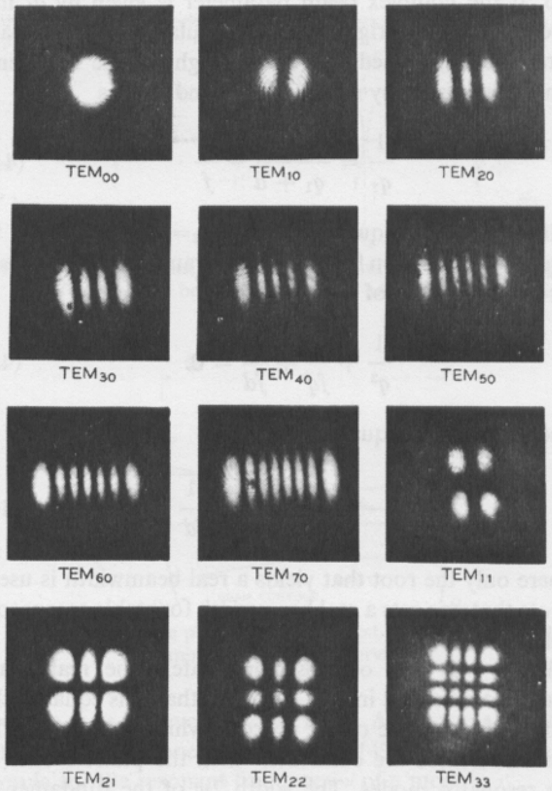


Fig. 7. Mode patterns of a gas laser oscillator (rectangular symmetry).

Kogelnik and Li, "Laser beams and resonators," Proc. IEEE, **54**, 1312-29 (1966)
Appl. Opt. **5**, 1550-1567 (1966)

High-order Modes: Laguerre-Gaussian beams

High-order solutions of the PWE in circular cylindrical coordinates are given by the Laguerre-Gaussian beams

Even LG	Odd LG	Helical LG

$$\mathrm{LG}_{n,l}(\mathbf{r}) = C_{n,l}\left(\frac{\mu^*}{\mu}\right)^{(2n+l)/2} \underbrace{\frac{1}{\mu}\exp\left(-\frac{r^2}{\mu w_0^2}\right)}_{\mathrm{GB}(\mathbf{r})} \left[\frac{\sqrt{2}r}{w(z)}\right]^l \mathrm{L}_n^l\left(\frac{2r^2}{w^2(z)}\right) \left\{\begin{array}{c} \cos l\theta \\ \sin l\theta \\ \dfrac{\exp(\pm il\theta)}{\sqrt{2}} \end{array}\right\}$$

where $\mu = \mu(z) = 1 + iz/z_R$ **and**

$$w(z) = w_0|\mu| = w_0\sqrt{1 + z^2/z_R^2}$$ **is the waist size**

$$C_{n,l} = \frac{1}{w_0}\left[\frac{4n!}{(1+\delta_{0,l})\,\pi\,(n+l)!}\right]^{1/2}$$ **is the normalization constant**

Important properties of LG beams

$$\mathrm{LG}^{\pm}_{n,l}(\mathbf{r}) = \frac{C_{n,l}}{\sqrt{2}} \left(\frac{\mu^*}{\mu}\right)^{n+l/2} \underbrace{\frac{1}{\mu}\exp\left(-\frac{r^2}{\mu w_0^2}\right)}_{\mathrm{GB}(\mathbf{r})} \left[\frac{\sqrt{2}r}{w(z)}\right]^l \mathrm{L}^l_n\left(\frac{2r^2}{w^2(z)}\right)\exp(\pm i l\theta)$$

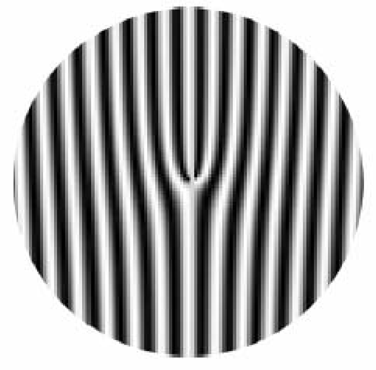
LG beams satisfy important mathematical and physical properties

1) Complete family of exact and orthogonal solutions of the PWE
2) They are eigenmodes of stable resonators
3) Their transverse shapes do not change under propapation (structurally stable)

Helical LG beams carry an Orbital Angular Momentum (OAM) of $l\hbar$ per photon [1].

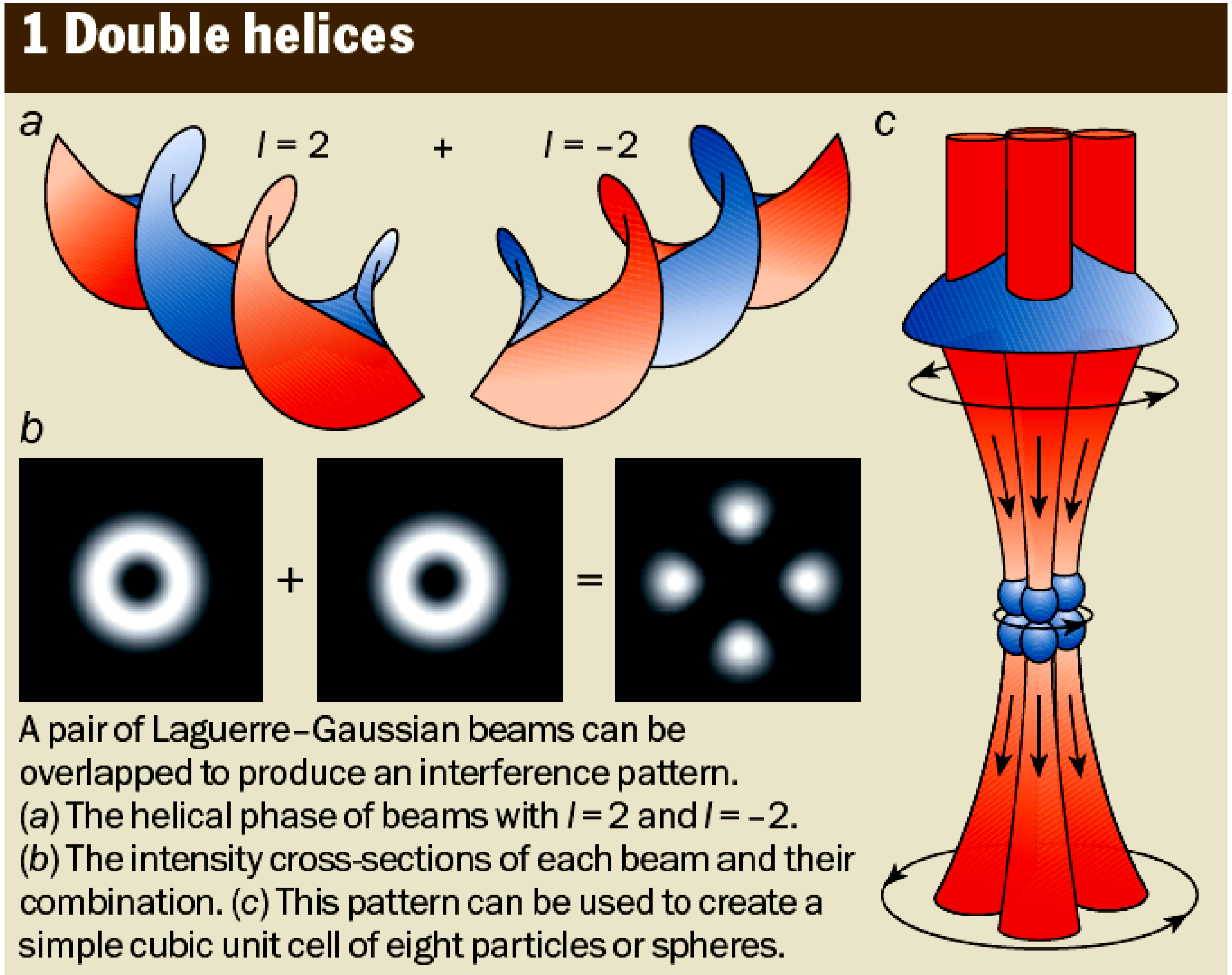
Vortex at $r = 0$ with a topological charge of l.

Applications in cylindrical-lens mode converters, optical tweezers, optical trapping, etc.



[1] L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw, and J. P.Woerdman, Phys. Rev. A **45**, 8185–8189 (1992).

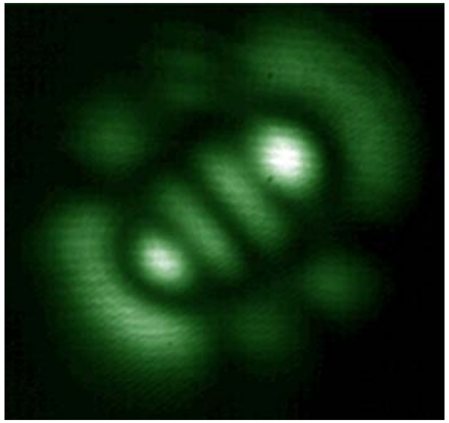
An aplicattion of the LG beams: Optical trapping



A pair of Laguerre–Gaussian beams can be overlapped to produce an interference pattern. (*a*) The helical phase of beams with $l = 2$ and $l = -2$. (*b*) The intensity cross-sections of each beam and their combination. (*c*) This pattern can be used to create a simple cubic unit cell of eight particles or spheres.

[1] Kishan Dholakia, Gabriel Spalding and Michael MacDonald, Phys. World, **45**, Oct. 2002.

Ince-Gaussian beams



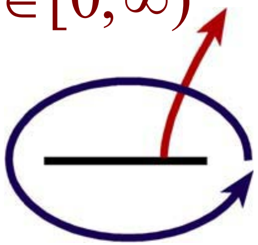
Definition of the elliptic coordinate system

$$(x, y) \leftrightarrow (\xi, \eta)$$
$$x = f \cosh\xi \cos\eta$$
$$y = f \sinh\xi \sin\eta$$
$$z = z$$

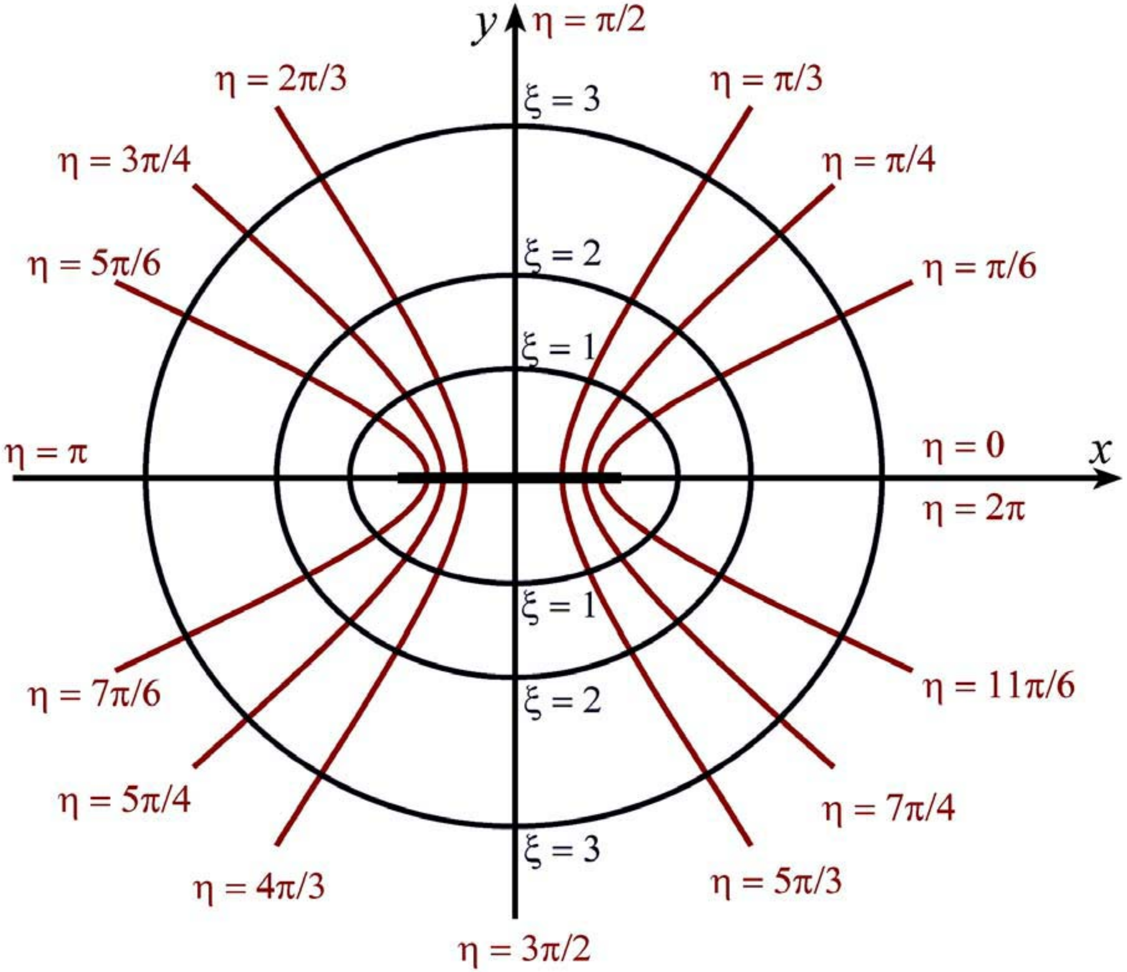
For a given ellipse

$$f^2 = a^2 - b^2$$
$$e = \frac{f}{a} = \frac{1}{\cosh\xi_0}$$

$\xi \in [0, \infty)$



$\eta \in [0, 2\pi)$



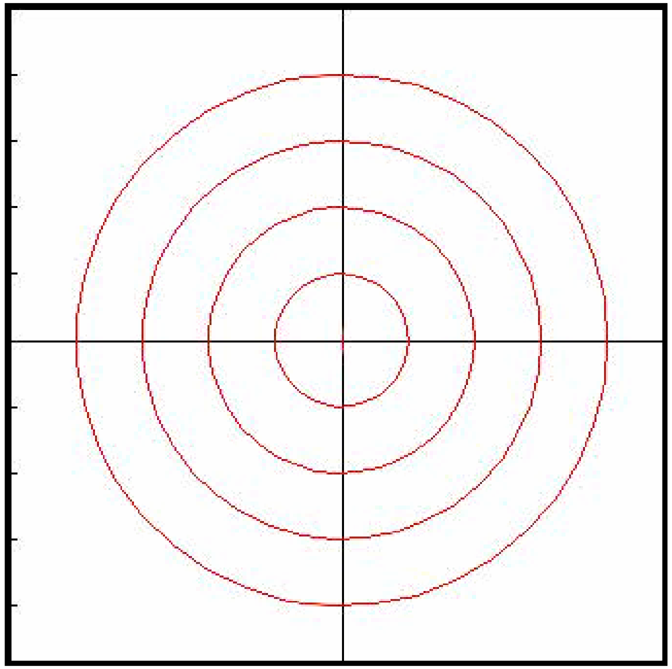
Limits of the elliptic coordinate system (ECS)

When $f = 0$, then ECS becomes the circular cylindrical system.

When $f \to \infty$, then ECS tends to the Cartesian coordinates

$$x = f \cosh \xi \cos \eta$$
$$y = f \sinh \xi \sin \eta$$
$$z = z$$

Anima_sistema_eliptico



Paraxial Wave Equation in elliptic coordinates

We start from the PWE:

$$\left[\nabla_t^2 + 2ik\,\frac{\partial}{\partial z}\right]\Psi\left(\mathbf{r}\right) = 0.$$

expressed in elliptic coordinates:

$$x = f(z)\cosh\xi\cos\eta$$
$$y = f(z)\sinh\xi\sin\eta$$
$$z = z$$

where we set the focal distance of the coordinate system as:

$$f(z) = f_0\,\frac{w(z)}{w_0}$$

$$f(z) = f_0\,\sqrt{1+\left(z/z_R\right)^2}$$

GO TO MATLAB

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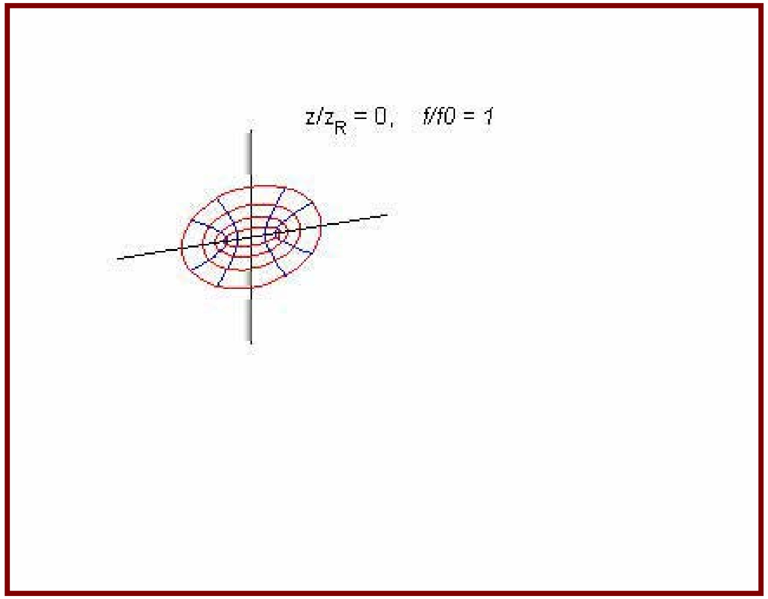


Paraxial Wave Equation in elliptic coordinates

$$\underbrace{\frac{1}{f^2\left(\cosh^2\xi-\cos^2\eta\right)}\left[\frac{\partial^2 U(\mathbf{r})}{\partial\xi^2}+\frac{\partial^2 U(\mathbf{r})}{\partial\eta^2}\right]}_{\nabla_t^2 U(\mathbf{r})}+2ik\frac{\partial U(\mathbf{r})}{\partial z}=0,$$

We seek SEPARABLE solutions of the form:

$$\mathrm{IG}(\mathbf{r})=E(\xi)\,N(\eta)\exp\left[iZ(z)\right]\ \mathrm{GB}(\mathbf{r})$$

where $\mathrm{GB}(\mathbf{r})=\frac{C}{\mu(z)}\exp\left[-\frac{1}{\mu(z)}\frac{r^2}{w_0^2}\right]$

is the fundamental Gaussian beam.

Separation of the PWE in elliptic coordinates: Ince equations

Separation of the PWE in elliptic coordinates leads to the Ince equations

$\eta \to i\xi$

$$\frac{\mathrm{d}^2 E}{\mathrm{d}\xi^2} - \varepsilon \sinh 2\xi \frac{\mathrm{d}E}{\mathrm{d}\xi} - (a - p\varepsilon \cosh 2\xi)\, E = 0,$$

$$\frac{\mathrm{d}^2 N}{\mathrm{d}\eta^2} + \varepsilon \sin 2\eta \frac{\mathrm{d}N}{\mathrm{d}\eta} + (a - p\varepsilon \cos 2\eta)\, N = 0,$$

$$-\left(\frac{z^2 + z_R^2}{z_R}\right) \frac{\mathrm{d}Z}{\mathrm{d}z} = p,$$

where p and a are separation constants, and $\varepsilon = 2f_0^2/w_0^2$ is the ellipticity parameter.

Solutions of Eq. (5) are known as the even and odd Ince polynomials of order p and degree m, they are denoted usually as $\mathrm{C}_p^m(\eta,\varepsilon)$ and $\mathrm{S}_p^m(\eta,\varepsilon)$ respectively, where $0 \le m \le p$ for even functions, $1 \le m \le p$ for odd functions, the indices (p,m) have always the same parity, i.e. $(-1)^{p-m} = 1$, and ε is the ellipticity parameter.[6]

Computing the Ince Polynomials

Ince polynomials can be calculated using Fourier series and the standard theory on periodic differential equations.

The number of terms of the series is finite !!

Even	Odd

$$C_{2n}^{m} = \sum_{r=0,1,2,...}^{p/2} A_r^{+} \cos 2r\eta, \qquad p = 2n = even, \qquad m = 0, 2, 4, .., p$$

$$C_{2n+1}^{m} = \sum_{r=0,1,2,..}^{(p-1)/2} A_r^{-} \cos(2r+1)\eta, \qquad p = 2n+1 = odd, \qquad m = 1, 3, 5, .., (p-1)/2$$

$$S_{2n}^{m} = \sum_{r=1,2,3...}^{p/2} B_r^{+} \sin 2r\eta, \qquad p = 2n = even, \qquad m = 2, 4, 6, .., p/2,$$

$$S_{2n+1}^{m} = \sum_{r=0,1,2,...}^{(p-1)/2} B_r^{-} \sin(2r+1)\eta, \qquad p = 2n+1 = odd, \qquad m = 1, 3, 5, .., (p-1)/2.$$

Classical book: F. M. Arscott, *Periodic differential equations*, (Pergamon Press, Oxford, 1964).

Behavior of the Ince Polynomials

Even	Odd

$C_3^{\,1}(\eta,\varepsilon)$ $\quad C_p^{\,4}(\eta,\varepsilon=3)$ $\quad S_6^{\,2}(\eta,\varepsilon)$ $\quad S_p^{\,3}(\eta,\varepsilon=3)$

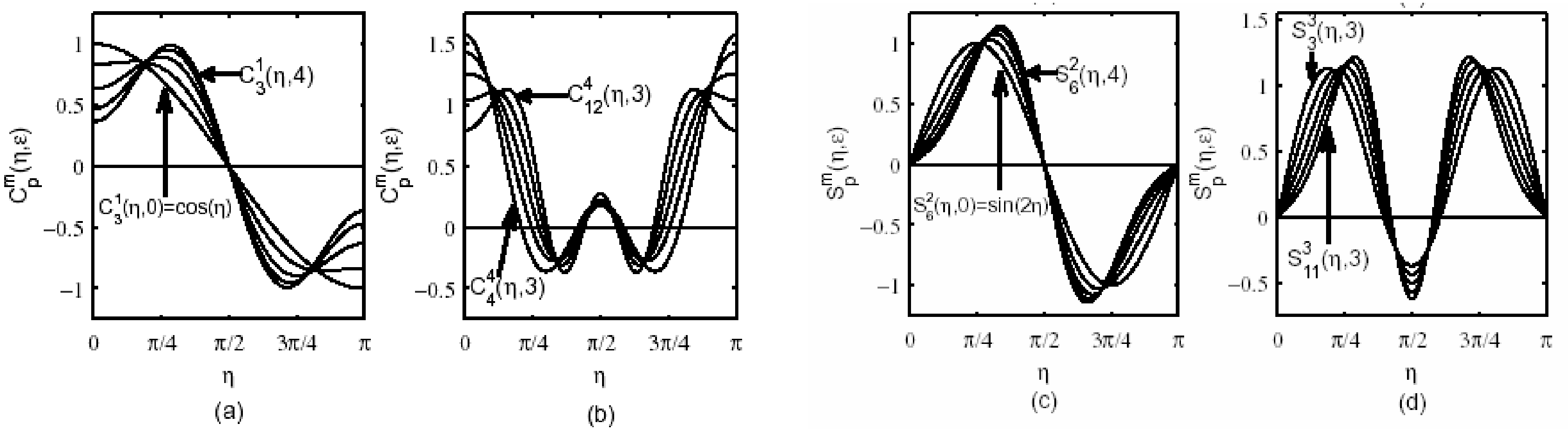


Fig. 6. Plots of Ince polynomials (a) $C_3^1(\eta,\varepsilon)$, $\varepsilon \in \{0,1,2,3,4\}$; (b) $C_p^4(\eta,3)$, $p \in \{4,6,8,10,12\}$; (c) $S_6^2(\eta,\varepsilon)$, $\varepsilon \in \{0,1,2,3,4\}$; (d) $S_p^3(\eta,3)$, $p \in \{3,5,7,9,11\}$.

$$\int_{-\pi}^{\pi} \mathrm{C}_p^m(\eta)\,\mathrm{C}_p^{m'}(\eta)\,\mathrm{d}\eta = \pi\delta_{mm'}$$

Mathematical form of the IG beams

Even IGB

$$\mathrm{IG}^{e,o}_{p,m}\left(\mathbf{r};\varepsilon\right) = \left\{ \begin{array}{c} \mathfrak{C}_{p,m} \\ \\ \mathfrak{S}_{p,m} \end{array} \right\} \left[\frac{\mu^*}{\mu}\right]^{p/2} \underbrace{\frac{1}{\mu}\exp\left(-\frac{r^2}{\mu w_0^2}\right)}_{\mathrm{GB}(\mathbf{r})} \left\{ \begin{array}{c} \mathrm{C}^m_p\left(i\xi,\varepsilon\right)\mathrm{C}^m_p\left(\eta,\varepsilon\right) \\ \\ \mathrm{S}^m_p\left(i\xi,\varepsilon\right)\mathrm{S}^m_p\left(\eta,\varepsilon\right) \end{array} \right\}$$

Even IGB

where $\mu = \mu\left(z\right) = 1 + iz/z_R$ $\qquad w(z) = w_0\left|\mu\right| = w_0\sqrt{1 + z^2/z_R^2}$

and $\mathfrak{C}_{p,m}$ and $\mathfrak{S}_{p,m}$ **are normalization constants**

$$x = f\left(z\right)\cosh\xi\cos\eta, \qquad y = f\left(z\right)\sinh\xi\sin\eta$$

$$f\left(z\right) = f_0\frac{w\left(z\right)}{w_0} = f_0\sqrt{1 + z^2/z_R^2}$$

$$\varepsilon = \frac{2f_0^2}{w_0^2}$$

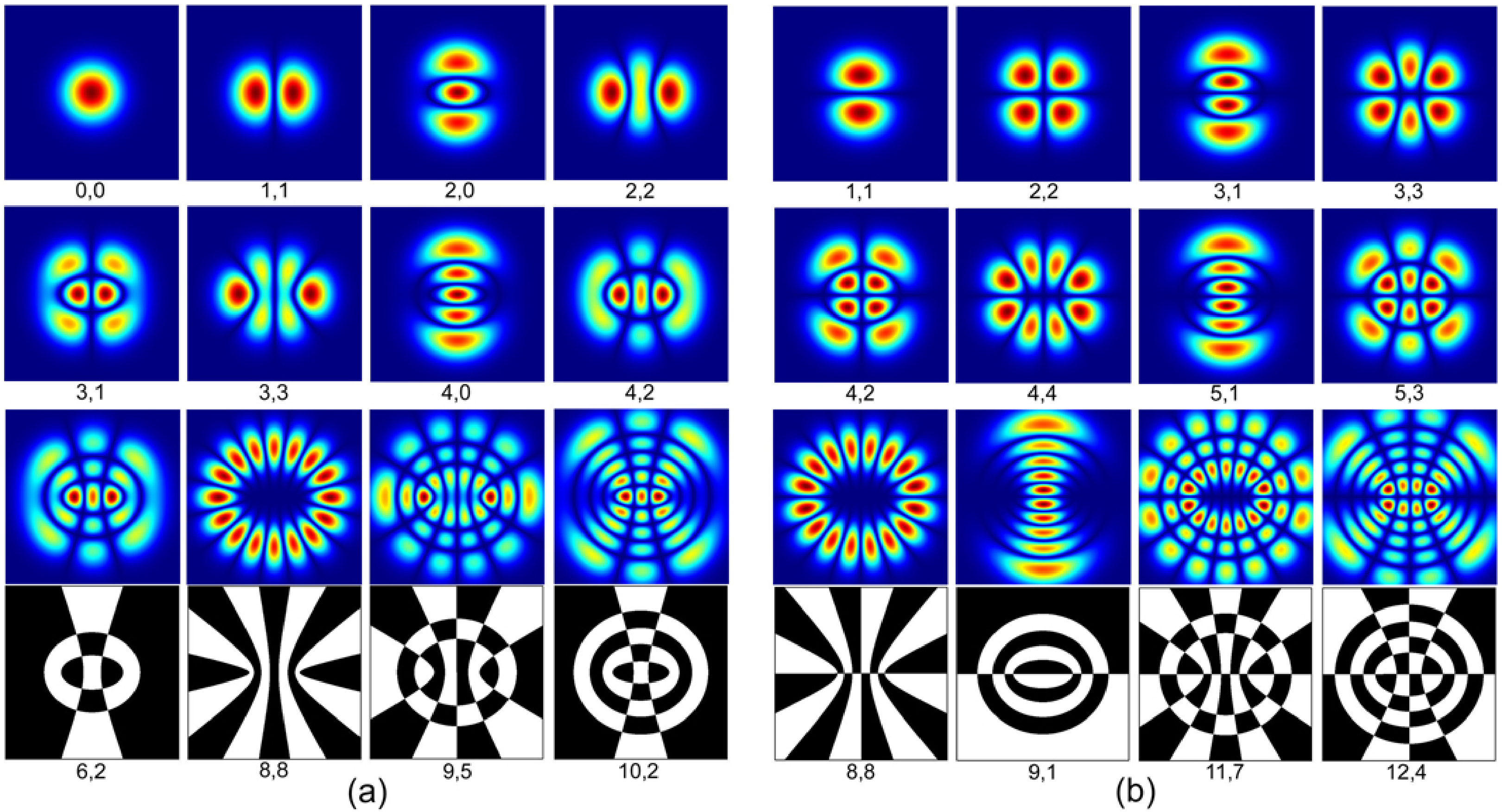
Transverse structure of the IG beams (z = 0)

Even IGB

$$\exp\left(-\frac{r^2}{w_0^2}\right) \mathrm{C}_p^m\left(i\xi,\varepsilon\right) \mathrm{C}_p^m\left(\eta,\varepsilon\right)$$

Odd IGB

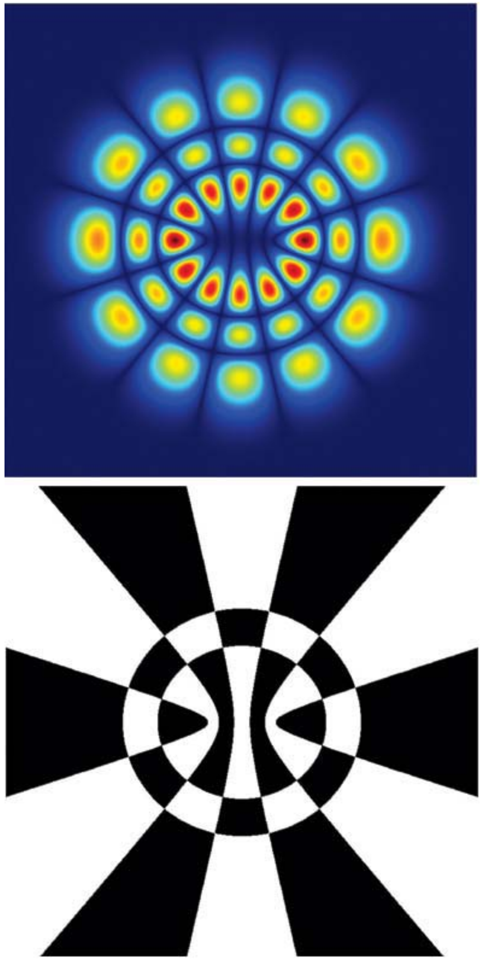
$$\exp\left(-\frac{r^2}{w_0^2}\right) \mathrm{S}_p^m\left(i\xi,\varepsilon\right) \mathrm{S}_p^m\left(\eta,\varepsilon\right)$$



Physical meaning of the indices p and m

Even IGB

$\mathrm{IG}^{e}_{p,m}(\mathbf{r},\ \epsilon)$



$m = 6$

Hyperbolic nodal lines

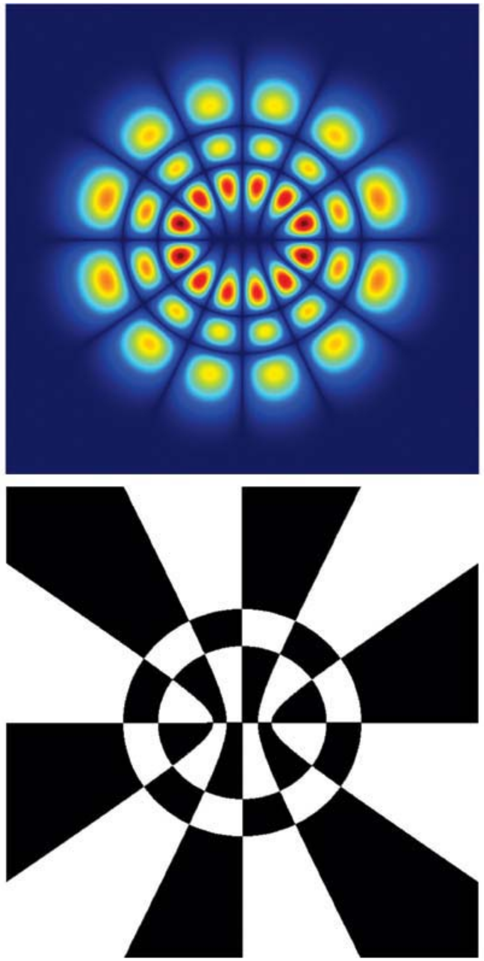
$p = 10$

Elliptic nodal lines

$$\frac{p-m}{2} = 2$$

Odd IGB

$\mathrm{IG}^{o}_{p,m}(\mathbf{r},\ \epsilon)$



Needed parameters

To fully describe the transverse distribution of High-order Gaussian beams at the waist plane we need to give

	Hermite	**Laguerre**	**Ince**
the wavenumber:	k	k	k
the mode (2 indices):	$n_x,\ n_y$	$n,\ l$	$p,\ m$
the parity:		*even* or *odd* *or helical* $(\pm)$	*even* or *odd* *or helical* $(\pm)$
the physical size:	w_0	w_0	$\left.\begin{matrix} w_0 \\ f_0 \\ \varepsilon \end{matrix}\right\}$ *two of these three parameters* $\varepsilon = \frac{2 f_0^2}{w_0^2}$

Mathematical and physical properties of IG beams

IG beams satisfy three important mathematical and physical properties

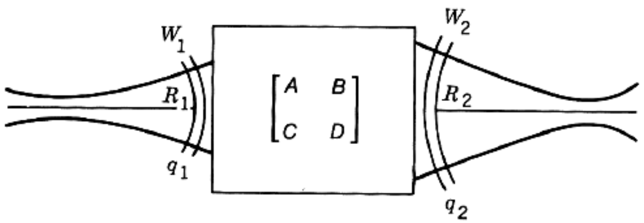
1) Complete family of exact and orthogonal solutions of the PWE
2) They are eigenmodes of stable resonators
3) Their transverse shapes do not change under propapation (structurally stable)

Orthogonal complete family:
$$\iint_{-\infty}^{\infty} \mathrm{IG}_{p,m}^{\sigma}(\mathbf{r})\, \overline{\mathrm{IG}}_{p',m'}^{\sigma'}(\mathbf{r})\ \mathrm{d}S = \delta_{pp'}\delta_{mm'}\delta_{\sigma\sigma'}$$

Gouy shift:
$$\Theta(z) = (p+1)\arctan(z/z_R)$$

As with Hermite and Laguerre Gaussian beams, it is easy to propagate an Ince-Gaussian beam through an paraxial ABCD system:

Bilineal propagator
$$q_{out} = \frac{Aq_{in}+B}{Cq_{in}+D}$$



Fourier transform of the IG beams

2D-FT

$$\widetilde{U}(k_x, k_y) = \iint U(x, y) \exp(-ik_x x - ik_y y)\; \mathrm{d}x\mathrm{d}y.$$

IG beams at *z* = 0

$$\mathrm{gIG}^{e}_{p,m}(\xi, \eta; \varepsilon) = \exp\left(-\frac{r^2}{w_0^2}\right) \mathrm{C}^m_p(i\xi, \varepsilon)\, \mathrm{C}^m_p(\eta, \varepsilon)$$

where $x = f_0 \cosh\xi \cos\eta, \quad y = f_0 \sinh\xi \sin\eta$, and $\varepsilon = 2(f_0/w_0)^2$.

The FT of the IG beams is shape invariant

$$\widetilde{\mathrm{IG}}^{e}_{p,m}(\xi, \eta; \varepsilon) = (-i)^p \pi w_0^2 \;\exp\left(-\frac{k_t^2 w_0^2}{4}\right)\; \mathrm{C}^m_p\left(i\widetilde{\xi}, \varepsilon\right) \mathrm{C}^m_p(\widetilde{\eta}, \varepsilon)$$

where $k_x = k_f \cosh\widetilde{\xi} \cos\widetilde{\eta}, \quad k_y = k_f \sinh\widetilde{\xi} \sin\widetilde{\eta}$, and $k_f = \dfrac{2f_0}{w_0^2} = \dfrac{\sqrt{2\varepsilon}}{w_0}$.

$$k_t = \sqrt{k_x^2 + k_y^2}$$

IG beams are self-fractional Fourier functions

Defining the Fourier transform as [1,2]

$$\left(\mathcal{F}^{1} f\right)\left(x^{\prime}, y^{\prime}\right)=\frac{1}{s^{2}} \iint_{-\infty}^{\infty} f(x, y) \exp \left[-i \frac{2 \pi\left(x x^{\prime}+y y^{\prime}\right)}{s^{2}}\right] \mathrm{d} S,$$

where x, y, x', y', and s all have dimensions of length and $s = w\pi^{1/2}$.

it is easy to see that the eigenvalue equation for the Fourier-transform operator in elliptical coordinates is given by

$$\mathcal{F}^{1}[\mathrm{IG}_{p,m}^{\sigma}(\xi, \eta; \epsilon)] = (-i)^{p}\,\mathrm{IG}_{p,m}^{\sigma}(\xi', \eta'; \epsilon),$$

In a similar way the FrFT operator F^{α} satisfies the eigenvalue equation

$$\mathcal{F}^{\alpha}[\mathrm{IG}_{p,m}^{\sigma}(\xi, \eta; \epsilon)] = (-i)^{p\alpha}\,\mathrm{IG}_{p,m}^{\sigma}(\xi', \eta'; \epsilon).$$

M. A. Bandrés and J. C. Gutiérrez-Vega, "Ince–Gaussian series representation of the two–dimensional fractional Fourier transform," Opt. Lett., **30**, 540-542, 2005.

[1] D. Mendlovic and H. M. Ozaktas, JOSA A, 10, 1875 (1993).
[2] H. M. Ozaktas and D. Mendlovic, JOSA A, 10, 2522 (1993).

Normalizing the IG beams … not so easy!

IG beams at z = 0

$$\mathrm{IG}^{e,o}_{p,m}(\mathbf{r};\varepsilon) = \left\{ \begin{array}{c} \mathfrak{C}_{p,m} \\ \\ \mathfrak{S}_{p,m} \end{array} \right\} \exp\left(-\frac{r^2}{w_0^2}\right) \left\{ \begin{array}{c} \mathrm{C}^m_p(i\xi,\varepsilon)\,\mathrm{C}^m_p(\eta,\varepsilon) \\ \\ \mathrm{S}^m_p(i\xi,\varepsilon)\,\mathrm{S}^m_p(\eta,\varepsilon) \end{array} \right\}$$

NORMALIZED IG beams at z = 0

$$\mathrm{IG}^e_{2n,m} = \left[\frac{2}{w_0\sqrt{\pi}} \frac{(p/2)!\; A_0^+}{\mathrm{C}^m_p(0,\varepsilon)\,\mathrm{C}^m_p(\pi/2,\varepsilon)}\right] \mathrm{C}^m_{2n}(i\xi,\varepsilon)\,\mathrm{C}^m_{2n}(\eta,\varepsilon) \exp\left(-\frac{r^2}{w_0^2}\right),$$

$$\mathrm{IG}^e_{2n+1,m} = \left[\frac{2}{w_0}\sqrt{\frac{\varepsilon}{\pi}} \frac{[(p+1)/2]!\; A_0^-}{\mathrm{C}^m_p(0,\varepsilon)\,\mathrm{C}'^m_p(\pi/2,\varepsilon)}\right] \mathrm{C}^m_{2n+1}(i\xi,\varepsilon)\,\mathrm{C}^m_{2n+1}(\eta,\varepsilon) \exp\left(-\frac{r^2}{w_0^2}\right)$$

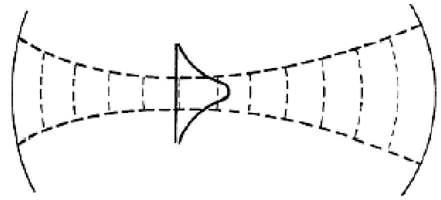
$$\mathrm{IG}^o_{2n,m} = \left[\frac{2}{w_0} \frac{\varepsilon}{\sqrt{\pi}} \frac{[(p+2)/2]!\; B_1^+}{\mathrm{S}'^m_p(0,\varepsilon)\,\mathrm{S}'^m_p(\pi/2,\varepsilon)}\right] \mathrm{S}^m_{2n}(i\xi,\varepsilon)\,\mathrm{S}^m_{2n}(\eta,\varepsilon) \exp\left(-\frac{r^2}{w_0^2}\right),$$

$$\mathrm{IG}^o_{2n+1,m} = \left[\frac{2}{w_0}\sqrt{\frac{\varepsilon}{\pi}} \frac{[(p+1)/2]!\; B_0^-}{\mathrm{S}^m_p(\pi/2,\varepsilon)\,\mathrm{S}'^m_p(0,\varepsilon)}\right] \mathrm{S}^m_{2n+1}(i\xi,\varepsilon)\,\mathrm{S}^m_{2n+1}(\eta,\varepsilon) \exp\left(-\frac{r^2}{w_0^2}\right),$$

IG beams are resonating modes of stable resonators

Eigenfunction of the self-consistency equation in stable resonators

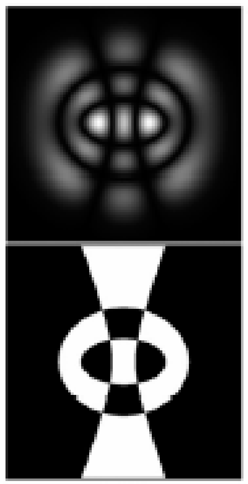
$$\gamma\ \mathrm{IG}\left(\xi_2,\eta_2\right) = \iint K\left(\xi_2,\eta_2,\xi_1,\eta_1\right)\mathrm{IG}\left(\xi_1,\eta_1\right)\ \mathrm{dS}_1$$



z

x

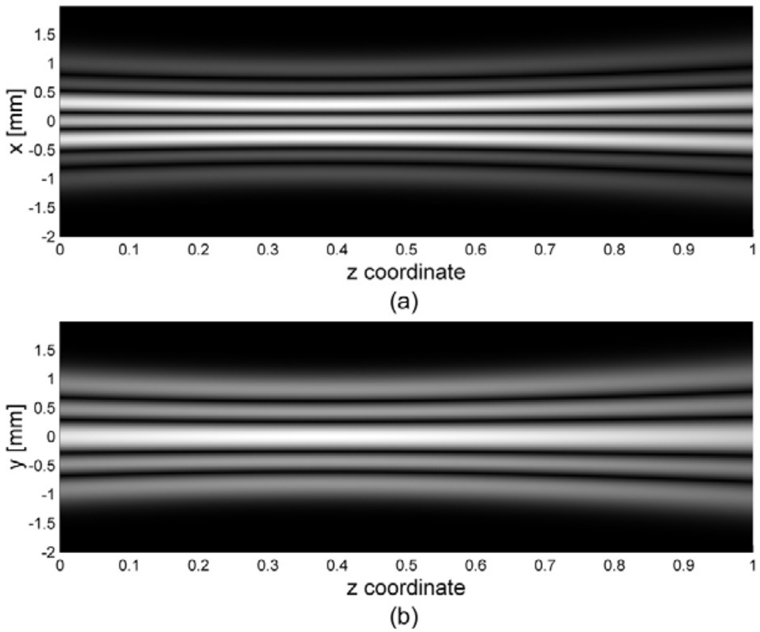
y



6,2

x

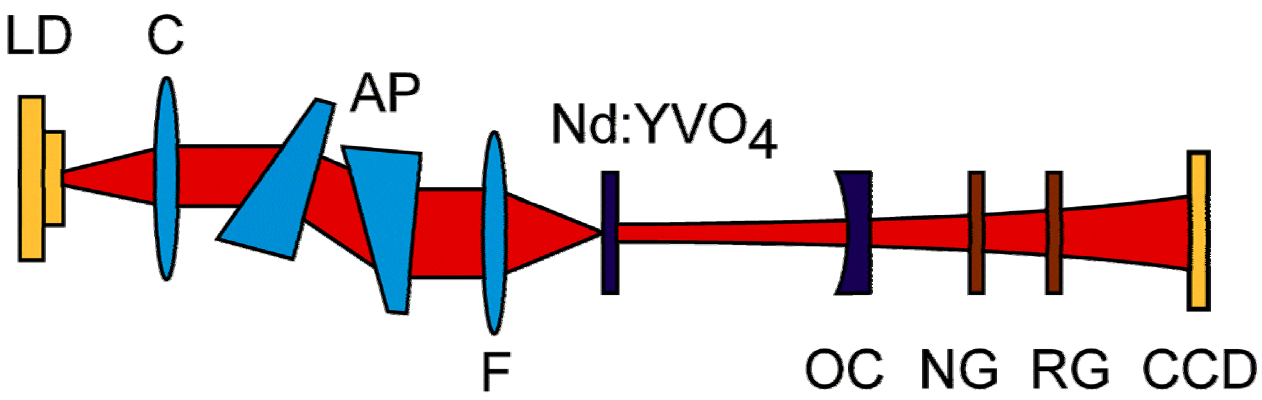
y



Passive intracavity field distribution inside the resonator

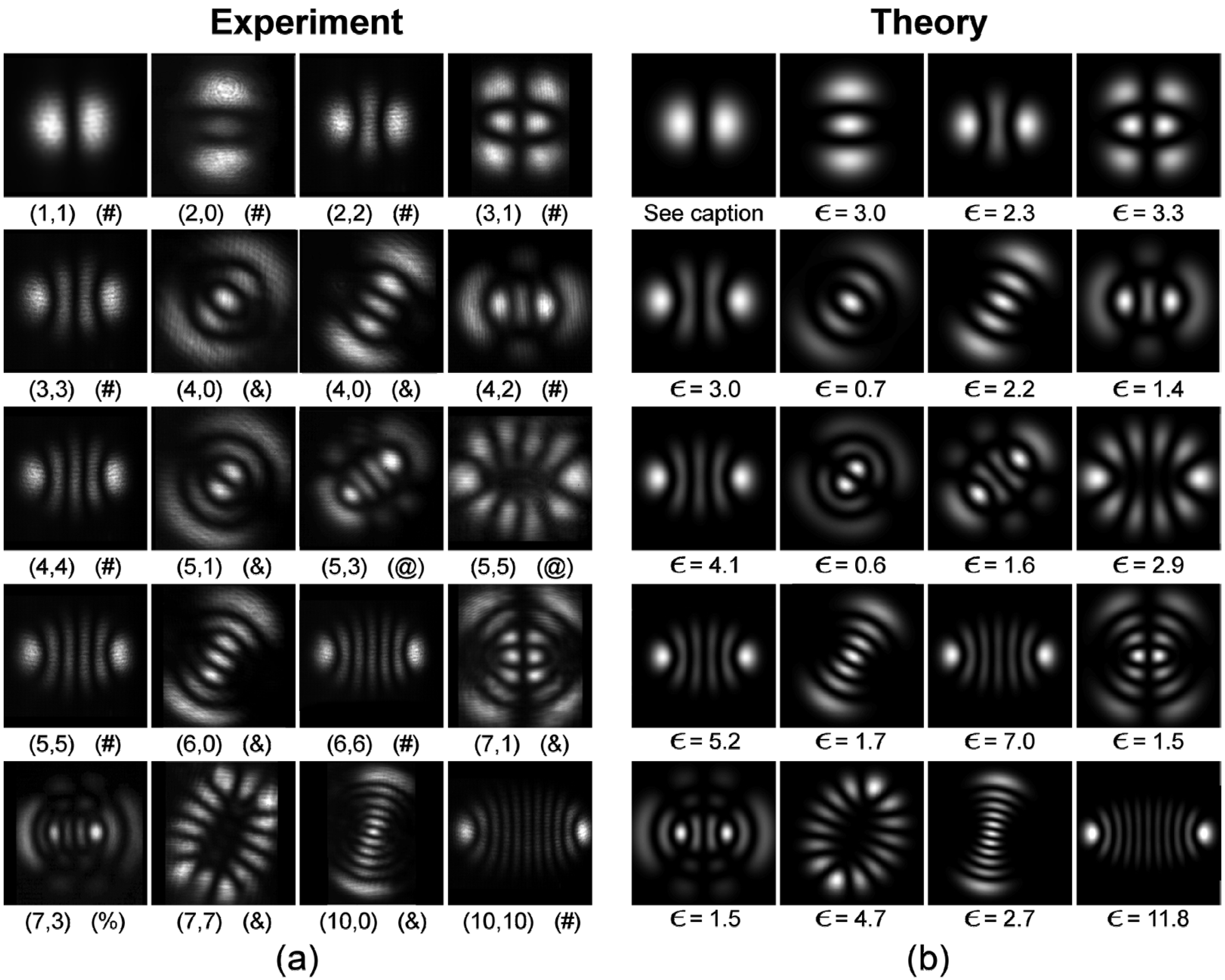
Observation of Ince-Gaussian beams in a stable resonator

- Diode pumped solid state laser.
- Active medium: Nd:YVO4.
- Pump: 100-300 mW at 808 nm.
- Output: 20 mW at 1064 nm.



LD: λ = 808 nm pump laser diode; C: collimating lens; AP: anamorphic prism pair; F: focusing lens; Nd:YVO4 laser crystal; OC: 97% output coupler; NG: neutral glas filter; RG: color glass filter; CCD camera.

U. T. Schwarz, M. A. Bandrés and J. C. Gutiérrez-Vega, "Observation of Ince-Gaussian modes in stable resonators," Opt. Lett., **29**, 1870-1872, (2004)



U. T. Schwarz, M. A. Bandrés and J. C. Gutiérrez-Vega, "Observation of Ince-Gaussian modes in stable resonators," Opt. Lett., **29**, 1870-1872, (2004)

Connecting the three BIG families…

$$\mathrm{HG}_{n_x,n_y}(\mathbf{r}) = C_{n_x,n_y} \; (\mu^*/\mu)^{(n_x+n_y)/2} \; \mathrm{GB}(\mathbf{r}) \; \mathrm{H}_{n_x}\left(\sqrt{2}\frac{x}{w(z)}\right)\mathrm{H}_{n_y}\left(\sqrt{2}\frac{y}{w(z)}\right)$$

$$\mathrm{LG}^e_{n,l}(\mathbf{r}) = C_{n,l} \; (\mu^*/\mu)^{(2n+l)/2} \; \mathrm{GB}(\mathbf{r}) \; \left[\sqrt{2}\frac{r}{w(z)}\right]^l \mathrm{L}^l_n\left(2\frac{r^2}{w^2(z)}\right)\cos l\theta$$

$$\mathrm{IG}^e_{p,m}(\mathbf{r};\varepsilon) = \mathfrak{C}_{p,m} \; (\mu^*/\mu)^{p/2} \; \mathrm{GB}(\mathbf{r}) \; \mathrm{C}^m_p(i\xi,\varepsilon)\,\mathrm{C}^m_p(\eta,\varepsilon)$$

$$\mu = \mu(z) = 1 + iz/z_R$$

TO BUILD UP AN INCE MODE WE MUST USE ONLY HERMITE OR LAGUERRE MODES WITH THE SAME GOUY SHIFT

$$n_x + n_y = 2n + l = p$$

AN INCE MODE ALWAYS TENDS TO A HERMITE OR LAGUERRE MODE WITH THE SAME GOUY SHIFT

Relation with Hermite and Laguerre Gaussian beams

Expansion in terms of Laguerre-Gaussian modes and vice versa:

$$\mathrm{LG}^{\sigma}_{n,l}\left(r,\varphi\right)=\sum_{m} D_m \mathrm{IG}^{\sigma}_{p=2n+l,m}\left(\xi,\eta,\varepsilon\right),$$

$$\mathrm{IG}^{\sigma}_{p,m}\left(\xi,\eta,\varepsilon\right)=\sum_{l,n} D_{l,n} \mathrm{LG}^{\sigma}_{n,l}\left(r,\varphi\right),$$

Overlap integral

$$\iint_{-\infty}^{\infty} \mathrm{LG}^{\sigma}_{n,l}\overline{\mathrm{IG}}^{\sigma'}_{p,m}\mathrm{d}S=\delta_{\sigma'\sigma}\delta_{p,2n+l}\left(-1\right)^{n+l+(p+m)/2}$$
$$\times\sqrt{\left(1+\delta_{0,l}\right)\Gamma\left(n+l+1\right)n!}A^{\sigma}_{(l+\delta_{o,\sigma})/2}\left(a^{m}_{p}\right),$$

where $A^{\sigma}_{(l+\delta_{o,\sigma})/2}\left(a^{m}_{p}\right)$ is the $\left(l+\delta_{o,\sigma}\right)/2$-th Fourier coefficient of the C^{m}_{p} or S^{m}_{p}

Subsets with tha same Gouy shift: Example p = 5

Linear relations:

$$\mathbf{I}_p^{\sigma} = \left[{}_{LI}\mathbf{T}_p^{\sigma}\right]\mathbf{L}_p^{\sigma},$$
$$\mathbf{H}_p^{\sigma} = \left[{}_{IH}\mathbf{T}_p^{\sigma}\right]\mathbf{I}_p^{\sigma},$$
$$\mathbf{L}_p^{\sigma} = \left[{}_{HL}\mathbf{T}_p^{\sigma}\right]\mathbf{H}_p^{\sigma},$$

$$\begin{bmatrix} IG_{5,1}^{o} \\ IG_{5,3}^{o} \\ IG_{5,5}^{o} \end{bmatrix} = \begin{bmatrix} 0.792 & 0.558 & 0.248 \\ -0.500 & 0.358 & 0.789 \\ 0.351 & -0.748 & 0.562 \end{bmatrix} \begin{bmatrix} LG_{2,1}^{o} \\ LG_{1,3}^{o} \\ LG_{0,5}^{o} \end{bmatrix}$$

$$\begin{bmatrix} IG_{5,1}^{o} \\ IG_{5,3}^{o} \\ IG_{5,5}^{o} \end{bmatrix} = \begin{bmatrix} 0.101 & 0.310 & 0.945 \\ -0.649 & -0.700 & 0.298 \\ 0.755 & -0.643 & 0.139 \end{bmatrix} \begin{bmatrix} HG_{4,1}^{o} \\ HG_{2,3}^{o} \\ HG_{0,5}^{o} \end{bmatrix}$$

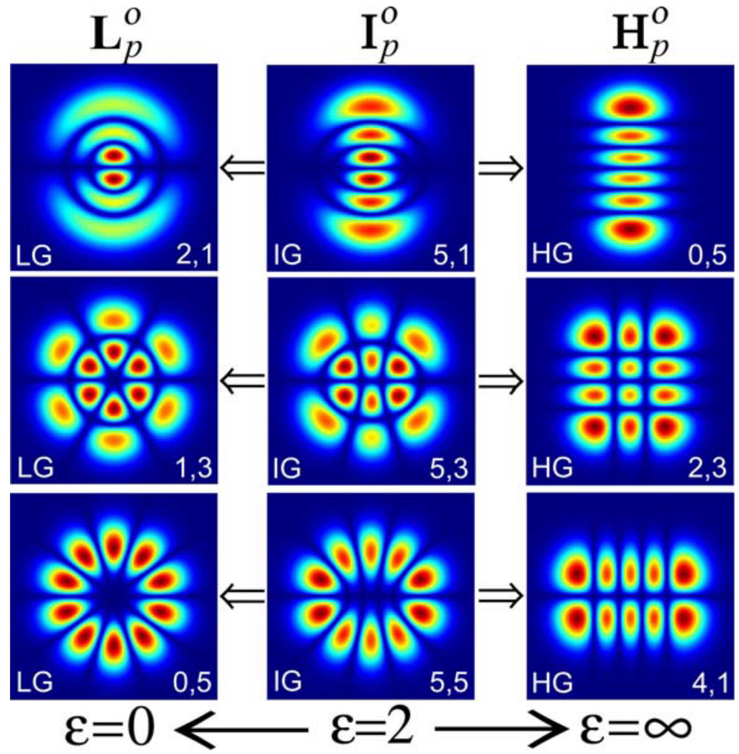
Each subset is composed by

$$N_p = \begin{cases} (p + 2\delta_{\sigma,e})/2, & \text{if } p \text{ is even} \\ (p+1)/2, & \text{if } p \text{ is odd} \end{cases}$$

degenerated modes whose

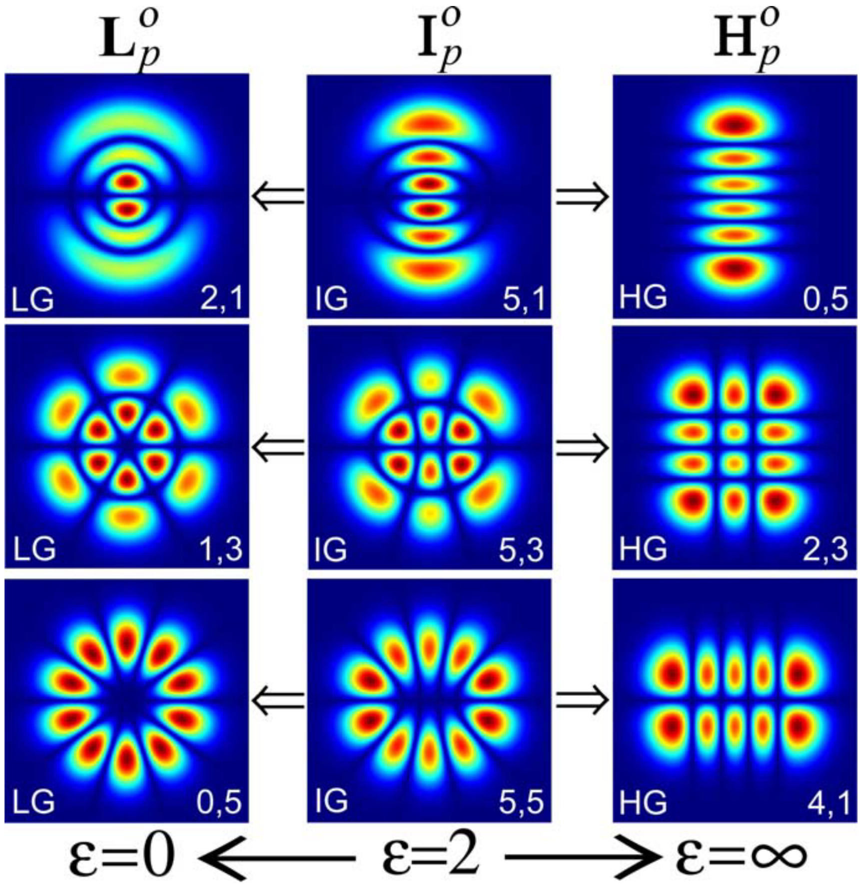
Gouy shift is $\psi_p(z) = (p+1)\,\psi_{GS}(z)$

$$n_x + n_y = 2n + l = p$$

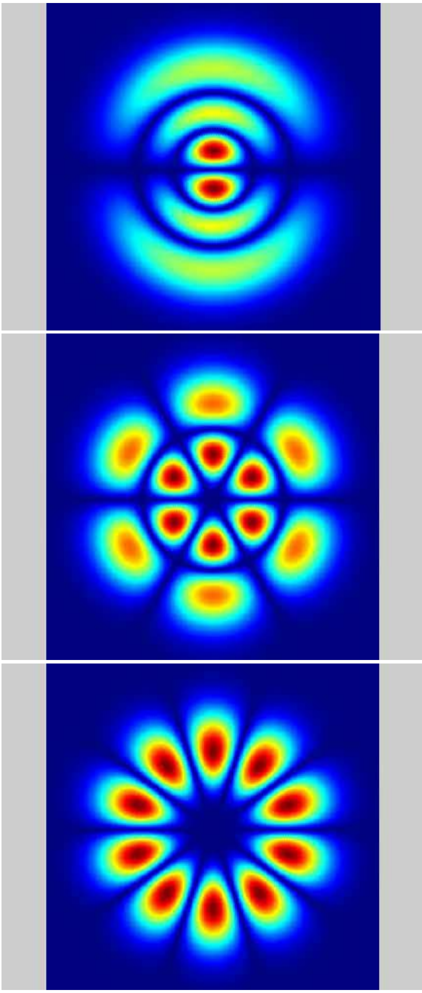


Limiting cases and superposition of modes

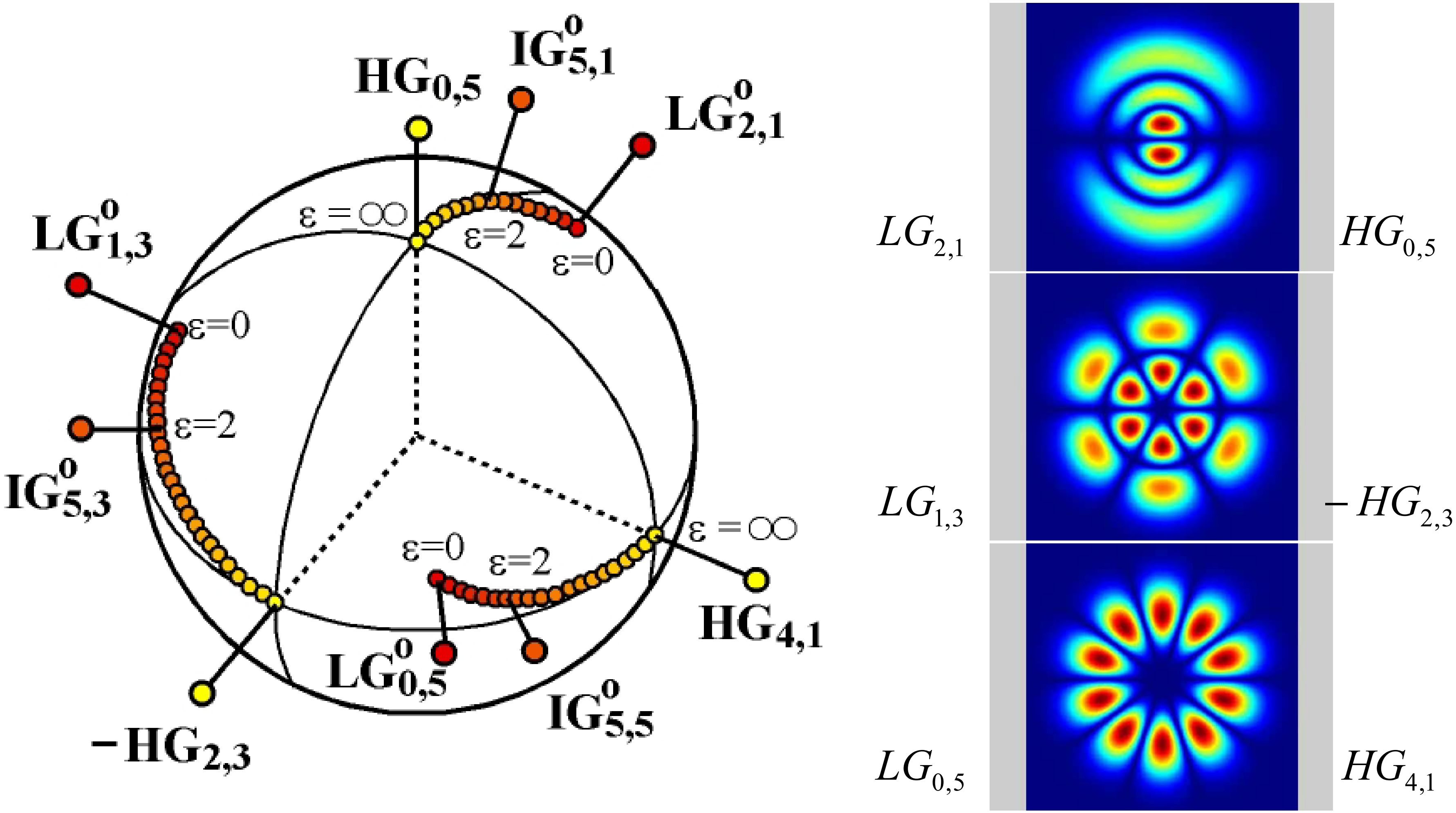
$$2n+l \quad = \quad p \quad = \quad n_x+n_y$$



Varying the ellipticity



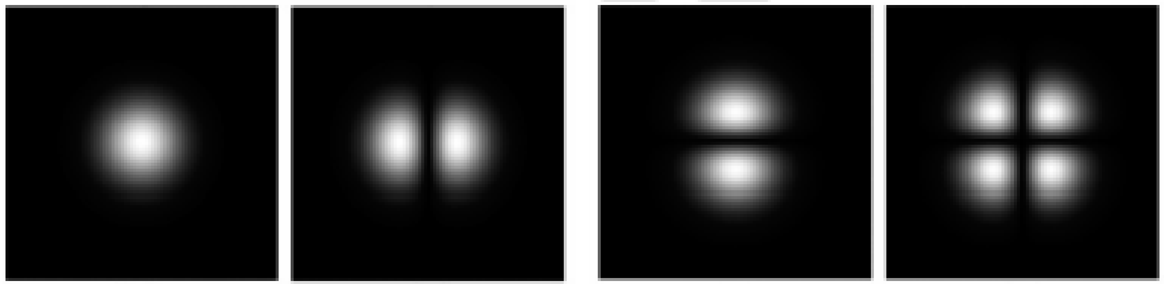
Subset p = 5: 3D representation in the vector space



Four ‘fundamental’ modes

Table 1. The Four “Fundamental” Modes

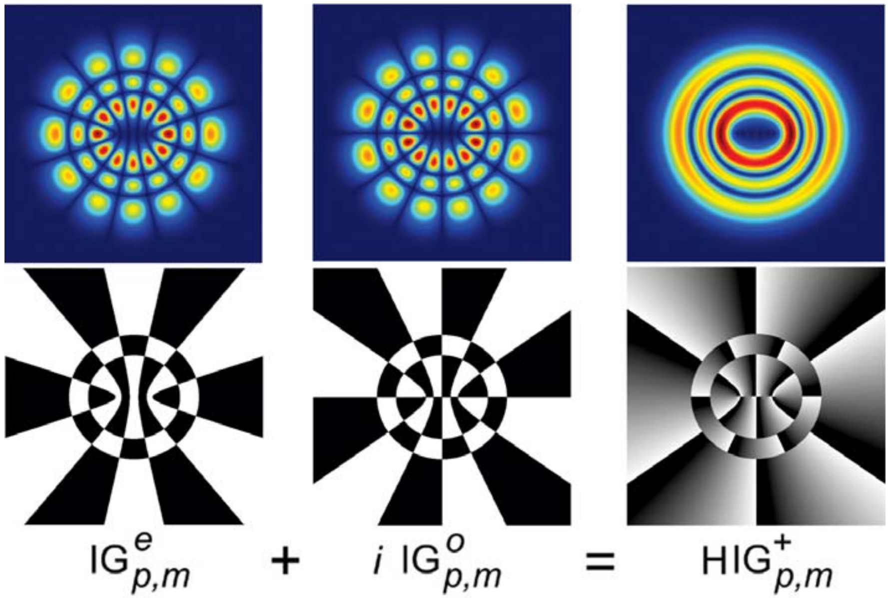
Shape	Parity	HG_{n_x,n_y}	$LG^{\sigma}_{n,l}$	$IG^{\sigma}_{p,m}$
○	$\sigma = e$	0,0	0,0	0,0
∞	$\sigma = e$	1,0	0,1	1,1
8	$\sigma = o$	0,1	0,1	1,1
⊕	$\sigma = 0$	1,1	0,2	2,2

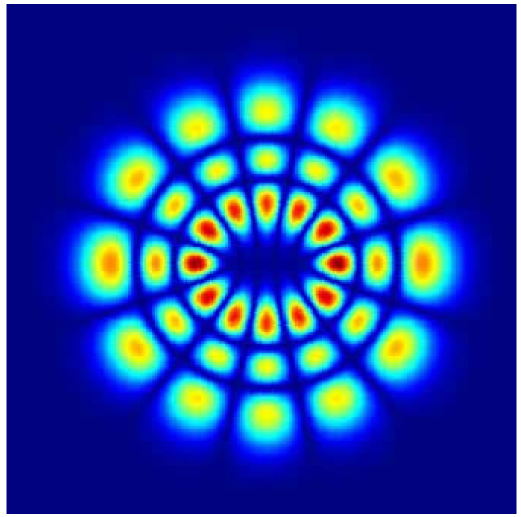


0,0 1,1 1,1 2,2

Helical (Rotating) Ince-Gaussian beams

$$HIG_{10,6} = IG^{e}_{10,6}(\xi,\eta,z;\varepsilon) + i\,IG^{o}_{10,6}(\xi,\eta,z;\varepsilon)$$

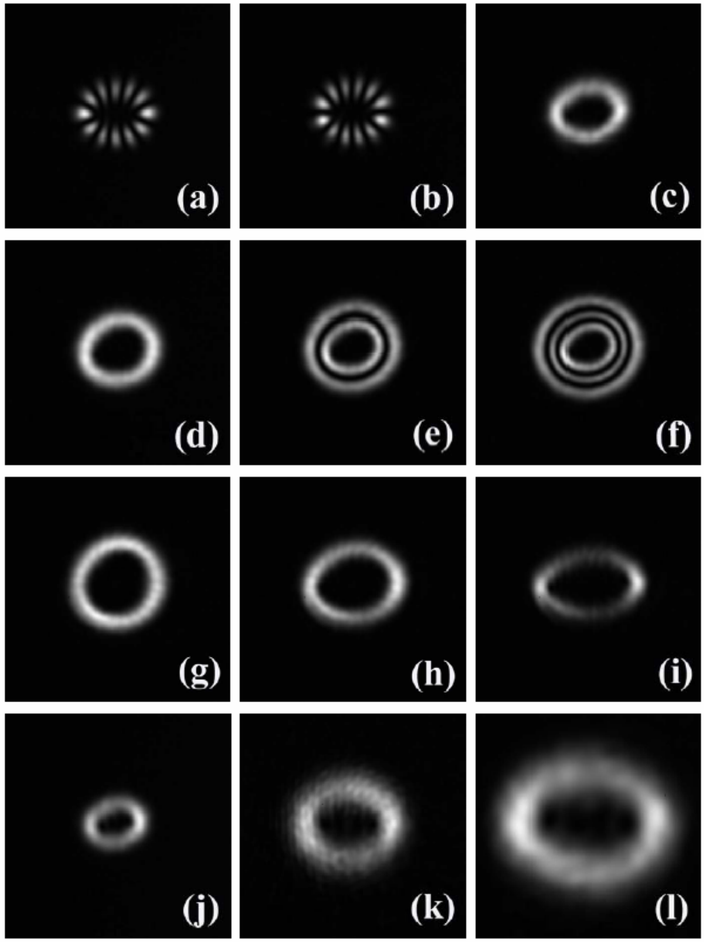




Real part of a HIG beam under propagation

1. Elliptical ring intensity patterns
2. Breakup of a single *m* vortex into a straight row of *m* unit vortices
3. Resulting orbital angular momentum
4. Analogies with Mathieu beams.

Helical Ince-Gaussian beams with a LCD



Even, odd and helical IGB 6,6,3

HIGB (8,8,2), (10,8,2), (12,8,2)

HIGB (12,12,0), (12,12,3), (12,12,6)

HIGB (4,4,2), at $z = 0$, $z = 0.8$, $z = 1$ m.

J. Bentley, J. Davis, M. A. Bandrés and J. C. Gutiérrez Vega, to be published in Opt. Lett. 2006

Ince-Gaussian beams in quadratic index media (QIM)

Index of refraction: $n(r) = n_0(1 - a^2 r^2/2).$

PWE in QIM: $$\left[\nabla_t^2 + 2ik\frac{\partial}{\partial z} - k^2 a^2 r^2\right]\Psi(\mathbf{r}) = 0,$$

Solution in elliptic coordinates yields IG beams of the form:

$$\mathrm{IG}^e_{p,m}(\mathbf{r},t) = \frac{\mathcal{C}w_0}{w(z)} C_p^m(i\xi,\epsilon)\, C_p^m(\eta,\epsilon)\exp\left[\frac{ikr^2}{2q(z)}\right] \times \exp\left[ikz - i(p+1)\arctan\left(\frac{\tan(az)}{az_R}\right) - i\omega t\right]$$

where $q(z) = \dfrac{1}{a}\dfrac{\sin(az) - iaz_R\cos(az)}{\cos(az) + iaz_R\sin(az)}$ and using $1/q = 1/R + i2/kw^2$

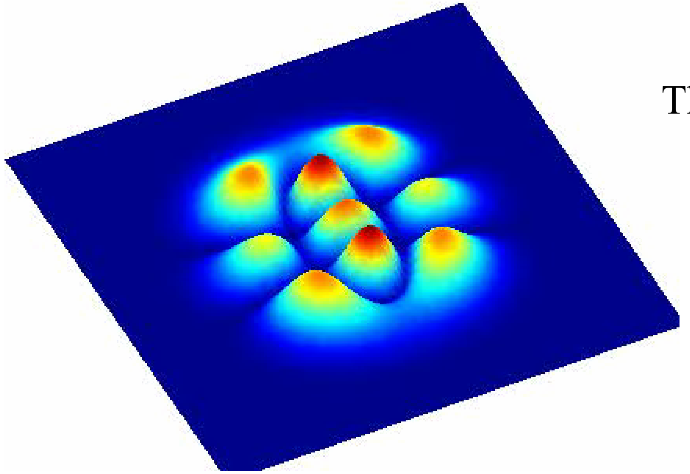
$$R(z) = \frac{\tan(az) + a^2 z_R^2\cot(az)}{a(1 - a^2 z_R^2)}$$

$$w(z) = w_0\left[1 + \beta\sin^2(az)\right]^{1/2}$$

$$\beta = (1 - a^2 z_R^2)/a^2 z_R^2.$$

J. C. Gutiérrez-Vega and M. A. Bandrés, "Ince-Gaussian beams in quadratic index medium," J. Opt. Soc. Am. A, **22**, 306-309, 2005

Ince-Gaussian beams in quadratic index media (QIM)



The width size is a periodic function of the propagation distance

$$w(z) = w_0 \left[1 + \beta \sin^2(az)\right]^{1/2}$$
$$\beta = (1 - a^2 z_R^2)/a^2 z_R^2.$$

IG eigenmodes with constant width can be obtained by satisfying the input condition

$$w_0 = (2/ak)^{1/2}.$$

These IG eigenmodes constitute a complete set of solutions of the two-dimensional Helmholtz equation in a quadratic-index medium and can be used to find the IG series representation of the two-dimensional fractional Fourier transform.

Analogy with the time dependent quantum 2D harmonic oscillator

J. C. Gutiérrez-Vega and M. A. Bandrés, "Ince-Gaussian beams in quadratic index medium," J. Opt. Soc. Am. A, **22**, 306-309, 2005

Gouy shift of the Ince-Gaussian beams in QIM

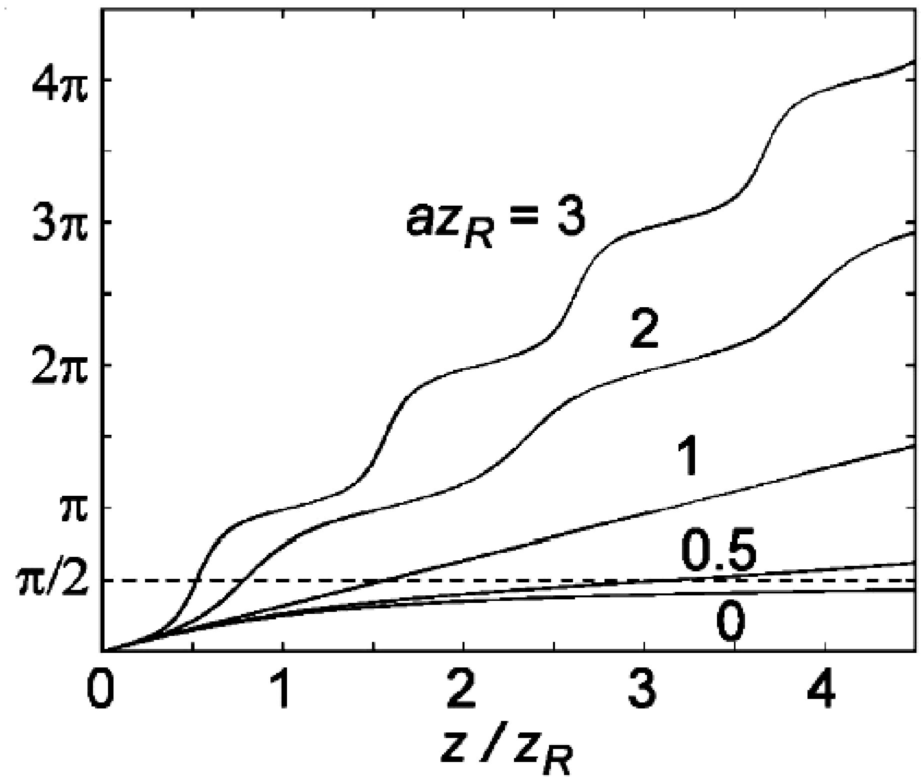


Fig. 2. Longitudinal phase-shift retardation $\phi(z) = \arctan[\tan(az)/az_R]$ as a function of the normalized propagation distance z/z_R for several values of az_R.

J. C. Gutiérrez-Vega and M. A. Bandrés, "Ince-Gaussian beams in quadratic index medium," J. Opt. Soc. Am. A, **22**, 306-309, 2005

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6. J. C. Gutiérrez-Vega and Miguel A. Bandrés, "Ince-Gaussian beams in quadratic index medium," J. Opt. Soc. Am. A, **22**, 306-309, 2005

7. Miguel A. Bandrés and J. C. Gutiérrez-Vega, "Ince–Gaussian series representation of the two–dimensional fractional Fourier transform," Opt. Lett., **30**, 540-542, 2005.

8. Joel B. Bentley, Jeffrey A. Davis, Miguel A. Bandres and J. C. Gutiérrez-Vega, "Generation of helical Ince-Gaussian beams with a liquid crystal display," to be published in Opt. Lett. 2006.

Thanks to my Ince-collegues

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