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WINTER COLLEGE
on
QUANTUM AND CLASSICAL ASPECTS
of
INFORMATION OPTICS

30 January - 10 February 2006

Optimal Quantum-State Estimations

&

Optimal Manipulations with Quantum Information

V. BUZEK
Slovak Academy of Sciences
Research Centre For Quantum Communication
Institute of Physics
Dubravska Cesta 9
84511 Bratislava
Slovak Republic

Three lectures on Q-Information

- Quantum state and process reconstruction
- Optimal manipulations with q-information and programmable q-processors
- Dynamics of open q-systems: Perspective of q-information theory

Further reading

- **V.Bužek and R.Derka: *"Quantum observations"*, in Coherence and Statistics of Photons and Atoms ed. J.Perina (John Wiley & Sons, New York, 2001) pp. 198—261**
- **V.Bužek: *"Quantum tomography from incomplete data via MaxEnt principle"* in Quantum Estimations: Theory and Experiment, eds. G.M.Paris and J. Rehacek (Springer-Verlag, Berlin, 2004), pp. 189 -- 234.**
- **V.Buzek, M.Hillery, M.Ziman, and M.Rosko: *"Programmable quantum processors: A review"* to appear in Quantum Information Processing (2006)**

RECONSTRUCTION OF QUANTUM STATES AND PROCESSES

06.02.2006

Vladimír Bužek

Quantum States of Light

Single mode field = quantum harmonic oscillator

$$\hat{E}(r,t) = \sqrt{2}\mathcal{E}_0 \left(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}\right) u(r)$$

- $u(r)$ the spatial field distribution
- $\mathcal{E}_0 = (\hbar\omega/2\epsilon_0 V)^{1/2}$electric field per photon
- $[\hat{a}, \hat{a}^\dagger] = 1$

$$\hat{q} = \sqrt{\frac{\hbar}{2}}\left[\hat{a} + \hat{a}^\dagger\right] \qquad \hat{p} = \frac{\sqrt{\hbar}}{i\sqrt{2}}\left[\hat{a} - \hat{a}^\dagger\right]$$

Description of states

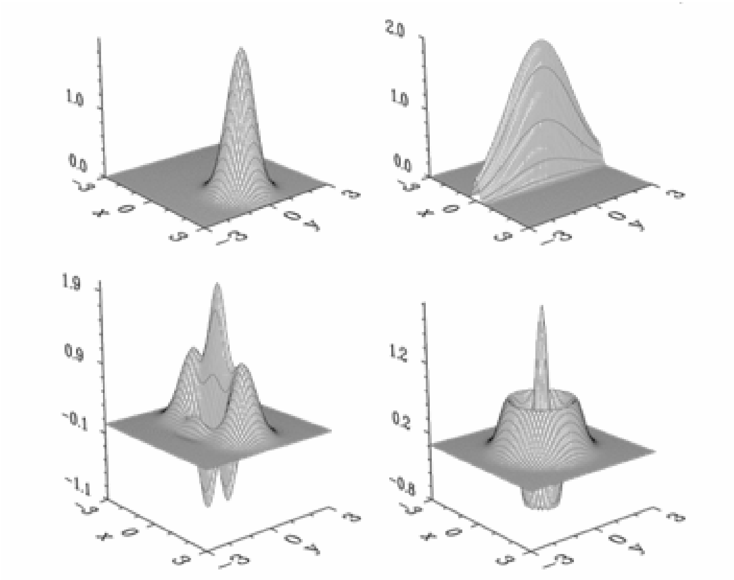
- State vector $|\Psi\rangle$
- density operator $\hat{\rho}$
- moments of system operators $\langle(\hat{a}^\dagger)^m \hat{a}^n\rangle$
- Wigner function

Wigner Functions of Light States

$$W(q,p) = \frac{1}{2\pi\hbar}\int C(q',p')\exp\left[-\frac{i(qp'-pq')}{\hbar}\right]dq'dp'$$

characteristic function $C_{\hat{\rho}}^{(W)}(q,p) = \mathrm{Tr}\left[\hat{\rho}\hat{D}(q,p)\right]$

displacement operator $\hat{D}(q,p) = \exp\left[\frac{i}{\hbar}(\hat{q}p - \hat{p}q)\right]$



Marginal distributions

$$P_{\hat{\rho}}(q) \equiv \frac{1}{\sqrt{2\pi\hbar}}\int dp\, W_{\hat{\rho}}(q,p) = \sqrt{2\pi\hbar}\langle q|\hat{\rho}|q\rangle$$

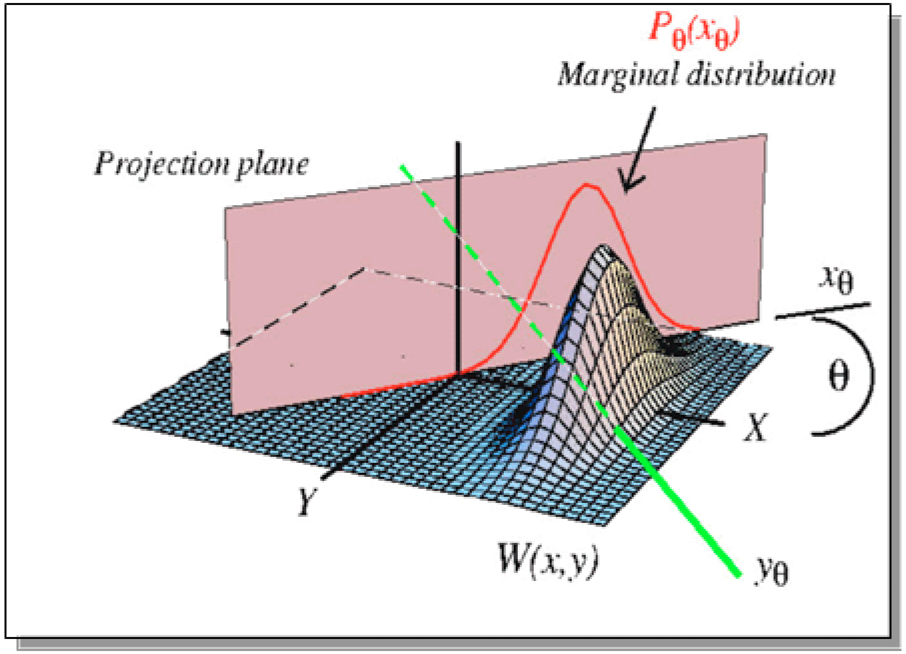
M.Hillery, R.F.O'Connell, M.O.Scully, and E.P.Wigner, *Phys. Rep.* 106, 121 (1984)

Quantum Tomography

- rotated quadratures

$$\hat{x}_\theta = \sqrt{\frac{\hbar}{2}}\left[\hat{a}\mathrm{e}^{-i\theta} + \hat{a}^\dagger \mathrm{e}^{i\theta}\right] \qquad \hat{x}_{\theta+\pi/2} = \frac{\sqrt{\hbar}}{i\sqrt{2}}\left[\hat{a}\mathrm{e}^{-i\theta} - \hat{a}^\dagger \mathrm{e}^{i\theta}\right]$$

- marginal distribution for $P_\theta(x_\theta)$



K.Vogel and H.Risken, *Phys. Rev. A* 40, 2847 (1987);

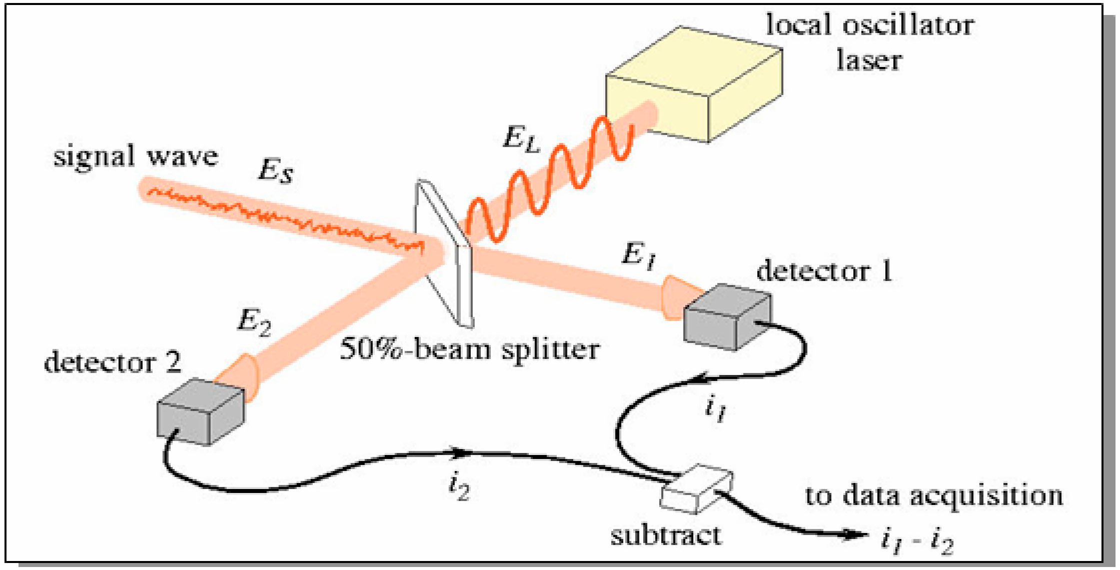
U.Leonhardt: *Measuring the quantum state of light* (Cambridge University Press, Cambridge, 1997).

Inverse Transformations

$$P_\theta(x_\theta)\forall\{-\infty \le x_\theta \le \infty; 0 \le \theta \le \pi\} \longrightarrow W(q,p)$$

- Inverse Radon transformation
- Transformation via sampling functions
- Pauli problem

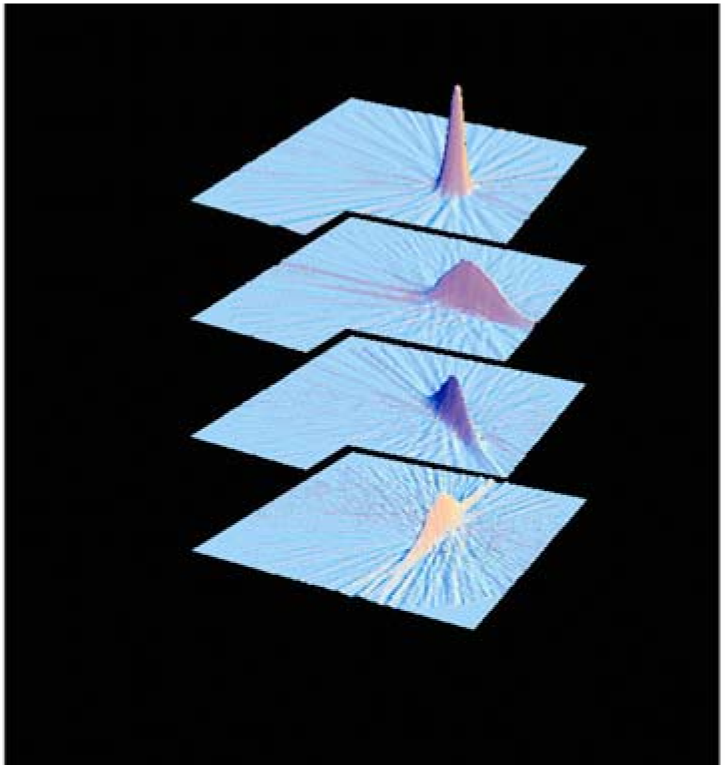
$$\rho_{mn} = \int_0^\pi \int_{-\infty}^\infty P_\theta(x_\theta) F_{mn}(x_\theta, \theta)\, dx_\theta\, d\theta$$



K.Vogel and H.Risken, *Phys. Rev. A* 40, 2847 (1987);
Th.Richter, *Phys. Lett. A* 211, 327 (1996);
G.M.D'Ariano, C.Macchiavelo, and M.G.A.Paris, *Phys. Rev. A* 50, 4298 (1994).

Experiments

- **M.G.Raymer – first tomographic reconstruction 1993**
- **J.Mlynek – WF of squeezed light 1996, 1997**



G.Breitenbach

D.T.Smithy, M.Beck, M.Besley, M.G.Raymer: *Phys. Rev. Lett.* 70, 1244 (1993)
G.Breitenbach, S.Schiller, J.Mlynek: *Nature* 387, 471 (1997)

Optimal State Reconstructions

...existing quantum theory must be supplemented with some principle that tells us how to translate, or encode, the results of measurements into a definite state description $\hat{\rho}$. Note that the problem is not to find $\hat{\rho}$ which correctly describes "true physical situation". That is unknown, and always remains so, because of incomplete information. In order to have a usable theory we must ask the much more modest question: **What $\hat{\rho}$ best describes our state of knowledge about the physical situation?**

E.T.Jaynes

E.T.Jaynes: "Information theory and statistical mechanics" in *1962 Brandeis Lectures*, p 181

Incomplete Observations

- set of observables $\hat{G}_\nu$
- measured meanvalues $G_\nu \equiv \langle \hat{G}_\nu \rangle$

How to find $\hat{\rho}$?

$$\mathrm{Tr}\,(\hat{\rho}_{\{\hat{G}\}} \hat{G}_\nu) = G_\nu\,, \quad \nu = 1, 2, \ldots, n$$

MaxEnt principle max *S*

$$S[\hat{\rho}_{\{\hat{G}\}}] = -\mathrm{Tr}(\hat{\rho}_{\{\hat{G}\}} \ln \hat{\rho}_{\{\hat{G}\}})$$

The MaxEnt principle is the most conservative assignment in the sense that it does not permit one to draw any conclusions not warranted by the data.

Solution – E.T.Jaynes

$$\hat{\rho}_{\{\hat{G}\}} = \frac{1}{Z_{\{\hat{G}\}}} \exp\,(-\sum_\nu \lambda_\nu \hat{G}_\nu)$$

$$Z_{\{\hat{G}\}}(\lambda_1, \ldots, \lambda_n) = \mathrm{Tr}[\exp(-\sum_\nu \lambda_\nu \hat{G}_\nu)]$$

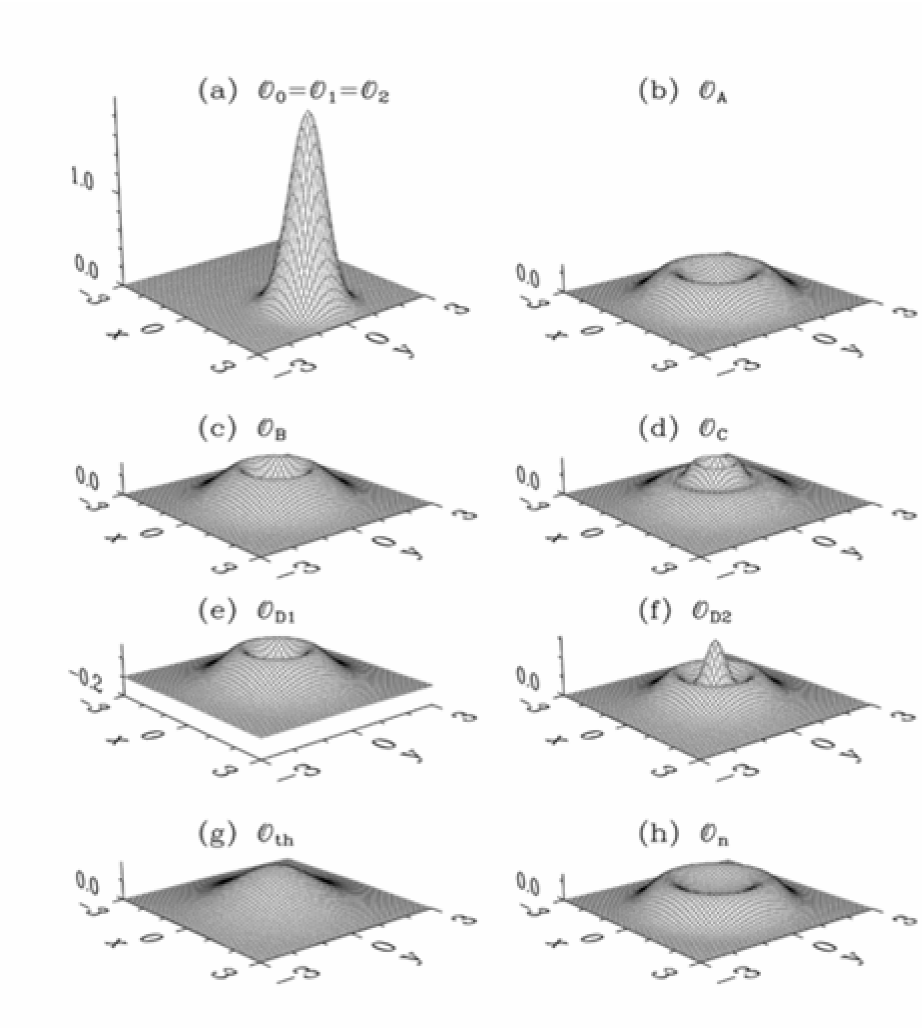
E.T.Jaynes, *Phys. Rev.* 180, 171 (1957);

E.T.Jaynes, *Am. J. Phys.* 31, 66 (1963).

Example I.

coherent state $|\alpha\rangle$

$$\mathcal{O}_0 \equiv \{(\hat{a}^\dagger)^k \hat{a}^l;\ \forall k, l\}$$
$$\mathcal{O}_{\rm th} \equiv \{\hat{a}^\dagger \hat{a}\}$$
$$\mathcal{O}_1 \equiv \{\hat{a}^\dagger \hat{a}, \hat{a}^\dagger, \hat{a}\}$$
$$\mathcal{O}_2 \equiv \{\hat{a}^\dagger \hat{a}, (\hat{a}^\dagger)^2, \hat{a}^2, \hat{a}^\dagger, \hat{a}\}$$
$$\mathcal{O}_{\rm A} \equiv \{\hat{P}_n = |n\rangle\langle n|;\quad \forall n\}$$
$$\mathcal{O}_{\rm n} \equiv \{\hat{n}, \hat{n}^2\}$$



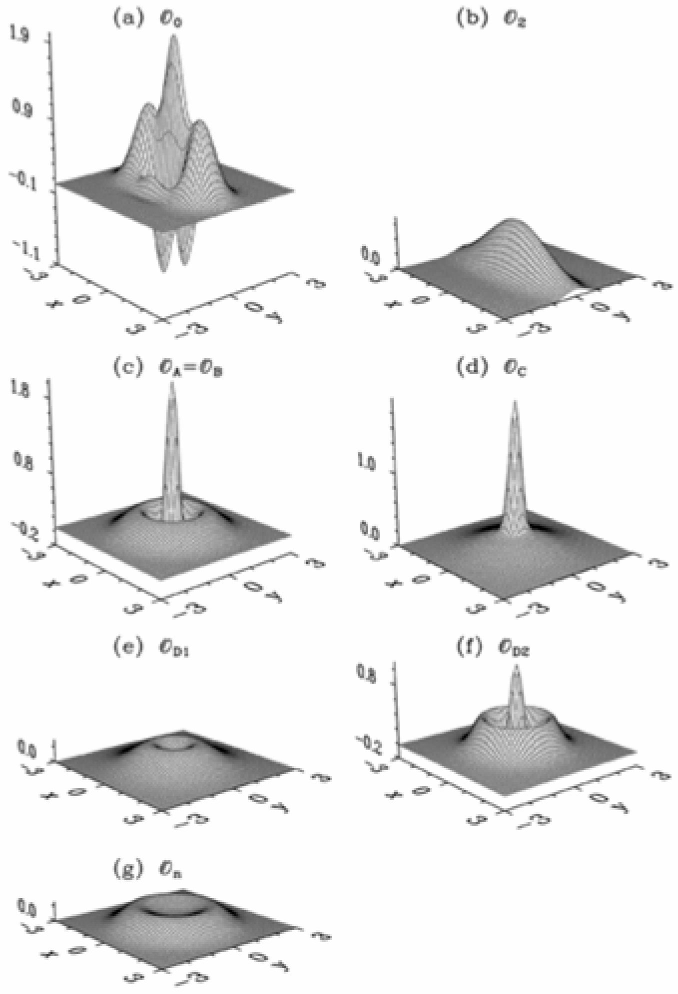
V.Buzek, G.Adam, and G.Drobny, *Ann. Phys. (N.Y.)* 245, 37 (1996)

Example II.

even coherent state

$$|\alpha_{\rm e}\rangle = N_e^{1/2}\left(|\alpha\rangle + |-\alpha\rangle\right)$$

$$N_e^{-1} = 2\left[1 + \exp(-2|\alpha|^2)\right]$$



V.Buzek, G.Adam, and G.Drobny, *Ann. Phys. (N.Y.)* 245, 37 (1996)

Q-Tomography & Incomplete Data

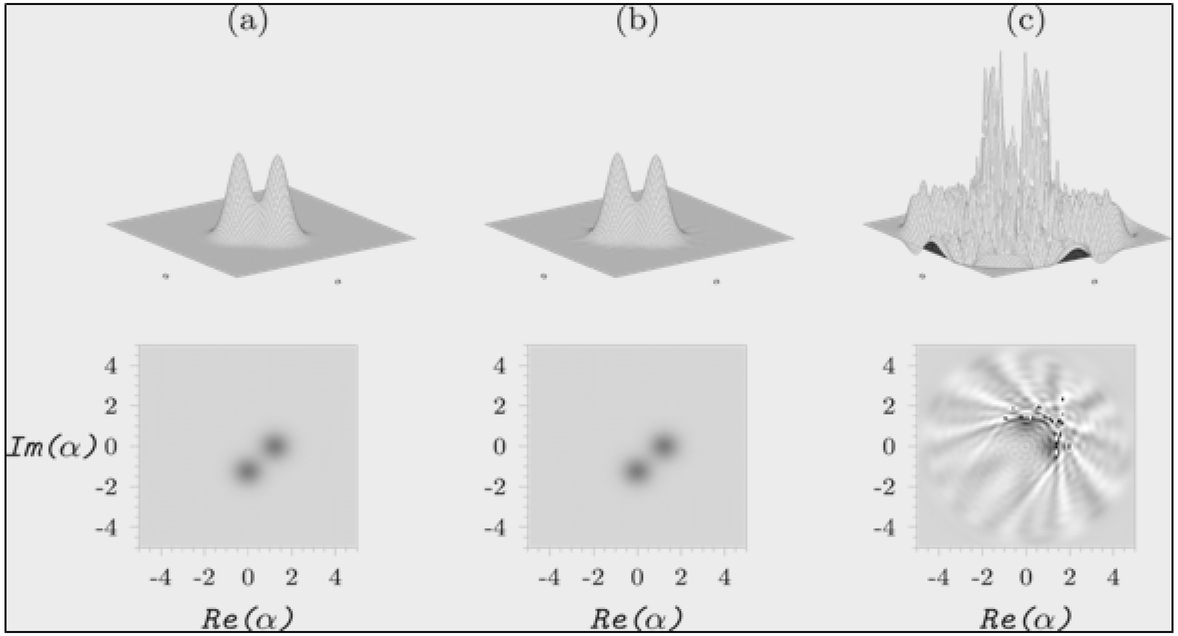
complete data:

$$\hat{\rho} = \frac{1}{Z_0} \exp\left[- \int_0^{\pi} d\theta \int_{-\infty}^{\infty} dx_\theta \, |x_\theta\rangle\langle x_\theta| \lambda(x_\theta) \right]$$

$$P_\theta(x_\theta) = \mathrm{Tr}\left[\hat{P}_\theta(x_\theta) \hat{\rho} \right]; \qquad \hat{P}_\theta(x_\theta) = |x_\theta\rangle\langle x_\theta|$$

incomplete data:

$$\hat{\rho} = \frac{1}{Z} \exp\left(\lambda_0 \hat{n} + \sum_{l=1}^{N_x} \sum_{m=1}^{N_\theta} \lambda_{l,m} |x_{\theta_m}^{(l)}\rangle\langle x_{\theta_m}^{(l)}| \right)$$



- $N_\theta = 5, \; N_x = 15$
- **error** $\Delta = \sum_{n_1,n_2} \left[(\rho_1)_{n_1,n_2} - (\tilde{\rho}_1)_{n_1,n_2} \right]^2$
- **MaxEnt** $\Delta = 1.0 \times 10^{-5}$
- **pattern functions D = 0.76**
- **number of cuts!**

V.Buzek, G.Drobny, R.Derka, G.Adam, and H.Wiedemann, *Chaos, Solitons & Fractals,* 10, 981 (1999)

WF via the parity operator

$$W(\alpha) = 2\ Tr\left[P\rho(\alpha)\right]$$

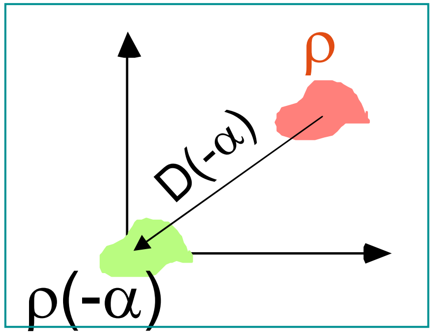
$$P|n\rangle = (-1)^n |n\rangle = \begin{cases} +|n=2k\rangle \\ -|n=2k+1\rangle \end{cases}$$

$$\rho(\alpha) = D^+(\alpha)\rho D(\alpha)$$

$$\rho(\alpha) = D(-\alpha)\rho D(\alpha)$$

$$D(-\alpha) = \exp\left(\alpha^* a - \alpha a^+\right)$$

Displacement operator

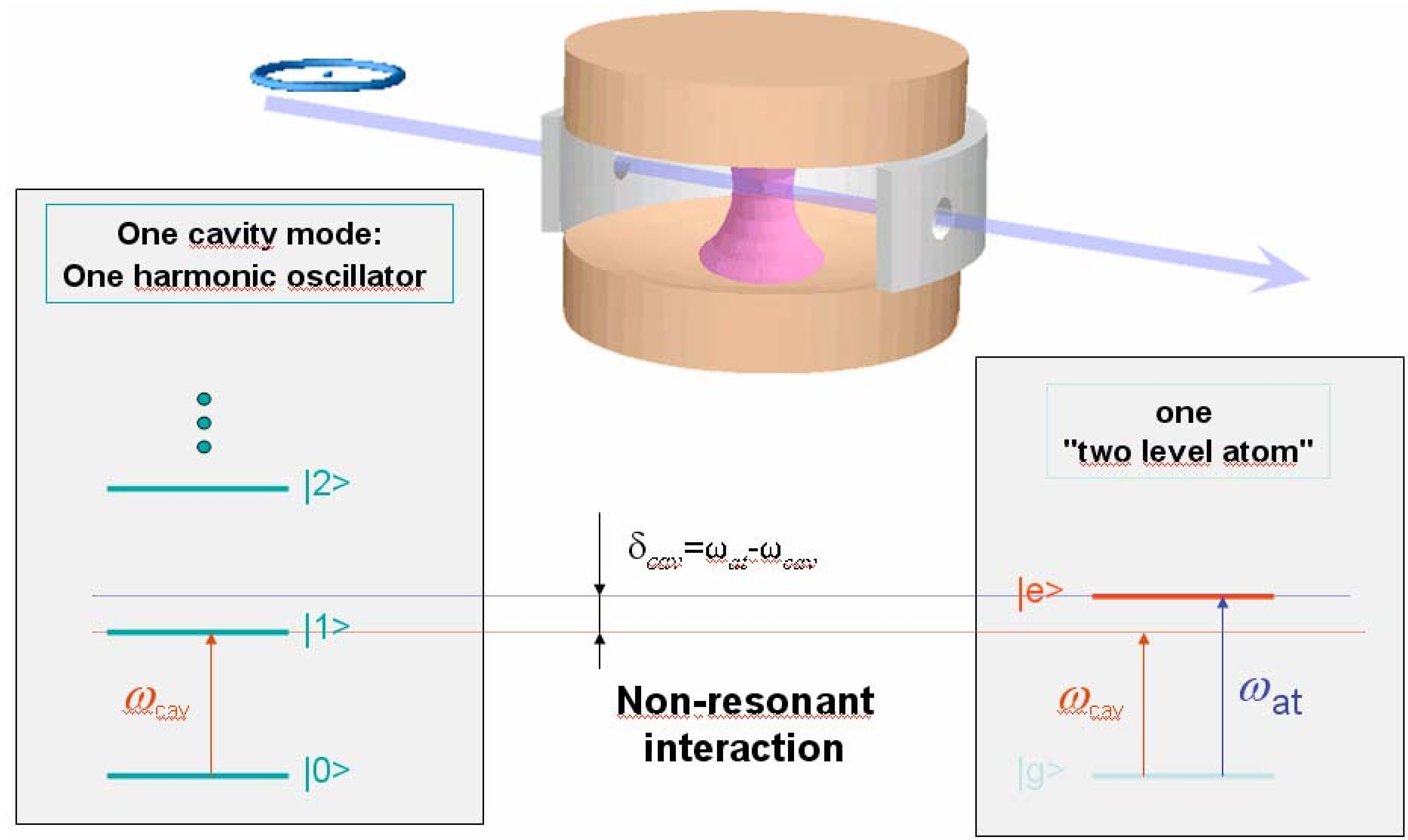


Realized with a classical source

The Wigner function is the average of the parity operator in the displaced state

L.G.Lutterbach & L.Davidovich, *PRL 78*, 2547 (1997)

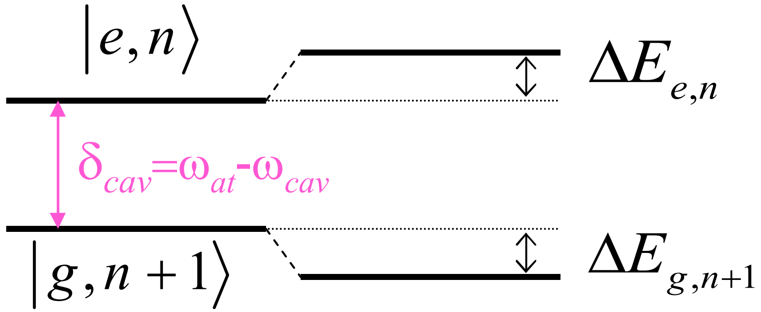
Measuring the field parity?



P.Bertet,Auffeves,.Maioli, S.Osnaghi, T.Meunier, M.Brune, J.M.Raimond, S.Haroche, *PRL.* 89, 200402(2002)

Non-resonant atom-field interaction

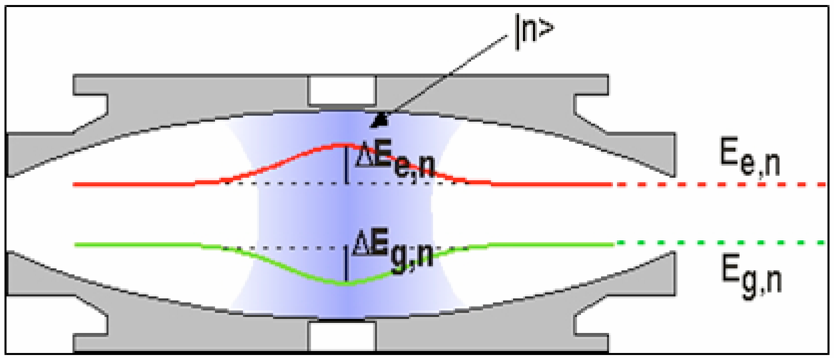
- **One atom interacting with *n* photons:**



- Coupling: $\langle e,n|\hat{V}|g,n+1\rangle = \Omega\sqrt{n+1}$

Ω: "Vacuum Rabi frequency"

- dispersive regime: $\delta \gg \Omega\sqrt{n+1}$

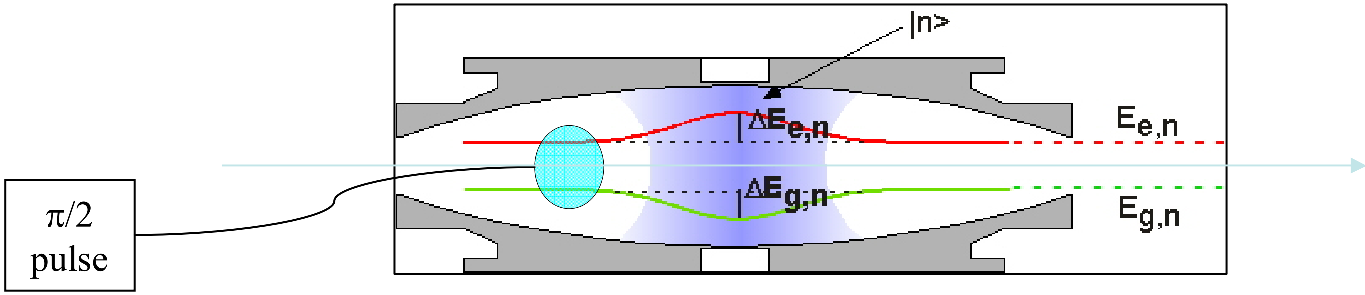


$$\Delta E_{e,n} \simeq \hbar\frac{\Omega^2}{4\delta}(n+1)$$

$$\Delta E_{g,n} \simeq -\hbar\frac{\Omega^2}{4\delta}n$$

- The cavity frequency is shifted: atom index of refraction
- atomic frequency: light shift and Lamb shift

Measuring the parity: Measuring a phase shift of the atomic state



$$\phi_0 = \frac{\Omega^2}{2\delta}.T_{eff}$$

$$\frac{1}{\sqrt{2}}\left(|e\rangle + |g\rangle\right)\otimes|n\rangle \longrightarrow \frac{1}{\sqrt{2}}e^{i\varphi_n}\left(|e\rangle + e^{i\Delta\Phi(n)}|g\rangle\right)\otimes|n\rangle \quad , \quad \Delta\Phi(n) = \phi_0\left(n+1/2\right)$$

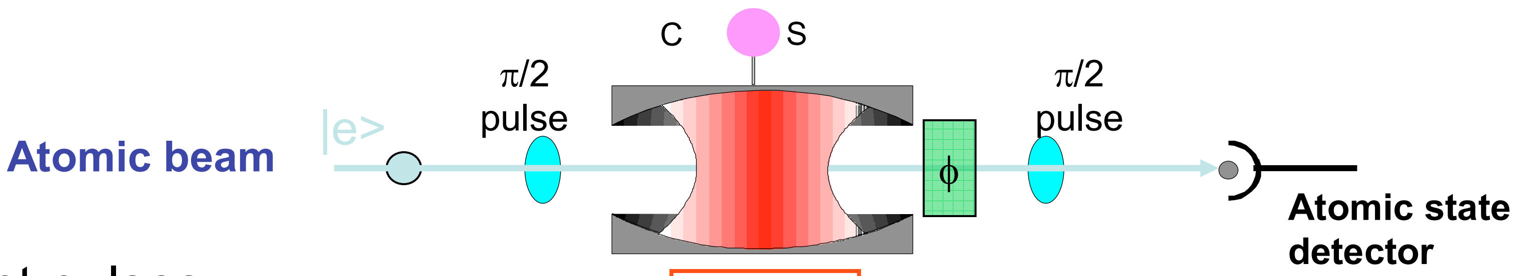
Atomic coherence is phase shifted proportionally to *n*

For $\phi_0 = \pi$

$$\frac{1}{\sqrt{2}}\left(|e\rangle + |g\rangle\right)\otimes|n\rangle \longrightarrow \frac{1}{\sqrt{2}}e^{i\varphi_n}\left(|e\rangle + i(-1)^n|g\rangle\right)\otimes|n\rangle$$

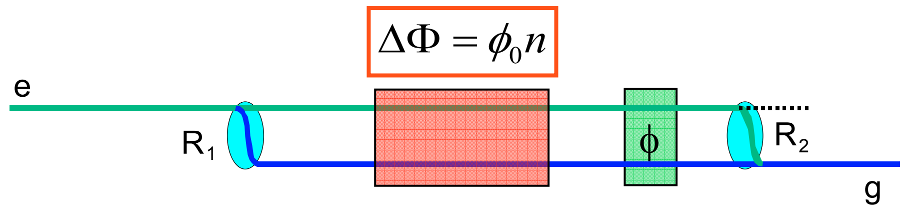
Even and odd number states are correlated to two orthogonal atomic states.

Measuring atomic phase shifts by Ramsey interferometry

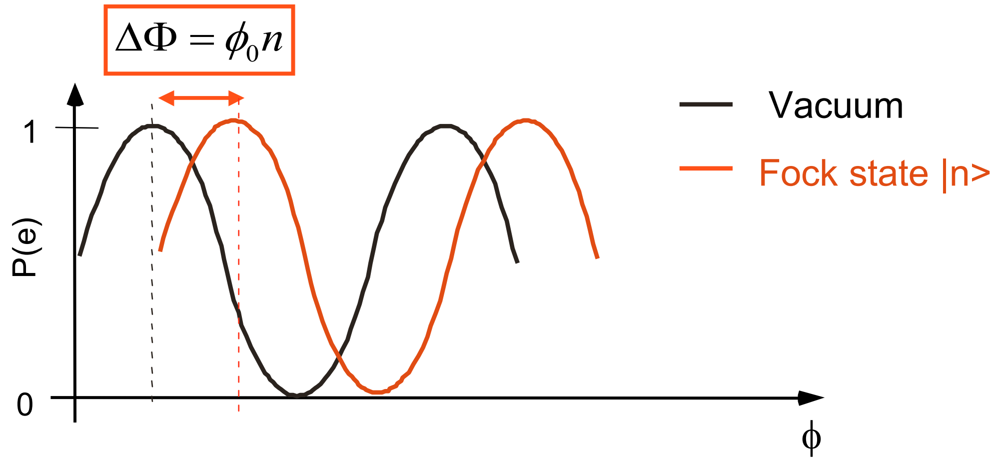


apply two $\pi/2$ resonant pulses R1 and R2 which acts as beam splitters

$$|e\rangle \longrightarrow \frac{1}{\sqrt{2}}\left(|e\rangle + |g\rangle\right)$$
$$|g\rangle \longrightarrow \frac{1}{\sqrt{2}}\left(-|e\rangle + |g\rangle\right)$$

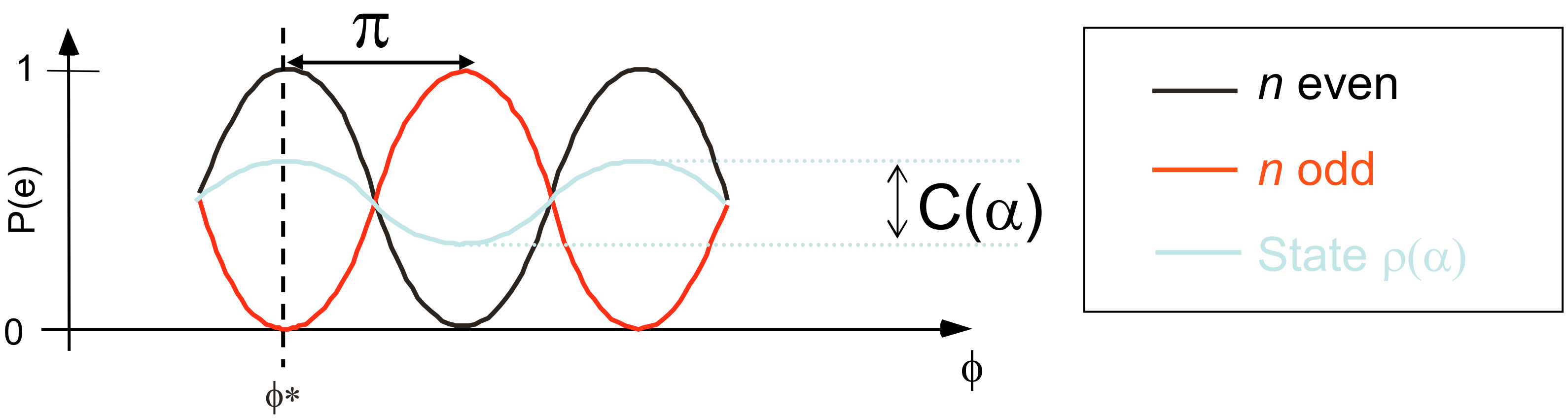


interferences between two undistinguishable quantum paths: "Ramsey fringes"



Measuring the parity of the photon number

$\Phi_0 = \pi$



C_{even}=+1 for state |2*n*>

C_{odd}=-1 for state |2*n*+*1*>

$$C(\alpha) = C_{even}.P_{even} + C_{odd}.P_{odd} = \left\langle (-1)^{\hat{N}} \right\rangle = \frac{W(\alpha)}{2}$$

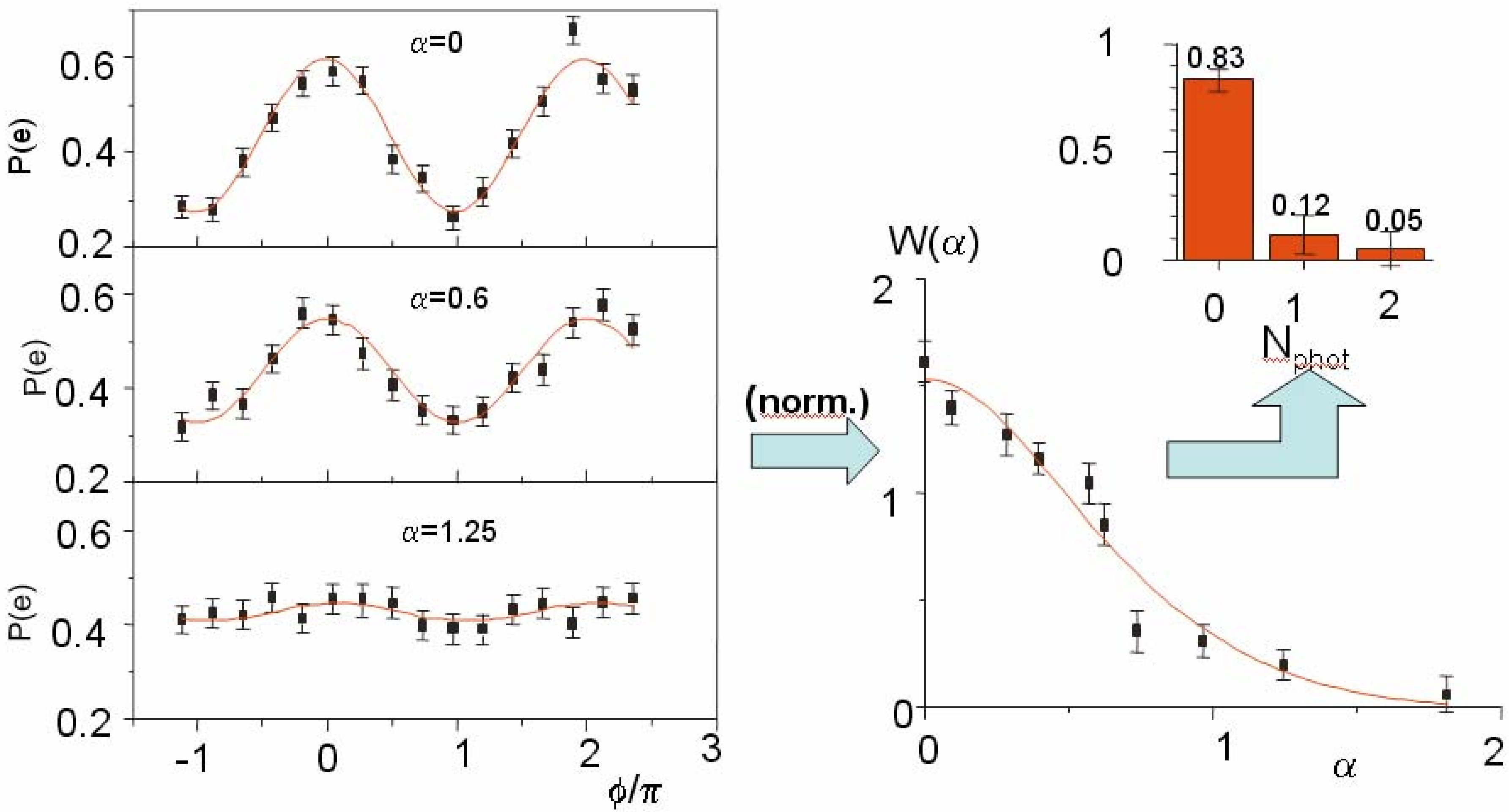
- Sensitivity to fringe contrast:

Assume: $C_{even} = -C_{odd} = \eta \leq 1$

Then: $$\frac{C(\alpha)}{\eta} = \frac{W(\alpha)}{2}$$

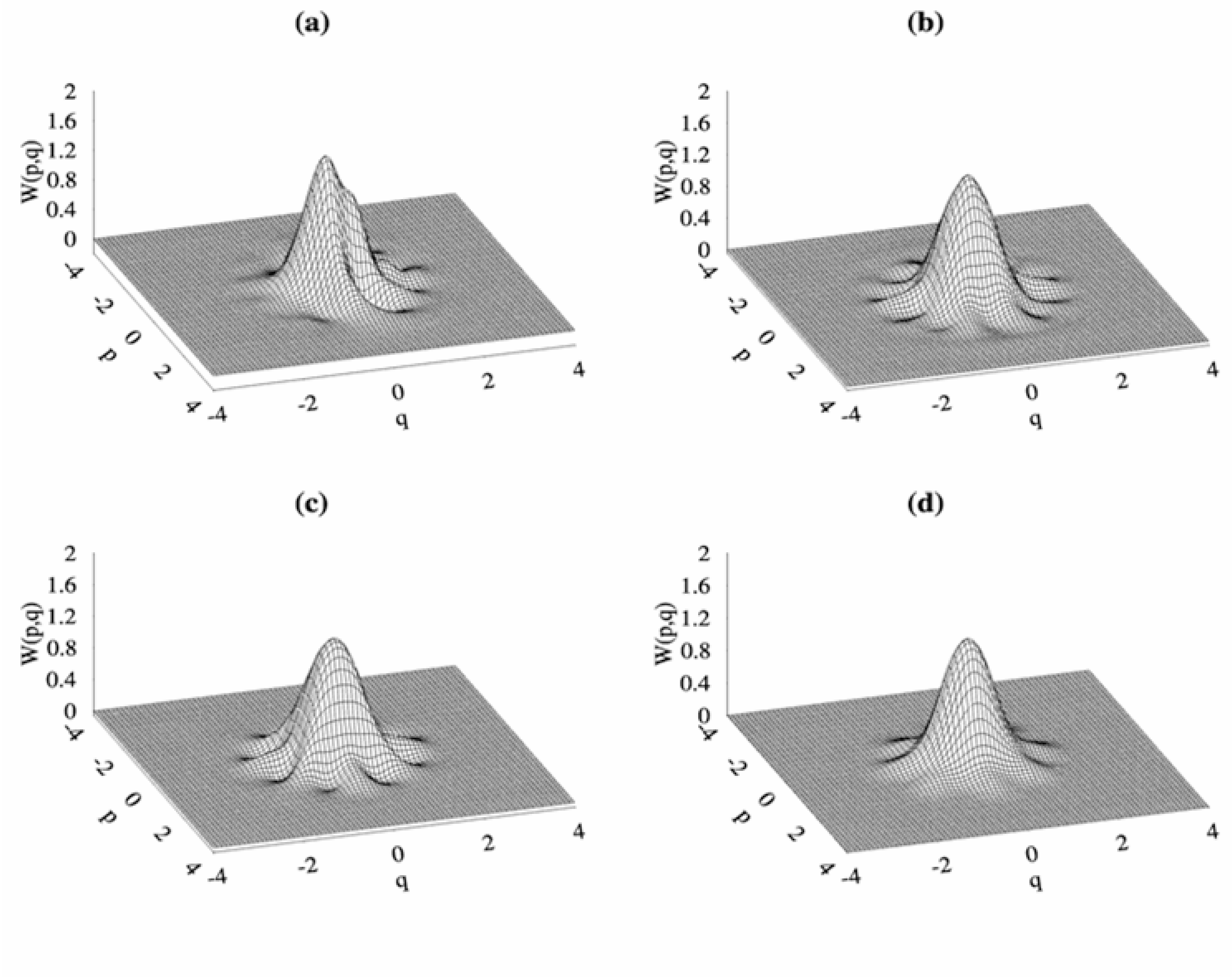
Finite fringe contrast only affects the signal to noise
One still measures W by renormalizing the signal

WF of the vacuum state

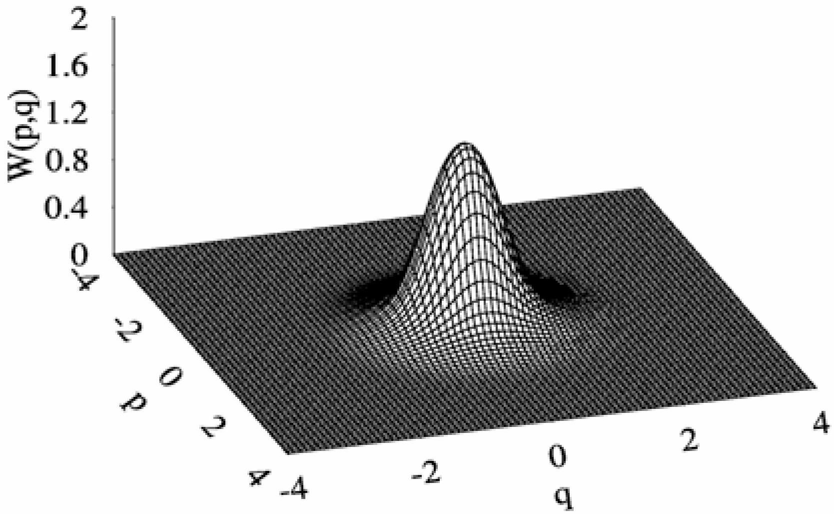


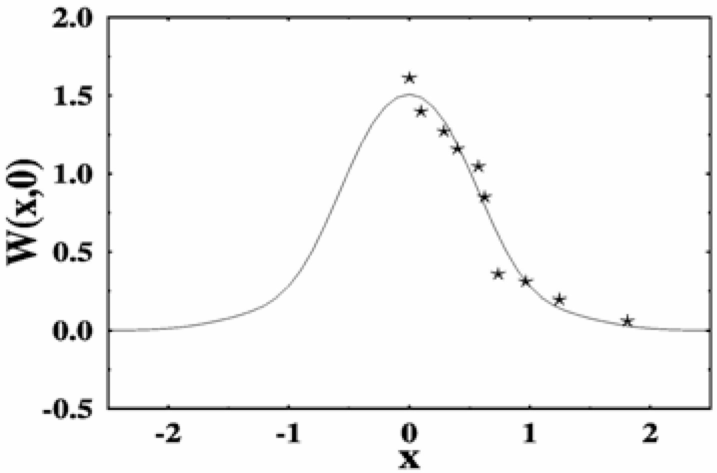
P.Bertet,Auffeves,.Maioli, S.Osnaghi, T.Meunier, M.Brune, J.M.Raimond, S.Haroche, *PRL.* 89, 200402(2002)

Reconstructed WF of the vacuum state I

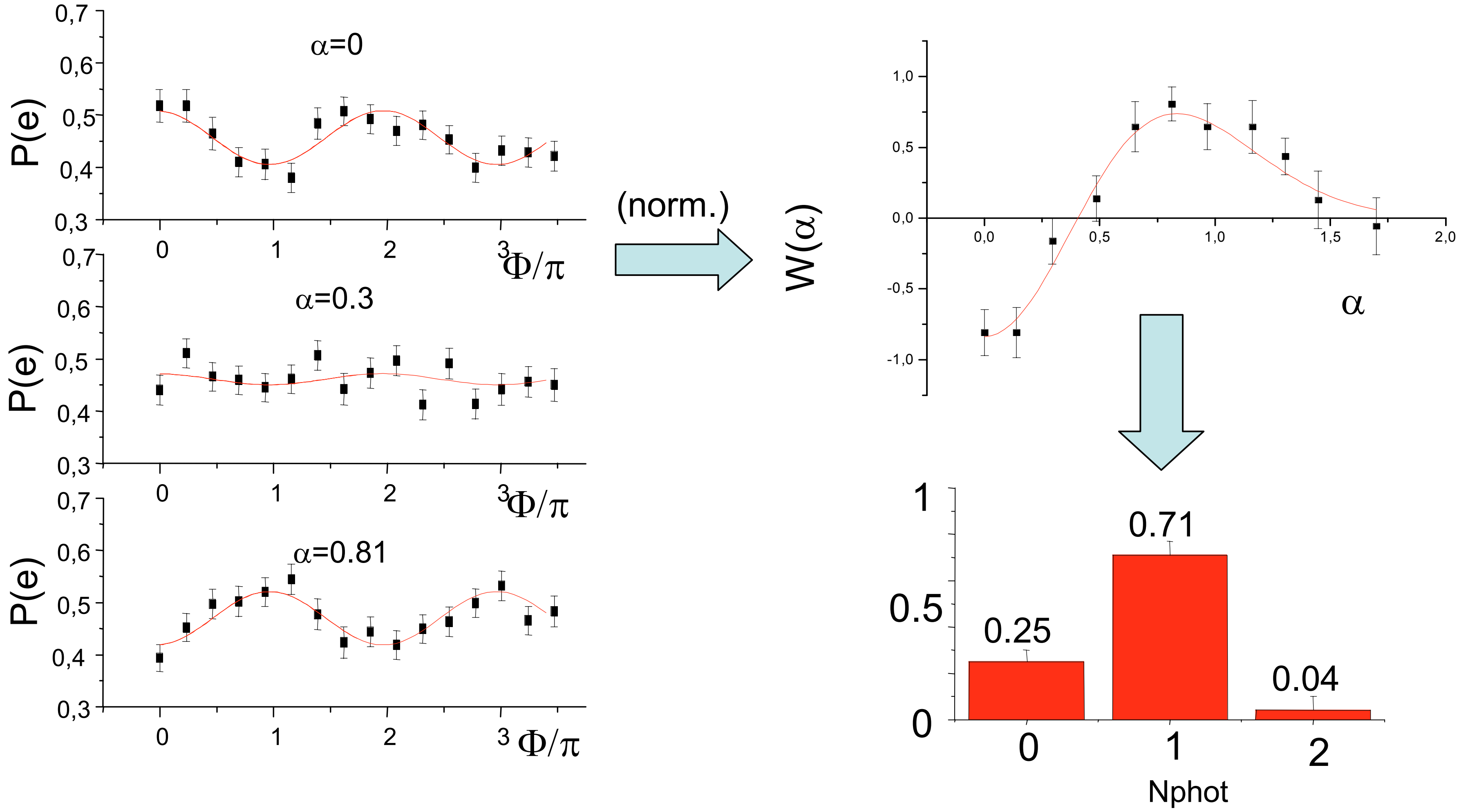


Wigner function of the vacuum state II

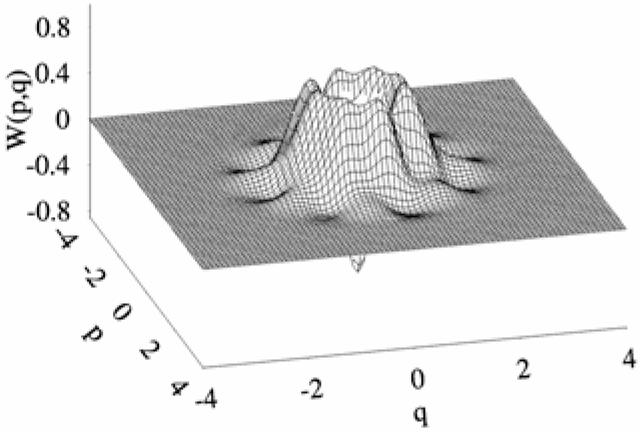


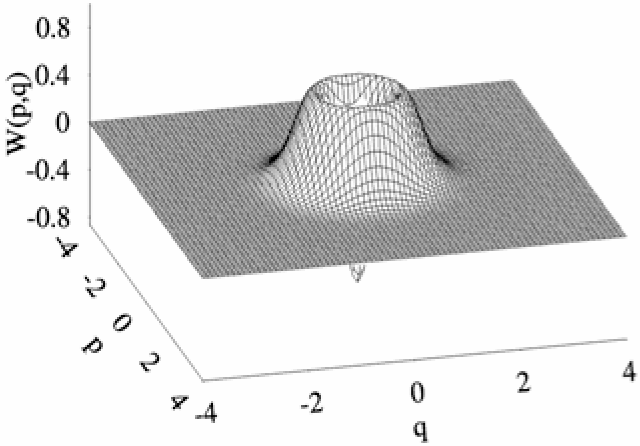


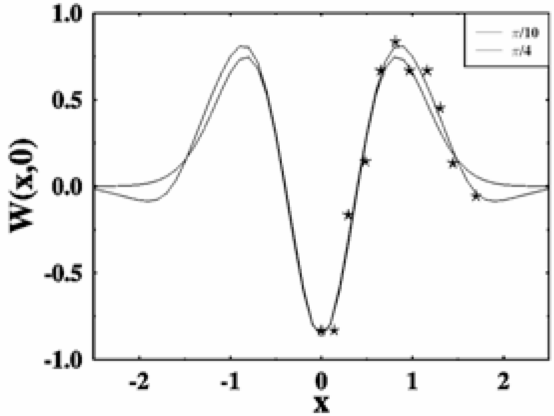
Reconstructed WF of the Fock state



Wigner function of the Fock state II







Qubit

Pure state of a spin -1/2 particle

$$|\psi\rangle = \cos\vartheta/2\,|1\rangle + e^{i\varphi}\sin\vartheta/2\,|0\rangle$$

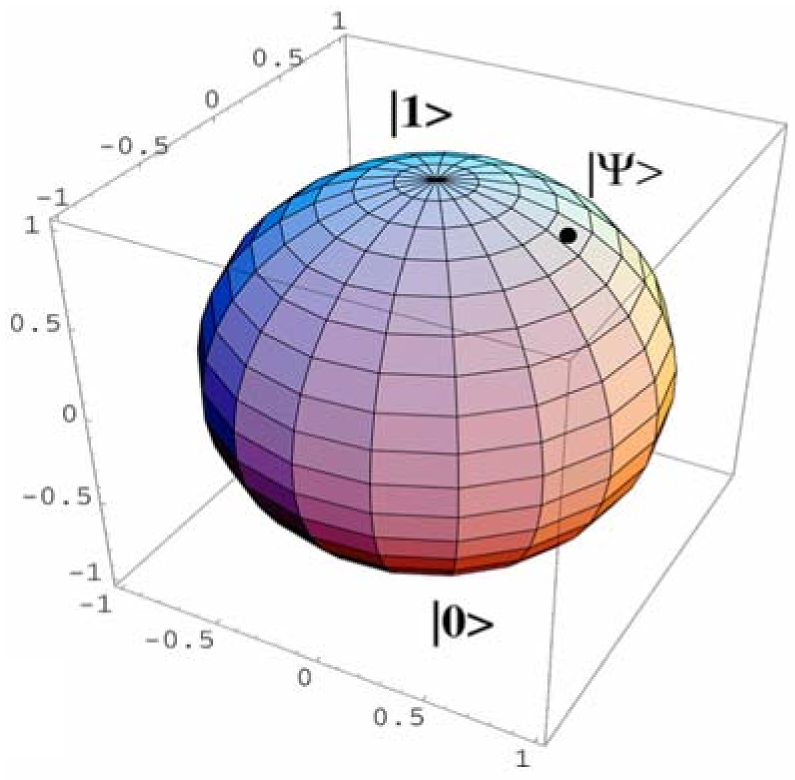
density operator

$$\hat{\rho} = \frac{1}{2}\left(\hat{I} + \vec{n}.\vec{\sigma}\right) = \frac{1}{2}\left(\hat{I} + n_x\hat{\sigma}_x + n_y\hat{\sigma}_y + n_z\hat{\sigma}_z\right)$$

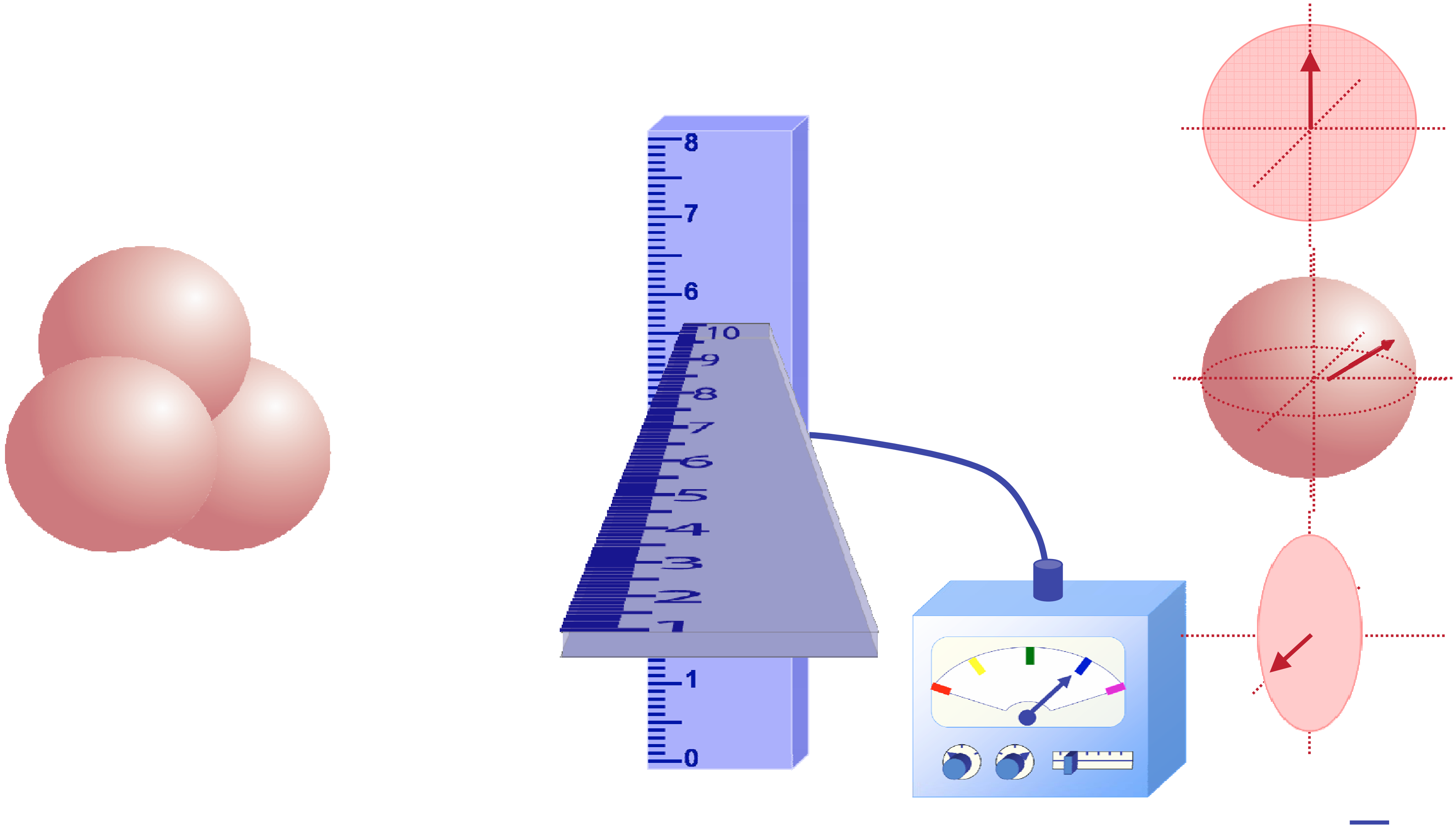
$$\rho = \frac{1}{2}\left(1+\vec{n}\bullet\vec{\sigma}\right) = \frac{1}{2}\begin{pmatrix} 1 \\ n_x \\ n_y \\ n \end{pmatrix} \leftrightarrow \vec{n} = \left(n_x, n_y, n_z\right)$$

State space – Bloch (Poincare) sphere

2-d Hilbert space



Complete State Measurement



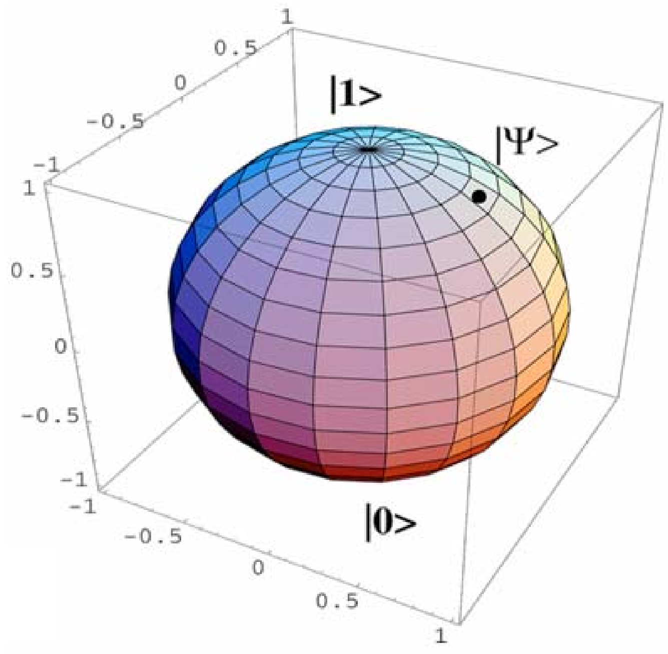
MaxEnt Reconstruction of Qubit

Pure state of a spin -1/2 particle

$$|\psi\rangle = \cos \vartheta/2\,|1\rangle + e^{i\varphi}\sin \vartheta/2\,|0\rangle$$

density operator

$$\hat{\rho} = \frac{1}{2}\left(\hat{I} + \vec{n}.\vec{\sigma}\right) = \frac{1}{2}\left(\hat{I} + n_x\hat{\sigma}_x + n_y\hat{\sigma}_y + n_z\hat{\sigma}_z\right)$$



MaxEnt reconstruction

OL	$\hat{\sigma}_z$	$\hat{\sigma}_x$	$\hat{\sigma}_y$	reconstructed density operator
$\mathcal{O}_{\mathrm{A}}^{(1)}$	•			$\hat{\rho}_{\mathrm{A}} = \frac{1}{2}\left(\hat{I} + n_z\hat{\sigma}_z\right)$
$\mathcal{O}_{\mathrm{B}}^{(1)}$	•	•		$\hat{\rho}_{\mathrm{B}} = \frac{1}{2}\left(\hat{I} + n_z\hat{\sigma}_z + n_x\hat{\sigma}_x\right)$
$\mathcal{O}_{\mathrm{comp}}^{(1)}$	•	•	•	$\hat{\rho}_{\mathrm{comp}} = \frac{1}{2}\left(\hat{I} + n_z\hat{\sigma}_z + n_x\hat{\sigma}_x + n_y\hat{\sigma}_y\right)$

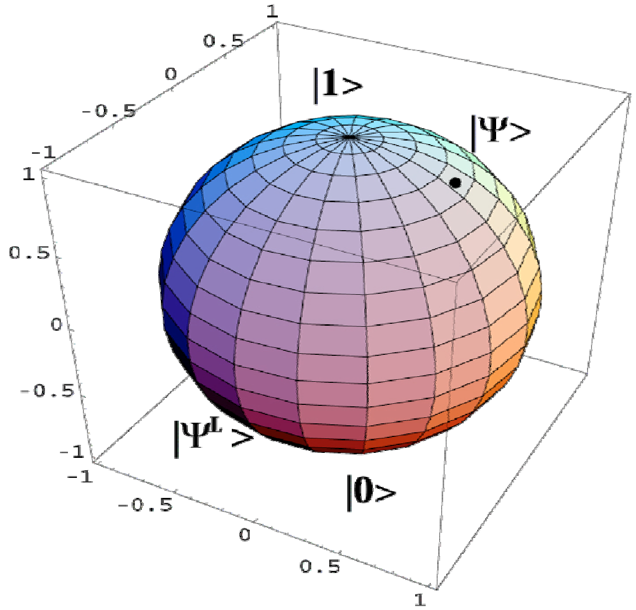
Reconstruction of Qubits

- **Pure state of a qubit**

$$|\psi\rangle = \cos\vartheta/2\,|1\rangle + e^{i\varphi}\sin\vartheta/2\,|0\rangle$$

density operator

$$\hat{\rho} = \frac{1}{2}\left(\hat{I} + \vec{n}.\vec{\sigma}\right) = \frac{1}{2}\left(\hat{I} + n_x\hat{\sigma}_x + n_y\hat{\sigma}_y + n_z\hat{\sigma}_z\right)$$



1) exact meanvalues – infinite ensembles

2) “What is the best a posteriori estimation of a quantum state when a measurement is performed on a finite (arbitrary small) number of elements of the ensemble?”

V.Bužek, G.Drobný, R.Derka, G.Adam, and H.Wiedeman: *Chaos, Solitons & Fractals* 10, 981 (1999)

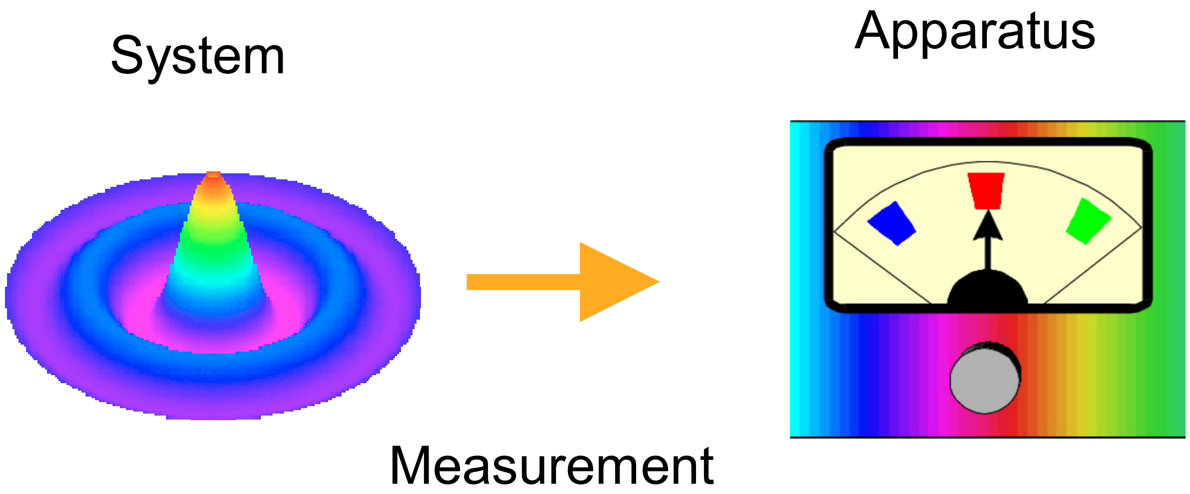
Quantum Clickology

- Measurement: conditional distribution on a discrete state space of the aparatus *A*: $\hat{O}$ observables with eigenvalues λ_j

$$p\left(\hat{O},\lambda_i \middle| \hat{\rho}\right) = \mathrm{Tr}\left(\hat{P}_{\lambda_i,\hat{O}}\,\hat{\rho}\right)$$

- a priori distribution $p_0\left(\hat{\rho}\right)$ on the state space of the system
- joint probability distribution

$$p\left(\hat{O},\lambda_i;\hat{\rho}\right) = p\left(\hat{O},\lambda_i \middle| \hat{\rho}\right) p_0\left(\hat{\rho}\right)$$



Quantum Bayesian inference

- Bayesian inversion from distribution on *A* to distribution on Ω

$$p\left(\hat{\rho}\middle|\hat{O},\lambda_i\right)=\frac{p\left(\hat{O},\lambda_i\middle|\hat{\rho}\right)p_0\left(\hat{\rho}\right)}{\int_\Omega p\left(\hat{O},\lambda_i;\hat{\rho}\right)\mathrm{d}\,\Omega}$$

- Reconstructed density operator given the result λ_i

$$\hat{\rho}_{est}=\int_\Omega \hat{\rho}\left(\vartheta,\varphi\right)p\left(\hat{\rho}\middle|\hat{O},\lambda_i\right)\mathrm{d}_\Omega$$

- $\mathrm{d}_\Omega -$ invariant integration measure

K.R.W. Jones, *Ann. Phys. (N.Y.)* 207, 140 (1991)
V.Bužek, R.Derka, G.Adam, and P.L.Knight, *Annals of Physics (N.Y.)*, 266, 454 (1998)

Single-qubit measurement

- Prior knowledge: state is pure

$$p_0(\hat{\rho}) = const.$$

- Density operator

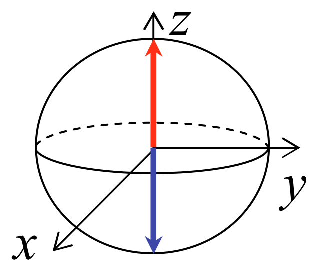
$$\hat{\rho} = \frac{1}{2}\left(\hat{I} + \sin\vartheta\cos\varphi\hat{\sigma}_x + \sin\vartheta\sin\varphi\hat{\sigma}_y + \cos\vartheta\hat{\sigma}_z\right)$$

- Invariant measure

$$\mathrm{d}_\Omega = \frac{1}{4\pi}\sin\vartheta\,\mathrm{d}\,\vartheta\,\mathrm{d}\,\varphi$$

- Projectors on eigenvectors of the apparatus

$$\hat{O}_z = \frac{1}{2}\left(\hat{I} + \lambda\hat{\sigma}_z\right); \quad \lambda = \pm 1$$



- Distribution on Ω

$$p\left(\hat{\rho}\middle|\hat{O},\lambda_i = +1\right) = \tfrac{1}{2}(1+\cos\vartheta)$$

$$\hat{\rho}_{est} = \int_\Omega \hat{\rho}(\vartheta,\varphi)\, p\left(\hat{\rho}\middle|\hat{O},\lambda\right)\mathrm{d}_\Omega = \frac{1}{2}\left(\hat{I} + \frac{1}{3}\hat{\sigma}_z\right)$$

EXAMPLE

	$\hat{\sigma}_z$	$\hat{\sigma}_x$	$\hat{\sigma}_y$	$\hat{\rho}$ **via pure-state reconstruction**	S	$\hat{\rho}$ **via mixture-state reconstruction**	S
1.	$\uparrow$			$\frac{1}{2}[\hat{1} + \frac{1}{3}\hat{\sigma}_z]$	0.637	$\frac{1}{2}[\hat{1} + \frac{1}{5}\hat{\sigma}_z]$	0.673
2.	$\uparrow^4$			$\frac{1}{2}[\hat{1} + \frac{2}{3}\hat{\sigma}_z]$	0.451	$\frac{1}{2}[\hat{1} + \frac{1}{2}\hat{\sigma}_z]$	0.562
3.	$\uparrow^5\downarrow$			$\frac{1}{2}[\hat{1} + \frac{1}{2}\hat{\sigma}_z]$	0.562	$\frac{1}{2}[\hat{1} + \frac{2}{5}\hat{\sigma}_z]$	0.611
4.	$\uparrow^{10}\downarrow^2$			$\frac{1}{2}[\hat{1} + \frac{4}{7}\hat{\sigma}_z]$	0.520	$\frac{1}{2}[\hat{1} + \frac{1}{2}\hat{\sigma}_z]$	0.562
5.	$\uparrow^{15}\downarrow^3$			$\frac{1}{2}[\hat{1} + \frac{3}{5}\hat{\sigma}_z]$	0.501	$\frac{1}{2}[\hat{1} + \frac{6}{11}\hat{\sigma}_z]$	0.536
6.	$\uparrow$	$\downarrow$		$\frac{1}{2}[\hat{1} - \frac{1}{3}\hat{\sigma}_x + \frac{1}{3}\hat{\sigma}_z]$	0.578	$\frac{1}{2}[\hat{1} - \frac{1}{5}\hat{\sigma}_x + \frac{1}{5}\hat{\sigma}_z]$	0.653
7.	$\uparrow^4$	$\uparrow^3\downarrow$		$\frac{1}{2}[\hat{1} + \frac{10}{37}\hat{\sigma}_x + \frac{26}{37}\hat{\sigma}_z]$	0.374	$\frac{1}{2}[\hat{1} + \frac{68}{309}\hat{\sigma}_x + \frac{158}{309}\hat{\sigma}_z]$	0.529
8.	$\uparrow^5\downarrow$	$\uparrow^4\downarrow^2$		$\frac{1}{2}[\hat{1} + \frac{704}{2601}\hat{\sigma}_x + \frac{1460}{2601}\hat{\sigma}_z]$	0.484	$\frac{1}{2}[\hat{1} + \frac{218}{1105}\hat{\sigma}_x + \frac{464}{1105}\hat{\sigma}_z]$	0.581
9.	$\uparrow^{10}\downarrow^2$	$\uparrow^8\downarrow^4$		$\frac{1}{2}[\hat{1} + \frac{1599844}{5073971}\hat{\sigma}_x + \frac{3143928}{5073971}\hat{\sigma}_z]$	0.427	$\frac{1}{2}[\hat{1} + \frac{513984}{2093401}\hat{\sigma}_x + \frac{1083360}{2093401}\hat{\sigma}_z]$	0.519
10.	$\uparrow$	$\downarrow$	$\uparrow$	$\frac{1}{2}[\hat{1} - \frac{1}{3}\hat{\sigma}_x + \frac{1}{3}(\hat{\sigma}_y + \hat{\sigma}_z)]$	0.518	$\frac{1}{2}[\hat{1} - \frac{1}{5}\hat{\sigma}_x + \frac{1}{5}(\hat{\sigma}_y + \hat{\sigma}_z)]$	0.632
11.	$\uparrow^4$	$\uparrow^3\downarrow$	$\uparrow^4$	$\frac{1}{2}[\hat{1} + \frac{831}{3503}\hat{\sigma}_x + \frac{2026}{3503}(\hat{\sigma}_y + \hat{\sigma}_z)]$	0.264	$\frac{1}{2}[\hat{1} + \frac{1051}{5253}\hat{\sigma}_x + \frac{2382}{5253}(\hat{\sigma}_y + \hat{\sigma}_z)]$	0.446
12.	$\uparrow^5\downarrow$	$\uparrow^4\downarrow^2$	$\uparrow^5\downarrow$	$\frac{1}{2}[\hat{1} + \frac{47109}{169636}\hat{\sigma}_x + \frac{99310}{169636}(\hat{\sigma}_y + \hat{\sigma}_z)]$	0.236	$\frac{1}{2}[\hat{1} + \frac{279193}{1446325}\hat{\sigma}_x + \frac{593708}{1446325}(\hat{\sigma}_y + \hat{\sigma}_z)]$	0.492
13.	$\uparrow^{10}\downarrow^2$	$\uparrow^8\downarrow^4$	$\uparrow^{10}\downarrow^2$	$\frac{1}{2}[\hat{1} + \frac{1222748838}{4026213681}\hat{\sigma}_x + \frac{2532792812}{4026213682}(\hat{\sigma}_y + \hat{\sigma}_z)]$	0.135	$\frac{1}{2}[\hat{1} + \frac{250224710127}{1073523481830}\hat{\sigma}_x + \frac{531888078934}{1073523481830}(\hat{\sigma}_y + \hat{\sigma}_z)]$	0.388
14.	$\uparrow^3\downarrow$	$\uparrow^2\downarrow^2$	$\uparrow^2\downarrow^2$	$\frac{1}{2}[\hat{1} + \frac{101}{161}\hat{\sigma}_z]$	0.481	$\frac{1}{2}[\hat{1} + \frac{413}{1389}\hat{\sigma}_z]$	0.648
15.	$\uparrow^6\downarrow^2$	$\uparrow^4\downarrow^4$	$\uparrow^4\downarrow^4$	$\frac{1}{2}[\hat{1} + \frac{88}{117}\hat{\sigma}_z]$	0.374	$\frac{1}{2}[\hat{1} + \frac{3125918}{8023325}\hat{\sigma}_z]$	0.615
16.	$\uparrow^9\downarrow^3$	$\uparrow^6\downarrow^6$	$\uparrow^6\downarrow^6$	$\frac{1}{2}[\hat{1} + \frac{10642815}{13619371}\hat{\sigma}_z]$	0.345	$\frac{1}{2}[\hat{1} + \frac{57056845292}{134078568484}\hat{\sigma}_z]$	0.600
17.	$\uparrow^{12}\downarrow^4$	$\uparrow^8\downarrow^8$	$\uparrow^8\downarrow^8$	$\frac{1}{2}[\hat{1} + \frac{10875098376}{13696058161}\hat{\sigma}_z]$	0.332	$\frac{1}{2}[\hat{1} + \frac{3073000318516432}{6928263111521097}\hat{\sigma}_z]$	0.591

Multi-qubit Measurement

- **Large *N* limit $\rightarrow$ complete reconstruction**
- **Adaptive measurements**
- **Optimal strategy?**

V.Bužek, R.Derka, G.Adam, and P.L.Knight, *Annals of Physics (N.Y.)*, 266, 454 (1998)

Generalized quantum measurements

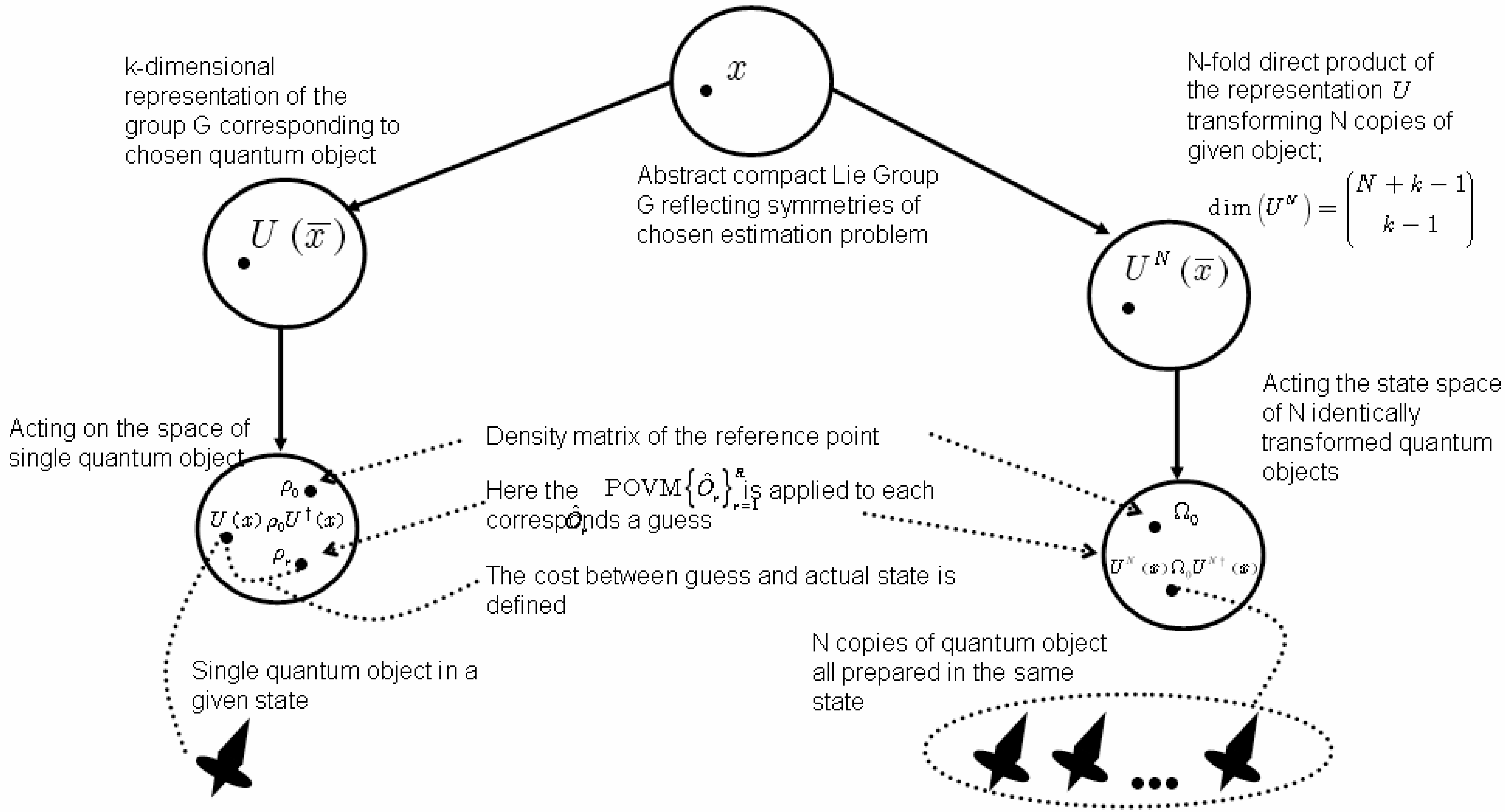
- Positive operators $\hat{O}_r$ – not projectors; $\sum_r \hat{O}_r = \mathbf{1}$

- Mean fidelity *F* via the cost function

$$F = \sum_r \int_\Omega \mathrm{dx}\, \mathrm{Tr}\left[\hat{O}_r \overbrace{U(\mathrm{x})\hat{\rho}_0 U^\dagger(\mathrm{x}) \otimes \ldots \otimes U(\mathrm{x})\hat{\rho}_0 U^\dagger(\mathrm{x})}^{\mathrm{N\ times}}\right] \mathrm{Tr}\left[U(\mathrm{x})\hat{\rho}_0 U^\dagger(\mathrm{x}) U_r \hat{\rho}_0 U_r^\dagger\right]$$

C.W.Holstrom, *Quantum detection and estimation theory* (Academic Press, New York, 1976)
A.S.Holevo, *Probabilistic and statistical aspects of quantum theory* (North Holland, Amsterdam, 1982)
M.G.A.Parisi and J.Rehacek, *Quantum estimations*, (Springer, Berlin, 2004)

Optimal quantum measurements



Optimal Reconstructions of Qubits

- average fidelity of estimation

$$F = \frac{N+1}{N+2}$$

- Estimated density operator on average

$$\hat{\rho}_{est} = s\hat{\rho} + \frac{1-s}{2}\hat{I}; \quad s = 2F-1 = \frac{N}{N+2}$$

- Construction of optimal (& finite-dimensional) POVM's – maximize the fidelity *F*
- POVM via von Neumann projectors – Naimark theorem
- Optimal decoding of information
- Optimal preparation of quantum systems
- Recycling of q-information

R.Derka, V.Bužek, and A.K.Ekert, *Phys. Rev. Lett* 80, 1571 (1998)
V.Bužek, R.Derka, and S.Massar, *Phys. Rev. Lett.* 82, 2207 (1999)
V.Bužek, P.L.Knight, and N.Imoto, *Phys. Rev. A* 62, 062309 (2000)

Black box Problem

- Having a black box (with no memory) processing one qubit in a time, how can we determine this channel?

C.W.Holstrom, *Quantum detection and estimation theory* (Academic Press, New York, 1976)
A.S.Holevo, *Probabilistic and statistical aspects of quantum theory* (North Holland, Amsterdam, 1982)
J.F.Poyatos and J.I.Cirac, PRL 78, 390 (1997)
V.Buzek, PRA 58, 1723 (1998).

General operations (maps, channels)

- The density operator

$$\rho = \frac{1}{2}\left(1 + \vec{r} \bullet \vec{\sigma}\right) = \frac{1}{2}\begin{pmatrix} 1 \\ x \\ y \\ z \end{pmatrix} \leftrightarrow \vec{r} = \left(x, y, z\right)$$

- The general operation is an affine transformation of Bloch sphere

$$\vec{r} \to \vec{r}' = T\vec{r} + \vec{t}$$

$$\rho' = \varepsilon\left[\rho\right] = \sum_l A_l \rho \, A_l^\dagger$$

$$\varepsilon = \begin{pmatrix} 1 & 0 \\ \vec{t} & \vec{T} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & \alpha_1 & \alpha_2 & \alpha_3 \\ y & \beta_1 & \beta_2 & \beta_3 \\ z & \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}$$

- With $x^2 + y^2 + z^2 \le 1$ and $\forall j \quad \left(x - \alpha_j\right) + \left(y - \beta_j\right) + \left(z - \gamma_j\right) \le 1$

Complete Positivity

- Every guess must be completely positive – in general it is hard to achieve analytically
- Check is done by applying an operation in the form

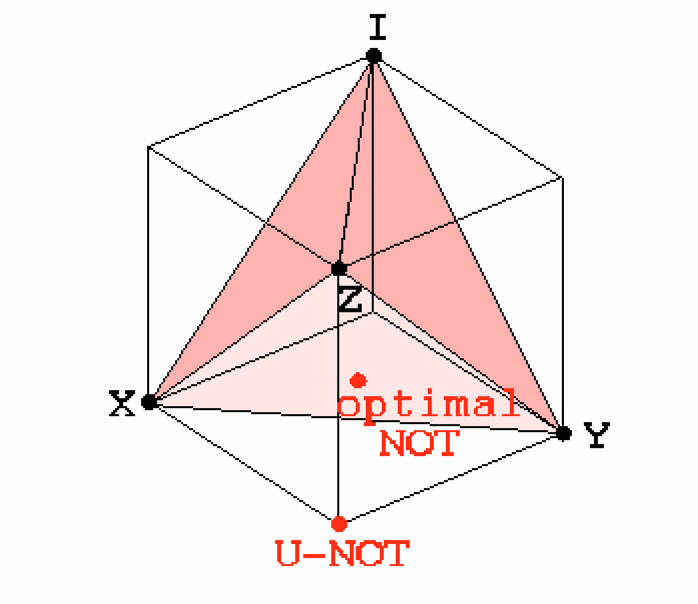
$$\Omega = I \otimes \varepsilon$$

on to the maximally entangled state $$|\phi_+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle\right)$$

Unital operations

- Displacement $\vec{t} = 0$
- Affine transformation specified as
- Positivity $\forall j \quad |\lambda_j| \leq 1$
- Complete positivity $|\lambda_1 \pm \lambda_2| \leq |1 \pm \lambda_3|$

$$\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & \vec{T} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix}$$



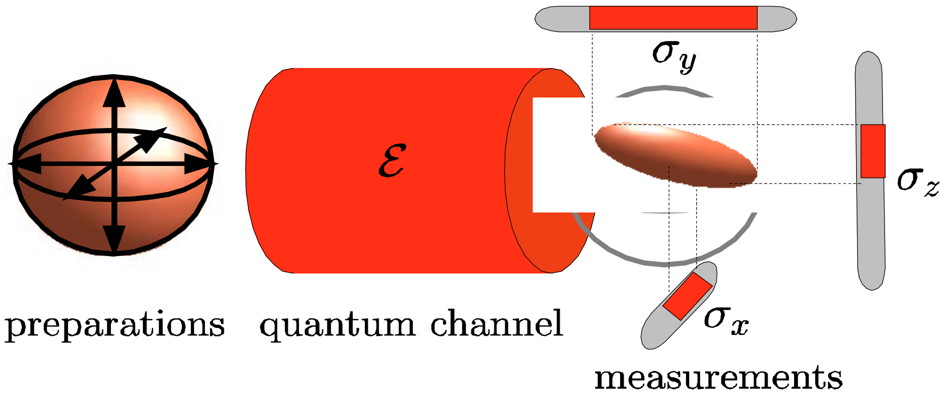
Unital CP maps are embeded in the set of all positive unital maps (cube). The CP maps form a tetrahedron with four unitary transformations in its corners (extremal points) I,x,y,z corresponding to the Pauli sigma-matrices.

The unphysical U-NOT operation $\lambda_1 = \lambda_2 = \lambda_3 = -1$ **and its optimal completely positive approximation quantum universal NOT gate** $\lambda_1 = \lambda_2 = \lambda_3 = -1/3$ **are shown.**

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \rightarrow |\psi^{\perp}\rangle = \beta^*|0\rangle - \alpha^*|1\rangle$$

M.Ziman & V.Buzek, PRA 72, 022110 (2005).

Process reconstruction



Complete reconstruction

- Linearly independent states
- Singlet state

Incomplete reconstruction - estimation

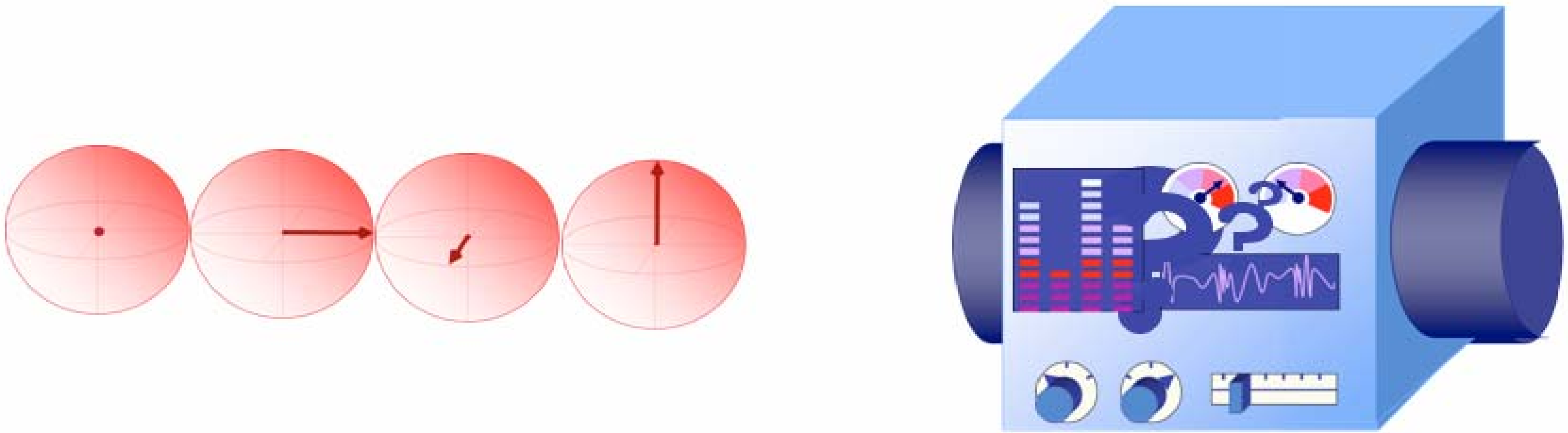
- The set of states is not complete
- Incomplete measurements are performed on outputs
- Each test state is represented by a finite ensemble

It is assumed that the preparation of test states is perfect

M.Ziman, M.Plesch, V.Bužek, & P.Stelmachovic, PRA 72, 022106 (2005)

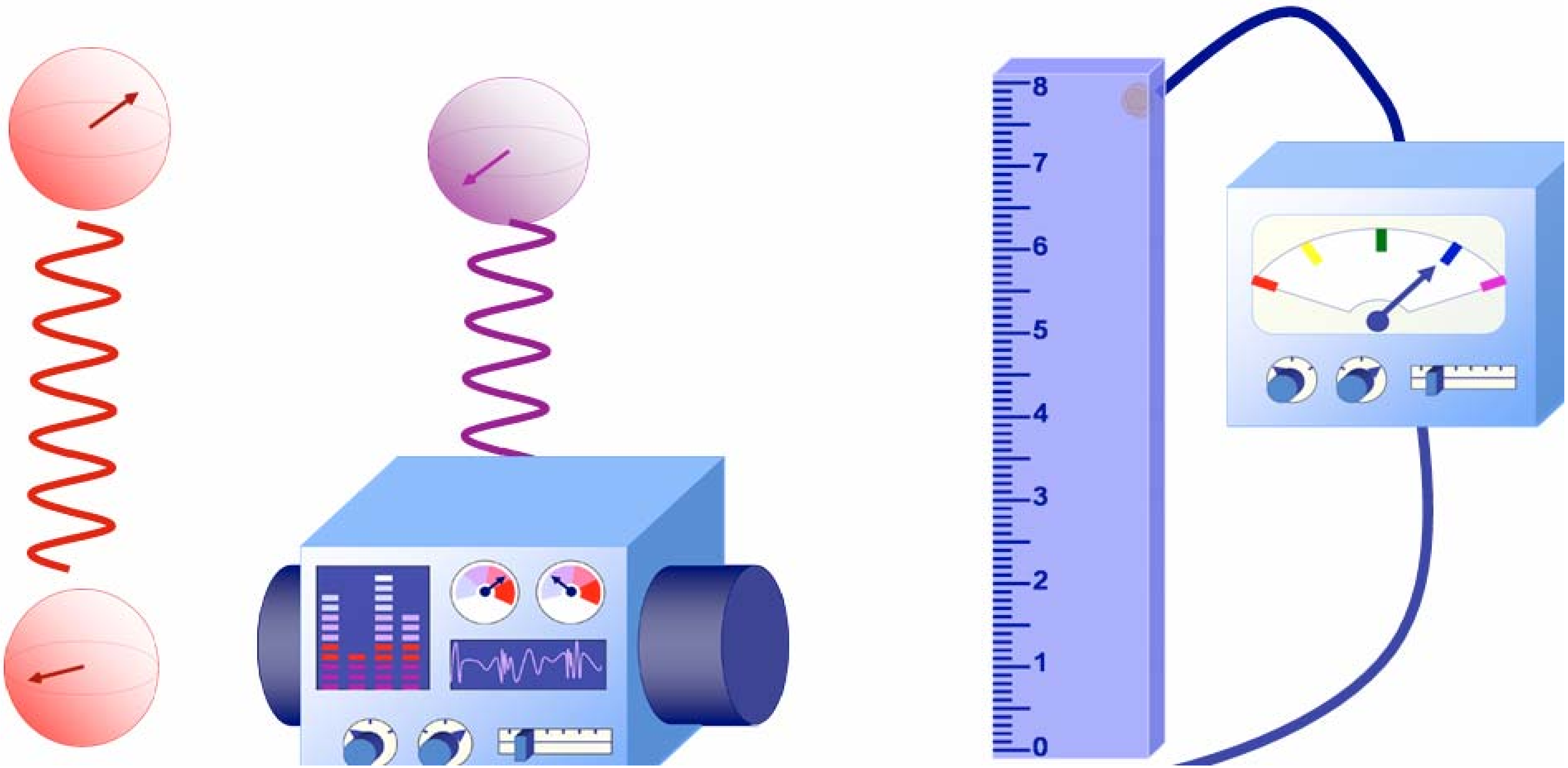
Complete Estimation

- For a complete estimation one needs four different states, which are linearly independent.



- Linearity of QM implies that the channel is determined by the action on a complete set of d^2 linearly independent basis (test) states
- Estimation of $d^2(d^2-1)$ parameters

Entangled States



Entangled States

- Using one completely entangled state in the form

$$|\phi_+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle\right)$$

one gets

$$\Omega = \left(\varepsilon \otimes I\right)\left[|\phi_+\rangle\langle\phi_+|\right] = \frac{1}{d}\sum_{j,k} \varepsilon\left[e_{jk}\right] \otimes e_{jk}$$

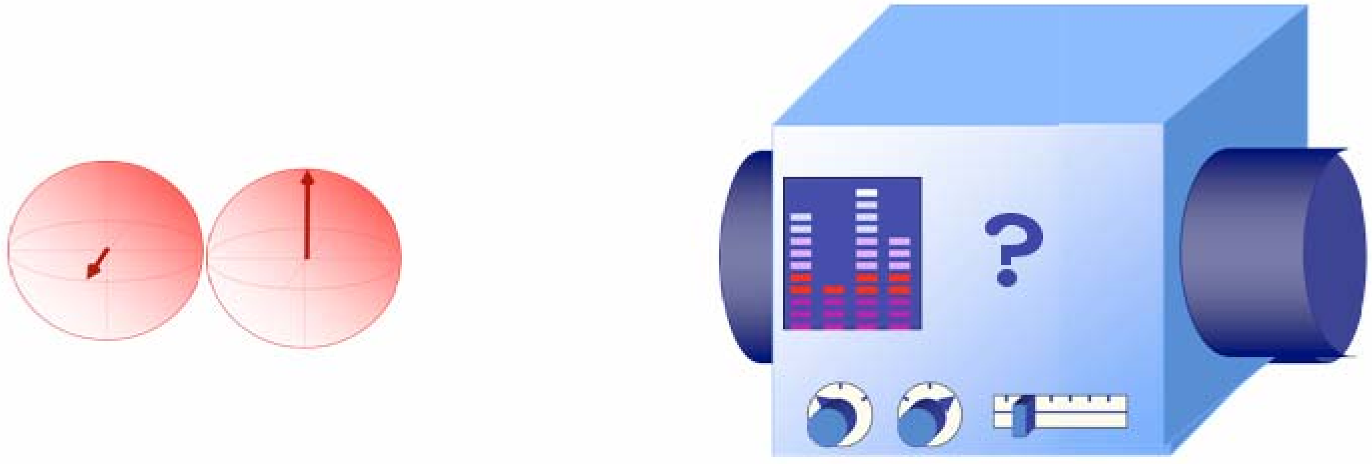
with $e_{jk} = |j\rangle\langle k|$

- By estimation of the output state we are able to completely determine the operation itself

G.M.D'Ariano & P.LoPresti, PRL 86, 4195 (2001)

Incomplete Estimation

- The case of only two different input states:



Incomplete information: Strategy

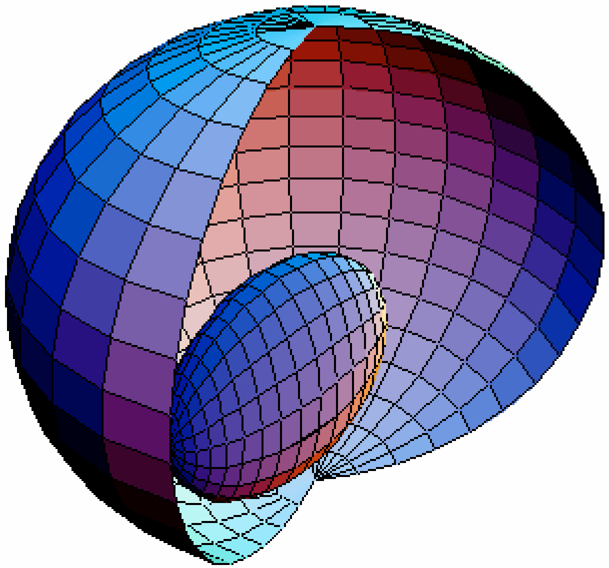
- The problem can be divided in to two relatively separate parts:
  - How is the identity transformed?
  - How are the remaining three pure basis vectors transformed?
- Priorities:
  1. Identity and pure states MUST be transformed according the existing data
  2. Transformation MUST be completely positive
  3. The Identity is transformed to the identity or as close as possible (measuring in distance)
  4. The remaining pure states are transformed to the same state as identity or as close as possible (measuring in distance)

Example: Specific Channel

- Let us assume a specific transformation – map, channel

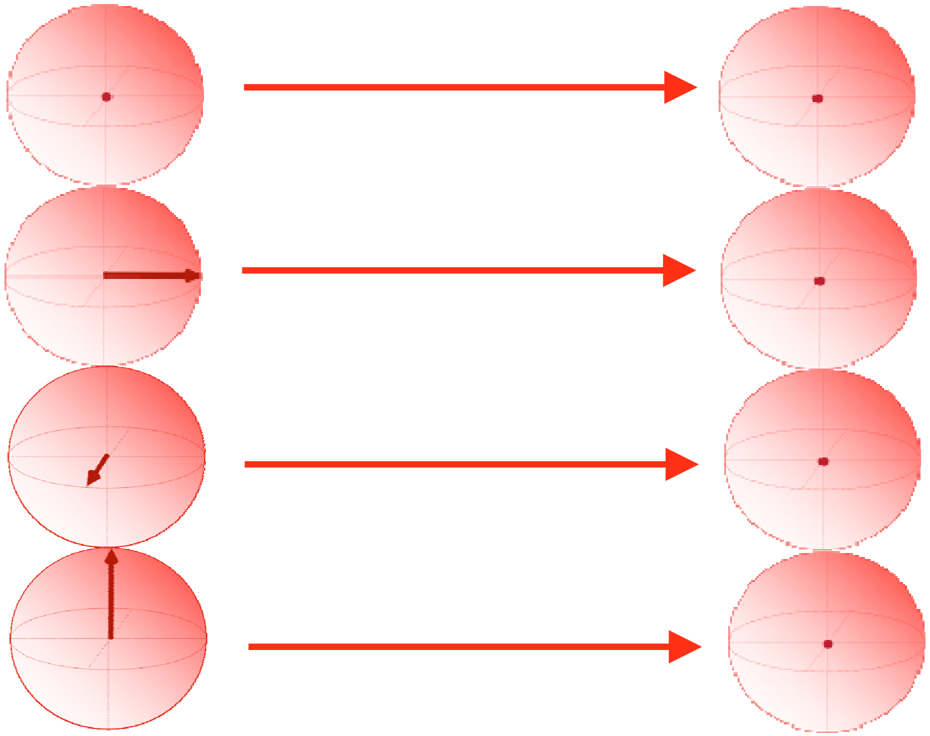
$$\rho' = \varepsilon[\rho] = \sum_l A_l \rho \, A_l^\dagger$$

$$\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 0.2 & -0.1 & 0.1 \\ 0 & 0.2 & 0 & -0.3 \\ 0 & 0 & 0.3 & 0.3 \end{pmatrix}$$



M.Ziman, M.Plesch, V.Buzek, & P.Stelmachovic, PRA 72, 022106 (2005).
M.Ziman, M.Plesch, & V.Buzek, EPJD 32, 215 (2005)
M.Ziman, M.Plesch, & V.Buzek, Foundations of Physics (2006)

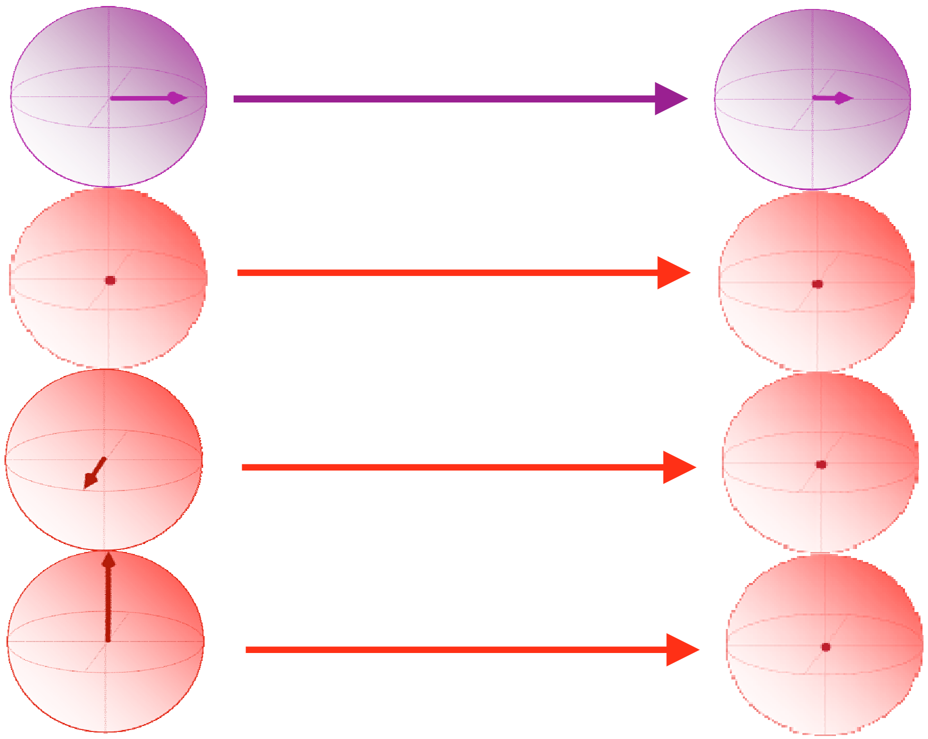
No Known State

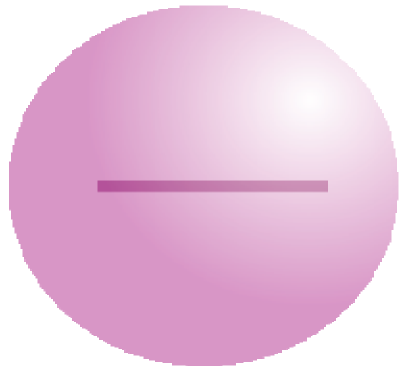


$$\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho' = \varepsilon[\rho] \equiv A[\rho] = \frac{1}{2} I$$

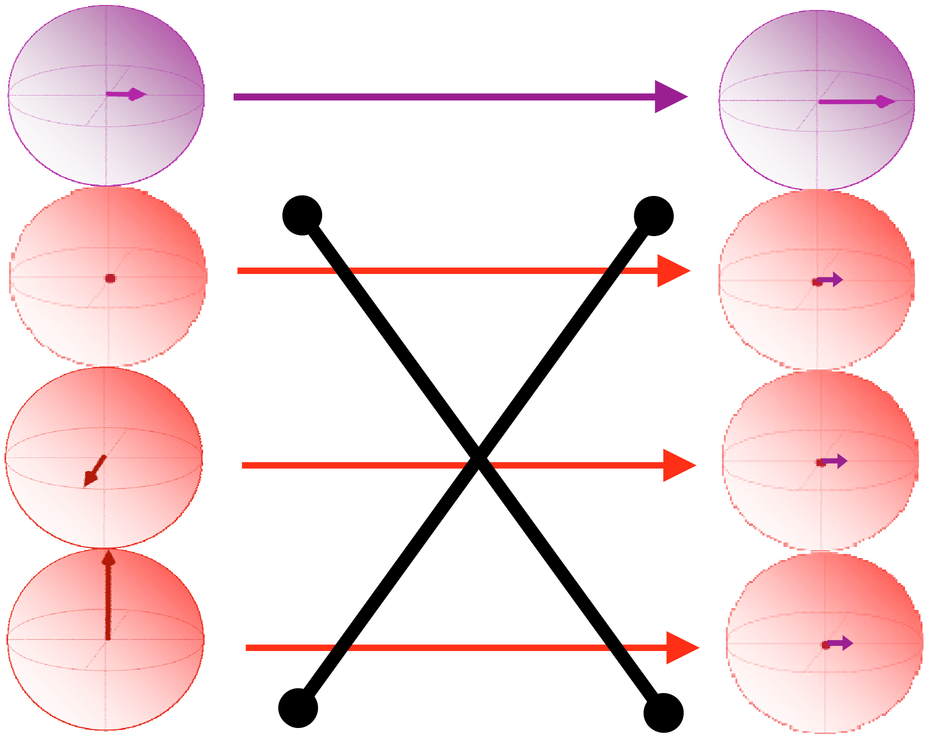
One Known State

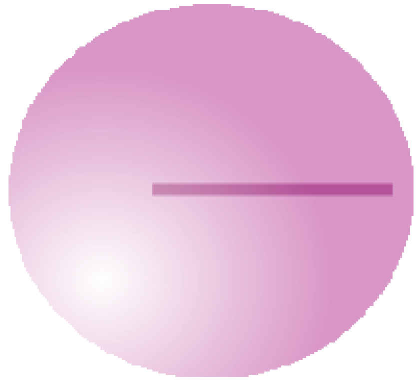




$$\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

One Known State



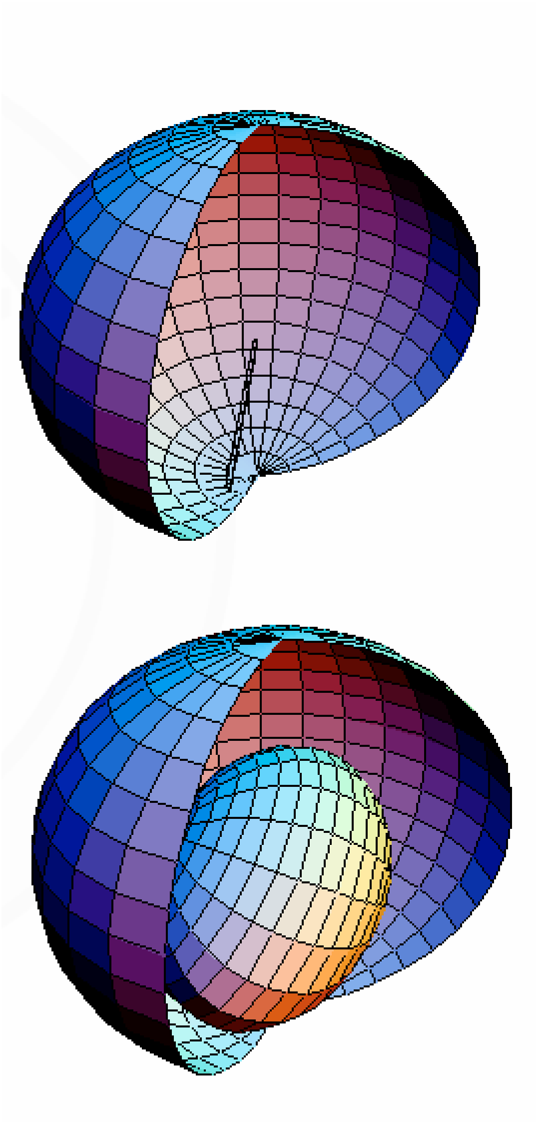


$$\mathcal{E} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{E} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ R & x-R & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

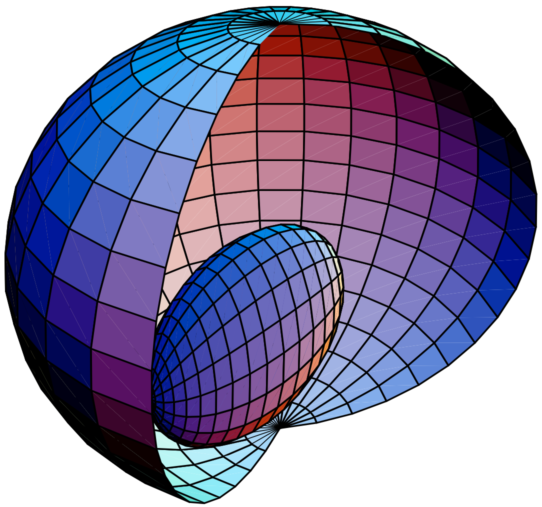
Two Known States

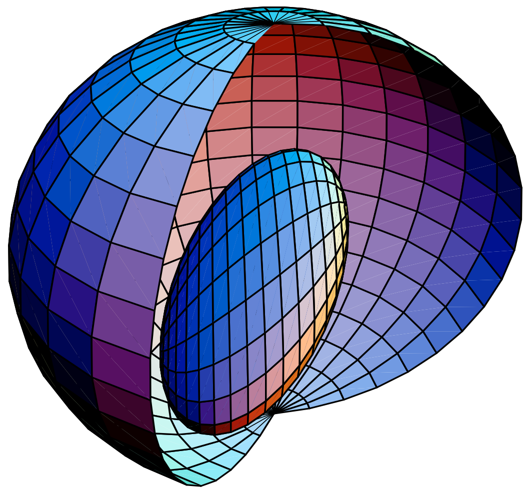
- If the states sum to identity, strategy is the same as in the previous case
- If not, a rather complex situation arises. In some cases…
  - … the Identity is not transformed and the third, perpendicular pure state is transformed to the same state as identity



Three Known States

- Here essentially enough data are already available
- If the three states sum to identity, we turn back to the previous case
- In other case, only numerical solutions are possible. The only open question is the transformation of identity, then all the perpendicular pure states are given

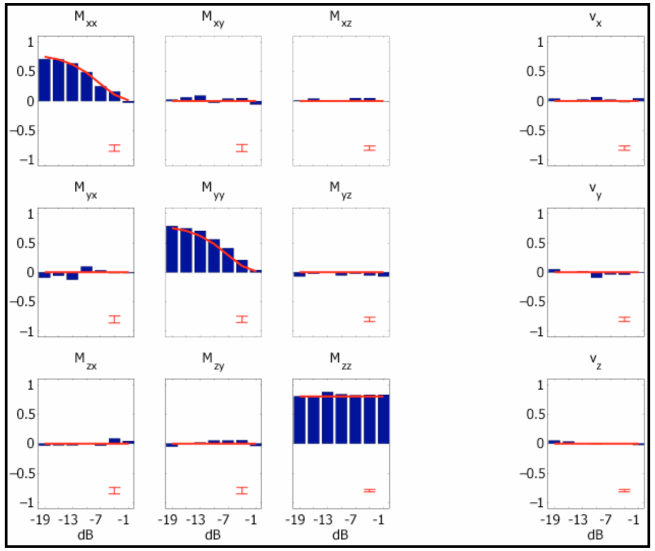


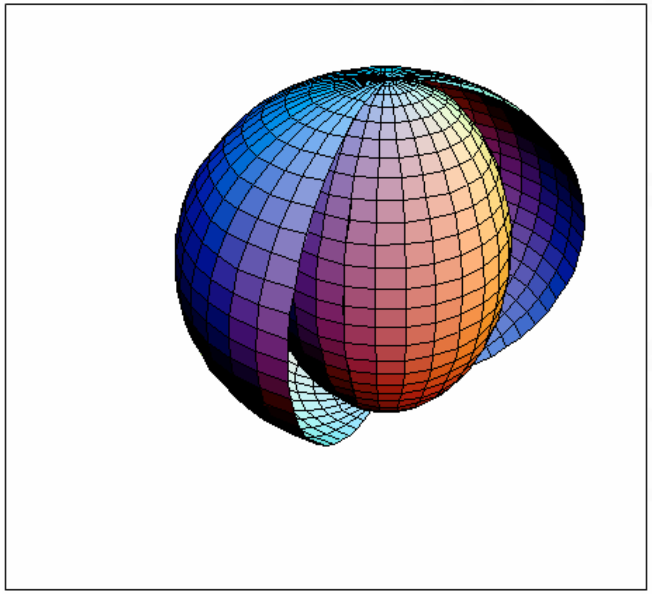


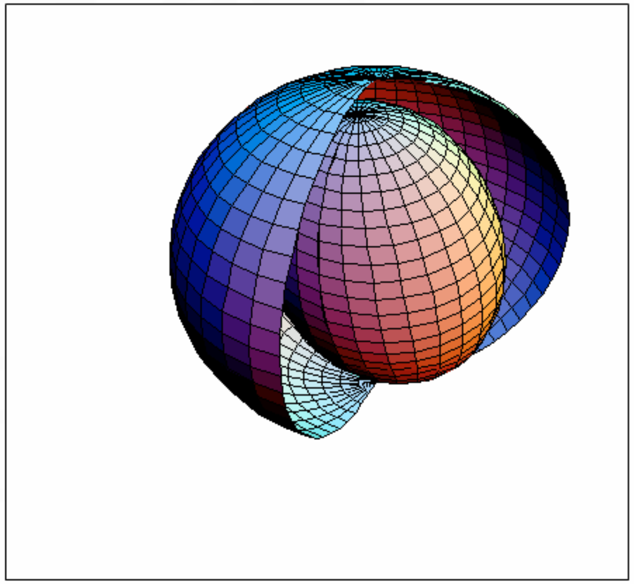
Experimental Data

- Data from the group of Ch. Wunderlich were analyzed
- Depolarization channel was expected

$$\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

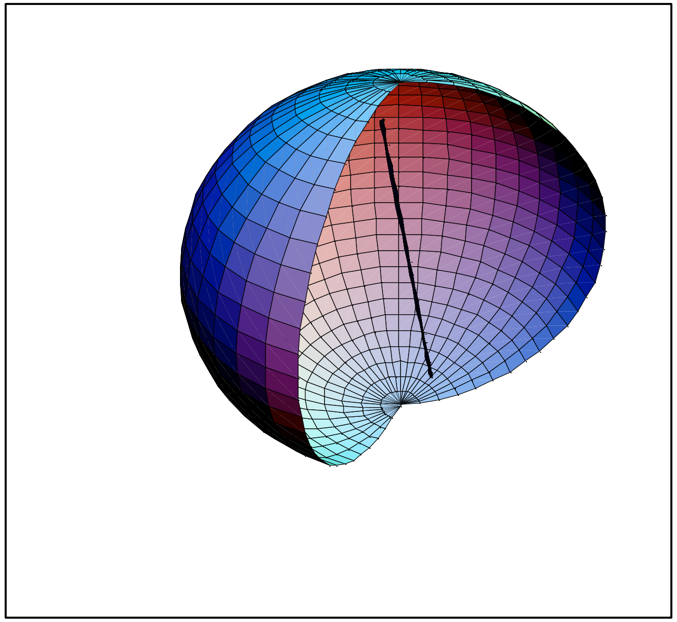




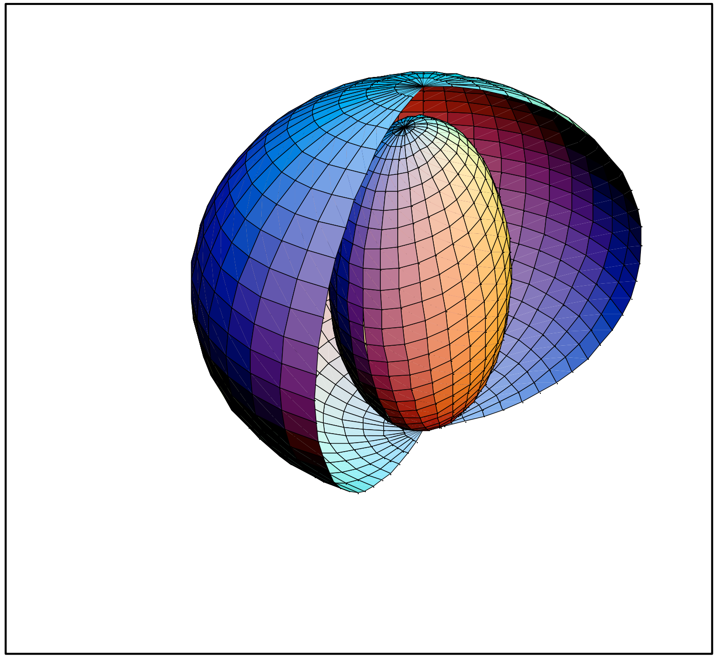


Experimental playground

$\uparrow_z$

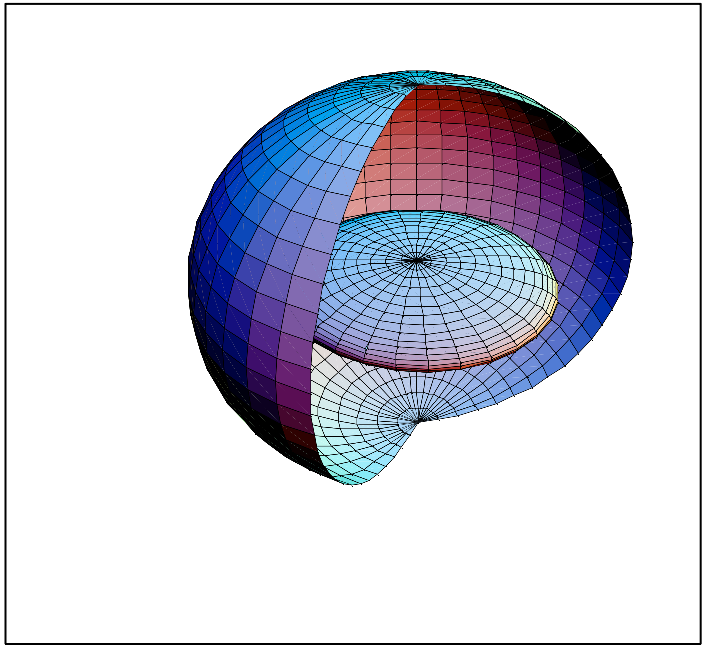


$\uparrow_z, \uparrow_x$

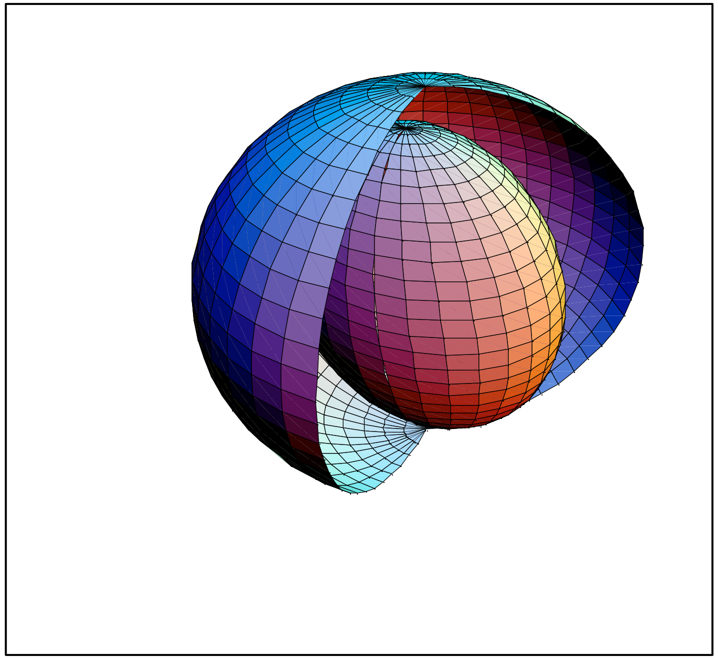


Experimental playground

$\uparrow_y, \uparrow_x$



$\uparrow_z, \uparrow_y$



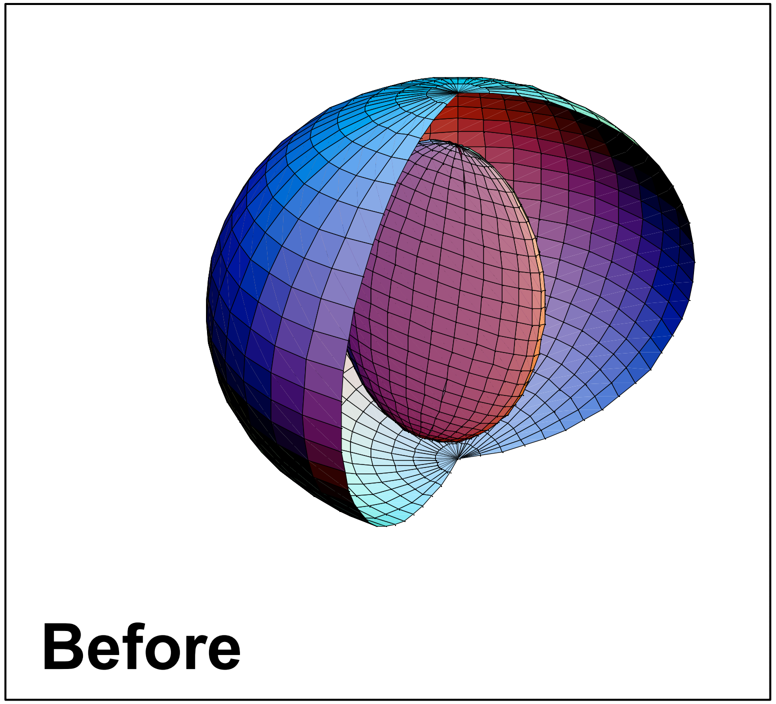
Regularization via noise

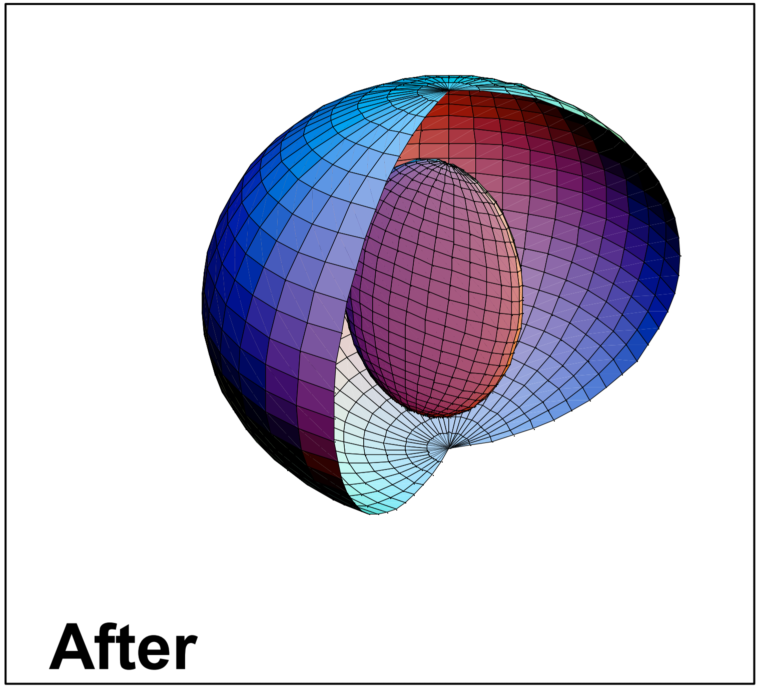
- In some cases, even for complete estimation the data were not consistent - violation of the contractivity condition

$$D(\rho,\zeta) \geq D(\varepsilon[\rho],\varepsilon[\zeta])$$

- in these cases add noise to preserve CP

$$\varepsilon_c = k\varepsilon + (1-k)A = \begin{pmatrix} 1 & 0 \\ k\vec{t} & k\vec{T} \end{pmatrix}$$





Maximum Likelihood

- Completely different approach, how to deal with potentially inconsistent data is to start the analysis from the very beginning – from the actual measurement outcomes – clicks
- Pluses:
  - Always fair and the best physical result
  - No pre – analysis needed
- Minuses:
  - Rather complicated numeric
  - Reliability of the result – post analysis needed

J.Fiurášek, Z.Hradil, Phys. Rev. A 63, 020101(R) (2001)
M.G.A.Parisi and J.Rehacek, *Quantum estimations*, (Springer, Berlin, 2004)

Clickology: Maximum Likelihood

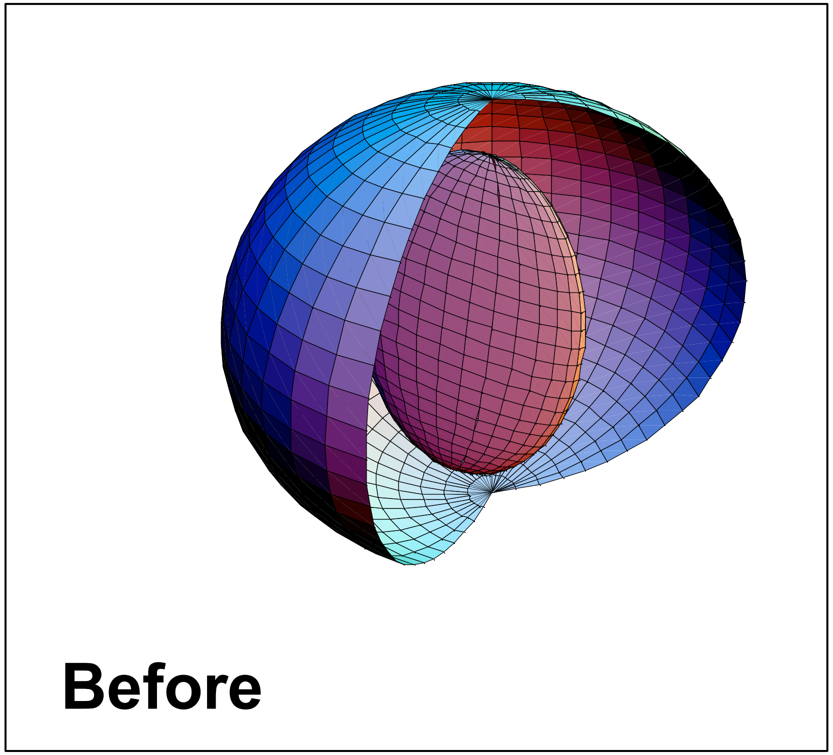
- ML works with finite sets of data, not with infinite ensembles
- In case of quantum operations, the related data are
  - Input state specification ρ_i
  - Measurement direction $|\psi\rangle_i$
  - Measurement outcome (binary) p_i
- We build a functional

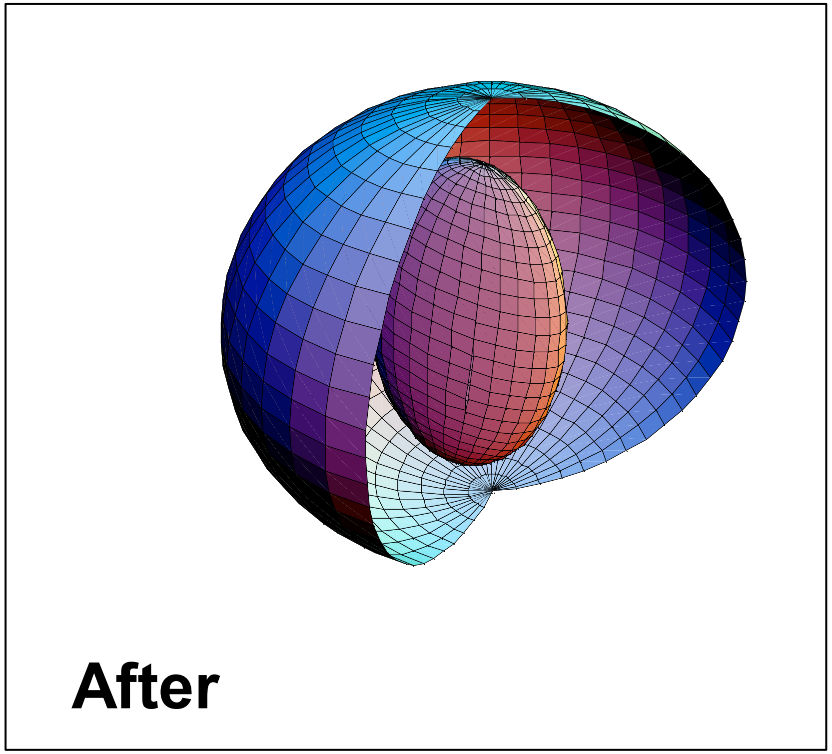
$$F = \prod_i \left[\langle \psi |_i \, \varepsilon(\rho_i) |\psi\rangle_i \, p_i + \left(1 - \langle \psi |_i \, \varepsilon(\rho_i) |\psi\rangle_i\right)(1 - p_i) \right]$$

- The numerical task is to find the ε , for which this functional reaches the maximum (using the logarithm of functional)
- Trace-preservation is obtained automatically from the parameterization, CP has to be checked in the algorithm

Maximum Likelihood on Experimental Data

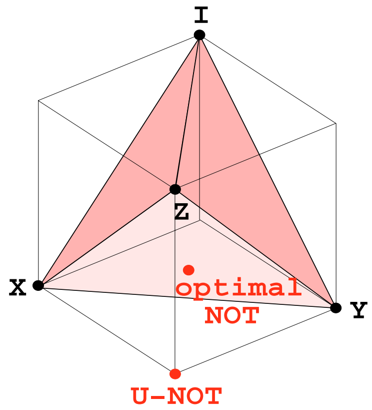
- The maximum likelihood method restores the CP condition in a different way than adding of noise





Approximation of non-physical maps I

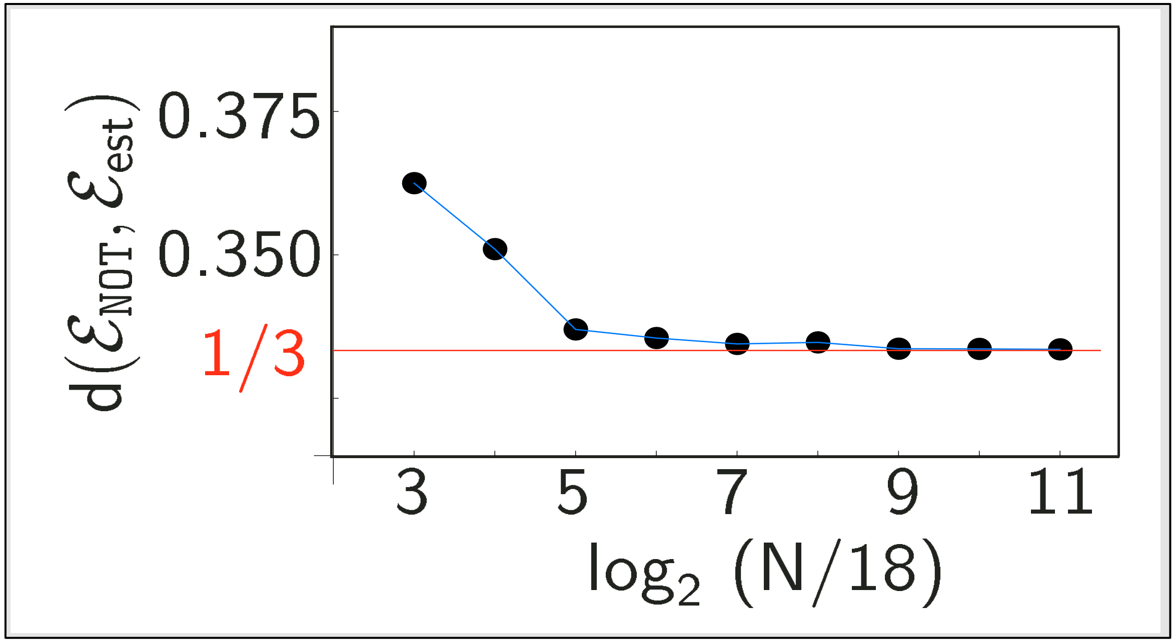
- **Universal NOT gate** $\varepsilon = \mathrm{diag}\,(1,-1,-1,-1)$
- **Best approximation** $\varepsilon = \mathrm{diag}\,(1,-1/3,-1/3,-1/3)$

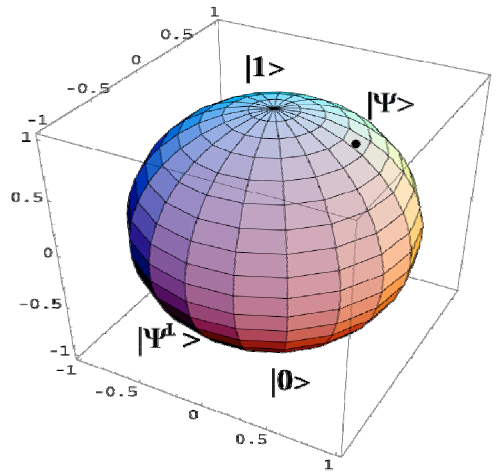


- **6 input states – eigenstates of** σ_j
- **3 measurements** σ_j
- **N=100 x 18 clicks**

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \rightarrow |\psi^{\perp}\rangle = \beta^*|0\rangle - \alpha^*|1\rangle$$

$$\hat{\rho}^{\perp}_{meas} = \frac{1}{3}\hat{\rho}^{\perp} + \frac{1}{3}\hat{I}$$





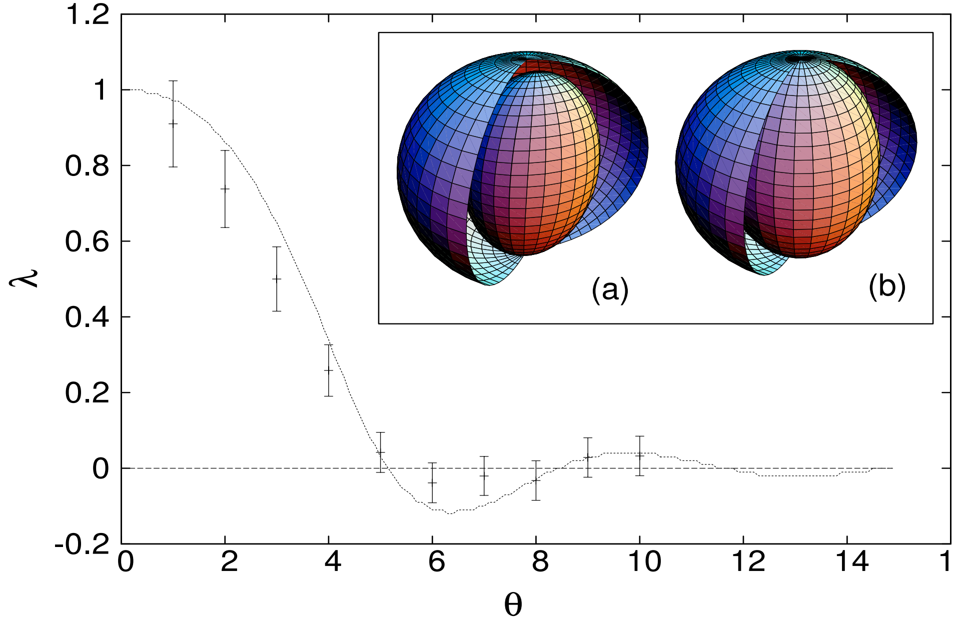
•**M.Ziman, M.Plesch, V.Buzek, P.Stelmachovic, Phys. Rev. A 72, 022106 (2005)**

Approximation of non-physical maps II

- **Nonlinear polarization rotation**
- 1800 input states
- 3 measurements

$$\varepsilon[\rho] = \exp\left(i\frac{\Theta}{2}\langle\sigma_z\rangle_\rho \sigma_z\right)\rho \exp\left(-i\frac{\Theta}{2}\langle\sigma_z\rangle_\rho \sigma_z\right)$$

$$\varepsilon = \mathrm{diag}\,(1,\lambda,\lambda,1)$$



•M.Ziman, M.Plesch, V.Buzek, P.Stelmachovic, Phys. Rev. A 72, 022106 (2005)
•M.Ziman, M.Plesch, V.Buzek, Foundations of Physics (2006)

Conclusions

- reliable reconstruction of states via MaxEnt principle
- incomplete quantum tomography via MaxEnt principle
- quantum Bayesian inference from finite ensembles
- optimal measurements of finite ensembles
- optimal coding and decoding of information
- recycling of quantum information
- Estimation of quantum channels
- Maximum likelihood and physical approximation of non-physical maps

http://www.quniverse.sk/buzek/

Optimal Manipulations with Quantum Information :
Programmable Quantum Processors

06.02.2006

Vladimír Bužek

Flipping a Bit – NOT Gate

0

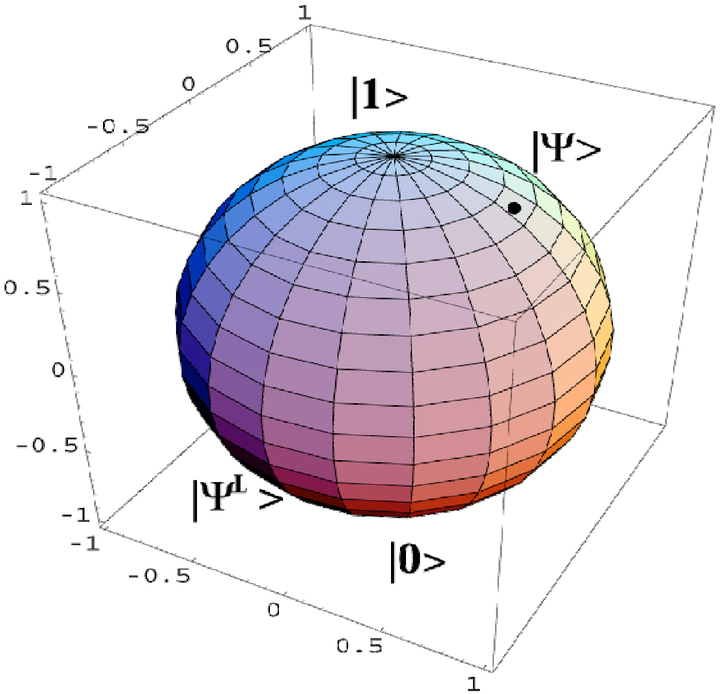
Flipping a Bit – NOT Gate

1

Universal NOT Gate

- **NOT gate in a computer basis:**

$$R|0\rangle = -|1\rangle;\ R|1\rangle = |0\rangle$$



Poincare sphere – state space

$\left|\psi^{\perp}\right\rangle$ **is antipode of** $\left|\psi\right\rangle$

$$\left\langle\psi\mid\psi^{\perp}\right\rangle = 0$$

$$\left|\psi\right\rangle = \alpha\left|0\right\rangle + \beta\left|1\right\rangle \rightarrow \left|\psi^{\perp}\right\rangle = \beta^{*}\left|0\right\rangle - \alpha^{*}\left|1\right\rangle$$

Universal NOT Gate: Problem

$\left|\psi^{\perp}\right\rangle$ is antipode of $\left|\psi\right\rangle$

- Spin flipping is an inversion of the Poincare sphere
- This inversion preserves angels
- The Wigner theorem - spin flip is either unitary or anti-unitary operation
- Unitary operations are equal to proper rotations of the Poincare sphere
- Anti-unitary operations are orthogonal transformations with det=-1
- Spin flip operation is anti-unitary and is not CP
- In the unitary world the ideal universal NOT gate which would flip a qubit in an arbitrary (unknown) state does not exist

Measurement-based vs q-Scenario

Measurement-based scenario: optimally measure and estimate the state then on a level of classical information perform flip and prepare the flipped state of the estimate

Quantum scenario (Stinespring-Kraus theorem) try to find a unitary operation on the qubit and ancillas that at the output generates the best possible approximation of the spin-flipped state. The fidelity of the operation should be state independent (universality of the U-NOT)

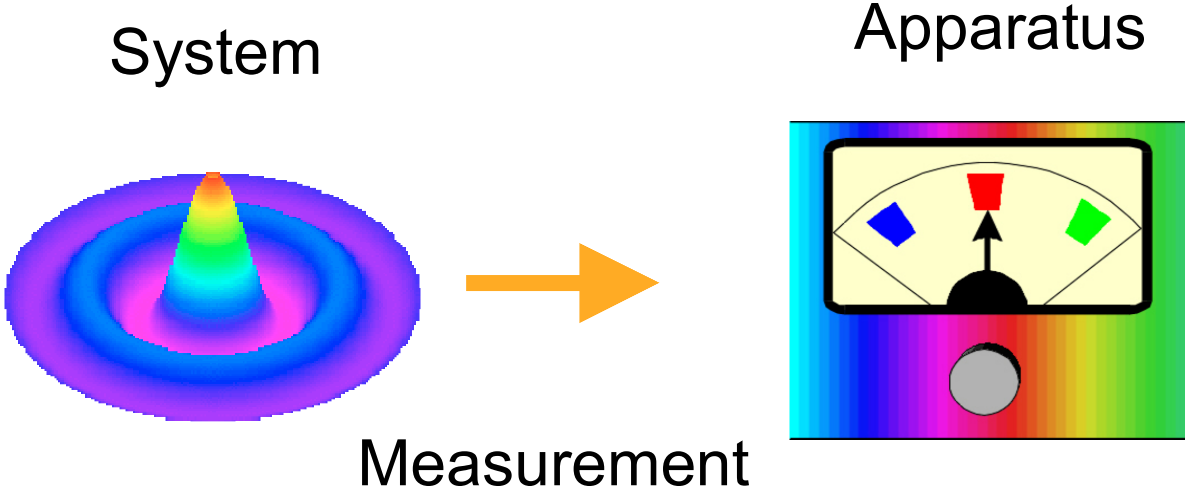
Quantum Clickology

- measurement $\rightarrow$ conditional distribution on a discrete state space of the apparatus A: $\hat{O}$ observables with eigenvalues λ_i

$$p\left(\hat{O},\lambda_i \middle| \hat{\rho}\right) = \mathrm{Tr}\left(\hat{P}_{\lambda_i,\hat{O}}\hat{\rho}\right)$$

- a priori distribution $p_0\left(\hat{\rho}\right)$ on the state space Ω of the system $\rightarrow$ joint probability distribution

$$p\left(\hat{O},\lambda_i;\hat{\rho}\right) = p\left(\hat{O},\lambda_i \middle| \hat{\rho}\right) p_0\left(\hat{\rho}\right)$$



Quantum Bayesian inference

- Bayesian inversion from distribution on A to distribution on Ω

$$p\left(\hat{\rho}\middle|\hat{O},\lambda_i\right) = \frac{p\left(\hat{O},\lambda_i\middle|\hat{\rho}\right)p_0\left(\hat{\rho}\right)}{\int_\Omega p\left(\hat{O},\lambda_i;\hat{\rho}\right)\mathrm{d}\,\Omega}$$

- Reconstructed density operator given the result λ_i

$$\hat{\rho}_{est} = \int_\Omega \hat{\rho}\left(\vartheta,\varphi\right)p\left(\hat{\rho}\middle|\hat{O},\lambda_i\right)\mathrm{d}_\Omega$$

- d_Ω – invariant integration measure

K.R.W. Jones, *Ann. Phys. (N.Y.)* 207, 140 (1991)
V.Bužek, R.Derka, G.Adam, and P.L.Knight, *Annals of Physics (N.Y.)*, 266, 454 (1998)

Optimal Reconstructions of Qubits

- average fidelity of estimation

$$F = \frac{M+1}{M+2}$$

- Estimated density operator on average

$$\hat{\rho}_{est} = s\hat{\rho} + \frac{1-s}{2}\hat{I}$$

$$s = 2F - 1 = \frac{M}{M+2}$$

- Construction of optimal POVM's – maximize the fidelity F

- POVM via von Neumann projectors – Naimark theorem

- Optimal decoding of information

- Optimal preparation of quantum systems

S.Massar and S.Popescu, *Phys. Rev. Lett.* 74, 1259 (1995)
R.Derka, V.Bužek, and A.K.Ekert, *Phys. Rev. Lett* 80, 1571 (1998)

Measurement-based Flipping of Qubit

- Estimated density operator when just a single qubit is available

$$\hat{\rho}_{est} = \frac{1}{3}\hat{\rho} + \frac{1}{3}\hat{I}$$

- Flipping based on this estimation

$$\hat{\rho}^{\perp}_{meas} = \frac{1}{3}\hat{\rho}^{\perp} + \frac{1}{3}\hat{I}$$

A.Holevo ***Probabilistic and Statistical Aspect of Quantum Theory (1982)***

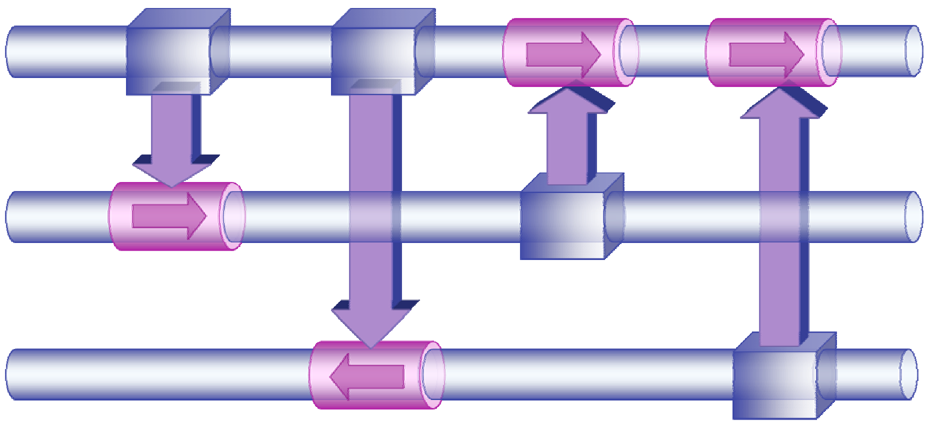
S.Massar and S.Popescu ***Phys. Rev. Lett.*** **74, 1259 (1995)**

Quantum Scenario: Universal NOT Gate

$$\left|\psi\right\rangle_a \rightarrow \left|\psi^{\perp}\right\rangle_a$$

$$\left|\psi\right\rangle_a$$

$$\left|\Xi^{\perp}\right\rangle_{bc}$$



$$\rho_a^{(out)} = \frac{1}{3}\rho^{\perp} + \frac{1}{3}I$$

C-NOT gate: $\left|k\right\rangle\left|l\right\rangle \rightarrow \left|k\right\rangle\left|(l+k) \bmod 2\right\rangle$

$$D_{ab} = \sum_{k,l=0}^{1} \left|k\right\rangle_a \left\langle k\right| \otimes \left|(l+k) \bmod 2\right\rangle_b \left\langle l\right|$$

V.Bužek, M.Hillery, and R.F.Werner, *J. Mod. Opt.* 47, 211 (2000)

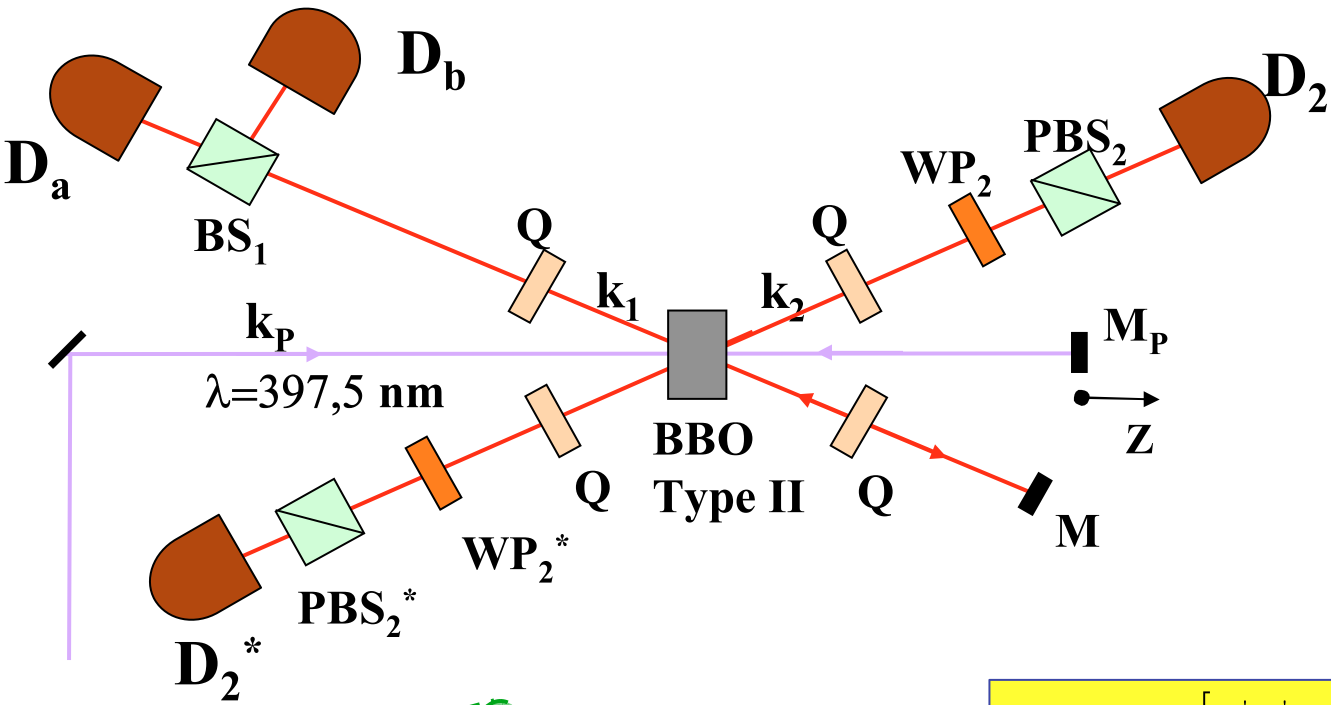
Theorem: Optimal Universal NOT Gate

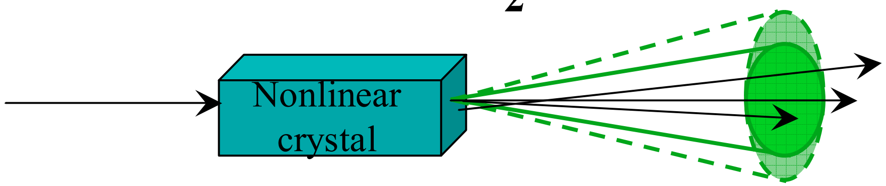
Theorem

Among all completely positive trace preserving maps $T : S\left(H_{+}^{\otimes N}\right) \rightarrow S(H)$
The measurement-based U-NOT scenario attains the highest possible fidelity, namely
$F = (N+1)/(N+2)$.

H.Bechmann-Pasquinucci and N.Gisin, Phys. Rev, A 59, 4238 (1999)
V.Bužek, M.Hillery, and R.F.Werner *Phys. Rev. A* 60, R2626 (1999)
N.Gisin and S.Popescu *Phys. Rev. Lett.* 83, 432 (1999)

U-NOT via Optical Parametric Amplifier





$$H_I = i\chi\hbar\left[a^\dagger_\psi b^\dagger_{\psi^\perp} - a^\dagger_{\psi^\perp} b^\dagger_\psi\right] + h.c.$$

$$\left|\psi\right\rangle = a^\dagger_\psi \left|0,0\right\rangle_{k_1} = \left|1,0\right\rangle_{k_1}$$

C.Simon, G.Weihs, and A.Zeilinger, *Phys. Rev. Lett.* 84, 2993 (2000)
A.Lamas-Linares, C.Simon, J.C.Howell, and D.Bouwmeester, *Science* 296, 712 (2002).
F.DeMartini, V.Buzek, F.Sciarrino, and C.Sias, *Nature* 419, 815 (2002)

Motivation: Bell Telephone & FLASH

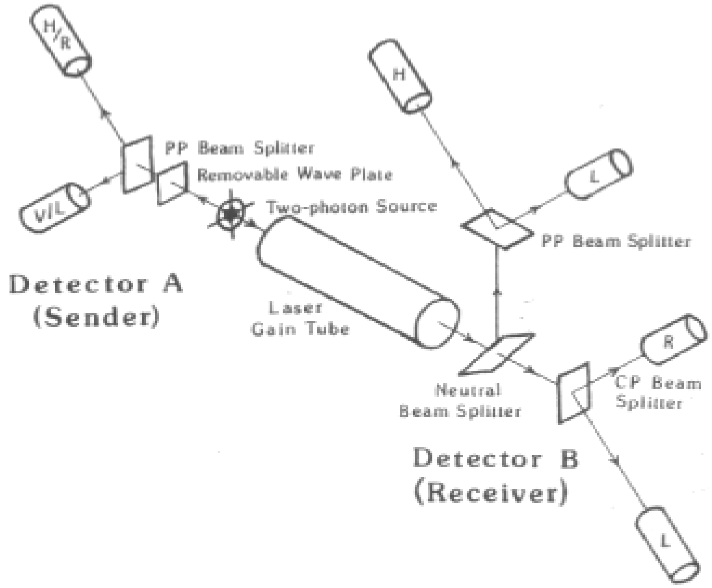


Fig. 1. The FLASH detection process. Photons in beam *B* (traveling to the right) are rendered either circularly unpolarized (CUP) or plane unpolarized (PUP) by positioning of the quarter wave plate in beam *A* (traveling to the left). Each *B* photon is amplified by a nonselective laser gain tube and the resulting isopolarized burst of light is examined for counting asymmetry in either the CP or PP channel.

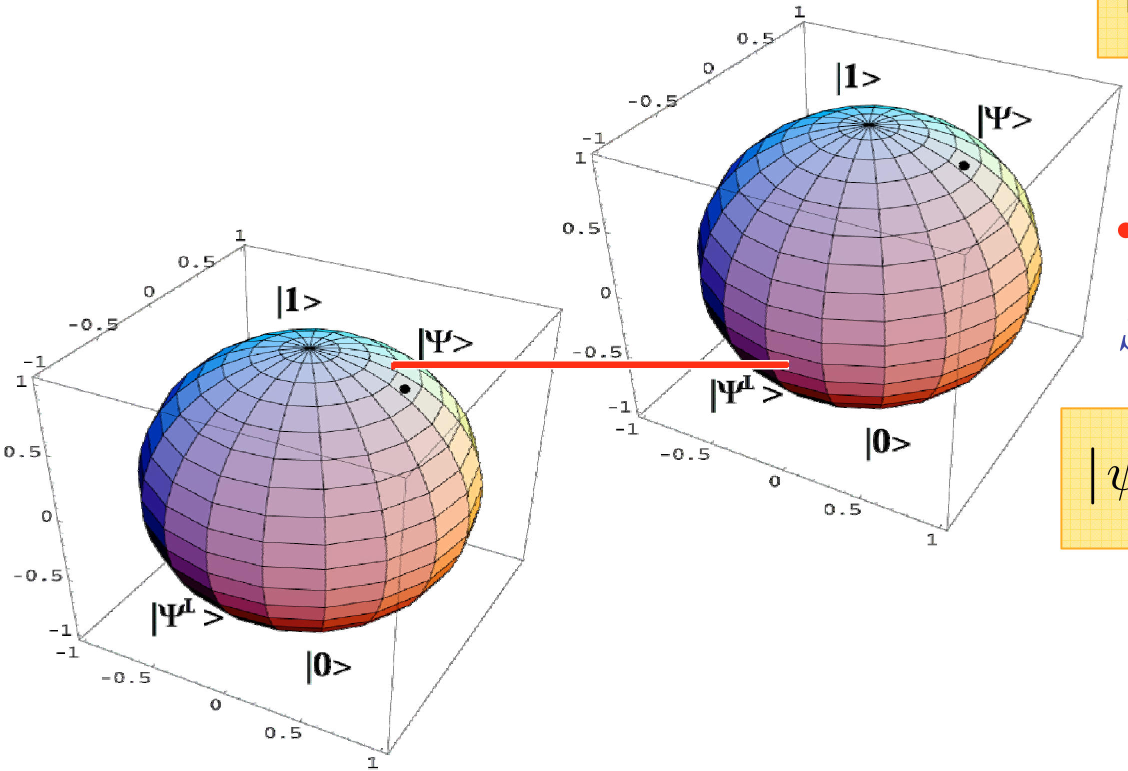
Can quantum nonlocality of entangled states be used for super-luminal communication?

N.Herbert, *Found. Phys.* 12, 1171 (1982)

Main Characters: Qubit & Entanglement

- Pure state of a qubit

$$|\psi\rangle = \cos\vartheta/2\,|1\rangle + e^{i\varphi}\sin\vartheta/2\,|0\rangle$$
$$|\psi^{\perp}\rangle = \cos\vartheta/2\,|0\rangle - e^{-i\varphi}\sin\vartheta/2\,|1\rangle$$



- Entangled state of two qubits: singlet state

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|0\rangle_A|1\rangle_B - |1\rangle_A|0\rangle_B\right)$$

- Invariance under local $SU(2) \otimes SU(2)$ transformation

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|\psi\rangle_A|\psi^{\perp}\rangle_B - |\psi^{\perp}\rangle_A|\psi\rangle_B\right)$$

$$\langle\psi|\psi^{\perp}\rangle = 0$$

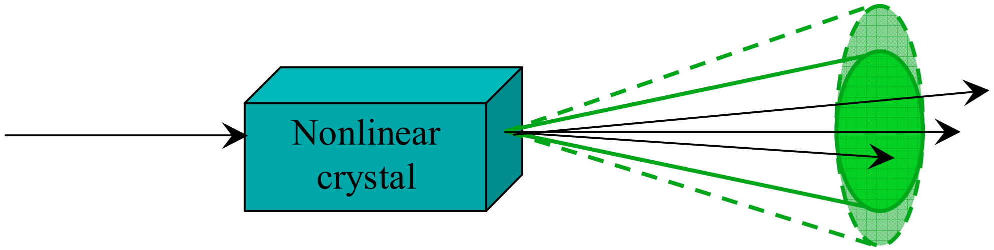
Generation of Entangled States

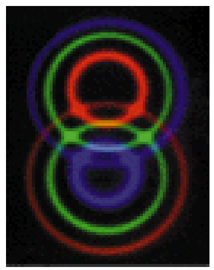
- Generation of polarization-entangled pairs of photons in a parametric down-conversion process in a nonlinear crystal.

Entangled photons are generated in a singlet state

$$|\Psi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B\right)$$

where $|\uparrow\rangle$ and $|\downarrow\rangle$ describe two polarization states of the photon in a given basis (e.g. horizontal/vertical polarization)





Alphabet in Bell Telephone & Flash

- Singlet states

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|\psi\rangle_A |\psi^\perp\rangle_B - |\psi^\perp\rangle_A |\psi\rangle_B\right)$$

exhibits perfect quantum correlations for polarization measurement along orthogonal but *arbitrary* axes.

- Alice and Bob have pair before any communication

Alice might like to send a message to Bob. She performs a measurement on her particle in one of the two bases

$$\{|\uparrow\rangle, |\downarrow\rangle\} \text{ and } \{|\leftarrow\rangle, |\rightarrow\rangle\}$$

After Alice performs her measurement in one of the bases, say $\{|\uparrow\rangle, |\downarrow\rangle\}$

Then she can predict with certainty what Bob's result would be if he performs a measurement in the same basis.

$$\text{logical zero} = \text{basis } \{|\uparrow\rangle, |\downarrow\rangle\}$$

$$\text{logical one} = \text{basis } \{|\leftarrow\rangle, |\rightarrow\rangle\}$$

- Infinite (continuous) alphabet: $\{|\psi\rangle, |\psi^\perp\rangle\}$

Question: **Can we discriminate (reconstruct) quantum states based on results of measurements performed on a single quantum object?**

Back to FLASH

- Optimal Quantum Measurement:

$$F = \frac{2}{3}$$

This does not allow for signaling – from a single shot measurement we are not able to discriminate between bases

$$\{|\uparrow\rangle, |\downarrow\rangle\} \quad \text{and} \quad \{|\leftarrow\rangle, |\rightarrow\rangle\}$$

Herbert:

"*a serious objection to FLASH concerns the noise… of the copying process*"

- Can we do better? Cloning quantum states?

  – active quantum detectors?

$$|\psi\rangle|0\rangle^{\otimes(N-1)} \to |\psi\rangle^{\otimes N}$$

$$F = \frac{N+1}{N+2}$$

N.Herbert, *Found. Phys.* **12**, 1171 (1982)

No-cloning Theorem

- Wigner 1961:

"the probability is zero for existence of self-reproducing states"

- Wootters & Zurek 1982:

"unkonwn pure states cannot be cloned perfectly"

- Condition for universal cloning

$$|\psi\rangle|0\rangle|S\rangle \xrightarrow{u} |\psi\rangle \otimes |\psi\rangle|S'\rangle$$

$$|\tilde{\psi}\rangle|0\rangle|S\rangle \xrightarrow{u} |\tilde{\psi}\rangle \otimes |\tilde{\psi}\rangle|S'\rangle$$

- Unitarity of the cloning operation:

$$\langle\tilde{\psi}|\psi\rangle = \left(\langle\tilde{\psi}|\psi\rangle\right)^2$$

- $\langle\tilde{\psi}|\psi\rangle = 0$ or $\left|\langle\tilde{\psi}|\psi\rangle\right| = 1$
  – states are ither orthogonal (distinguishable) or identical

Distinguishable states can be copied perfectly

E.Wigner, in The Logic of Personal Knowledge (The Free Press, 1961), p.231.
W.K.Wootters and W.H.Zurek, *Nature* 299, 802 (1982).
H.Yuen, *Phys. Lett. A* 113, 405 (1986)

Universal Quantum Cloners

- Input: $|\psi\rangle$
- Outputs are identical $\rho_a^{(out)} = \rho_b^{(out)}$
- $F\left(\rho_x^{(out)};\rho_x^{(id)}\right) = \max\left\{F^{(U)}\left(\rho_x^{(out)};\rho_x^{(id)}\right);\forall U\right\}$

$$|\psi\rangle_a |\Xi\rangle_{bc} \rightarrow \sqrt{\frac{2}{3}}|\psi,\psi\rangle_{ab}\left|\psi^{\perp}\right\rangle_c - \frac{1}{\sqrt{3}}\left|\{\psi^{\perp},\psi\}\right\rangle_{ab}|\psi\rangle_c$$

$$\left|\{\psi^{\perp},\psi\}\right\rangle_{ab} = \left(|\psi\rangle_a\left|\psi^{\perp}\right\rangle_b + \left|\psi^{\perp}\right\rangle_a|\psi\rangle_b\right)/\sqrt{2}$$

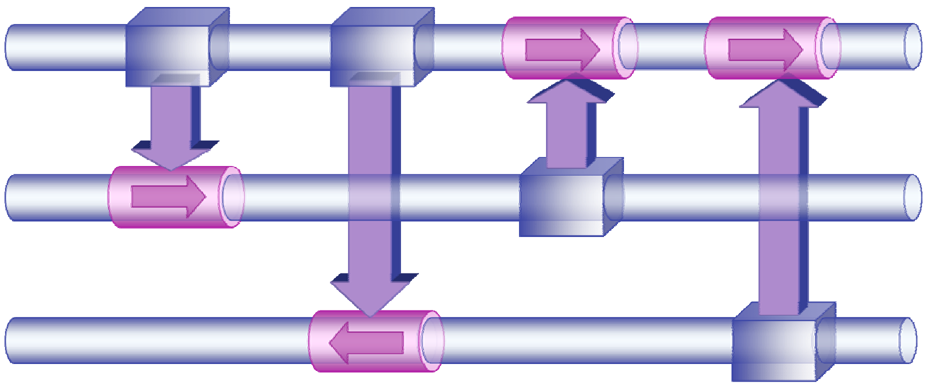
$$\rho_j^{(out)} = s\rho + \frac{1-s}{2}I;\ \ s = \frac{2}{3}$$

No-Cloning Theorem & U-QCM

$$|\psi\rangle_a \rightarrow |\psi\rangle_a \otimes |\psi\rangle_b$$

$|\psi\rangle_a$

$|\Xi\rangle_{bc}$



$$\rho_a^{(out)} = \frac{2}{3}\rho + \frac{1}{6}I$$

$$\rho_b^{(out)} = \frac{2}{3}\rho + \frac{1}{6}I$$

$$\rho_c^{(out)} = \frac{1}{3}\rho^{\perp} + \frac{1}{3}I$$

W.Wootters and W.H.Zurek, *Nature* 299, 802 (1982)
V.Bužek and M.Hillery, *Phys. Rev. A* 54, 1844 (1996)
S.L.Braunstein, V.Bužek, M.Hillery, and D.Bruss, *Phys. Rev. A* 56, 2153 (1997)

Bounds On Cloning Due To No-signaling

- Input qubit:

$$\rho_a = \frac{1}{2}(1 + \vec{\sigma}\cdot\vec{m}) = |+\vec{m}\rangle\langle+\vec{m}|$$

- Linearity → implies no-signaling; output: $\rho_{ab}(\vec{m})$ linear in $\vec{m}$

- Universality (covariance) condition

$$\rho_{ab}(U\vec{m}) = U \otimes U \rho_{ab}(\vec{m}) U^{\dagger} \otimes U^{\dagger}$$

- U: $\forall$ single-qubit unitary operations

- Basis:

$$\{I \otimes I, I \otimes \sigma_i, \sigma_i \otimes I, \sigma_i \otimes \sigma_k\}$$

$$\rho_{ab}(\vec{m}) = \frac{1}{4}\left(I \otimes I + \eta_1 \vec{m}\vec{\sigma} \otimes I + \eta_2 I \otimes \vec{m}\vec{\sigma} + t\vec{\sigma} \otimes \vec{\sigma} + t_{xy}\vec{m}(\vec{\sigma} \wedge \vec{\sigma})\right)$$

Bounds On Cloning Due To No-signaling

$$\rho_{ab}(\vec{m}) = \frac{1}{4}\Big(I \otimes I + \eta_1 \vec{m}\vec{\sigma} \otimes I + \eta_2 I \otimes \vec{m}\vec{\sigma} + t\vec{\sigma} \otimes \vec{\sigma} + t_{xy}\vec{m}(\vec{\sigma} \wedge \vec{\sigma})\Big)$$

$1 \to 2$ cloning

- $\eta_1, \eta_2, t, t_{xy}$ are real parameters
- $\rho(\vec{m})$ – non-negative eigenvalues

$$1 + t \pm (\eta_1 + \eta_2) \geq 0$$

$$1 - t \pm \sqrt{4t^2 + 4t_{xy}^2 + (\eta_1 - \eta_2)^2} \geq 0$$

- Optimize the fidelity

  $F = \mathrm{Tr}\big[\rho_{ab}(\vec{m}) P_{\vec{m}} \otimes I\big]$

- $P_{\vec{m}} = |+\vec{m}\rangle\langle+\vec{m}|$,

  assuming $\eta_1 = \eta_2 \equiv \eta$

- Optimal values

  $t_{xy} = 0,\ t = 1/3,\ \eta = 2/3 \to F = \frac{5}{6}$

- Generalization to 1→N cloning

No-signaling and QM give the same fidelity!

N.Gisin, *Phys.Lett. A* 143, 1 (1990)

C.Simon, V.Bužek, and N.Gisin, *Phys. Rev. Lett.* 87, 170405 (2001)

There is Something in This Network

$$\left|\psi\right\rangle_1 = \sum_{k=0}^{N-1} c_k \left|x_k\right\rangle_1$$

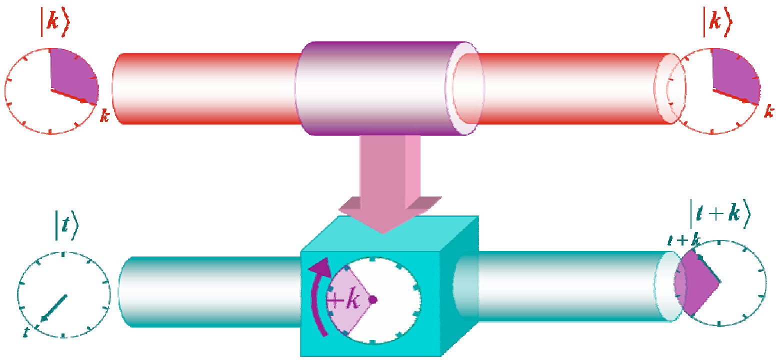
$$\hat{x}\left|x_k\right\rangle = x_k\left|x_k\right\rangle \qquad x_k = L\sqrt{\frac{2\pi}{N}}k$$

$$\hat{p}\left|p_l\right\rangle = p_l\left|p_l\right\rangle$$

$$p_l = \frac{1}{L}\sqrt{\frac{2\pi}{N}}l$$

$$\left|x_k\right\rangle = \frac{1}{\sqrt{N}}\sum_{l=0}^{N-1}\exp\left(-i\frac{2\pi}{N}kl\right)\left|p_l\right\rangle$$

$$\left|\left\langle x_k \mid p_l\right\rangle\right|^2 = \frac{1}{N}$$

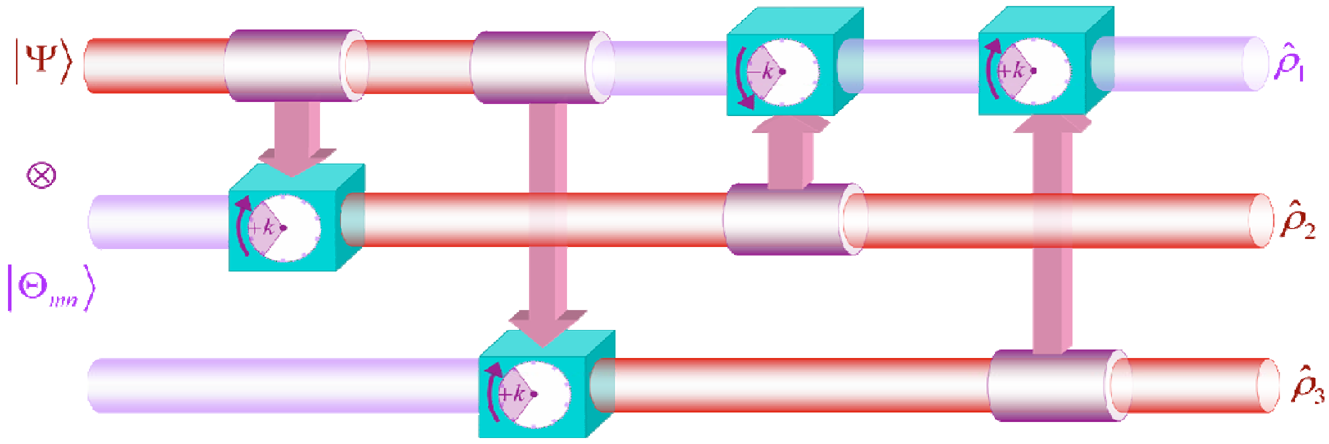


$$\widehat{D}_{ab} : \left|k\right\rangle_a \left|l\right\rangle_b \rightarrow \left|k\right\rangle_a \left|(l+k) \bmod N\right\rangle_b$$

$$\widehat{D}_{ab} = \exp\left[-i\hat{x}_a \hat{p}_b\right]$$

S.L.Braunstein, V.Bužek, and M.Hillery, *Phys. Rev. A* 63, 052313 (2001)

Quantum Information Distributor



$$\rho_1^{(out)} = \left(\alpha^2 + \frac{2\alpha\beta}{N}\right)\rho + \frac{\beta^2}{N} I$$

$$\rho_2^{(out)} = \left(\beta^2 + \frac{2\alpha\beta}{N}\right)\rho + \frac{\alpha^2}{N} I$$

$$\rho_3^{(out)} = \frac{2\alpha\beta}{N}\rho^T + \frac{(N - 2\alpha\beta)}{N^2} I$$

$$\widehat{U} = \exp\left[-i(\hat{x}_3 - \hat{x}_2)\hat{p}_1\right]\exp\left[-i\hat{x}_1(\hat{p}_2 - \hat{p}_3)\right]$$

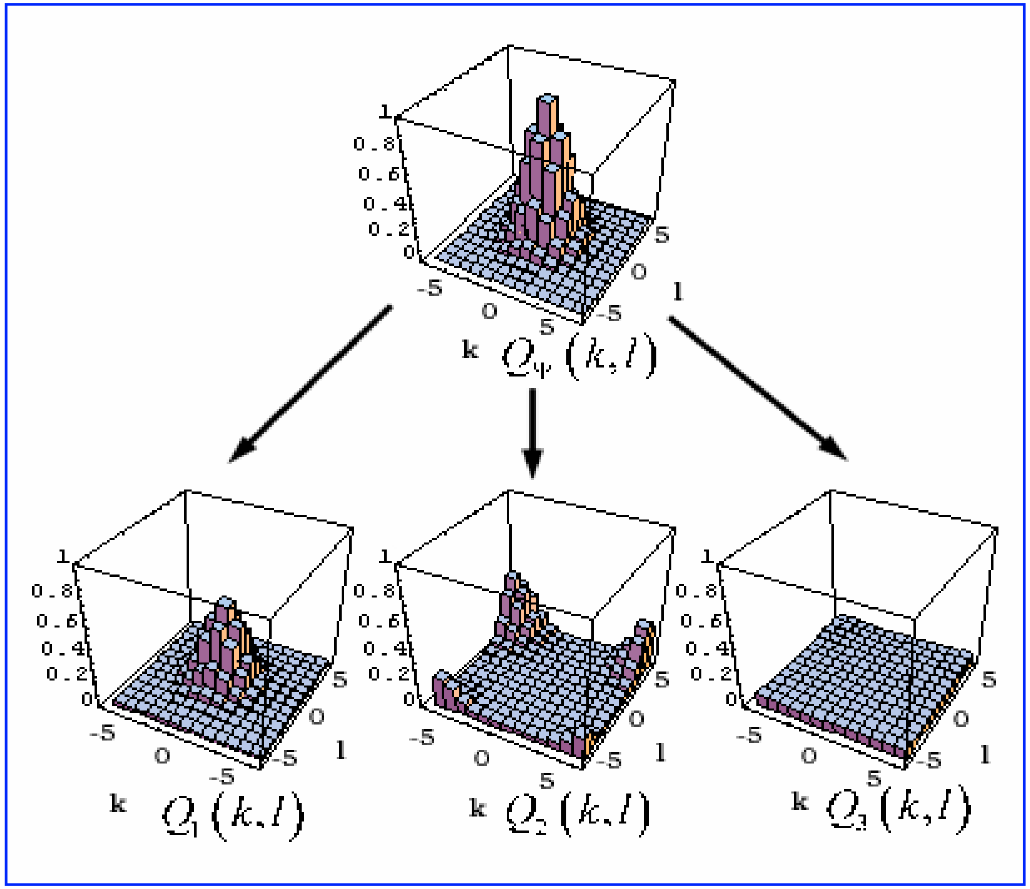
$$|\Theta(s)\rangle_{23} = \alpha \frac{1}{\sqrt{N}} \sum_k^{N-1} |x_k\rangle_2 |x_k\rangle_3 + \beta |x_0\rangle_2 |p_0\rangle_3$$

$$\alpha^2 + \beta^2 + \frac{2\alpha\beta}{N} = 1$$

- Covariant device with respect to SU(N) operations
- POVM measurements
- eavesdropping

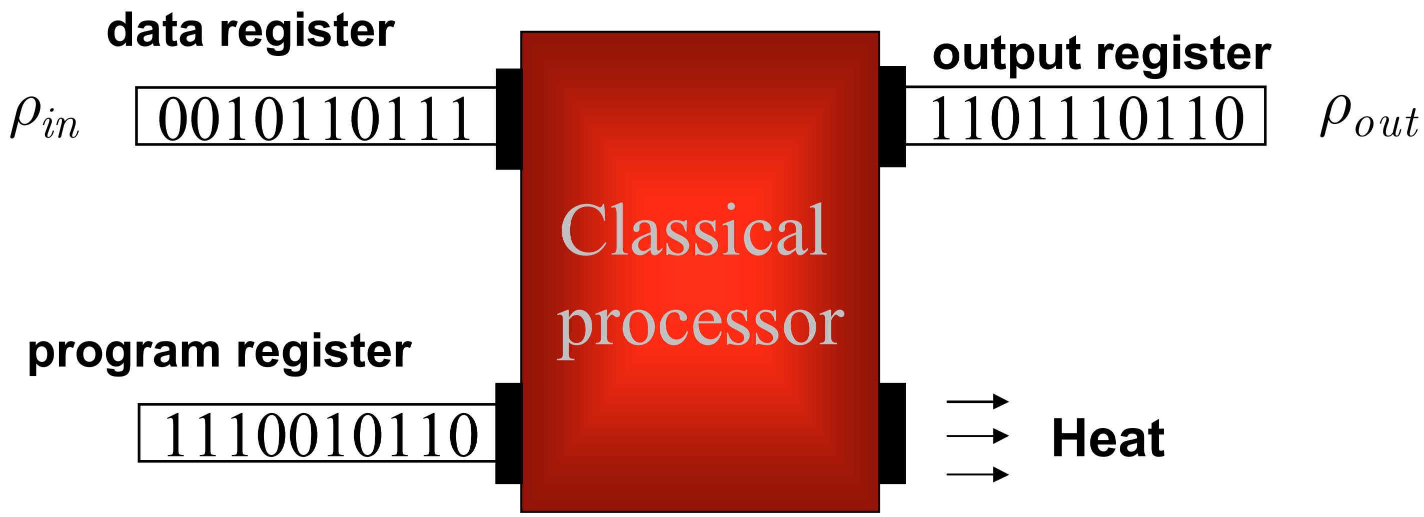
S.L.Braunstein, V.Bužek, and M.Hillery, ***Phys. Rev. A*** **63, 052313 (2001)**

POVM Measurement



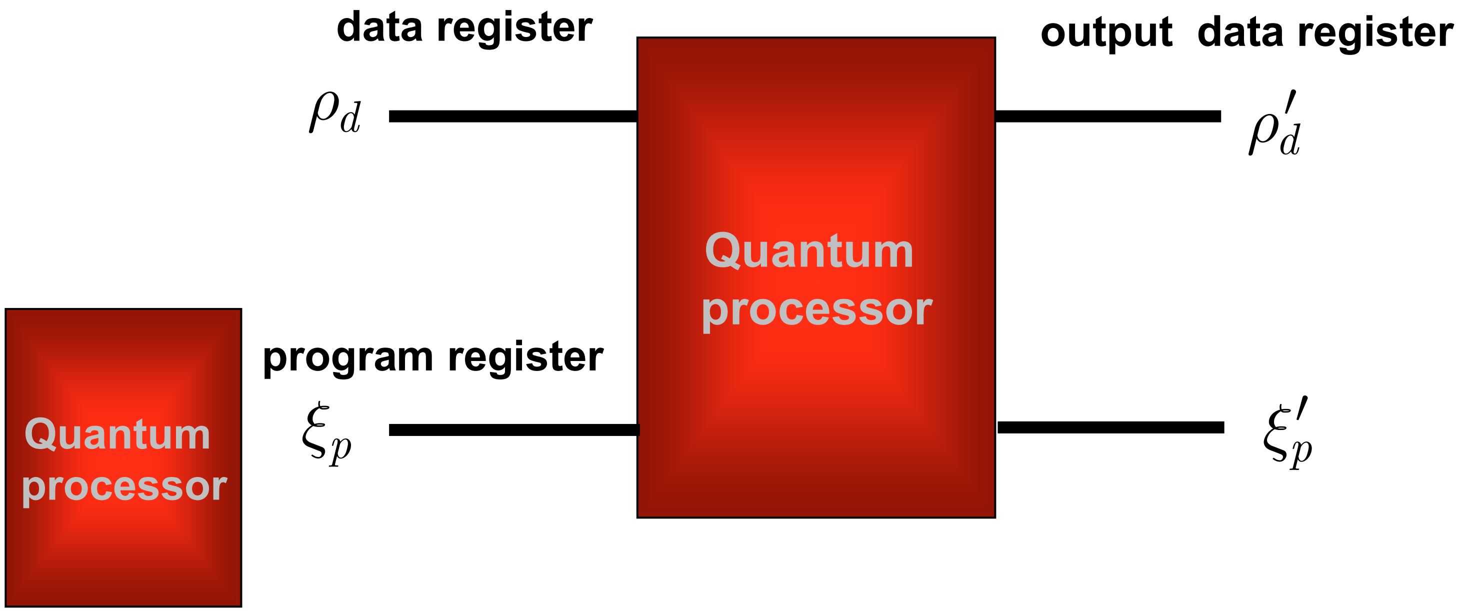
V.Bužek, M.Roško, and M.Hillery, *Phys. Rev. A (2003).*

Model of Classical Processor



$$\rho_{out} = T[\rho_{in}]$$

Quantum Processor



Quantum processor – fixed unitary transformation U_{dp}

$\mathcal{H}_d$ – data system, $S(\mathcal{H}_d)$ – data states

$\mathcal{H}_p$ – program system, $S(\mathcal{H}_p)$ – program states

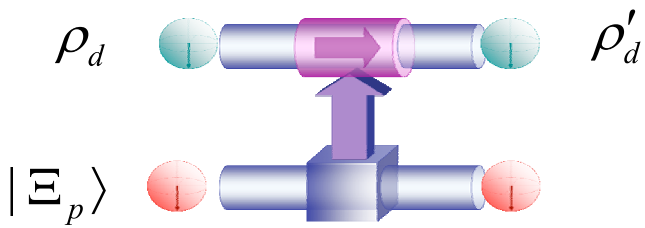
Two Scenarios

- **Measurement-based strategy - estimate the state of program**

$$F = \frac{N+1}{N+M}$$

- **Quantum strategy – use the quantum program register**
  conditional (probabilistic) processors
  unconditional processors

C-NOT as Unconditional Quantum Processor



$$\mathrm{CNOT}\ |\psi\rangle|0\rangle = |\psi\rangle|0\rangle$$

$$\mathrm{CNOT}\ |\psi\rangle|1\rangle = \sigma_x|\psi\rangle \otimes |1\rangle$$

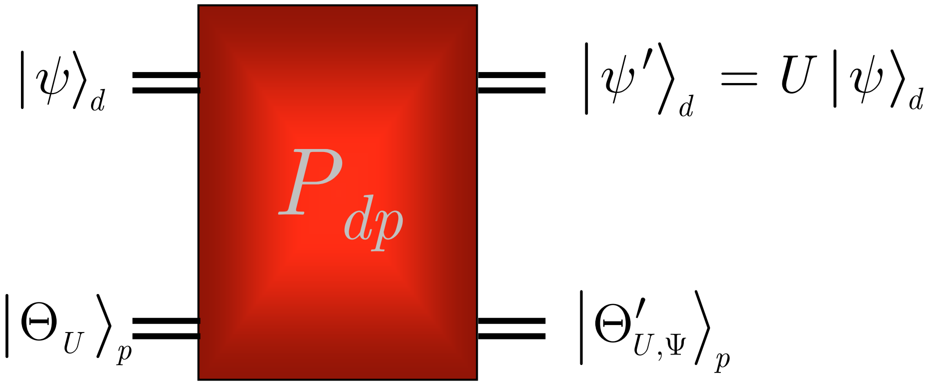
- program state $|0\rangle \Rightarrow \mathbf{1}$ implemented, i.e. $\rho_d \rightarrow \rho'_d = \rho_d$
- program state $|1\rangle \Rightarrow \sigma_x$ implemented, i.e. $\rho_d \rightarrow \rho'_d = \sigma_x \rho_d \sigma_x$
- general pure state $|\Xi_p\rangle = \alpha|0\rangle_p + \beta|1\rangle_p \Rightarrow \rho_d \mapsto \rho'_d = |\alpha|^2 \rho_d + |\beta|^2 \sigma_x \rho_d \sigma_x$
- unital operation, since $\Phi[\mathbf{1}] = |\alpha|^2 \mathbf{1} + |\beta|^2 \sigma_x \mathbf{1} \sigma_x = \mathbf{1}$
- program state is **2-d** and we can apply **2** unitary operations

Question

Is it possible to build a *universal* programmable quantum gate array which take as input a quantum state specifying a quantum program and a data register to which the unitary operation is applied ?

on a qubit an ∞ number of operations can be performed

No-go Theorem



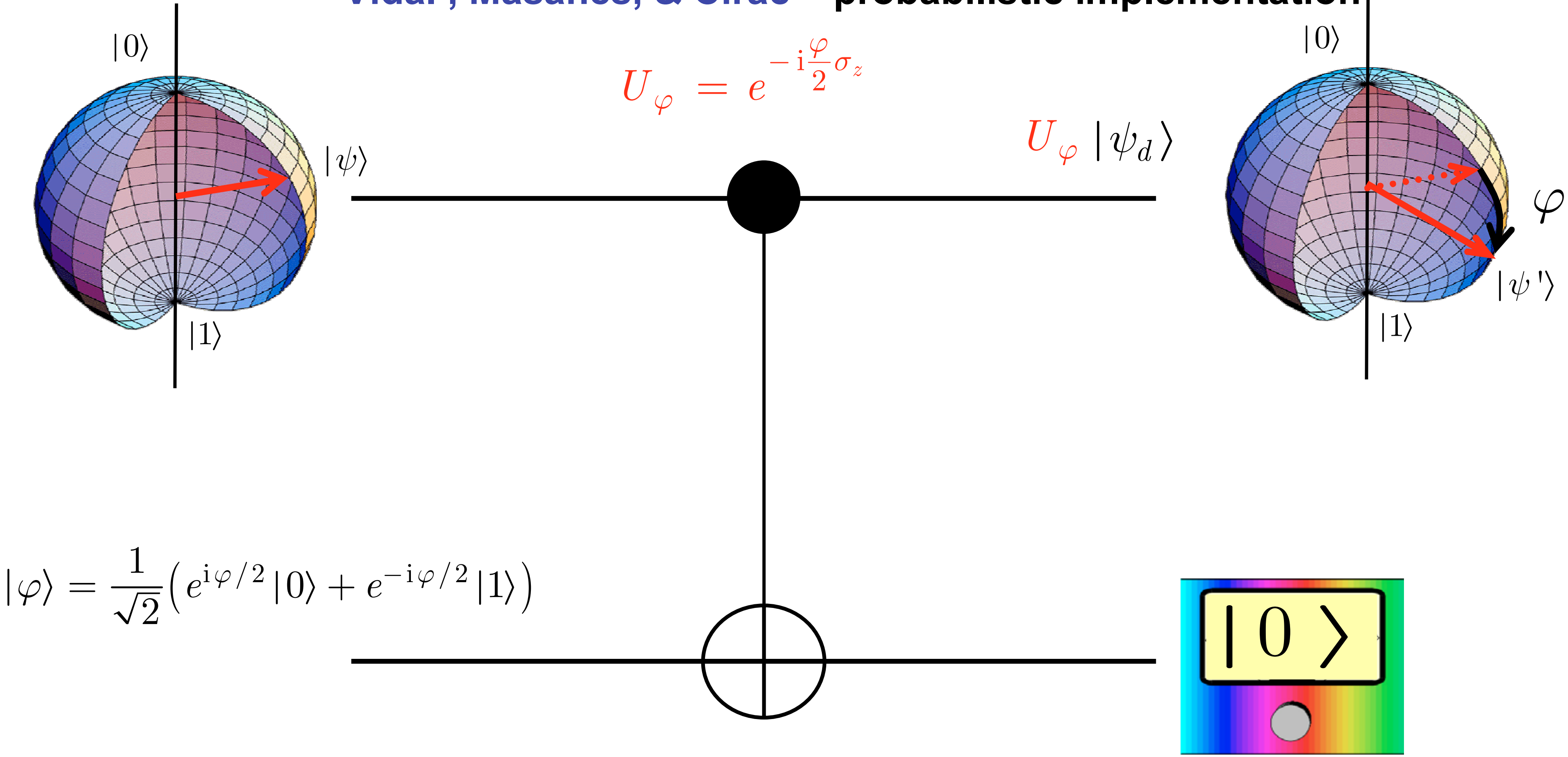
$$P_{dp}\left(|\psi\rangle_d \otimes |\Theta_U\rangle_p\right) = (U|\psi\rangle) \otimes \left|\Theta'_{U,\psi}\right\rangle$$

- no universal deterministic quantum array of finite extent can be realized
- on the other hand – a program register with *d* dimensions can be used to implement *d* unitary operations by performing an appropriate sequence of controlled unitary operations

M.A.Nielsen & I.L.Chuang, *Phys. Rev. Lett* 79, 321 (1997)

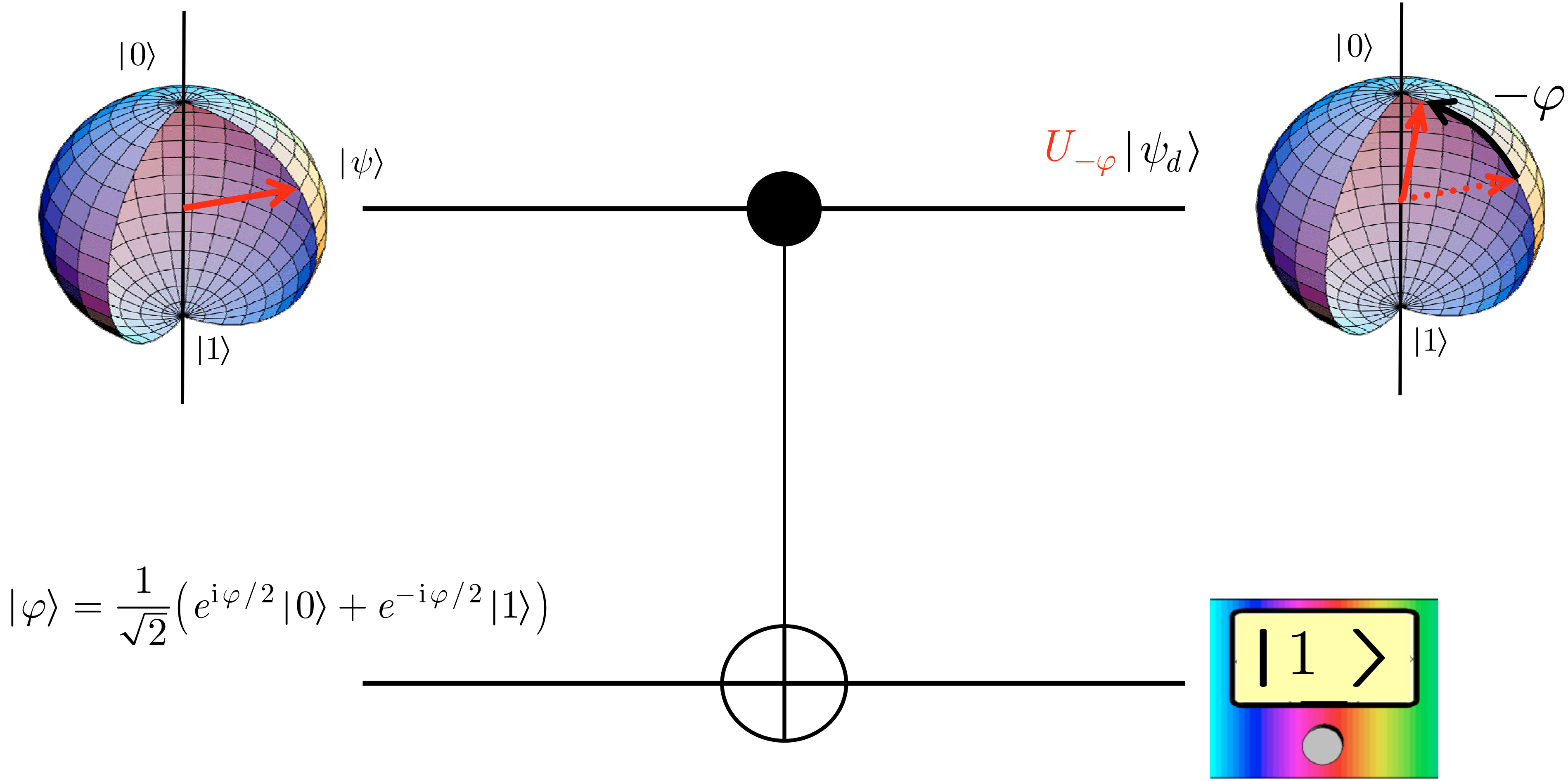
C-NOT as Probabilistic Q-Processor

•Vidal , Masanes, & Cirac – probabilistic implementation



G.Vidal, L.Masanes, and J.I.Cirac, *Phys. Rev. Lett.* 80, 047905 (2002).

C-NOT as Probabilistic Q-Processor



Correction of the error – $|2\varphi\rangle; |4\varphi\rangle;$ ***vs*** $|\varphi\rangle^{\otimes N}$

Description of Quantum Processors

- **definition of U_{dp} via "Kraus operators"** $A_{kl} := {}_p\langle l|U_{dp}|k\rangle_p$

$$U_{dp}\left(\,|\psi\rangle_d \otimes |k\rangle_p\right) = \sum_l \left(A_{kl}|\psi\rangle_d\right) \otimes |l\rangle_p$$

- **normalization condition** $\sum_l A^\dagger_{k_1 l} A_{k_2 l} = \delta_{k_1 k_2} \mathbf{1}_d$
- **induced quantum operation** $\rho_d \mapsto \rho'_d = \Phi_k[\rho_d] = \sum_l A_{kl}\rho_d A^\dagger_{kl}$
- **general pure program state** $|\Xi\rangle_p = \sum_k \alpha_k |k\rangle_p$

$$\rho_d \mapsto \rho'_d = \Phi_\Xi[\rho_d] = \sum_l A_l(\Xi)\rho_d A^\dagger_l(\Xi)$$

$$A_l(\Xi) = {}_p\langle l|U_{dp}|\Xi\rangle_p = \sum_k \alpha_k A_{kl}$$

- **can be generalized for mixed program states**

Universal Probabilistic Processor

- **Quantum processor** U_{dp}
- **Data register** ρ_d, $\dim H_d = D$
- **Quantum programs** U_k = **program register** ρ_p, $\dim H_p = N = D^2$

- **Nielsen & Chuang:**
  - N programs $\Rightarrow$ N orthogonal states
  - Universal quantum processors do not $\exists$
- **Probabilistic implementation**
  - $\{U_k\}$ operator basis,

$$U = \sum_k \alpha_k U_k, \quad \alpha_k = \frac{1}{D}\mathrm{Tr}U_k^{+}U$$

  - program state

$$|\psi_U\rangle = \sum_k \alpha_k |\psi_k\rangle$$

Example:

Data register = qudit, program register = 2 qudits

$$U_k \equiv U^{(mn)} = \sum_{s=0}^{D-1} \exp\left(-\frac{2\pi ism}{N}\right)|s-n\rangle\langle s|$$

$$|\psi_k\rangle \equiv |\Xi_{mn}\rangle = \frac{1}{\sqrt{D}}\sum_{s=0}^{D-1} \exp\left(-\frac{2\pi ism}{N}\right)|s\rangle|s-n\rangle$$

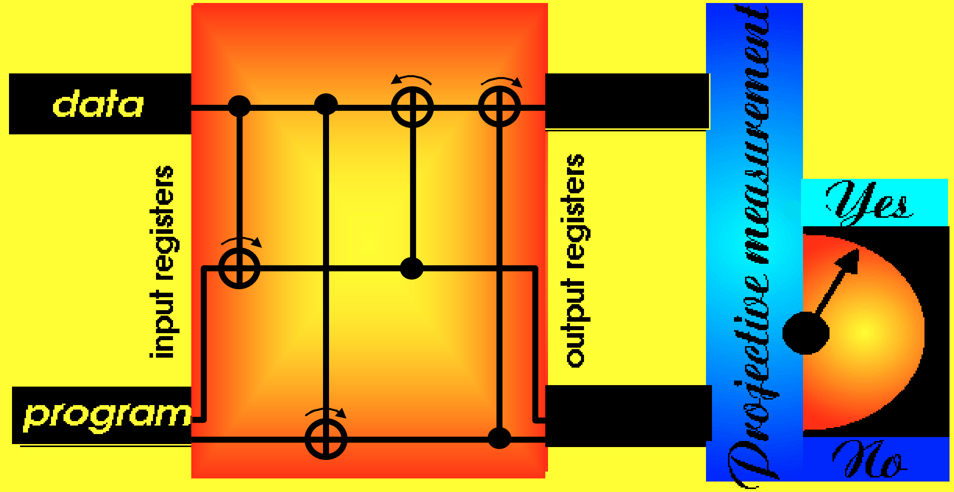
"universal" processor

$$\mathbf{U}_{dp} = \sum_{k=1}^{D^2} \mathbf{U}_k \otimes |\psi_k\rangle\langle\psi_k|, \quad \frac{1}{D}\mathbf{Tr}\mathbf{U}_k^{+}\mathbf{U}_l = \langle\psi_k|\psi_l\rangle = \delta_{kl}$$

projective yes/no measurement

$$\mathbf{M} = yes|\Phi\rangle\langle\Phi| + no\left(I - |\Phi\rangle\langle\Phi|\right), \quad |\Phi\rangle_p = \frac{1}{D}\sum_{k=1}^{D^2}|\psi_k\rangle$$

probability of success: $\mathbf{P}_{\text{success}} = \frac{1}{D^2}$



Conclusions & Open Questions

- programmable quantum computer – programs via quantum states programs can be outputs of another QC
- some CP maps via unconditional quantum processors
- arbitrary CP maps via probabilistic programming
- controlled information distribution (eavesdropping)
- simulation of quantum dynamics of open systems
- set of maps induced by a given processor (loops)
- quantum processor for a given set of maps
- quantum multi-meters

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