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WINTER COLLEGE
on
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of
INFORMATION OPTICS

30 January - 10 February 2006

OPTICAL COHERENCE:
The Classical Insight

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International Centre for Theoretical Physics

United Nations
Educational, Scientific
and Cultural Organization

International Atomic
Energy Agency

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I. Foundations and Background
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References:

- M. Born and E. Wolf, *Principles of Optics* (Canbridge University Press, UK, 1999)
- L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, UK, 1996)

Reviews:

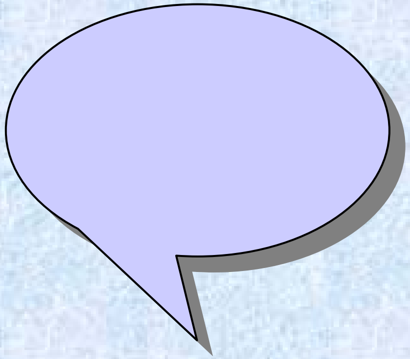
❑ A.T. Friberg, ”Partial polarization in arbitrary 3D electromagnetic fields”, in *Free and Guided Beams* (World Scientific, Singapore, 2004)

❑ A.T. Friberg, ”Electromagnetic theory of optical coherence”, in *Tribute to Emil Wolf: Science and Engineering Legacy of Physical Optics* (SPIE, Bellingham, WA, 2004)

Electromagnetic Optics

Near-Field, Micro/Nano-Optics

Nano-Photonics



Light

ALTERED FIELD CORRELATIONS:

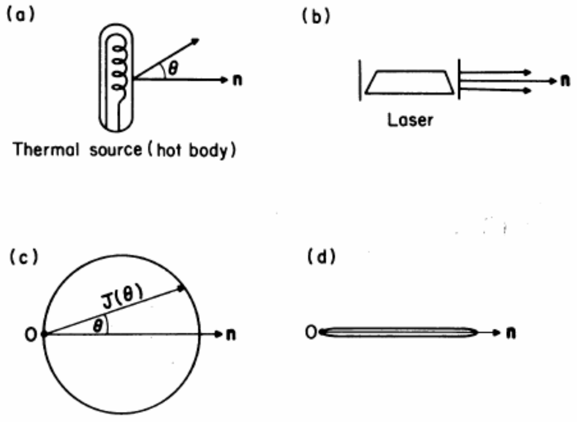
Changes in

- **Coherence properties**
- **Polarization state**
- **Spectrum**
- **Entropy**

RADIATION
PROPAGATION
CONFINEMENT

- light seeps out (evanescent waves, surface plasmons)

Foundations and Background



Polar diagrams of radiant intensity $J(\theta)$:

(c) From thermal source (d) From laser

FIG. 2. Comparison of the angular distribution of radiant intensity from a thermal source and from a laser.

Incoherent light

- random, chaotic
  (~ croud in open-air market)

Coherent light

- orderly, deterministic
  (~ marching band)

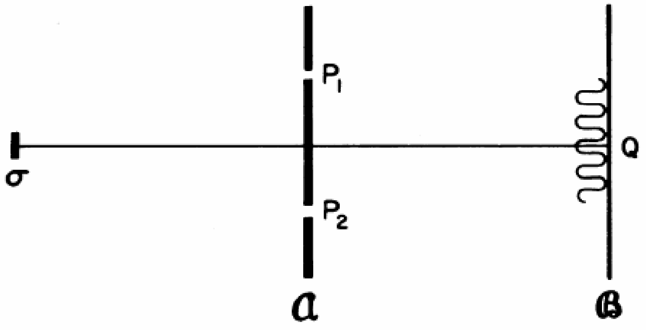


FIG. 5. Spatial coherence illustrated by means of Young's interference experiment.

Spatial coherence
(Young interferometer)

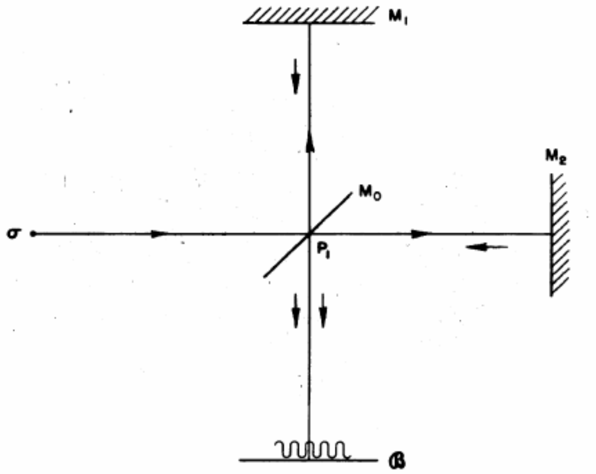


FIG. 4. Temporal coherence illustrated by means of a Michelson interferometer.

Temporal coherence
(Mickelson interferometer)
=> spectrum of light

STAR
(thermal source,
incoherent,
spontaneous
emission)

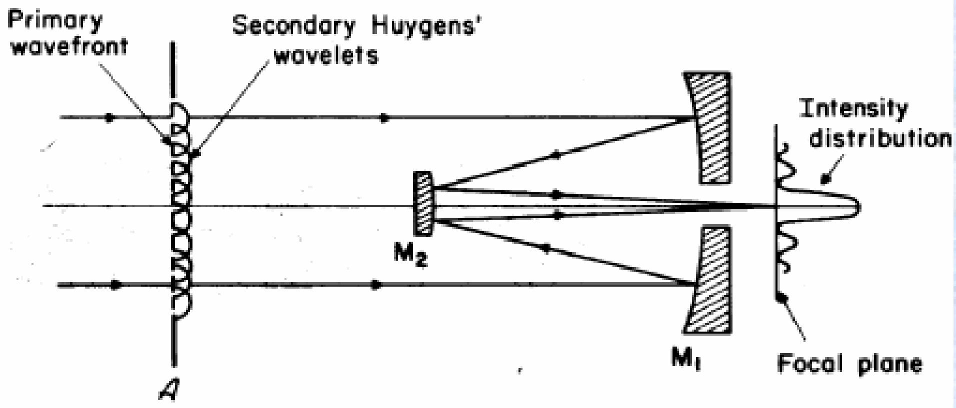


FIG. 7. Formation of the diffraction image in the focal plane of a telescope.

- Light accumulates coherence on propagation (van Cittert-Zernike theorem)

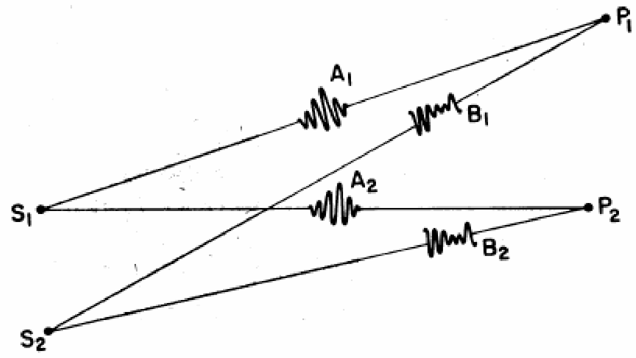


FIG. 8. Illustrating the generation of spatial coherence from two uncorrelated point sources.

$$\langle A_i B_j \rangle = 0 \qquad (i,j = 1,2).$$

$$A_2 \approx A_1,$$

$$B_2 \approx B_1.$$

$$\text{at } P_1: \qquad V_1 = A_1 + B_1,$$

$$\text{at } P_2: \qquad V_2 = A_2 + B_2.$$

$$V_1 \approx V_2.$$

[E. Wolf, ”Coherence and radiometry”, JOSA **68**, 7-17 (1978) – Ives Medal Address]

Fundamentals

Complex analytic signal (Gabor, 1946)

$$E(\mathbf{r},t) = \int_0^\infty \widetilde{E}(\mathbf{r},\omega)e^{-i\omega t}\mathrm{d}\omega$$

$$E(\mathbf{r},t) = \frac{1}{2}[E^{(r)}(\mathbf{r},t) + iE^{(i)}(\mathbf{r},t)]$$

Positive-frequency part
Real & Imaginary parts
⇔ Hilbert transform pair

Time average $$\left\langle E(\mathbf{r},t)\right\rangle_{\mathrm{time}} \equiv \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}E(\mathbf{r},t')\mathrm{d}t'$$

Note: $\left\langle [E^{(r)}(\mathbf{r},t)]^2\right\rangle_{\mathrm{time}} = \left\langle |E(\mathbf{r},t)|^2\right\rangle_{\mathrm{time}}$

Stationarity $\left\langle E(\mathbf{r},t)\right\rangle_{\mathrm{time}}$ independent of time t

$\left\langle E^*(\mathbf{r}_1,t_1)E(\mathbf{r}_2,t_2)\right\rangle_{\mathrm{time}}$ depends on $\tau = t_2 - t_1$

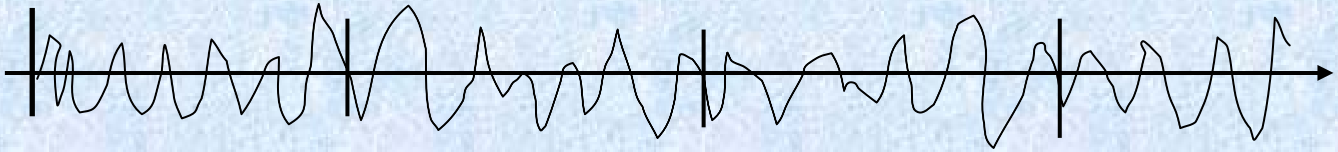
Ensemble average
(statistical)

$$\left\langle E(\mathbf{r},t)\right\rangle_{\mathrm{ensemble}} \equiv \int E\, p(E;\mathbf{r})\mathrm{d}E$$

$$\left\langle E^*(\mathbf{r}_1,t_1)E(\mathbf{r}_2,t_2)\right\rangle_{\mathrm{ensemble}} = \int\int E_1^* E_2\, p_2(E_1,E_2;\mathbf{r}_1,\mathbf{r}_2;\tau)\mathrm{d}E_1\mathrm{d}E_2$$

Ergodicity $\left\langle \ldots\right\rangle_{\mathrm{time}} = \left\langle \ldots\right\rangle_{\mathrm{ensemble}}$

p = probablity density
p_2 = joint probablility density



(realizations)

Partial coherence

Mutual coherence function (space-time domain)

$$\Gamma(\mathbf{r}_1,\mathbf{r}_2,\tau)=\left\langle V^*(\mathbf{r}_1,t)V(\mathbf{r}_2,t+\tau)\right\rangle$$

- Scalar (complex-valued)
- Obeys 2 wave equations
- Intensity (optical) $I(\mathbf{r}) = <|V(\mathbf{r},t)|^2>$
- Complex degree of coherence

$$\nabla_1^2\Gamma-\frac{1}{c^2}\frac{\partial^2\Gamma}{\partial\tau^2}=0$$

$$\nabla_2^2\Gamma-\frac{1}{c^2}\frac{\partial^2\Gamma}{\partial\tau^2}=0$$

$$\boxed{\gamma(\mathbf{r}_1,\mathbf{r}_2,\tau)\equiv\frac{\Gamma(\mathbf{r}_1,\mathbf{r}_2,\tau)}{\sqrt{I(\mathbf{r}_1)I(\mathbf{r}_2)}}}$$

$$0\le\left|\gamma(\mathbf{r}_1,\mathbf{r}_2,\tau)\right|\le 1$$

incoherent partially coherent coherent

- ➢ Monochromatic $\Gamma(\mathbf{r}_1,\mathbf{r}_2,\tau)=V^*(\mathbf{r}_1)V(\mathbf{r}_2)e^{-i\omega\tau}$

(= fully coherent.
Also in reverse: a field that is fully coherent in a domain is monochromatic)

- ➢ Quasi-monochromatic

$$\Gamma(\mathbf{r}_1,\mathbf{r}_2,\tau)\approx\left\langle V^*(\mathbf{r}_1)V(\mathbf{r}_2)\right\rangle e^{-i\omega\tau}$$

(= partially coherent)

[E. Wolf, Proc. Royal Soc. (London) A **230**, 246 (1955);
F. Zernike, Physica **5**, 785 (1938)]

Cross-spectral density function

(space-frequency domain)

$$\left\langle \tilde{V}^*(\mathbf{r}_1,\omega)\tilde{V}(\mathbf{r}_2,\omega')\right\rangle = W(\mathbf{r}_1,\mathbf{r}_2,\omega)\;\delta(\omega-\omega')$$

- Wiener-Khintchine theorem:
  - ✓ different Fouerier components are uncorrelated
  - ✓ mutual coherenre and cross-spectral density functions are Fourier conjugates

$$W(\mathbf{r}_1,\mathbf{r}_2,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma(\mathbf{r}_1,\mathbf{r}_2,\tau)e^{i\omega\tau}\,\mathrm{d}\tau$$

$$\Gamma(\mathbf{r}_1,\mathbf{r}_2,\tau) = \int_{0}^{\infty}W(\mathbf{r}_1,\mathbf{r}_2,\omega)e^{-i\omega\tau}\,\mathrm{d}\omega$$

- Cross-spectral density W is ”complex analytic signal”

NOTE: *Stationary field at given frequency is spatially partially coherent !!*
(Monochromatic field is fully coherent.)

Also: *W by ensemble averages of monochromatic functions* $\left\{V(\mathbf{r},\omega)e^{-i\omega\tau}\right\}$

- Instensity (spectrum) $I(\mathbf{r},\omega) = W(\mathbf{r},\mathbf{r},\omega)$
- Spectral (spatial) degree of coherence ⟶

$$\boxed{\mu(\mathbf{r}_1,\mathbf{r}_2,\omega) \equiv \frac{W(\mathbf{r}_1,\mathbf{r}_2,\omega)}{\sqrt{I(\mathbf{r}_1,\omega)I(\mathbf{r}_2,\omega)}}}$$

$$0 \le \left|\mu(\mathbf{r}_1,\mathbf{r}_2,\omega)\right| \le 1$$

incoherent ↗ partially coherent ↖ coherent

- ➢ coherent $W(\mathbf{r}_1,\mathbf{r}_2,\omega) = V^*(\mathbf{r}_1,\omega)V(\mathbf{r}_2,\omega)$
- ➢ incoherent $W(\mathbf{r}_1,\mathbf{r}_2,\omega) \approx I(\mathbf{r}_1,\omega)\;\delta(\mathbf{r}_1-\mathbf{r}_2)$

[L. Mandel and E. Wolf, JOSA **66**, 529 (1976); Opt. Commun. **36**, 247 (1981)]

Spectral interference law

$$V(\mathbf{r},\omega)=K_1\,e^{i\omega s_1/c}\,V(\mathbf{r}_1,\omega)$$
$$+K_2\,e^{i\omega s_2/c}\,V(\mathbf{r}_2,\omega)$$

$$W(\mathbf{r},\mathbf{r},\omega)=\left\langle V^*(\mathbf{r},\omega)V(\mathbf{r},\omega)\right\rangle$$
$$=|K_1|^2\,W(\mathbf{r}_1,\mathbf{r}_2,\omega)+|K_1|^2\,W(\mathbf{r}_1,\mathbf{r}_2,\omega)$$
$$+K_1^*K_2\,e^{-i\omega(s_1-s_2)/c}\,W(\mathbf{r}_1,\mathbf{r}_2,\omega)+K_1K_2^*\,e^{i\omega(s_1-s_2)/c}\,W(\mathbf{r}_2,\mathbf{r}_1,\omega)$$

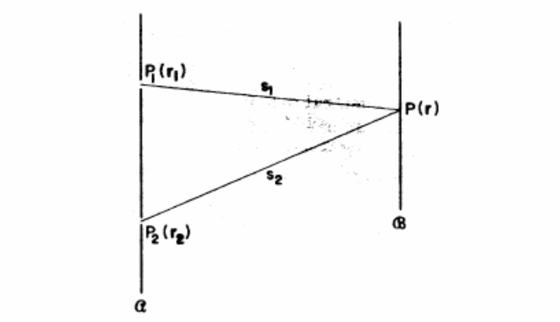


FIG. 1. Outline of the geometry for the interference experiment.

Intensities from holes 1 & 2 alone: $I^{(1)}(\mathbf{r},\omega)=|K_1|^2\,W(\mathbf{r}_1,\mathbf{r}_2,\omega)$ $I^{(2)}(\mathbf{r},\omega)=|K_2|^2\,W(\mathbf{r}_1,\mathbf{r}_2,\omega)$

$$\longrightarrow\qquad I(\mathbf{r},\omega)=I^{(1)}(\mathbf{r},\omega)+I^{(2)}(\mathbf{r},\omega)$$
$$+2[I^{(1)}(\mathbf{r},\omega)I^{(2)}(\mathbf{r},\omega)]^{1/2}\,\mathrm{Re}\{\mu(\mathbf{r}_1,\mathbf{r}_2,\omega)e^{i[\alpha-\omega(s_1-s_2)/c]}\}$$

Let $\mu(\mathbf{r}_1,\mathbf{r}_2,\omega)=\left|\mu(\mathbf{r}_1,\mathbf{r}_2,\omega)\right|e^{i\beta(\mathbf{r}_1,\mathbf{r}_2,\omega)}$

$$\boxed{\begin{aligned}I(\mathbf{r},\omega)&=I^{(1)}(\mathbf{r},\omega)+I^{(2)}(\mathbf{r},\omega)\\&\quad+2[I^{(1)}(\mathbf{r},\omega)I^{(2)}(\mathbf{r},\omega)]^{1/2}\left|\mu(\mathbf{r}_1,\mathbf{r}_2,\omega)\right|\cos[\beta(\mathbf{r}_1,\mathbf{r}_2,\omega)-\delta]\end{aligned}}\qquad[\delta=\omega(s_1-s_2)/c-\alpha]$$

- Fringe visibility $\boxed{\mathrm{V}=\dfrac{I_{\max}-I_{\min}}{I_{\max}+I_{\min}}=\left|\mu(\mathbf{r}_1,\mathbf{r}_2,\omega)\right|}$ [if $I^{(1)}=I^{(2)}$]

- Spectral changes
- Cross-spectral purity

- Fringe position determined by $\beta(\mathbf{r}_1,\mathbf{r}_2,\omega)$

Van Cittert-Zernike theorem

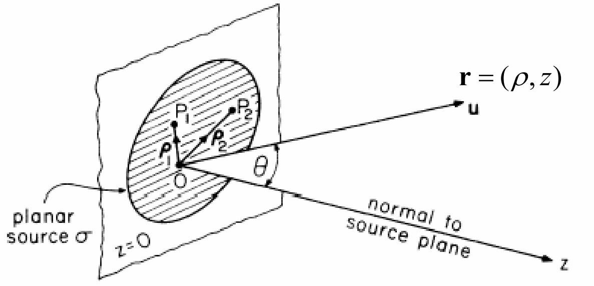


FIG. 1. Illustration of the notation.

- Incoherent source $W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega) = I(\boldsymbol{\rho}_1, \omega)\, \delta(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)$
- Huygens wavelet $\approx (-i/\lambda) \dfrac{e^{ikR}}{R} \qquad (k = \dfrac{\omega}{c} = \dfrac{2\pi}{\lambda})$

$$V(\mathbf{r}) = -\frac{i}{\lambda} \int V(\boldsymbol{\rho}) \frac{e^{ik|\mathbf{r}-\boldsymbol{\rho}|}}{|\mathbf{r}-\boldsymbol{\rho}|} \mathrm{d}^2\rho$$

$$W(\mathbf{r}_1, \mathbf{r}_2) = \left\langle V^*(\mathbf{r}_1) V(\mathbf{r}_2) \right\rangle = \left(1/\lambda r\right)^2 \iint \left\langle V^*(\boldsymbol{\rho}_1) V(\boldsymbol{\rho}_2) \right\rangle e^{-ik[|\mathbf{r}_1-\boldsymbol{\rho}_1|-|\mathbf{r}_2-\boldsymbol{\rho}_2|]} \mathrm{d}^2\rho_1 \mathrm{d}^2\rho_2$$

$$= \left(1/\lambda r\right)^2 \int I(\boldsymbol{\rho})\, e^{-ik[|\mathbf{r}_1-\boldsymbol{\rho}|-|\mathbf{r}_2-\boldsymbol{\rho}|]} \mathrm{d}^2\rho$$

$$|\mathbf{r}_1 - \boldsymbol{\rho}| - |\mathbf{r}_2 - \boldsymbol{\rho}| = \sqrt{(\boldsymbol{\rho}_1 - \boldsymbol{\rho})^2 + z_1^2} - \sqrt{(\boldsymbol{\rho}_2 - \boldsymbol{\rho})^2 + z_2^2} \approx z\left(1 + \frac{\boldsymbol{\rho}_1^2 - 2\boldsymbol{\rho}_1 \cdot \boldsymbol{\rho} + \boldsymbol{\rho}^2}{2z}\right) - z\left(1 + \frac{\boldsymbol{\rho}_2^2 - 2\boldsymbol{\rho}_2 \cdot \boldsymbol{\rho} + \boldsymbol{\rho}^2}{2z}\right) = \frac{\rho_1^2 - \rho_2^2}{2z} - \frac{(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)\cdot\boldsymbol{\rho}}{z}$$

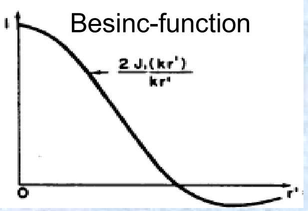
$$W_z(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \left(1/\lambda r\right)^2 e^{-ik[(\rho_1^2-\rho_2^2)/2z]} \int I(\boldsymbol{\rho})\, e^{ik[(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)\cdot\boldsymbol{\rho}/z]} \mathrm{d}^2\rho$$

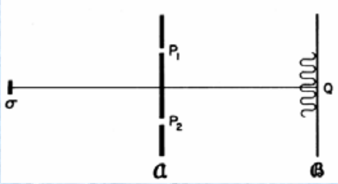
$$\boxed{\left|\mu_z(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)\right| = \left| \int I(\boldsymbol{\rho})\, e^{ik[(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)\cdot\boldsymbol{\rho}/z]} \mathrm{d}^2\rho \right|}$$

Degree of coherence
= Fourier transform of
source intensity

- Example:
  Circular source
  of radius *a*

$$\boxed{\left|\mu_z(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)\right| = \left| \frac{2J_1(ka\,|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|/z)}{ka\,|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|/z} \right|}$$





Application: Stellar interferometry

Coherent-mode decomposition

Cross-spectral density (in domain D)

- Hermitian $W(\mathbf{r}_2,\mathbf{r}_1,\omega)=W^*(\mathbf{r}_1,\mathbf{r}_2,\omega)$
- Non-negative definite $\iint W(\mathbf{r}_1,\mathbf{r}_2,\omega)f(\mathbf{r}_1)f(\mathbf{r}_2)\mathrm{d}r_1\mathrm{d}r_2 \geq 0$
- Finite (Hilbert-Schmidt kernel) $\iint |W(\mathbf{r}_1,\mathbf{r}_2,\omega)|^2\mathrm{d}r_1\mathrm{d}r_2 < \infty$

$\longrightarrow$

Mercer's theorem:

$$\boxed{\begin{aligned}&W(\mathbf{r}_1,\mathbf{r}_2,\omega)=\sum_n \lambda_n(\omega)\phi_n^*(\mathbf{r}_1,\omega)\phi_n(\mathbf{r}_2,\omega)\\&\int W(\mathbf{r}_1,\mathbf{r}_2,\omega)\phi_n(\mathbf{r}_1,\omega)\,\mathrm{d}r_1=\lambda_n(\omega)\phi_n(\mathbf{r}_2,\omega)\end{aligned}}$$

where

$\lambda_n(\omega)\geq 0$

$\phi_n(\mathbf{r},\omega)$ orthonormal in D (not necessarily complete)

Hence $W(\mathbf{r}_1,\mathbf{r}_2,\omega)=\sum_n W_n(\mathbf{r}_1,\mathbf{r}_2,\omega)$

(Natural modes of oscillation)

Each term *W*$_n$ is spatially fully coherent $|\mu_n|=1$
(factors in **r**$_1$ and **r**$_2$)

[E. Wolf, Opt. Commun. **38**, 3 (1981)
E. Wolf, JOSA **72**, 343 (1982); JOSA A **3**, 76 (1986)]

Coherent-mode decomposition

Operator approach: Associate with $W(\mathbf{r}_1,\mathbf{r}_2)$ a linear Hilbert-space operator $\hat{W}$

- Hermitian $\left\langle \varphi | \hat{W} | \psi \right\rangle = \left\langle \psi | \hat{W} | \varphi \right\rangle^*$
- Non-negative definite $\left\langle \varphi | \hat{W} | \varphi \right\rangle \geq 0 \qquad (\forall \varphi)$
- Hilbert-Schmidt $\mathrm{Tr}\left\{ \hat{W}^2 \right\} < \infty$

Spectral theorem: **(a)** $\hat{W} = \sum_n \lambda_n \, | n >< n | \qquad \lambda_n \geq 0 \qquad < m | n > = \delta_{m,n}$

(b) $\hat{W} \, | n > = \lambda_n \, | n >$

For scalar fields $W(\mathbf{r}_1,\mathbf{r}_2) = < \mathbf{r}_1 | \hat{W} | \mathbf{r}_2 > \qquad < n | \mathbf{r} > \equiv \phi_n(\mathbf{r})$

(a) =>

$$\boxed{W(\mathbf{r}_1,\mathbf{r}_2) = \sum_n \lambda_n \, \phi_n^*(\mathbf{r}_1) \phi_n(\mathbf{r}_2)}$$

(b) => $< \mathbf{r}_2 | \hat{W} | n >^* = \lambda_n < \mathbf{r}_2 | n >^* \longrightarrow < n | \int \mathrm{d}r_1 \, | \mathbf{r}_1 >< \mathbf{r}_1 | \hat{W} | \mathbf{r}_2 > = \lambda_n < n | \mathbf{r}_2 >$

$$\boxed{\int W(\mathbf{r}_1,\mathbf{r}_2) \, \phi_n(\mathbf{r}_1) \, \mathrm{d}r_1 = \lambda_n \, \phi_n(\mathbf{r}_2)}$$

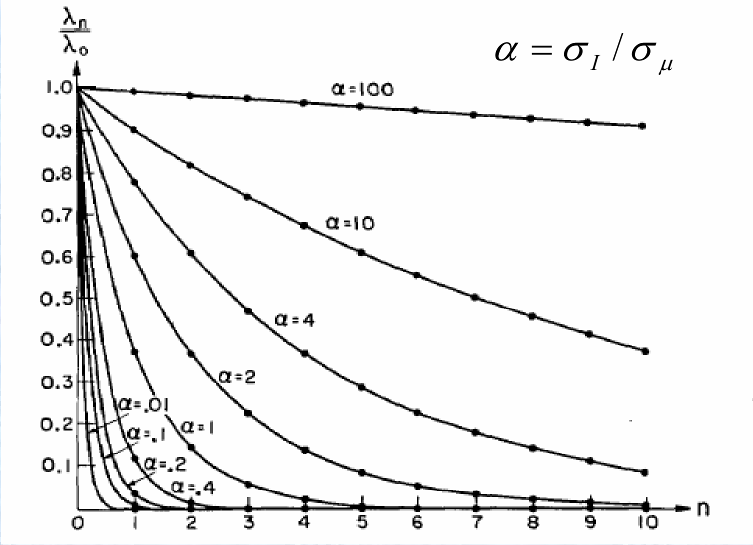
[F. Gori, M. Santarsiero, R. Simon, G. Piquero, R. Borghi, and G. Guattari, JOSA A **20**, 78-84 (2003)]

Example: Coherent modes of Gaussian Schell-model fields

$$W(\mathbf{r}_2,\mathbf{r}_1,\omega)=\sqrt{I(\mathbf{r}_1,\omega)I(\mathbf{r}_2,\omega)}\;\mu(\mathbf{r}_1,\mathbf{r}_2,\omega)$$

- Schell-model $\mu(\mathbf{r}_1,\mathbf{r}_2,\omega)=g(\mathbf{r}_1-\mathbf{r}_2,\omega)$
- Gaussian $I(\mathbf{r})=I_0\exp[-\mathbf{r}^2/2\sigma_I^2]$ $\mu(\mathbf{r}')=\exp[-\mathbf{r}'^2/2\sigma_\mu^2]$

Factors in x and y

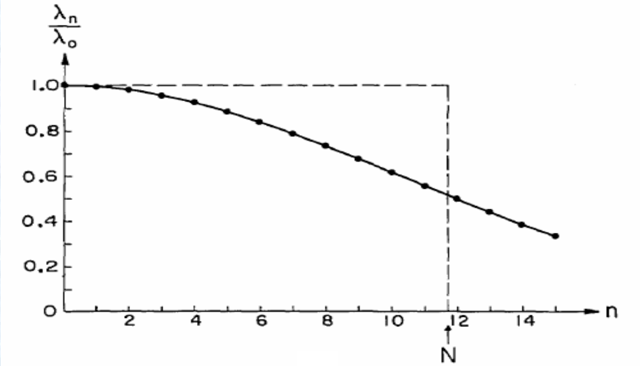


$$\varphi_n(x)=\left(\frac{2c}{\pi}\right)^{1/4}\frac{1}{(2^n n!)^{1/2}}H_n(x\sqrt{2c})e^{-cx^2}$$

$$\lambda_n=I_0\left(\frac{\pi}{a+b+c}\right)^{1/2}\left(\frac{b}{a+b+c}\right)^n$$

$$a\equiv\frac{1}{4\sigma_I{}^2},\qquad b\equiv\frac{1}{2\sigma_\mu{}^2},\qquad c\equiv(a^2+2ab)^{1/2},$$

- Effective number N
  of degrees of freedom



[A. Starikov & E. Wolf, JOSA **72**, 923 (1982)
A. Starikov, JOSA **72**, 1538 (1982)]

Polarization & Coherence of Electromagnetic Field

- Light is vector-field pair {***E***, ***H***} governed by Maxwell's eaquations

Space-time coherence
(EM coherence tensors)

$$\begin{aligned}\mathcal{E}_{jk}(\mathbf{r}_1,\mathbf{r}_2,\tau) &= \langle E_j^*(\mathbf{r}_1,t)E_k(\mathbf{r}_2,t+\tau)\rangle,\\ \mathcal{H}_{jk}(\mathbf{r}_1,\mathbf{r}_2,\tau) &= \langle H_j^*(\mathbf{r}_1,t)H_k(\mathbf{r}_2,t+\tau)\rangle,\\ \mathcal{M}_{jk}(\mathbf{r}_1,\mathbf{r}_2,\tau) &= \langle E_j^*(\mathbf{r}_1,t)H_k(\mathbf{r}_2,t+\tau)\rangle,\\ \mathcal{N}_{jk}(\mathbf{r}_1,\mathbf{r}_2,\tau) &= \langle H_j^*(\mathbf{r}_1,t)E_k(\mathbf{r}_2,t+\tau)\rangle.\end{aligned}$$

$$\begin{aligned}W_{jk}^{(e)}(\mathbf{r}_1,\mathbf{r}_2,\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{E}_{jk}(\mathbf{r}_1,\mathbf{r}_2,\tau)e^{i\omega\tau}\,d\tau, \qquad (\omega\geq 0)\\ \mathcal{E}_{jk}(\mathbf{r}_1,\mathbf{r}_2,\tau) &= \int_{0}^{\infty}W_{jk}^{(e)}(\mathbf{r}_1,\mathbf{r}_2,\omega)e^{-i\omega\tau}\,d\omega.\end{aligned}$$

(Wiener-Khintchine theorem)

$$\begin{aligned}\langle\tilde{E}_j^*(\mathbf{r}_1,\omega)\tilde{E}_k(\mathbf{r}_2,\omega')\rangle &= W_{jk}^{(e)}(\mathbf{r}_1,\mathbf{r}_2,\omega)\delta(\omega-\omega'),\\ \langle\tilde{H}_j^*(\mathbf{r}_1,\omega)\tilde{H}_k(\mathbf{r}_2,\omega')\rangle &= W_{jk}^{(h)}(\mathbf{r}_1,\mathbf{r}_2,\omega)\delta(\omega-\omega'),\\ \langle\tilde{E}_j^*(\mathbf{r}_1,\omega)\tilde{H}_k(\mathbf{r}_2,\omega')\rangle &= W_{jk}^{(m)}(\mathbf{r}_1,\mathbf{r}_2,\omega)\delta(\omega-\omega'),\\ \langle\tilde{H}_j^*(\mathbf{r}_1,\omega)\tilde{E}_k(\mathbf{r}_2,\omega')\rangle &= W_{jk}^{(n)}(\mathbf{r}_1,\mathbf{r}_2,\omega)\delta(\omega-\omega').\end{aligned}$$

Space-frequency coherence
(EM cross-spectral tensors)

- **Hermiticity**
- **Non-negative definiteness**
- **Progagation relations (from Maxwell's equations)**

❑ Ensemble averages of monochromatic fields $\{\mathbf{E}(\mathbf{r},\omega),\mathbf{H}(\mathbf{r},\omega)\}$

MAIN OBJECT OF INTEREST

$$\boxed{\mathsf{W}(\mathbf{r}_1,\mathbf{r}_2,\omega) \equiv \left\langle \mathbf{E}^*(\mathbf{r}_1,\omega)\mathbf{E}^{\mathrm{T}}(\mathbf{r}_2,\omega) \right\rangle}$$

ω = frequency

< > = ensemble average

$$\mathsf{W}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \begin{bmatrix} W_{xx}(\mathbf{r}_1,\mathbf{r}_2,\omega) & W_{xy}(\mathbf{r}_1,\mathbf{r}_2,\omega) & W_{xz}(\mathbf{r}_1,\mathbf{r}_2,\omega) \\ W_{yx}(\mathbf{r}_1,\mathbf{r}_2,\omega) & W_{yy}(\mathbf{r}_1,\mathbf{r}_2,\omega) & W_{yz}(\mathbf{r}_1,\mathbf{r}_2,\omega) \\ W_{zx}(\mathbf{r}_1,\mathbf{r}_2,\omega) & W_{zy}(\mathbf{r}_1,\mathbf{r}_2,\omega) & W_{zz}(\mathbf{r}_1,\mathbf{r}_2,\omega) \end{bmatrix}$$

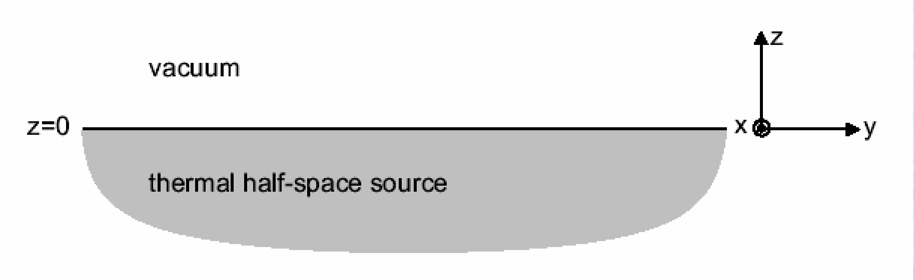
- ✓ *Hermitian & non-negative definite conditions (correlation functions)*
- ✓ *Propagation relations (Maxwell's equations)*

Electric cross-spectral density tensor → EM coherence

"Diagonal element" $(\mathbf{r}_1 = \mathbf{r}_2)$ → Polarization

Tr $\mathsf{W}(\mathbf{r},\mathbf{r},\omega)$ → Spectrum

Half-space properties



R. Carminati et al., PRL **82**, 1660 (1999)
C. Hankel et al., Opt. Comm. **186**, 57 (2000)

Fluctuation-dissipation theorem

$$\boxed{\begin{aligned}&\langle j_m(\mathbf{r},\omega) j_n^*(\mathbf{r}',\omega')\rangle = \\ &\frac{\omega}{\pi}\epsilon_o \epsilon''(\omega)\Theta(\omega,T)\delta(\mathbf{r}-\mathbf{r}')\delta_{mn}\delta(\omega-\omega'),\end{aligned}}$$

$$\Theta(\omega,T) = \hbar\omega/2 + \hbar\omega/[\exp(\hbar\omega/kT) - 1]$$

Green dyadic

$$\boxed{\mathbf{E}(\mathbf{r},\omega) = i\mu_o\omega\int_V \overleftrightarrow{\mathbf{G}}(\mathbf{r},\mathbf{r}',\omega)\cdot\mathbf{j}(\mathbf{r}',\omega)\,d^3\mathbf{r}',}$$

Procedure: Expand **E**-field (Green tensor) in vectorial plane waves,
refraction at z=0 using Fresnel coefficients,
propagate to observation points,
and integrate **=>** $W_{jk}(\mathbf{r}_1,\mathbf{r}_2,\omega)$

PLANE GEOMETRY

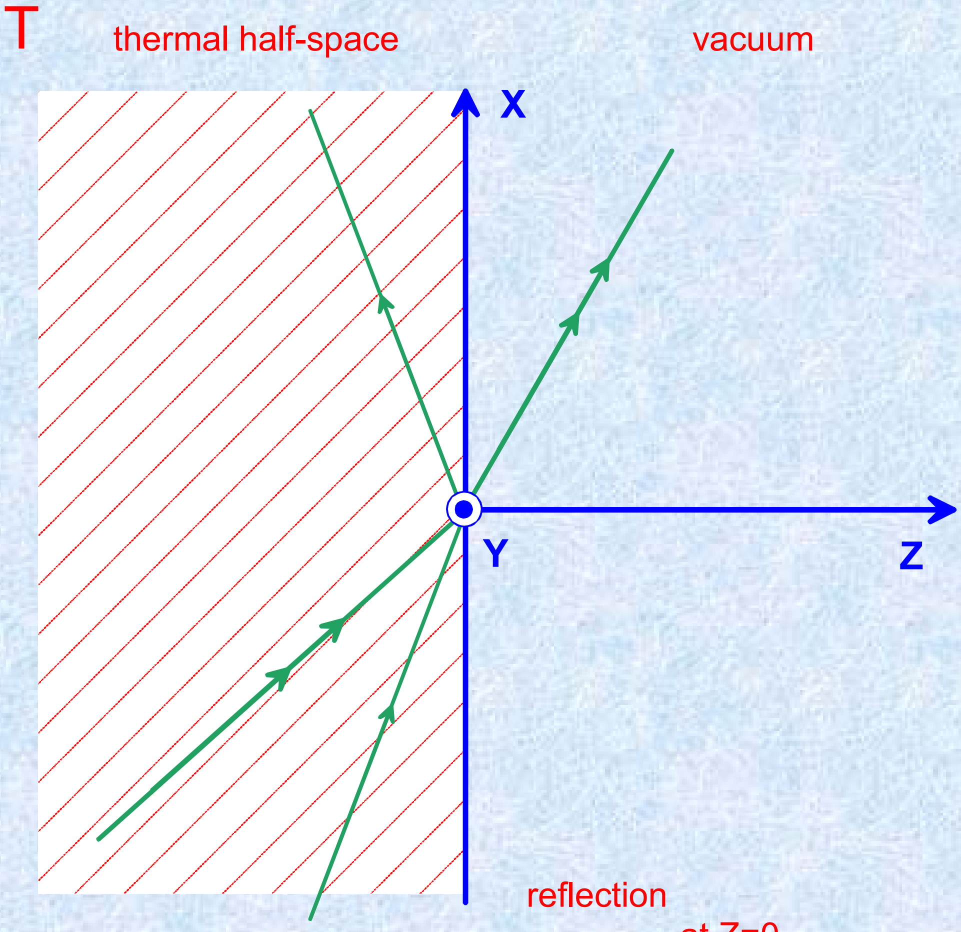
3D Half-space:

Surface polaritons

Plasmons
Electron (density) waves in metal

Phonons
Phonon (lattice vibration) waves in polar material

$$\mathrm{Re}\left\{\varepsilon(\omega)\right\} < -1$$



plane-wave expansion

reflection
at Z=0
refraction

Fresnel coefficients
(s, p polarization)

Conventional wisdom on coherence

Field correlations at least $\approx \lambda$
[cf. Ponomarenko et al., PRE **65**, 016602 (2001)]

R. Carminati et al.,
PRL **82**, 1660 (1999)

Results:

- Lossy glass: $\approx \sin(kr)/kr$
  (~ blackbody radiation)
- Tungsten W:
  coherence length $<< \lambda$
  (~ skin depth)

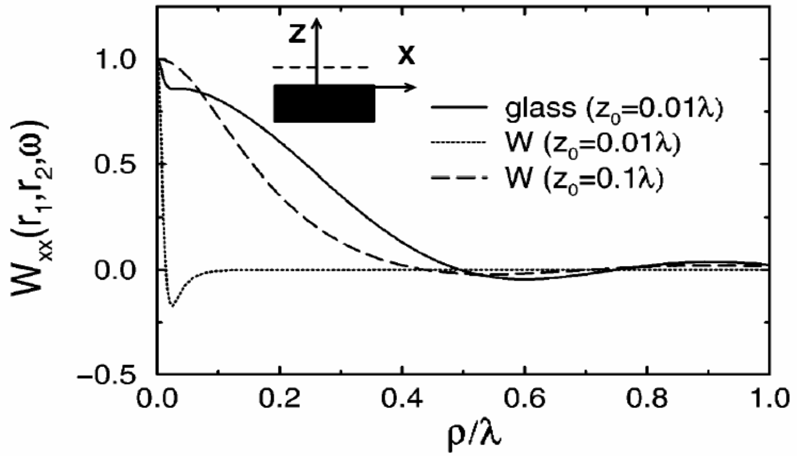


FIG. 1. $W_{xx}(\mathbf{r}_1, \mathbf{r}_2, \omega)$ in the plane $z = z_0$ versus $\rho = |\mathbf{r}_1 - \mathbf{r}_2|$. $\mathbf{r}_1$ and $\mathbf{r}_2$ are on the x axis. $\lambda = 500$ nm. Two materials are considered: lossy glass ($z_0 = 0.01\lambda$) and tungsten ($z_0 = 0.01\lambda$ and $z_0 = 0.1\lambda$). All curves are normalized by their maximum value at $\rho = 0$.

If coherence length $<<< \lambda$, what about quantum optics ?
(e.g. dipole approximation, 2-level atom, Rabi oscillations, ...)

Surface plasmons

Confined EM modes (collective electron motion), correspond to poles in p-polarized transmission

R. Carminati et al.,
PRL **82**, 1660 (1999)

Results:

Surface waves =>
coherence length $>> \lambda$

- plasmon, Au $\lambda = 620 nm$
- phonon, SiC $\lambda = 11.36 \mu m$

✓Grating coupler =>
directed thermal radiation
[Nature **416**, 61 (2002)]

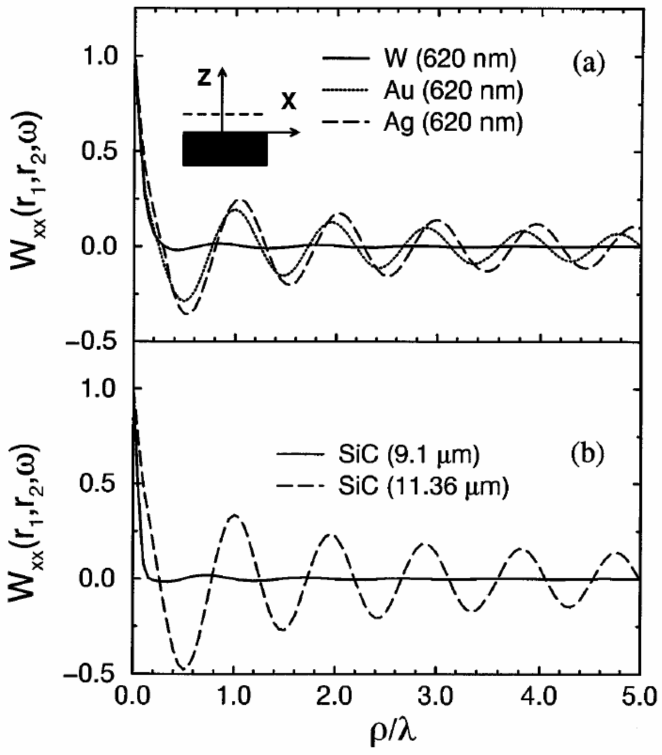


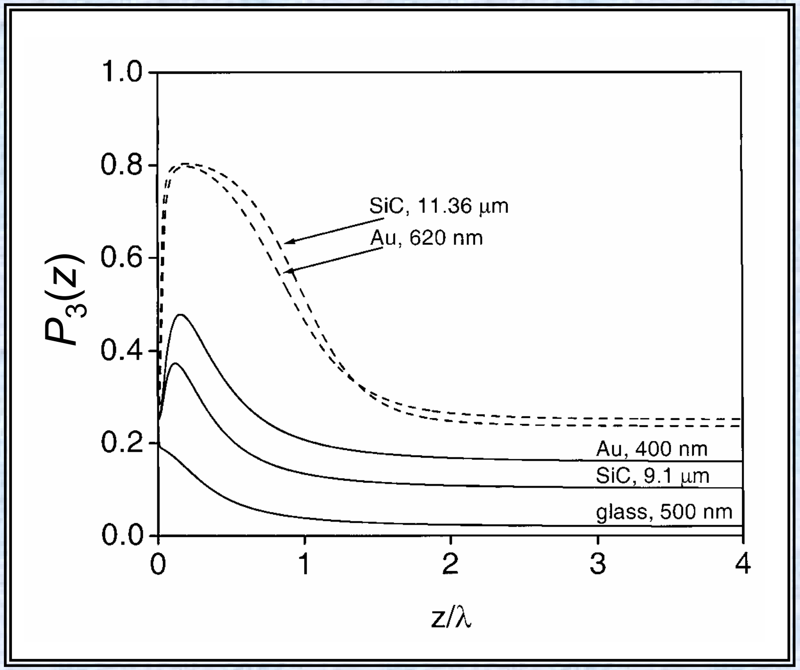
FIG. 2. Same as Fig. 1, with $z_0 = 0.05\lambda$. (a) The case of three metals (tungsten, gold, and silver), $\lambda = 620$ nm. (b) The case of SiC with $\lambda = 9.1$ μm and $\lambda = 11.36$ μm.

Near-field degree of polarization

The 3D polarization theory applied to random electromagnetic near fields emitted by thermal half-space sources

[T. Setälä, et. al.,
PRL **88**, 123902 (2002)]

- The results demonstrate the effects of evanescent waves and of resonant surface waves, such as surface plasmons or phonons, on the degree of polarization.
  (the 3D degree of polarization is introduced later)

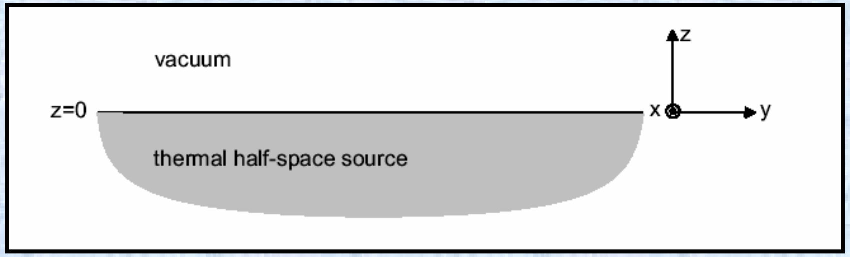


Behavior of the 3D degree of polarization, $P_3(z)$, as a fucntion of distance z in the optical near field of some thermal half-space sources

Spectral changes

[E. Wolf and D.F.V. James,
Rep. Progr. Phys. **59**, 771 (1996)]

- A.V. Shchegrov et al.,
  PRL **85**, 1548 (2000)



Surface polaritons:

- *Phonons*
  (SiC at $\lambda = 11.36\mu m$)
- *Plasmons*
  (Ag, Au at $\lambda = 620 nm$)

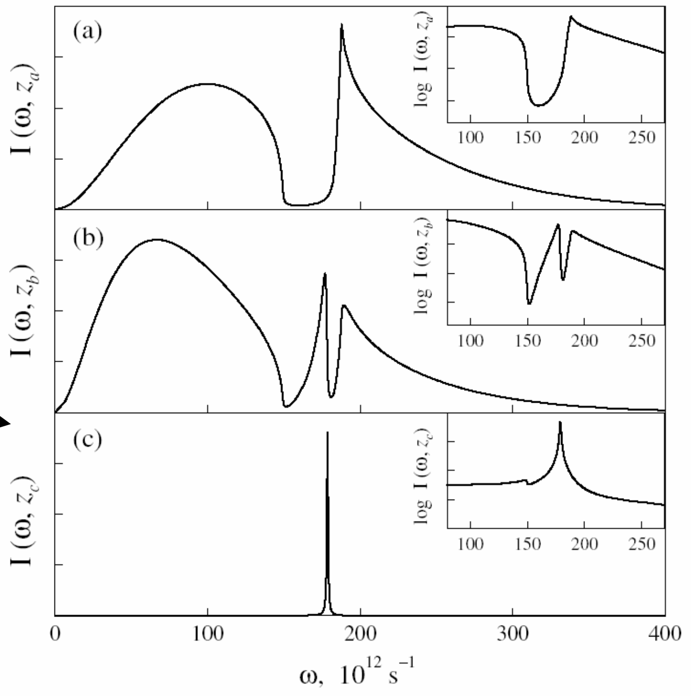


FIG. 1. Spectra of thermal emission of a semi-infinite sample of SiC at $T = 300$ K and three different heights above the surface: (a) $z_a = 1000\ \mu$m, (b) $z_b = 2\ \mu$m, (c) $z_c = 0.1\ \mu$m. The insets magnify the spectra plotted on a semilog scale in the range of strong contribution from evanescent surface modes.

$(z_0 >> \lambda)$

$(z_0 << \lambda)$

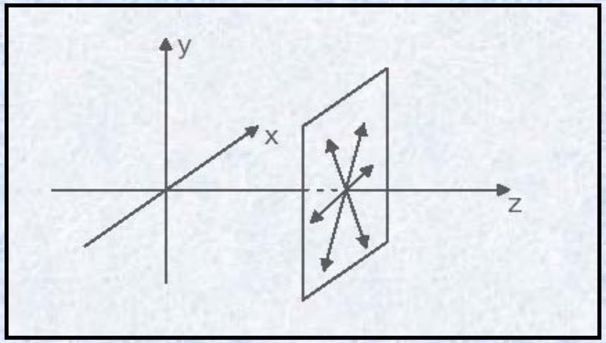
✓ Nano-spectroscopy
✓ Micro-particle manipulation
(tweezers, spanners)

Polarization

- ❑ ***Partial polarization:*** 2D "coherency" matrix, degree of polarization [E. Wolf, *Nuovo Cimento* **12**, 884 (1954), *ibid* **13**, 1165 (1959)]

$$\mathbf{J} = \begin{bmatrix} \langle E_x^* E_x \rangle & \langle E_x^* E_y \rangle \\ \langle E_y^* E_x \rangle & \langle E_y^* E_y \rangle \end{bmatrix} \qquad P_2 = \left[1 - \frac{4 \det \mathbf{J}}{\mathrm{tr}^2 \mathbf{J}} \right]^{1/2}$$

  - No polarization modulation (uniform plane waves)



Summary

Existing measures for polarization and coherence:

Polarization ⟶

Coherence ⟶

	1D	2D	3D
$\mathbf{r}_1 = \mathbf{r}_2$	–	P_2	??
$\mathbf{r}_1 \neq \mathbf{r}_2$	γ	??	??

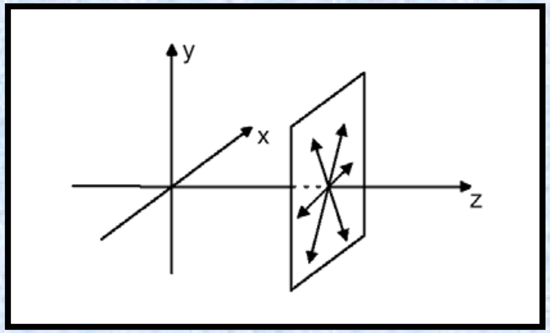
2D & 3D DEGREE OF POLARIZATION

Coherence (”polarization”) matrix

$$\Phi(\mathbf{r},\omega) = \mathsf{W}(\mathbf{r},\mathbf{r},\omega) = \begin{bmatrix} \langle E_x^* E_x \rangle & \langle E_x^* E_y \rangle & \langle E_x^* E_z \rangle \\ \langle E_y^* E_x \rangle & \langle E_y^* E_y \rangle & \langle E_y^* E_z \rangle \\ \langle E_z^* E_x \rangle & \langle E_z^* E_y \rangle & \langle E_z^* E_z \rangle \end{bmatrix}$$

Hermitian, non-negative definite matrix

- **Beam (planar, 2D) fields:**
  $\Phi_2(\mathbf{r},\omega)$
  Eigenvalues $\lambda_1 \geq \lambda_2$



- **Arbitrary (3D) fields:** $\Phi_3(\mathbf{r},\omega)$
  Eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$

Degree of Polarization

Planar (2D) fields

- All information about the polarization state at a given point is in Φ_2 (2D) or in Φ_3 (3D)

Unambiguously:

$$\Phi_2 = \Phi_2^{unpol} + \Phi_2^{pol} = A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} B & D \\ D^* & C \end{pmatrix}$$

$$A \geq 0 \quad B \geq 0 \quad C \geq 0$$

$$BC - DD^* = 0$$

$$\boxed{P_2 = \frac{\mathrm{tr}\,(\Phi_2^{pol})}{\mathrm{tr}\,(\Phi_2)} = \sqrt{1 - \frac{4\det\Phi_2}{(\mathrm{tr}\,\Phi_2)^2}}}$$

This is the traditional definition of degree of polarization for 2D fields!

- Also $P_2 = \dfrac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}$

Another way …

Entropy

~ a measure of disorder or randomness

- **2D electric field at single point:** $\mathbf{E}(\mathbf{r},\omega)=\begin{bmatrix} E_x(\mathbf{r},\omega) \\ E_y(\mathbf{r},\omega) \end{bmatrix}$ $\quad \Phi_2(\mathbf{r},\omega)=\mathsf{W}(\mathbf{r},\mathbf{r},\omega)=\begin{bmatrix} \langle E_x^* E_x \rangle & \langle E_x^* E_y \rangle \\ \langle E_y^* E_x \rangle & \langle E_y^* E_y \rangle \end{bmatrix}$

Probability density function: $p(\mathbf{E};\mathbf{r})$

Shannon entropy: $\boxed{S(\mathbf{r})=-\int p(\mathbf{E};\mathbf{r})\log[p(\mathbf{E};\mathbf{r})]\,dE}$

Gaussian statistics $\quad p(\mathbf{E};\mathbf{r})=\dfrac{1}{\pi^2\det\Phi_2(\mathbf{r})}\exp[-\mathbf{E}^\dagger(\mathbf{r})\Phi_2^{-1}(\mathbf{r})\mathbf{E}(\mathbf{r})]$

→ $\boxed{S(\mathbf{r})=\log\{\pi^2 e^2\det\Phi_2(\mathbf{r})\}=2\log(e\pi/2)+2\log I(\mathbf{r})+\log[1-P_2^2(\mathbf{r})]}$ **(~ degree of polarization)**

- **Scalar field at two points:** $\mathsf{E}(\mathbf{r}_1,\mathbf{r}_2,\omega)=\begin{bmatrix} V(\mathbf{r}_1,\omega) \\ V(\mathbf{r}_2,\omega) \end{bmatrix}$ $\quad \mu(\mathbf{r}_1,\mathbf{r}_2,\omega)=\dfrac{W(\mathbf{r}_1,\mathbf{r}_2)}{\sqrt{W(\mathbf{r}_1,\mathbf{r}_1)W(\mathbf{r}_2,\mathbf{r}_2)}}=\dfrac{\langle V^*(\mathbf{r}_1)V(\mathbf{r}_2)\rangle}{\sqrt{I(\mathbf{r}_1)I(\mathbf{r}_2)}}$

$$p(\mathsf{E};\mathbf{r}_1,\mathbf{r}_2)=\frac{1}{\pi^2\det\Phi(\mathbf{r}_1,\mathbf{r}_2)}\exp[-\mathsf{E}^\dagger(\mathbf{r}_1,\mathbf{r}_2)\Phi^{-1}(\mathbf{r}_1,\mathbf{r}_2)\mathsf{E}(\mathbf{r}_1,\mathbf{r}_2)]$$

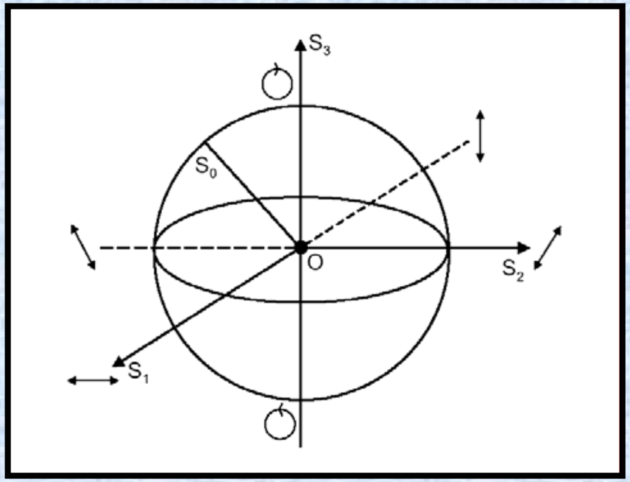
→ $\boxed{S(\mathbf{r}_1,\mathbf{r}_2)=2\log(e\pi)+\log[I(\mathbf{r}_1)I(\mathbf{r}_2)]+\log[1-|\mu(\mathbf{r}_1,\mathbf{r}_2)|^2]}$ **(~ degree of coherence)**

- **General case:** 2D or 3D electric field at pair of points – under research !

[Ph. Réfrégier, F. Goudail, P. Chavel, and A.T. Friberg, JOSA A **21**, 2124 (2004)]

Poincaré Sphere

Jones calculus
Mueller calculus



Jones vector $\mathbf{E} = \begin{pmatrix} E_x \\ E_y \end{pmatrix}$

Jones matrix (2x2) $\mathbf{E}_{\text{out}} = \underline{A} \cdot \mathbf{E}_{\text{in}}$

(Normally to coherent light only!)

$$\Phi_2 = \left\langle \mathbf{E}^*_{\text{out}} \mathbf{E}^{\text{T}}_{\text{out}} \right\rangle = \underline{A}^* \cdot \left\langle \mathbf{E}^*_{\text{in}} \mathbf{E}^{\text{T}}_{\text{in}} \right\rangle \cdot \underline{A}^{\text{T}}$$

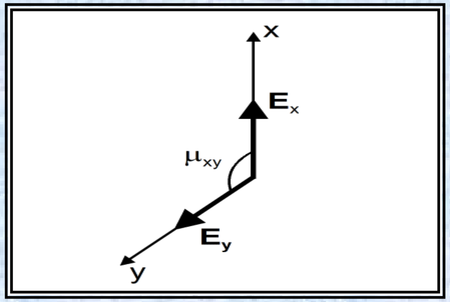
Stokes vector $S = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}$

Mueller matrix (4x4) $S_{\text{out}} = \underline{M} \cdot S_{\text{in}}$

(Applies to fully & partially polarized light!)

[E.L. O’Neill, *Introduction to Statistical Optics* (1963); E. Hecht, *Optics* (2002)]

Polarization



- 2x2 coherence matrix Φ_2

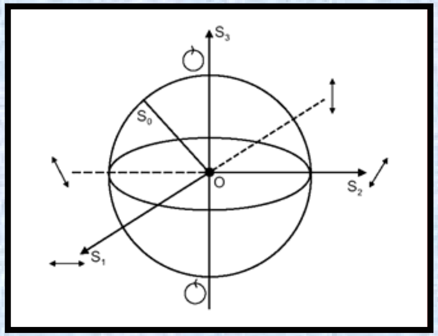
$$\Phi_2 = \frac{1}{2}\sum_{j=0}^{3} S_j \sigma_j$$

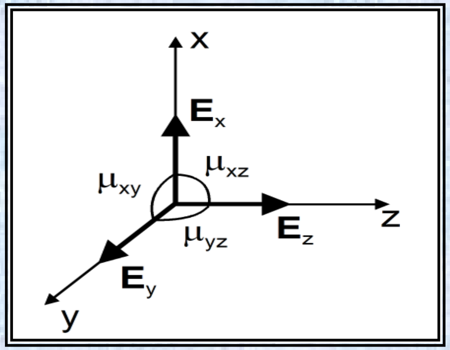
$\sigma_0 =$ unit matrix

$\sigma_j \ (j = 1,\ldots,3)$ **Pauli matrices** SU(2)

Stokes parameters $S_0, \ldots, S_3$

Poincaré sphere





- 3x3 coherence matrix Φ_3

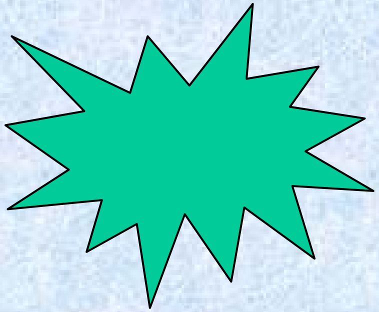
$$\Phi_3 = \frac{1}{3}\sum_{j=0}^{8} \Lambda_j \lambda_j$$

$\lambda_0 =$ unit matrix

$\lambda_j \ (j = 1,\ldots,8)$ **Gell-Mann matrices** SU(3)

Generalized Stokes parameters

$\Lambda_1, \ldots, \Lambda_8$



(8-dimensional
sphere)

Degree of polarization

(2D)

$$\boxed{P_2^2 = 1 - \frac{4\det(\Phi_2)}{\mathrm{tr}^2(\Phi_2)} = \frac{S_1^2 + S_2^2 + S_3^2}{S_0^2}}$$

The degrees of polarization P_2 and P_3 characterize, fully analogously, the correlations between the **E**-field components E_x and E_y (in 2D) and E_x, E_y, and E_z (in 3D)

$$\boxed{P_2 \geq \left|\mu_{xy}\right|}$$

μ_{jk} = correlation coefficient

(3D)

$$\boxed{P_3^2 = \frac{1}{3}\frac{\sum_{j=1}^{8}\Lambda_j^2}{\Lambda_0^2}}$$

$$\boxed{P_3^2 = \frac{3}{2}\left[\frac{\mathrm{tr}\left(\Phi_3^2\right)}{\mathrm{tr}^2(\Phi_3)} - \frac{1}{3}\right]}$$

- P_3 invariant in unitary transformations
- P_3 bounded $0 \leq P_3 \leq 1$

$$\boxed{P_3^2 \geq \frac{\left|\mu_{xy}\right|^2 \phi_{xx}\phi_{yy} + \left|\mu_{xz}\right|^2 \phi_{xx}\phi_{zz} + \left|\mu_{yz}\right|^2 \phi_{yy}\phi_{zz}}{\phi_{xx}\phi_{yy} + \phi_{xx}\phi_{zz} + \phi_{yy}\phi_{zz}}}$$

- T. Setälä et al., PRE **66**, 016615 (2002)
- T. Setälä et al., PRL **88**, 123902 (2002)
- T. Setälä et al., Opt. Lett. **28**, 1069 (2003)

o J.C. Samson & J.V. Olson, Geophys. J. R. Astron. Soc. **61**, 115 (1980)

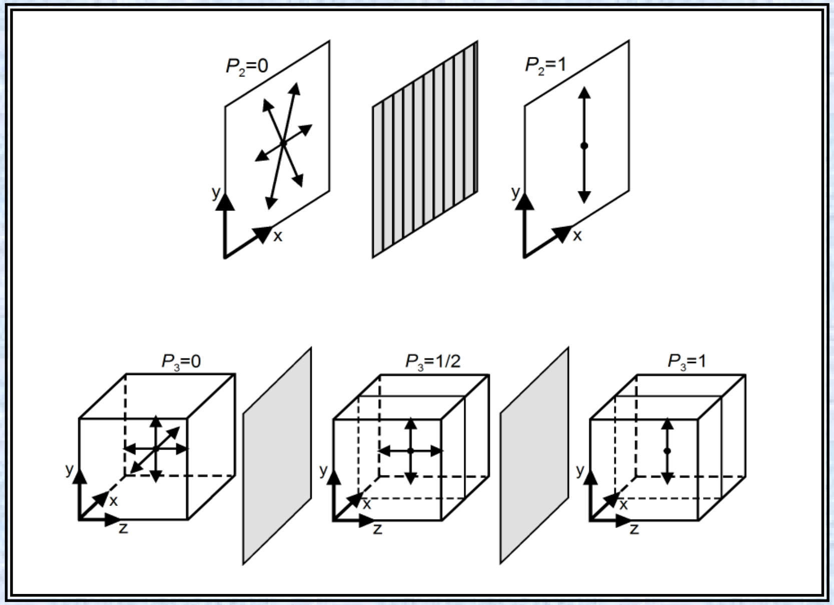
o R. Barakat, Optica Acta **30**, 1171 (1983)

(2D) **Comparisons** (3D)

- Fully unpolarized,
  if $\phi_{xx} = \phi_{yy}$ and no correlations
- Fully polarized, if $|\mu_{xy}| = 1$

- Fully unpolarized,
  if $\phi_{xx} = \phi_{yy} = \phi_{zz}$ and no correlations
- Fully polarized, if $|\mu_{xy}| = |\mu_{xz}| = |\mu_{yz}| = 1$

For plane waves, the value of 3D degree of polarization is between $1/2 \le P_3 \le 1$ Thus, a plane wave that is fully unpolarized in the 2D formalism is partially polarized in 3D analysis !?



2D & 3D fields consisting of propagating plane waves only

Application of 3D formalism to near-field polarization was shown earlier.

Gaussian intensity fluctuations

$$\frac{\left\langle \left[\Delta I(\mathbf{r},t)\right]^2 \right\rangle}{\left\langle I(\mathbf{r},t)\right\rangle^2} = \begin{cases} 1 & \text{in 1D} \\ \frac{1}{2}\left[1+P_2^2(\mathbf{r})\right] & \text{in 2D} \\ \frac{1}{3}\left[1+2P_3^2(\mathbf{r})\right] & \text{in 3D} \end{cases}$$

Physical Consequences

LOGIC: While the degree of polarization P_d may depend on the dimensionality of the analysis, the intensity fluctuation $\left\langle [\Delta I]^2 \right\rangle / \langle I \rangle^2$ will not (it's a physical characteristic).

(I) 1D field (fully polarized) $P_1(\mathbf{r}) = P_2(\mathbf{r}) = P_3(\mathbf{r}) = 1$

(II) 2D field (any polarization in 2D sense)

$$\boxed{P_3^2(\mathbf{r}) = \frac{1}{4} + \frac{3}{4}P_2^2(\mathbf{r})}$$

- T. Setälä et al., Opt. Lett. **29**, 2587 (2004)

COMPARISONS (a) Polarization

- E. Wolf [J. Ellis et al., Opt. Commun. **248**, 333 (2005)]
- A. Luis ["Degree of polarization for three-dimensional fields as a distance between correlation matrices", Opt. Commun. (in press)]
- Ph. Réfrégier [Opt. Lett. **30**, 1090 (2005); similarity of pdf's]

$$\Phi_3 = \Phi_3^{3D,pol} + \Phi_3^{2D,unpol} + \Phi_3^{3D,unpol}$$

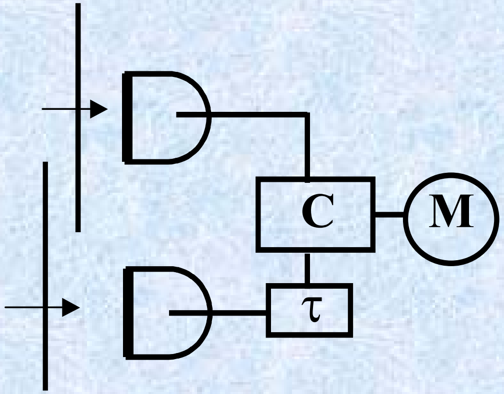
$$P_3 = \frac{\mathrm{Tr}\,(\Phi_3^{3D,pol})}{\mathrm{Tr}\,(\Phi_3)} = \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2 + \lambda_3}$$

BUT, $\Phi_3^{2D,unpol}$ **is partially polarized in 3D !!!**

- Compare: for **Setälä** et al.

$$P_3 = \sqrt{\frac{(\lambda_1 - \lambda_2)^2 + (\lambda_2 - \lambda_3)^2 + (\lambda_1 - \lambda_3)^2}{2(\lambda_1 + \lambda_2 + \lambda_3)^2}}$$

Degree of coherence (I)



- Let us consider beams of Gaussian statistics: (Hanbury Brown – Twiss experiment)

SCALAR FIELD

Instantaneous intensity $I(\mathbf{r},t)=U^*(\mathbf{r},t)U(\mathbf{r},t)$

Intensity fluctuation $\Delta I(\mathbf{r},t)=I(\mathbf{r},t)-\langle I(\mathbf{r},t)\rangle$

$$\langle \Delta I(\mathbf{r}_1,t)\Delta I(\mathbf{r}_2,t+\tau)\rangle=\langle U^*(\mathbf{r}_1,t)U(\mathbf{r}_1,t)U^*(\mathbf{r}_2,t+\tau)U(\mathbf{r}_2,t+\tau)\rangle-\langle I(\mathbf{r}_1)\rangle\langle I(\mathbf{r}_2)\rangle$$

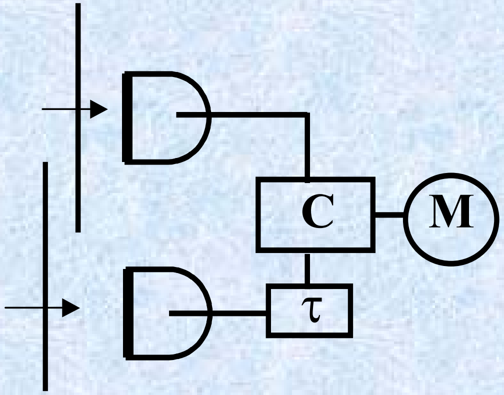
⇒

$$\frac{\langle \Delta I(\mathbf{r}_1,t)\Delta I(\mathbf{r}_2,t+\tau\rangle}{\langle I(\mathbf{r}_1)\rangle\langle I(\mathbf{r}_2)\rangle}=\frac{\left|\Gamma(\mathbf{r}_1,\mathbf{r}_2,\tau)\right|^2}{\langle I(\mathbf{r}_1)\rangle\langle I(\mathbf{r}_2)\rangle}=\left|\gamma(\mathbf{r}_1,\mathbf{r}_2,\tau)\right|^2$$

(by moment theorem)

(Complex degree of coherence)

Degree of coherence (II)



ELECTROMAGNETIC FIELD

$\mathbf{E}=(E_x,E_y)$ **2D Gaussian vector wave**

Instantaneous intensity $I(\mathbf{r},t)=\mathbf{E}^*(\mathbf{r},t)\cdot\mathbf{E}(\mathbf{r},t)$

Intensity fluctuation $\Delta I(\mathbf{r},t)=I(\mathbf{r},t)-\langle I(\mathbf{r})\rangle$

$$\langle \Delta I(\mathbf{r}_1,t)\Delta I(\mathbf{r}_2,t+\tau)\rangle=\left\langle \sum_j E_j^{\,*}(\mathbf{r}_1,t)E_j(\mathbf{r}_1,t)\sum_k E_k^{\,*}(\mathbf{r}_2,t+\tau)E_k(\mathbf{r}_2,t+\tau)\right\rangle-\langle I(\mathbf{r}_1)\rangle\langle I(\mathbf{r}_2)\rangle$$

$$\frac{\langle \Delta I(\mathbf{r}_1,t)\Delta I(\mathbf{r}_2,t+\tau)\rangle}{\langle I(\mathbf{r}_1)\rangle\langle I(\mathbf{r}_2)\rangle}=\frac{\sum_{jk}\left|\Gamma_{jk}(\mathbf{r}_1,\mathbf{r}_2,\tau)\right|^2}{\langle I(\mathbf{r}_1)\rangle\langle I(\mathbf{r}_2)\rangle}\equiv\gamma_{EM}^2(\mathbf{r}_1,\mathbf{r}_2,\tau)$$

(E_x,E_y jointly Gaussian random processes)

(Electromagnetic degree of coherence)

Degree of coherence (III)

- In space-frequency domain
- For any EM field (1-3D, not just Gaussian statistics)

$$\mu_{\mathrm{EM}}^2(\mathbf{r}_1,\mathbf{r}_2,\omega)=\frac{\sum_{jk}|\,\mathrm{W}_{jk}(\mathbf{r}_1,\mathbf{r}_2,\omega)|^2}{\sum_j \mathrm{W}_{jj}(\mathbf{r}_1,\mathbf{r}_1,\omega)\sum_k \mathrm{W}_{kk}(\mathbf{r}_2,\mathbf{r}_2,\omega)}=\frac{\mathrm{Tr}\left[\mathsf{W}(\mathbf{r}_1,\mathbf{r}_2,\omega)\cdot\mathsf{W}(\mathbf{r}_2,\mathbf{r}_1,\omega)\right]}{S(\mathbf{r}_1,\omega)\,S(\mathbf{r}_2,\omega)}$$

W = electric cross-spectral density tensor

S = Tr **W** = spectrum

- ✓ Frobenius (or Euclidean) norm
- ✓ Treats all components of **W** equally
- ✓ **Valid in exactly the same form in 1D (scalar), 2D and 3D (EM) !!**
- ✓ Measureable in 2D by polarizers (in 3D by molecular scattering, SNOM)

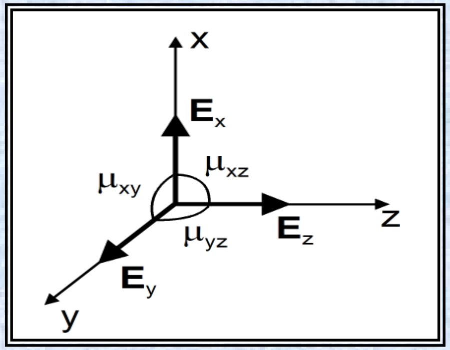
$$\boxed{0\leq\mu_{\mathrm{EM}}(\mathbf{r}_1,\mathbf{r}_2,\omega)\leq 1}$$

- J. Tervo et al., Opt. Express **11**, 1137 (2003)
- T. Setälä et al., Opt. Lett. **29**, 328 (2004)

Interpretation

- **Intensity-weighted average correlation**

$$\mu^2_{\mathrm{EM}}(\mathbf{r}_1,\mathbf{r}_2,\omega)=\frac{\sum_{jk}|\mu_{jk}(\mathbf{r}_1,\mathbf{r}_2,\omega)|^2\,\mathrm{W}_{jj}(\mathbf{r}_1,\mathbf{r}_1,\omega)\mathrm{W}_{kk}(\mathbf{r}_2,\mathbf{r}_2,\omega)}{\sum_{jk}\mathrm{W}_{jj}(\mathbf{r}_1,\mathbf{r}_1,\omega)\mathrm{W}_{kk}(\mathbf{r}_2,\mathbf{r}_2,\omega)}$$



Full coherence

$$\mu_{\mathrm{EM}}(\mathbf{r}_1,\mathbf{r}_2,\omega)=1 \quad \text{iff} \quad |\mu_{jk}(\mathbf{r}_1,\mathbf{r}_2,\omega)|=1 \quad \text{for all } (j,k)$$

- ➢ The following properties of EM fields are **equivalent**:

i) $\boxed{\mu_{\mathrm{EM}}(\mathbf{r}_1,\mathbf{r}_2,\omega)=1}$

ii) $\boxed{\mathsf{W}(\mathbf{r}_1,\mathbf{r}_2,\omega)=\mathsf{E}^*(\mathbf{r}_1,\omega)\mathsf{E}^{\mathsf{T}}(\mathbf{r}_2,\omega)}$

Factorization is essential for full coherence !!

In space-time domain:

$$\Gamma(\mathbf{r}_1,\mathbf{r}_2,\tau)=\mathsf{E}^*(\mathbf{r}_1)\mathsf{E}^{\mathsf{T}}(\mathbf{r}_2)e^{-i\omega\tau}$$

- T. Setälä et al., Opt. Lett. **29**, 328 (2004)
- T. Setälä et al., Opt. Commun. **238**, 229 (2004)

[cf., quantum coherence, R.J. Glauber, Phys. Rev. **130**, 2529 (1963)]

Equal-point EM coherence (~ polarization)

- **In scalar-wave theory μ(r, r, ω) = 1, by definition.**

As $\mathbf{r}_2 \rightarrow \mathbf{r}_1$

1D: $\mu_{\mathrm{EM}}(\mathbf{r},\mathbf{r},\omega) = 1$ *(= scalar case)*

2D: $\mu_{\mathrm{EM}}(\mathbf{r},\mathbf{r},\omega) = \sqrt{\dfrac{P_2^2(\mathbf{r},\omega)+1}{2}}$ *(EM paraxial waves)*

3D: $\mu_{\mathrm{EM}}(\mathbf{r},\mathbf{r},\omega) = \sqrt{\dfrac{2P_3^2(\mathbf{r},\omega)+1}{3}}$

* $\mu_{\mathrm{EM}}(\mathbf{r},\mathbf{r},\omega) = 1$ iff $P_{2(3)} = 1$ (fully polarized)

* $\mu_{\mathrm{EM}}(\mathbf{r},\mathbf{r},\omega) \geq \dfrac{1}{\sqrt{D}}$ (since x, y, and z components fully self-correlate)

➢ Electromagnetic coherence is fundamentally different from usual scalar-field coherence !!

Coherent-mode representation

- Superposition of completely coherent, mutually uncorrelated, elementary oscillations (scalar fields, Gori 1980, Wolf 1981)
- Each **E**-field component has different eigenfunctions, so component-by-component approach will not work (off-diagonal elements of W?)
- Use vector-valued functions and spectral theorem of Hilbert-Schmidt operators (*):

$$\hat{W} \xrightarrow{\text{(vector fields)}} |\,\alpha,\mathbf{r}>$$

$$<n\,|\,\alpha,\mathbf{r}> = \psi_{n,\alpha}(\mathbf{r})$$

$$\hat{W} = \sum_n \lambda_n\,|\,n><n\,|$$

$$<\alpha,\mathbf{r}_1\,|\,\hat{W}\,|\,\beta,\mathbf{r}_2> = W_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2)$$

$$\hat{W}\,|\,n> = \lambda_n\,|\,n>$$

$$\boxed{\begin{aligned} \mathbf{W}(\mathbf{r}_1,\mathbf{r}_2,\omega) &= \sum_n \lambda_n(\omega)\,\boldsymbol{\psi}_n^*(\mathbf{r}_1,\omega)\boldsymbol{\psi}_n^{\mathrm{T}}(\mathbf{r}_2,\omega) \\ \int_D \boldsymbol{\psi}_n^{\mathrm{T}}(\mathbf{r}_1,\omega)\mathbf{W}(\mathbf{r}_1,\mathbf{r}_2,\omega)\,\mathrm{d}^3 r_1 &= \lambda_n(\omega)\,\boldsymbol{\psi}_n^{\mathrm{T}}(\mathbf{r}_2,\omega) \end{aligned}}$$

⇨ **Modes fully coherent (& fully polarized), since the W$_n$ factor !!**

(*)
- F. Gori et al., JOSA A **20**, 78 (2003)
- J. Tervo et al., JOSA A **21**, 2205 (2004)

Example Partially polarized Gaussian Schell-model beam

$$\mathbf{W}(x_1,x_2,\omega)=\mathbf{J}(\omega)\exp\left[-\frac{x_1^2+x_2^2}{4w_0^2(\omega)}\right]\exp\left[-\frac{(x_1-x_2)^2}{2\sigma_0^2(\omega)}\right]$$

$\mathbf{J}(\omega)=2x2$ pol.matrix$=\mathbf{J}_1(\omega)+\mathbf{J}_2(\omega)$ (uncorrelated) (Mandel, 1963)

$\mathbf{J}_1(\omega)=\mathrm{Tr}\,\mathbf{J}_1(\omega)\ \hat{s}_1^*(\omega)\hat{s}_1^{\mathrm{T}}(\omega)$ ($\hat{s}_1\cdot\hat{s}_2=0,$ orthog. pol.)

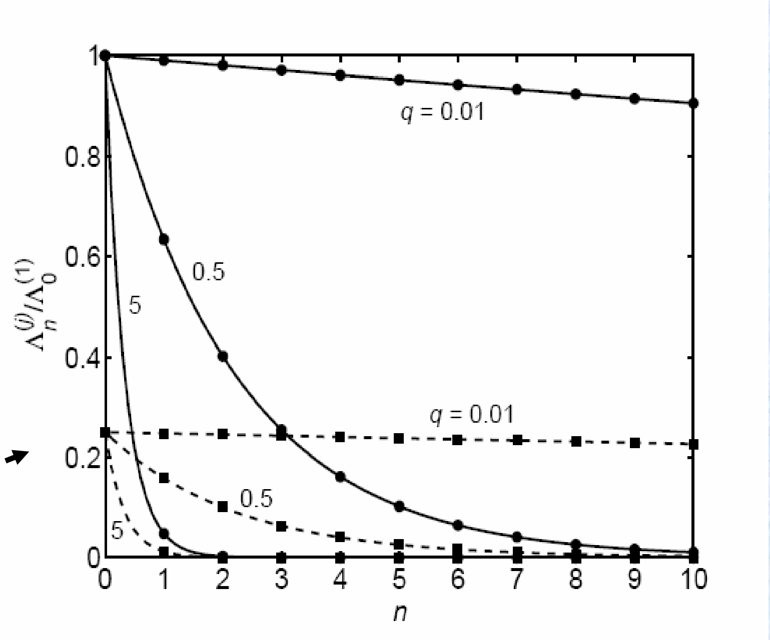
$$\mathbf{W}(x_1,x_2,\omega)=\sum_{n=0}^{\infty}\Lambda_n^{(1)}(\omega)\,\boldsymbol{\varphi}_n^{(1)*}(x_1,\omega)\boldsymbol{\varphi}_n^{(1)\mathrm{T}}(x_2,\omega)$$

$$+\sum_{n=0}^{\infty}\Lambda_n^{(2)}(\omega)\,\boldsymbol{\varphi}_n^{(2)*}(x_1,\omega)\boldsymbol{\varphi}_n^{(2)\mathrm{T}}(x_2,\omega)$$

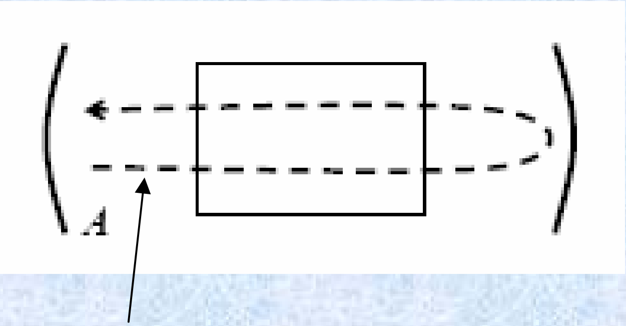
$$\Lambda_n^{(1)}(\omega)=\mathrm{Tr}\,\mathbf{J}_1(\omega)\,\lambda_n(\omega)$$

$$\boldsymbol{\varphi}_n^{(1)}(x,\omega)=\phi_n(x,\omega)\,\hat{s}_1(\omega)$$

$$\frac{\Lambda_n^{(2)}(\omega)}{\Lambda_n^{(1)}(\omega)}=\frac{1-P(\omega)}{1+P(\omega)}$$



Laser Modes



- EM coherence theory of open resonators

$$\mathbf{E}_{j+1}(\boldsymbol{\rho},\omega) = \int_A \mathsf{L}(\boldsymbol{\rho},\boldsymbol{\rho}',\omega)\mathbf{E}_j(\boldsymbol{\rho}',\omega)\mathrm{d}^2\rho'$$

L = 2x2 tensor

$$\iint_A \mathsf{L}^*(\boldsymbol{\rho}_1,\boldsymbol{\rho}_1',\omega)\cdot\mathbf{W}(\boldsymbol{\rho}_1',\boldsymbol{\rho}_2',\omega)\cdot\mathsf{L}^{\mathrm{T}}(\boldsymbol{\rho}_2,\boldsymbol{\rho}_2',\omega)\mathrm{d}^2\rho_1'\mathrm{d}^2\rho_2' = \sigma(\omega)\mathbf{W}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)$$

Use bi-orthogonal vector expansion

⇒ **Single mode**

$$\mathbf{W}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) = \lambda_k(\omega)\,\boldsymbol{\psi}_k^*(\boldsymbol{\rho}_1,\omega)\boldsymbol{\psi}_k^{\mathrm{T}}(\boldsymbol{\rho}_2,\omega)$$

Spatially fully coherent (and polarized) !!

Multi-mode
(at frequency ω)

Coherent-mode represenation
Spatially partially coherent

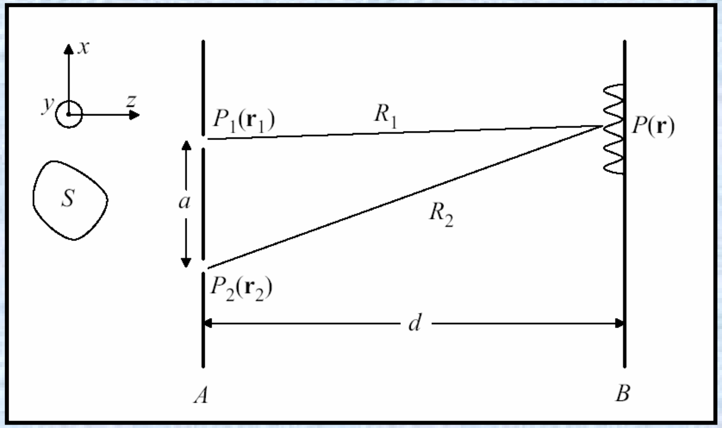
- T. Saastamoinen et al., JOSA A **22**, 103 (2005)
- J. Tervo et al., Proc. SPIE **5456**, 28 (2004)

COMPARISONS (b) Coherence

- E. Wolf [Phys. Lett. A **312**, 263 (2003)]
- A. Luis [Vector-space distance between coherence matrices]
- Ph. Réfrégier [Opt. Lett. (in press, 2005);
  & F. Goudail, Opt. Express **13**, 6051 (2005);
  mutual information, joint pdf's $p_2(\mathbf{E}_1,\mathbf{E}_2;\mathbf{r}_1,\mathbf{r}_2)$]

EM fringe visibility

Two-pinhole spectral interference law:



$$S(\mathbf{r},\omega) = 2S^{(1)}(\mathbf{r},\omega)$$

$$\times\left[1+\boxed{|\,\mu(\mathbf{r}_1,\mathbf{r}_2,\omega)\,|}\cos\alpha(\mathbf{r}_1,\mathbf{r}_2,\omega)+\delta\right] \qquad \left(\delta=\frac{R_2-R_1}{c}\right)$$

fringe visibility $V(\omega)$ **fringe location**

$$\boxed{\mu(\mathbf{r}_1,\mathbf{r}_2,\omega)=\frac{\mathrm{tr}W(\mathbf{r}_1,\mathbf{r}_2,\omega)}{\left[\mathrm{tr}W(\mathbf{r}_1,\mathbf{r}_1,\omega)\,\mathrm{tr}W(\mathbf{r}_2,\mathbf{r}_2,\omega)\right]^{1/2}}=V(\mathbf{r},\omega)}$$

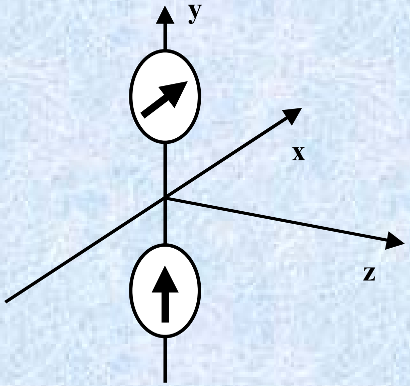
✓ Correct 1D (scalar) limit !!

BUT

✓ Fringes depend on **coherence** & **polarization**

✓ $|\mu| = 1$ => trW factors ??

✓ Single-mode laser (eg., radially or azimuthally polarized **E**-field) would **not** be **fully coherent** ??



Visibility is **not** an appropriate measure of EM degree of coherence !!!

(”intrinsic” degrees of coherence – in progress)

- B. Karczewski, Phys. Lett. **5**, 191 (1963)
- B. Karczewski, Nuovo Cimento **30**, 906 (1963)
- S. Ponomarenko & E. Wolf, Opt. Commun. **227**, 73 (2003)

Summary and conclusions

❖ *Reviewed:* Scalar coherence & 2D partial polarization (entropy)
Effects of surface waves on: Coherence
Polarization
Energy

❖ *Introduced:* 3D degree of polarization
Full polarization
EM degree of coherence
Complete coherence
Coherent modes

❖ *Concluded:* Complete coherence → factorization
Complete coherence → complete polarization
(i.e., complete correlations ≡ factorization → full polarization)

❖ *Further:* EM degree of coherence ≠ fringe visibility

❖ *Hence:* Coherence ≠ similarity
(coherence = ability of becoming similar)

❖ *Discussed:* Other definitions of 3D degree of polarization
& EM degree of coherence
(Comparisons with some criticism)

❖ *Many open questions remain !*