The Abdus Salam
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Educational, Scientific
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International Atomic
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SMR.1745- 1

SPRING SCHOOL ON SUPERSTRING THEORY
AND RELATED TOPICS

27 March - 4 April 2006

Integrability in AdS/CFT

PART I

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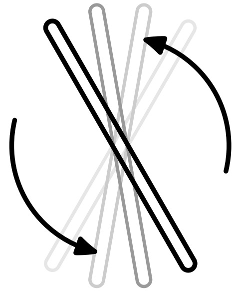
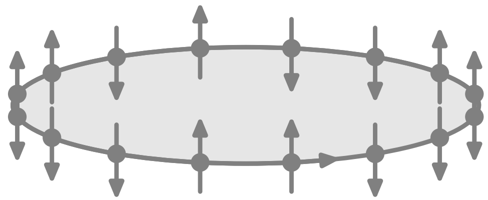
Integrability in AdS/CFT

Lecture I: Overview





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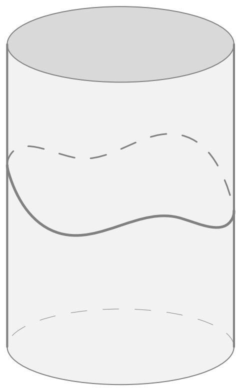
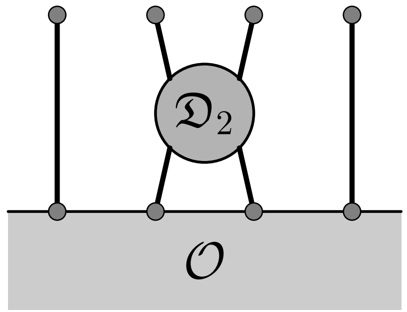




These lectures are about . . .

. . . efficient methods for the planar spectrum of AdS/CFT.

- Understanding the spectrum of classical strings on $AdS_5 \times S^5$.
- Avoiding higher-loop calculations in planar $\mathcal{N} = 4$ gauge theory.

⋆ **Lecture I: Introduction and Overview**

- Spectra of string/gauge theories
- Computing energies by brute force
- Overview of integrable methods

⋆ **Lecture II: Classical String Theory and Spectral Curves**

- From a classical solution to a spectral curve

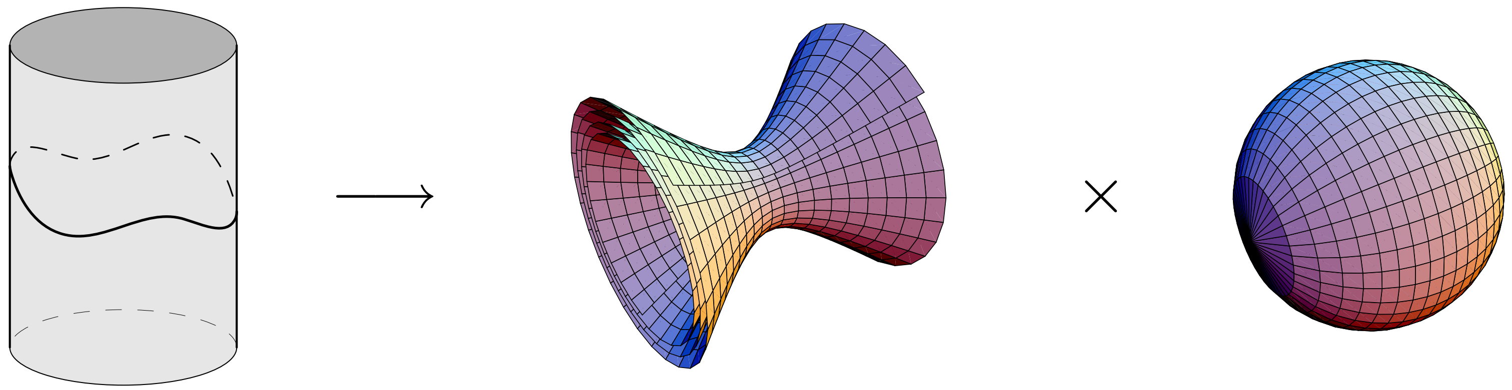
⋆ **Lectures III & IV: Gauge Theory and the Bethe ansatz**

- Integrable spin chains
- Bethe ansatz; S-matrix
- R-matrix formalism

AdS/CFT Correspondence

Strings on $AdS_5 \times S^5$

IIB superstrings on the curved $AdS_5 \times S^5$ superspace



$\times$ fermi

Coset space

$$AdS_5 \times S^5 \times \text{fermi} = \frac{\mathrm{PSU}(2,2|4)}{\mathrm{Sp}(1,1) \times \mathrm{Sp}(2)}\,.$$

Sigma model action on a cyclinder (Green-Schwarz superstring) [Metsaev Tseytlin]

$$S_\sigma = \frac{\sqrt{\lambda}}{2\pi} \int d\tau\, d\sigma \sqrt{-\gamma}\, \Big(\tfrac{1}{2}(\partial_a \vec{X})^2 - \tfrac{1}{2}(\partial_a \vec{Y})^2 \Big) + \ldots\,.$$

$\mathcal{N} = 4$ Gauge Theory

$\mathrm{U}(N)$ gauge field $\mathcal{A}_\mu$, 4 adjoint fermions Ψ^a_α, 6 adjoint scalars Φ_m

$$S_{\mathcal{N}=4} = N \int \frac{d^4x}{4\pi^2} \, \mathrm{Tr}\Big(\tfrac{1}{4}(\mathcal{F}_{\mu\nu})^2 + \tfrac{1}{2}(\mathcal{D}_\mu \Phi_m)^2 - \tfrac{1}{4}[\Phi_m, \Phi_n]^2 + \ldots \Big).$$

Some remarkable properties:

- Unique action due to maximal supersymmetry,
- single coupling constant $g \sim \sqrt{\lambda} \sim g_{\mathrm{YM}}\sqrt{N}$ (plus top. θ-angle),
- all fields adjoints: $N \times N$ matrices for $\mathrm{U}(N)$ gauge group,
- all fields massless (pure gauge),
- “finite” theory: beta-function exactly zero, no running coupling,
- unbroken conformal symmetry,
- superconformal symmetry $\mathrm{PSU}(2,2|4)$.

And some more mysterious ones. . .

AdS/CFT Correspondence

Conjectured exact duality of [Maldacena hep-th/9711200][Gubser Klebanov Polyakov][Witten hep-th/9802150]

- IIB string theory on $AdS_5 \times S^5$ and
- $\mathcal{N} = 4$ gauge theory (CFT).

Symmetry groups match: $\mathrm{PSU}(2,2|4)$.

Holography: Boundary of AdS_5 is conformal $\mathbb{R}^4$.

Prospects:

- More general AdS/CFT may explain aspects of QCD strings.
- Study aspects of quantum gravity with QFT means.

No proof yet!

Many qualitative comparisons. Quantitative tests missing.

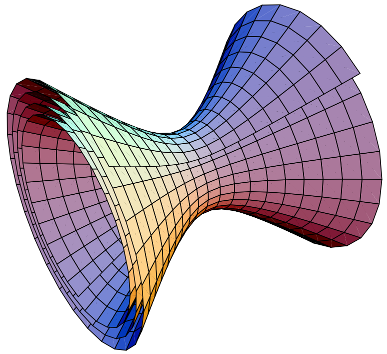
Would like to verify quantitatively.

One prediction: **Matching of spectra.**

Central motivation for these lectures. Goal: Obtain spectra on both sides.

Spectra of String and Gauge Theory

String Theory:



States: Solutions of classical equations of motion
plus quantum corrections.

Energy: Charge for translation along AdS-time
(rotations along unwound circle in figure)

Gauge Theory:

States: Local operators. Local combinations of the fields, e.g.

$$\mathcal{O} = \operatorname{Tr} \varPhi_1 \varPhi_2 (\mathcal{D}_1 \mathcal{D}_2 \varPhi_2)(\mathcal{D}_1 \mathcal{F}_{24}) + \ldots$$

Energy: Scaling dimensions, e.g. two-point function in conformal theory

$$\big\langle \mathcal{O}(x)\, \mathcal{O}(y) \big\rangle = C|x-y|^{-2D(\lambda)}$$

Matching: String energies and gauge dimensions match, $E(\lambda) = D(\lambda)$.

Strong/Weak Duality

Problem: Strong/weak duality.

- Perturbative regime of strings at $\lambda \to \infty$

$$E(\lambda) = \sqrt{\lambda}\, E_0 + E_1 + E_2/\sqrt{\lambda} + \ldots$$

E_ℓ: Contribution at ℓ (world-sheet) loops. Limit: 1 or 2 loops.

- Perturbative regime of gauge theory at $\lambda \approx 0$.

$$D(\lambda) = D_0 + \lambda D_1 + \lambda D_2 + \ldots$$

D_ℓ: Contribution at ℓ (gauge) loops. Limit: 3 or 4 loops.

Tests impossible unless quantities are known at finite λ.

Cannot compare, not even approximately.

Integrability may help.

Spinning Strings

Classical Spinning Strings

Strings in a flat background are simple...

Mode decomposition leads to free oscillators. Easily quantised.

Curved background: $AdS_5 \times S^5$ embedded in $\mathbb{R}^{2,4} \times \mathbb{R}^6$ as $\vec{Y}^2 = \vec{X}^2 = 1$.

String sigma model action [Metsaev Tseytlin]

$$S_\sigma = \frac{\sqrt{\lambda}}{2\pi} \int d\tau\, d\sigma \sqrt{-\gamma} \left(\tfrac{1}{2}(\partial_a \vec{X})^2 - \tfrac{1}{2}(\partial_a \vec{Y})^2 + \text{fermi.} \right).$$

Equations of motion, Virasoro constraints (variation by $\vec{X}, \vec{Y}, \gamma_{ab}$)

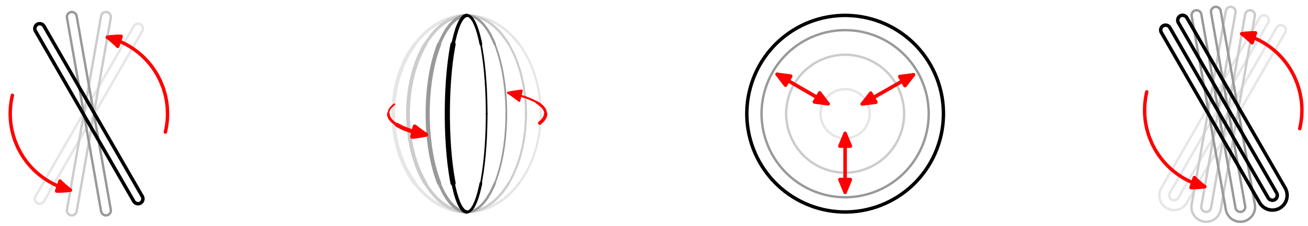
$$\partial^2 \vec{X} = \Lambda \vec{X}, \qquad \partial^2 \vec{Y} = \Lambda' \vec{Y}, \qquad (\partial_\pm \vec{X})^2 = (\partial_\pm \vec{Y})^2.$$

Let's find a classical string solution.

For classical solutions may drop fermions.

Spinning Strings Ansatz

Many examples investigated: [Gubser Klebanov Polyakov] [Frolov Tseytlin] [Minahan hep-th/0209047] [Frolov Tseytlin] . . .

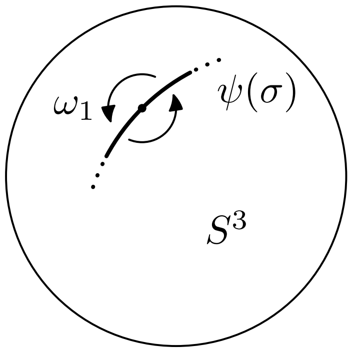


folded | circular | pulsating | higher modes

A particular ansatz for a spinning strings on $\mathbb{R} \times S^3$: [Frolov, Tseytlin hep-th/0306143]

- uniform motion in AdS-time: WS energy ε
- uniform rotation in 12-plane and 34-plane: WS frequencies $\omega_{1,2}$
- stretched in 12/34 plane: profile $\psi(\sigma)$.

$$t(\sigma, \tau) = \varepsilon \tau, \qquad \vec{X}(\sigma, \tau) = \begin{pmatrix} \cos\psi(\sigma)\cos\omega_1\tau \\ \cos\psi(\sigma)\sin\omega_1\tau \\ \sin\psi(\sigma)\cos\omega_2\tau \\ \sin\psi(\sigma)\sin\omega_2\tau \end{pmatrix}.$$



Solving a Spinning String

Equation of motion (pendulum with $\vartheta = 2\psi$, $g/L = \omega_1^2 - \omega_2^2$)

$$\psi'' = (\omega_1^2 - \omega_2^2) \cos\psi \sin\psi.$$

Virasoro constraint (integrated EOM)

$$\psi'^2 = \varepsilon^2 - \omega_1^2 + (\omega_1^2 - \omega_2^2) \sin^2\psi.$$

Solution (elliptic amplitude am, modulus k, integration constant σ_0)

$$\psi(\sigma) = \mathrm{am}\big(b(\sigma - \sigma_0), k\big).$$

Relation between (ω_1, ω_2) and (b, k)

$$\varepsilon^2 - \omega_1^2 = b^2, \qquad \omega_2^2 - \omega_1^2 = b^2 k^2.$$

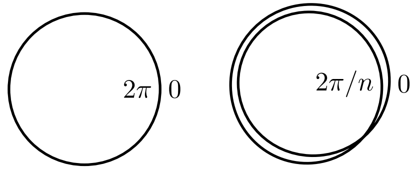
Periodicity

Closed string must be periodic: $\psi(\sigma+2\pi) \equiv \psi(\sigma) \pmod{2\pi}$.

Understand elliptic functions or reconsider Virasoro

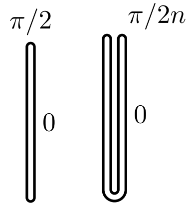
$$\frac{d\psi}{d\sigma} = b\sqrt{1-k^2\sin^2\psi}.$$

Circular string, $k<1$, n windings



$$2\pi = \int_0^{2\pi} d\sigma = n\int_0^{2\pi} d\psi\,\frac{d\sigma}{d\psi} = \frac{4n}{b}\int_0^{\pi/2}\frac{d\psi}{\sqrt{1-k^2\sin^2\psi}} = \frac{4n}{b}\mathrm{K}(k).$$

Folded string, $k>1$, n folds (folded at $\pm\psi_0$ with $\sin\psi_0 = 1/k$)



$$2\pi = \int_0^{2\pi} d\sigma = 4n\int_0^{\psi_0} d\psi\,\frac{d\sigma}{d\psi} = \frac{4n}{b}\Big(\mathrm{K}(k) + i\,\mathrm{K}(\sqrt{1-k^2})\Big).$$

New integer parameter: mode number n. Parameter b fixed.

Charges

AdS-Energy

$$E = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma\, \dot{t} = \sqrt{\lambda}\,\varepsilon$$

Charge for rotation in 12-plane (circular string)

$$\begin{aligned} J_1 &= \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma\,\big(\dot{X}_2 X_1 - \dot{X}_1 X_2\big) = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma\,\omega_1\cos^2\psi \\ &= \frac{\sqrt{\lambda}\,\omega_1}{\mathrm{K}(k)}\int_0^{\pi/2}\frac{d\psi\,\cos^2\psi}{\sqrt{1-k^2\sin^2\psi}} = \sqrt{\lambda}\,\omega_1\,\frac{\mathrm{E}(k)-(1-k^2)\,\mathrm{K}(k)}{k^2\,\mathrm{K}(k)}\,. \end{aligned}$$

Similar for rotation in 34-plane. Invert relations

$$\varepsilon = \frac{E}{\sqrt{\lambda}}, \quad \omega_1 = \frac{J_1}{\sqrt{\lambda}}\,\frac{k^2\,\mathrm{K}(k)}{\mathrm{E}(k)-(1-k^2)\,\mathrm{K}(k)}, \quad \omega_2 = \frac{J_2}{\sqrt{\lambda}}\,\frac{k^2\,\mathrm{K}(k)}{\mathrm{K}(k)-\mathrm{E}(k)}\,.$$

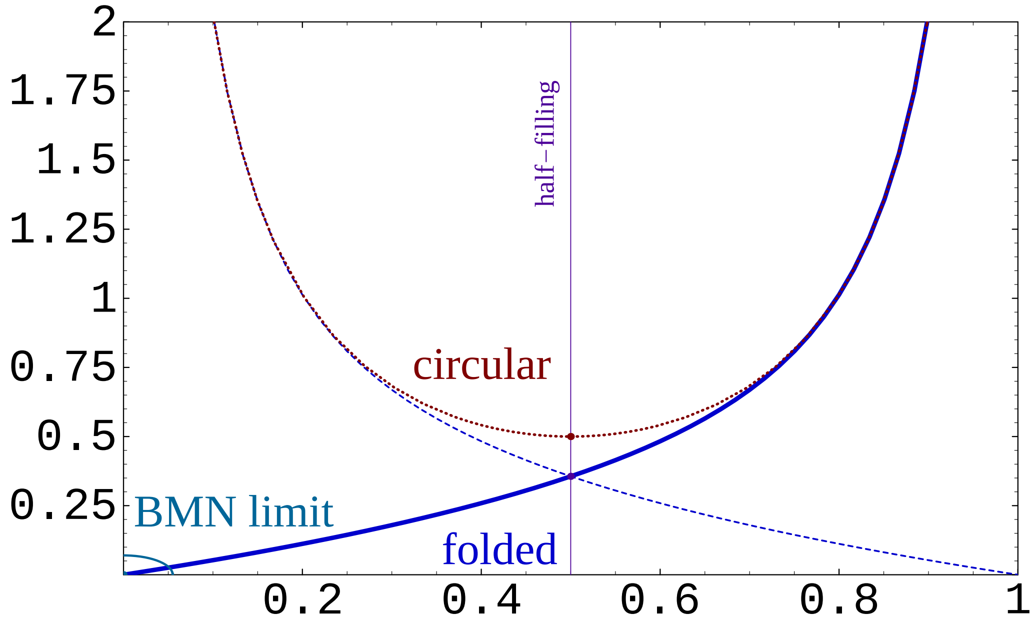
Large Spin Limit

Use relation between (ω_1, ω_2) and (b, k) to solve for $E(J_1, J_2)$.

Expansion (total spin $J = J_1 + J_2$, spin ratio $\alpha = J_2/J$).

$$E = J\left(1 + \frac{\lambda}{J^2}E_1(\alpha) + \frac{\lambda^2}{J^4}E_2(\alpha) + \ldots\right)$$

Comparison to perturbative gauge theory?! (No: λ is large!) [Frolov,Tseytlin hep-th/0306143]



Plot of $E_1(\alpha)$ vs. α

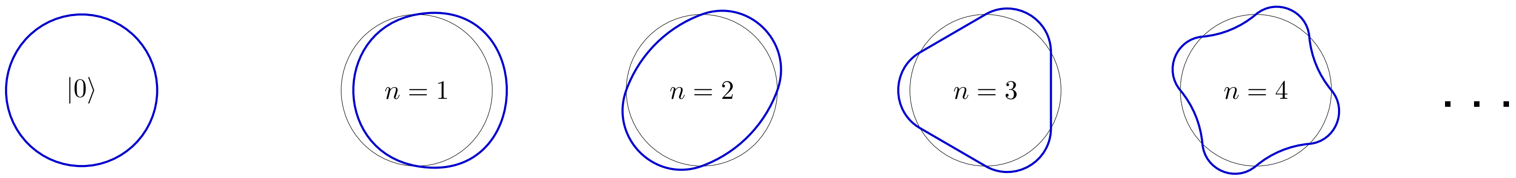
Quantum Corrections

Fluctuations

[Frolov, Tseytlin hep-th/0306130]

Classical solution is saddle point of action. Quantum states accumulate.

Find modes by diagonalising second variation of action: $S_n = \delta^2 S/(\delta\vec{X})^2$.



Energies of string modes similar to $e_n \approx \sqrt{1 + (\lambda/J^2)n^2}$.

[Berenstein Maldacena Nastase]

Energy Shifts

[Frolov, Tseytlin hep-th/0306130] [Frolov Park Tseytlin] [NB, Tseytlin hep-th/0509084]

Generically $E = JE^{(0)}(\sqrt{\lambda}/J) + E^{(1)}(\sqrt{\lambda}/J) + \dfrac{1}{J}E^{(2)}(\sqrt{\lambda}/J) + \ldots$.

One-loop energy shift $E^{(1)}$: Sum over string modes

$$E^{(1)} = \frac{1}{2}\sum_n e_n^{\mathrm{b}} - \frac{1}{2}\sum_n e_n^{\mathrm{f}}.$$

Open Questions

Obtained perturbative energy for one state, but:

- hard to find suitable ansätze (trial and error),
- hard to solve more general ansaetze.
- no two-loop results available.

More generic questions:

- What is the structure of the spectrum?
- Why elliptic functions $\mathrm{am}, \mathrm{K}, \mathrm{E}$?

Situation improved by **integrability:**

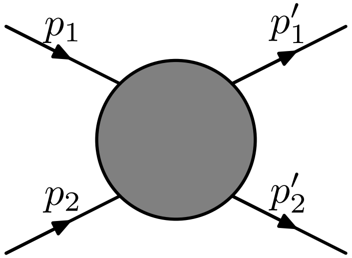
- Generic solution in terms of spectral curves.
- Ansätze based on physical properties.
- Understand structure of the spectrum.

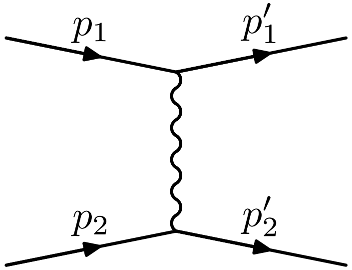
Details in Lecture II.

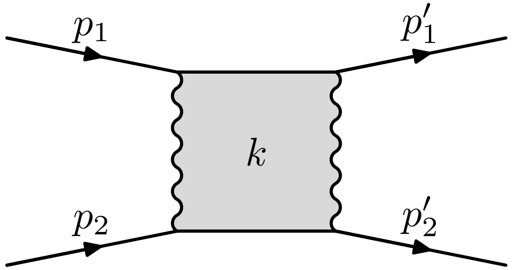
Anomalous Dimensions

Momentum Space

Usually: Scatter objects with definite momenta



$= g^2$


$+ g^4 \int d^d k$


$+ \dots$

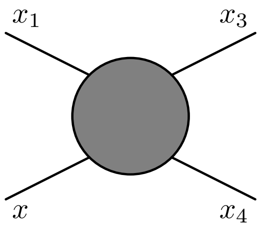
- Integration over loop momenta.
- Field propagator

$$\Delta_m(p) = \frac{1}{p^2 + m^2}\,.$$

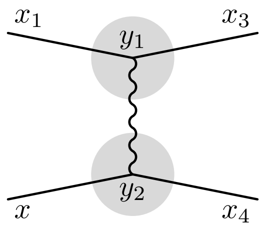
- Order in perturbation theory $g^{2\ell+E-2C}$: Loops are suppressed.
- Gauge theory: External legs on-shell and spins transverse.

Position Space

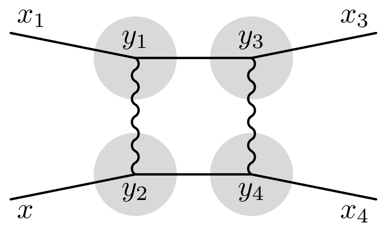
Dual perspective: Scatter objects with definite positions



$$= g^2 \int d^{2d}y$$



$$+ g^4 \int d^{4d}y$$



$$+ \dots$$

- Integration over vertex positions.
- Massive/massless propagator (useful for massless theory, CFT)

$$\Delta_m(x,y) = \frac{\mathrm{K}_{d/2-1}(m|x-y|)}{2\pi(2\pi|x-y|/m)^{d/2-1}}, \qquad \Delta(x,y) = \frac{\Gamma(d/2-1)}{4\pi^{d/2}|x-y|^{d-2}}.$$

- Order in perturbation theory $g^V = g^{2\ell+E-2C}$: Vertices are suppressed. One loop is equivalent to two vertices. One "loop" means $\mathcal{O}(g^2)$!
- Gauge theory: Correlators of gauge invariant local operators $\mathcal{O}(x)$

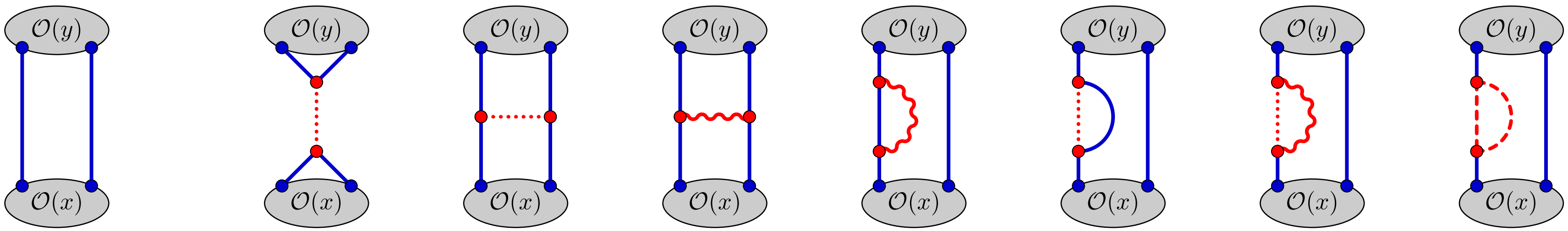
$$F(x_1, x_2, x_3, \dots) = \big\langle \mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\,\mathcal{O}_3(x_3)\dots\big\rangle.$$

A Sample Operator

Local, gauge invariant combination of the fields, e.g.

$$\mathcal{O}^{\text{bare}}_{kl}(x) = \operatorname{Tr} \Phi_k(x)\, \Phi_l(x).$$

Two-point function at one loop. Diagrams:



Correlator (in dimensional reduction scheme) [Bianchi, Kovacs Rossi, Stanev]

$$\left\langle \mathcal{O}^{\text{bare}}_{kl}(x)\, \mathcal{O}^{\text{bare}}_{mn}(y) \right\rangle = \frac{2(1-1/N^2)}{|x-y|^{4-4\epsilon}} \left(\delta_{k\{m}\delta_{n\}l} - \frac{6g^2\, \delta_{kl}\delta_{mn}}{\epsilon |x-y|^{-2\epsilon}} + \ldots \right).$$

Diverges as regulator $\epsilon \to 0$. Renormalisation!

Renormalisation and Mixing

Correlator

$$\left\langle \mathcal{O}^{\mathrm{bare}}_{kl}(x)\,\mathcal{O}^{\mathrm{bare}}_{mn}(y)\right\rangle = \frac{2(1-1/N^2)}{|x-y|^{4-4\epsilon}}\left(\delta_{k\{m}\delta_{n\}l} - \frac{6g^2\,\delta_{kl}\delta_{mn}}{\epsilon|x-y|^{-2\epsilon}} + \ldots\right).$$

Renormalisation: coefficients divergent & unphysical

$$\mathcal{O}_{kl} = \mathcal{O}^{\mathrm{bare}}_{kl} + \frac{g^2}{2\epsilon}\,\delta_{kl}\delta_{mn}\mathcal{O}^{\mathrm{bare}}_{mn} + \ldots$$

Mixing: Correlotor is non-diagonal $\left\langle \mathcal{O}_{11}(x)\,\mathcal{O}_{22}(y)\right\rangle \neq 0$

$$\mathcal{Q}_{kl} = \mathcal{O}_{kl} - \tfrac{1}{6}\delta_{kl}\delta_{mn}\mathcal{O}_{mn}, \qquad \mathcal{K} = \delta_{mn}\mathcal{O}_{mn}.$$

Here: Mixing resolved by representation of $\mathfrak{so}(6)$; 20-plet $\mathcal{Q}_{kl}$, singlet $\mathcal{K}$.

Usually: Mixing among many states with equal quantum numbers.

Anomalous Dimensions

Protected operator. Scaling dimension $D = 2$ is exact (CFT) [D'Hoker Freedman Skiba]

$$\langle \mathcal{Q}_{kl}(x)\, \mathcal{Q}_{mn}(y) \rangle = \frac{2(1 - 1/N^2)}{|x - y|^4}\,, \qquad D = 2.$$

Generic non-protected operator

$$\begin{aligned}
\langle \mathcal{K}(x)\, \mathcal{K}(y) \rangle &= \frac{12(1 - 1/N^2)}{|x - y|^4} \left(1 + 6g^2 \, \frac{1 - |x - y|^{2\epsilon}}{\epsilon} + \ldots \right) \\
&= \frac{12(1 - 1/N^2)}{|x - y|^4} \left(1 + 6g^2 \log \frac{1}{|x - y|^2} + \ldots \right) \\
&= \frac{12(1 - 1/N^2)}{|x - y|^{2(2 + 6g^2 + \ldots)}}
\end{aligned}$$

Scaling dimension $D = 2 + 6g^2 + \ldots$ receives quantum correction. [Anselmi Grisaru Johansen]

Fields and Charges

We need large charge J of $\mathfrak{so}(6)$ (on S^5) to compare to strings.

Charges of the fields:

field	D_0	$\mathfrak{so}(3,1)$	$\mathfrak{so}(6)$
Φ	1	0	1
Ψ	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\mathcal{D}$	1	1	0

We should consider local operators consisting of many fields. [Berenstein Maldacena Nastase][Frolov Tseytlin]

Huge combinatorial problem to

- enumerate all operators which mix,
- evaluate Feynman diagrams (even at tree level),
- resolve mixing.

Some simplification from planar limit.

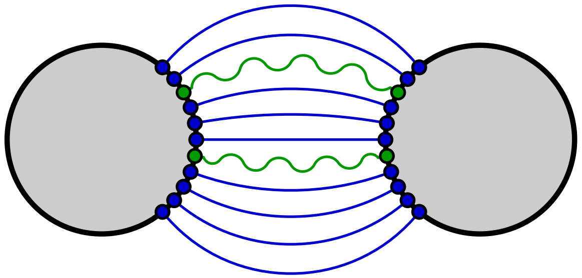
Loop Expansion, Genus Expansion

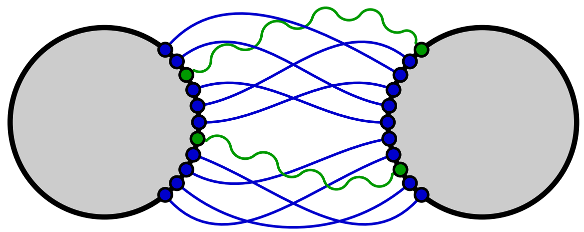
't Hooft large-N limit, genus expansion $1/N^{2h}$ for genus h diagram.
Feynman diagrams expanded in g and $1/N$:

planar $\mathcal{O}(1/N^0)$
no crossing lines

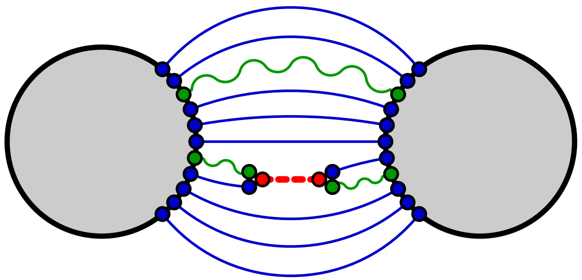
non-planar $\mathcal{O}(1/N^{2h})$
$1/N^2$ per crossing

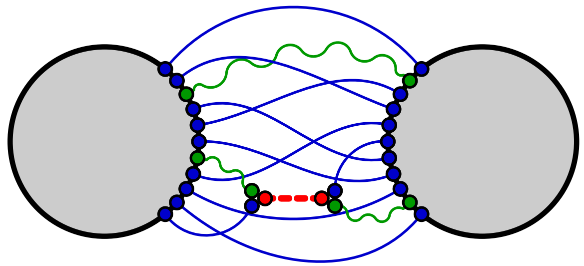
tree level $\mathcal{O}(g^0)$
no vertices





higher loop $\mathcal{O}(g^{2\ell})$
g per 3-vertex
g^2 per 4-vertex





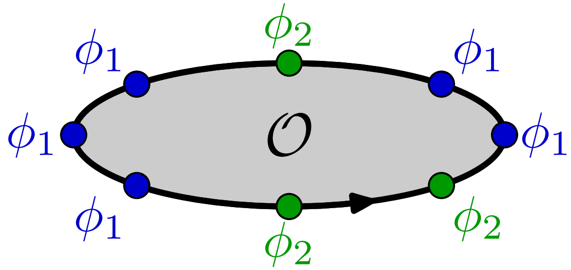
Consider only planar $\mathcal{O}(1/N^0)$ graphs at arbitrary loop order $\mathcal{O}(g^{2\ell})$.

Spin Chains

Single-trace operator, two complex scalars ϕ_1, ϕ_2 ($\mathfrak{su}(2)$ sector) [Minahan Zarembo]

$$\mathcal{O} = \operatorname{Tr} \phi_1\phi_1\phi_2\phi_1\phi_1\phi_1\phi_2\phi_2$$

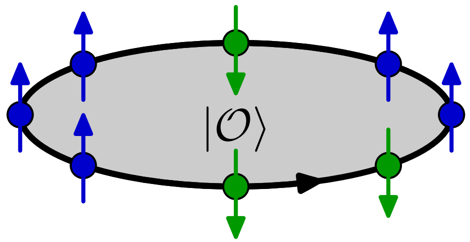
Length L: # of fields



Identify $\phi_1 = |\uparrow\rangle$, $\phi_2 = |\downarrow\rangle$

$$|\mathcal{O}\rangle = |\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\downarrow\downarrow\rangle$$

Length L: # of sites



Mixing of states $|\mathcal{O}\rangle = \psi_1|\mathcal{O}_1\rangle + \psi_2|\mathcal{O}_2\rangle + \ldots$ with equal $n_\uparrow, n_\downarrow$.
Multi-trace operators: $\mathcal{O} = \mathcal{O}_1\,\mathcal{O}_2 \ldots$ [not important in planar limit].

Dilatation Generator

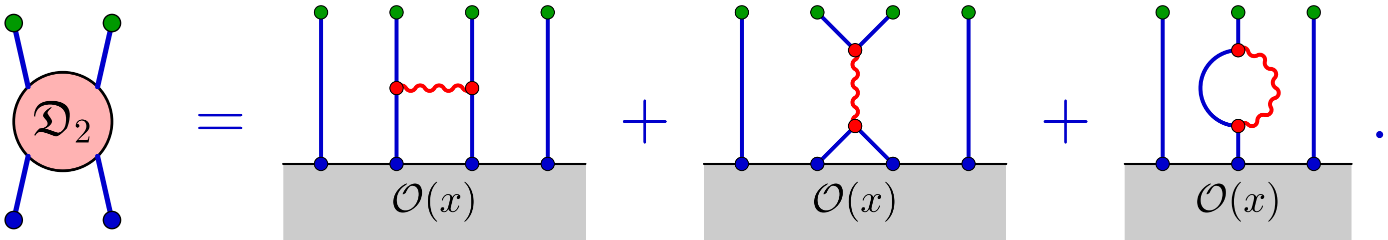
Scaling dimensions $D_{\mathcal{O}}(g)$ as eigenvalues of the dilatation generator $\mathfrak{D}(g)$

$$\mathfrak{D}(g)\,\mathcal{O} = D_{\mathcal{O}}(g)\,\mathcal{O}.$$

Spin chain picture: Hamiltonian $\delta\mathfrak{D} = g^2\mathcal{H}$ & energies $\delta D = g^2 E$.

At **leading order** (one loop): Interactions of nearest-neighbours

$\mathcal{H}_0 =$



Regularised action of $\delta\mathfrak{D}$ in $\mathfrak{su}(2)$ sector: Heisenberg $\mathrm{XXX}_{1/2}$ chain [Minahan Zarembo]

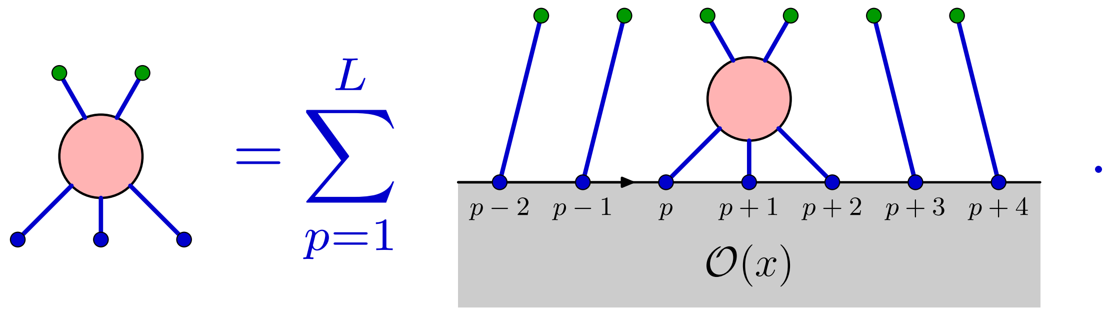
$$\mathcal{H} = \sum_{p=1}^{L} \left(\mathcal{I}_{p,p+1} - \mathcal{P}_{p,p+1}\right) = \sum_{p=1}^{L} \tfrac{1}{2}\left(1 - \vec{\sigma}_p \cdot \vec{\sigma}_{p+1}\right).$$

Higher-Loop Dilatation Generator

Quantum corrections to the dilatation generator: [NB Kristjansen Staudacher][NB hep-th/0310252]

$$\mathfrak{D}(g) = \mathfrak{D}_0 + g^2 \mathfrak{D}_2 + g^3\mathfrak{D}_3 + g^4\mathfrak{D}_4 + \ldots .$$

Interaction with I in legs & O out legs is of order $\mathcal{O}(g^{I+O-2})$.



- Action is homogeneous (along spin chain),
- local (in perturbation theory for sufficiently long chains),
- long-ranged (range grows with order; long-ranged at finite coupling g)
- dynamic (sites can be created or annihilated).

Application of Dilatation Generator

Scalars without derivatives: $\mathfrak{D}_{2(12)} = \mathcal{I}_{(12)} - \mathcal{P}_{(12)} + \frac{1}{2}\mathcal{K}_{(12)}$. [Minahan Zarembo]

Example: Two scalars $\mathcal{O}_{kl} = \operatorname{Tr} \Phi_k \Phi_l$:

$$\mathfrak{D}_2 \mathcal{O}_{kl} = 2\mathcal{O}_{kl} - 2\mathcal{O}_{lk} + \delta_{kl}\delta_{mn}\mathcal{O}_{mn} = \delta_{kl}\delta_{mn}\mathcal{O}_{mn}$$

Eigenvalue $D_2 = 0$: Eigenstate: $\mathcal{Q}_{kl} = \mathcal{O}_{kl} - \frac{1}{6}\delta_{mn}\mathcal{O}_{mn}$,
Eigenvalue $D_2 = 6$: Eigenstate: $\mathcal{K} = \delta_{mn}\mathcal{O}_{mn}$.

Example: State of $\mathfrak{su}(2)$ spin chain $\operatorname{Tr} {\phi_1}^2 {\phi_2}^2 + \ldots$:

$$\mathfrak{D}_2 \operatorname{Tr} \phi_1\phi_1\phi_2\phi_2 = +2 \operatorname{Tr} \phi_1\phi_1\phi_2\phi_2 - 2 \operatorname{Tr} \phi_1\phi_2\phi_1\phi_2,$$

$$\mathfrak{D}_2 \operatorname{Tr} \phi_1\phi_2\phi_1\phi_2 = -4 \operatorname{Tr} \phi_1\phi_1\phi_2\phi_2 + 4 \operatorname{Tr} \phi_1\phi_2\phi_1\phi_2.$$

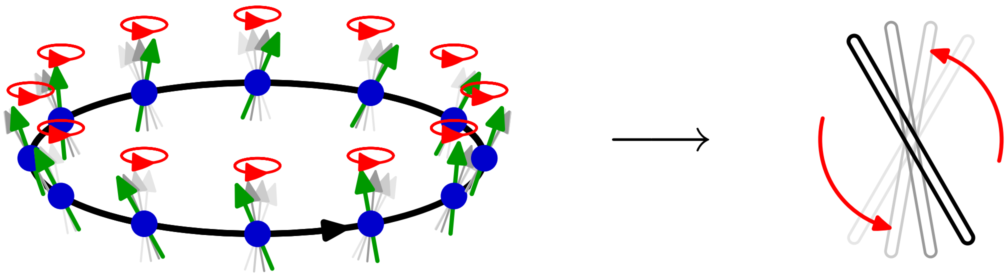
Eigenvalue $D_2 = 0$: Eigenstate: $\mathcal{O} = 2 \operatorname{Tr} \phi_1\phi_1\phi_2\phi_2 + \operatorname{Tr} \phi_1\phi_2\phi_1\phi_2$,
Eigenvalue $D_2 = 6$: Eigenstate: $\mathcal{O} = \operatorname{Tr} \phi_1\phi_1\phi_2\phi_2 - \operatorname{Tr} \phi_1\phi_2\phi_1\phi_2$.

Duality to Strings

Duality to classical strings in thermodynamic limit: [Frolov,Tseytlin hep-th/0304255][NB, Minahan Staudacher Zarembo]

- Number of spin sites $L \to \infty$.
- Number of spin flips $K \to \infty$.
- Ratio K/L fixed.
- Coherent spins.

Coherent spins for states $|\uparrow\rangle, |\downarrow\rangle$ specified by points on S^3 [Kruczenski hep-th/0311203]



Thermodynamic limit: Spinning string on S^3.

Effective theory in thermodynamic limit: Landau-Lifshitz sigma model.

Reliable & exact description: Integrability & Bethe equations.

Summary Gauge Theory

Spectrum via two-point functions

- Hard combinatorics.
- Loop expansion tedious.

Spectrum via dilatation generator/spin chain Hamiltonian

- Combinatorics improved, but still hard for long operators.
- Loop expansion improved, but can be constructed to some extent.

Spectrum via effective Hamiltonians from Landau-Lifshitz model

- No more combinatorics, but only approximate results for long operators.
- Loop expansion simpler, but needs input.

Situation improved by **integrability:**

- Bethe equations to replace combinatorics.
- Loop expansion trivial. Bethe equations for finite coupling may exist.

Details in Lectures III & IV.

Outlook: Bethe Equations

Spin Flips as Excitations

Identify $\phi_1 = |\downarrow\rangle$, $\phi_2 = |\uparrow\rangle$. Vacuum state: [Bethe, Z. Phys. A71, 205 (1931)]

$$|0\rangle = |\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\rangle.$$

A spin flip as a particle (momentum p or rapidity u):

$$|p\rangle = \sum\nolimits_k e^{ipk}\,|\downarrow\downarrow\cdots\overset{\overset{k}{\downarrow}}{\uparrow}\cdots\downarrow\downarrow\rangle.$$

Dispersion relation

$$\mathcal{H}|p\rangle = e(p)|p\rangle.$$

Additive energy (anomalous dimension)

$$\mathcal{H}|p_1,\ldots,p_K\rangle = E|p_1,\ldots,p_K\rangle, \qquad E = \sum_{j=1}^{K} e(p_j)$$

Asymptotic Bethe Equations in a Sector

Higher-loop Bethe equations for sector of $\{\phi_1, \phi_2\}$. [NB, Dippel Staudacher][Staudacher hep-th/0412188]

$$1 = \frac{x(u_k - \frac{i}{2})^L}{x(u_k + \frac{i}{2})^L} \prod_{\substack{j=1 \\ j\neq k}}^{K} \frac{u_k - u_j + i}{u_k - u_j - i}, \qquad x(u) = \tfrac{1}{2}u + \tfrac{1}{2}u\sqrt{1 - 2g^2/u^2}\,.$$

Momentum constraint (cyclity of trace) and higher-loop scaling dimension:

$$\prod_{j=1}^{K} \frac{x(u_j - \frac{i}{2})}{x(u_j + \frac{i}{2})} = 1, \qquad D = L + g^2 \sum_{j=1}^{K} \left(\frac{i}{x(u_j + \frac{i}{2})} - \frac{i}{x(u_j - \frac{i}{2})} \right).$$

Homework: Check that the equations for $L = 4$, $K = 2$ are solved by

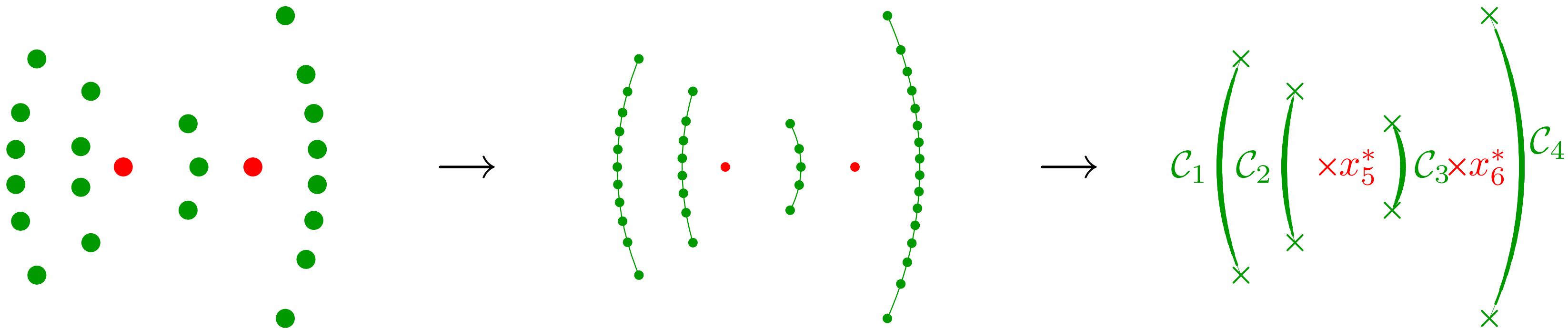
$$u_{1,2} = \pm \frac{1}{\sqrt{12}} \left(1 + 4g^2 - 5g^4 + \ldots\right), \quad D = 4 + 6g^2 - 12g^4 + 42g^6 + \ldots$$

Thermodynamic Limit of Bethe equations

Make contact to strings when: [Frolov,Tseytlin hep-th/0304255][NB, Minahan Staudacher Zarembo]

- Number of spin sites $L \to \infty$.
- Number of spin flips $K \to \infty$.
- Ratio K/L fixed.
- Coherent spins.

Rapidities distribute along lines in complex plane [Sutherland PRL 74,816][NB, Minahan Staudacher Zarembo]

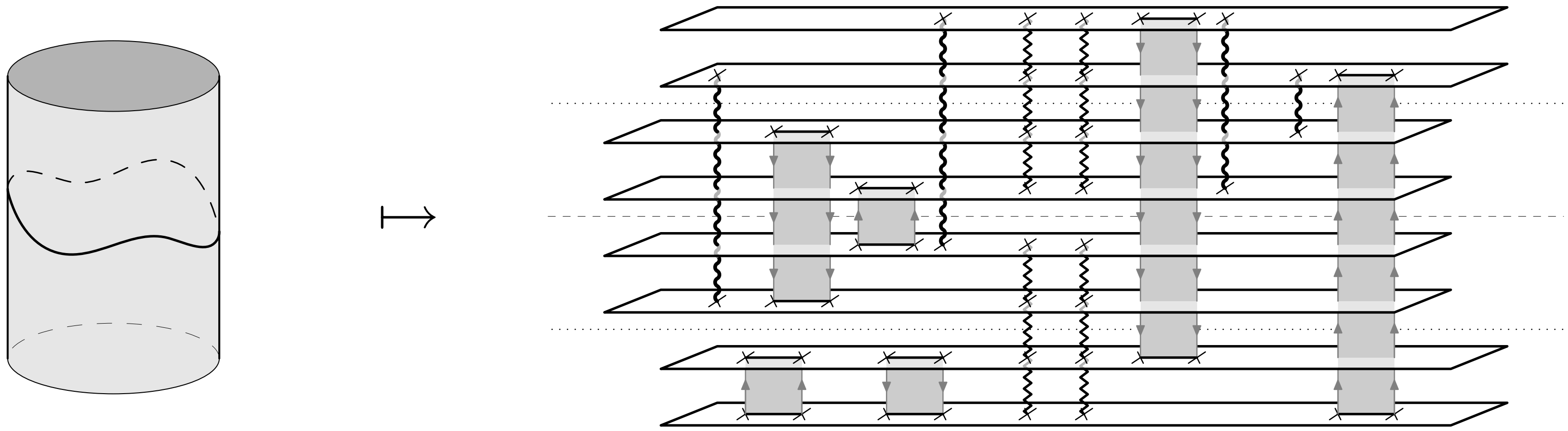


Condensation of roots into branch cuts.

Outlook: Spectral Curves

Spectral Transformation

From embedding of world-sheet to spectral curve [Kazakov, Marshakov Minahan, Zarembo]



Spectral curve encodes conserved charges of a string solution.

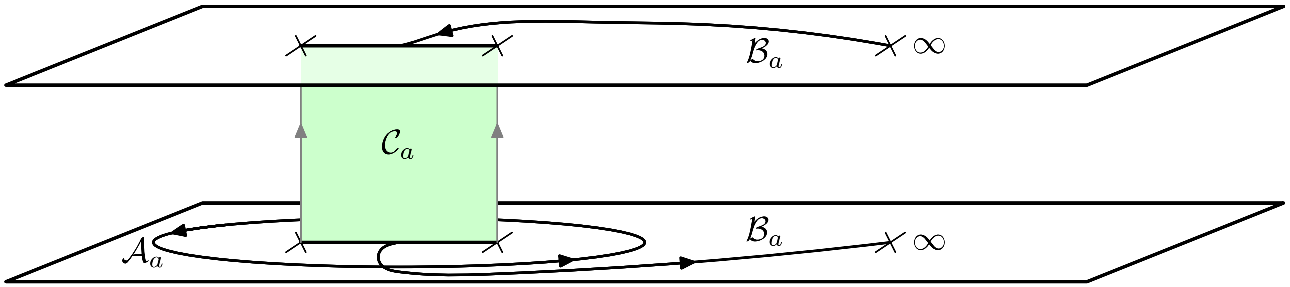
Can read off Noether charges (spins & energy) from curve.

Study spectral curves to study the spectrum of classical strings.

Cycles and Modes

Branch cuts: "mode number" $n_a \in \mathbb{Z}$ and "amplitude" $K_a \in \mathbb{R}$

$$\oint_{\mathcal{A}_a} dp = 0, \quad n_a = \frac{1}{2\pi}\int_{\mathcal{B}_a} dp, \quad K_a = -\frac{1}{2\pi i}\oint_{\mathcal{A}_a}\left(1-\frac{1}{x^2}\right)p(x)\,dx.$$

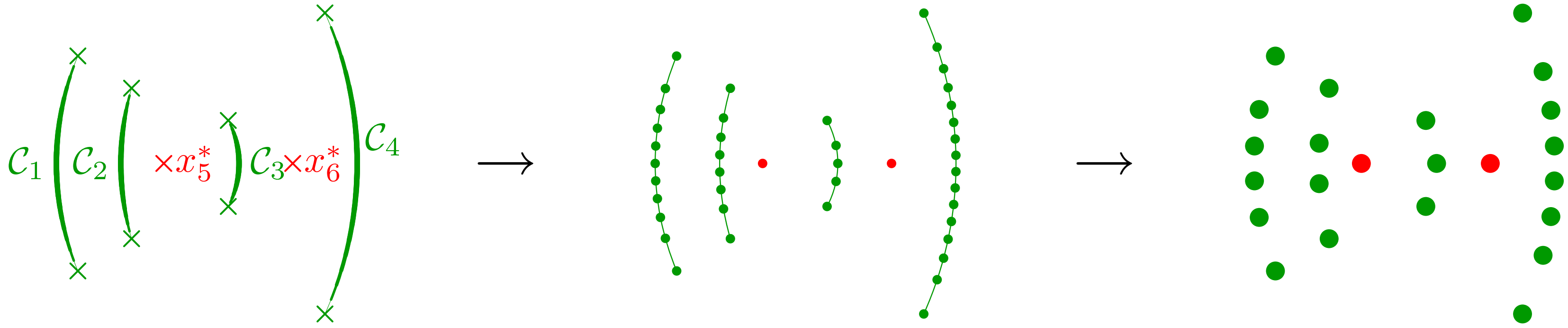


To construct the curve for any solution:

- Make a generic ansatz with a branch cut for each excited string mode.
- Fix the mode number on each cut.
- Fix the amplitude for each cut.
- Read off charges and energy.

Quantisation/Discretisation

In quantum theory: Branch cuts/poles discretise into a set of Bethe roots



Some educated guesses for Bethe equations exist. [Arutyunov Frolov Staudacher]

Complete Asymptotic Bethe Equations

Asymptotic Bethe Equations for $\mathcal{N}=4$ SYM

Asymptotic Bethe equations derived from S-matrix. [NB, Staudacher hep-th/0504190] [NB hep-th/0511082]

coupling constant

$$g^2 = \frac{\lambda}{8\pi^2}$$

transformation between u and x

$$x(u) = \tfrac{1}{2}u + \tfrac{1}{2}u\sqrt{1-2g^2/u^2}\,, \qquad u(x) = x + \frac{g^2}{2x}$$

$x^\pm$ parameters

$$x^\pm = x(u \pm \tfrac{i}{2})$$

spin chain momentum

$$e^{ip} = \frac{x^+}{x^-}$$

local charges

$$q_r(x) = \frac{1}{r-1}\left(\frac{i}{(x^+)^{r-1}} - \frac{i}{(x^-)^{r-1}}\right), \qquad Q_r = \sum_{j=1}^{K_4} q_r(x_{4,j})$$

anomalous dimension

$$\delta D = g^2 Q_2 = g^2 \sum_{j=1}^{K_4}\left(\frac{i}{x^+_{4,j}} - \frac{i}{x^-_{4,j}}\right)$$

Bethe equations

$$1 = \prod_{j=1}^{K_4} \frac{x^+_{4,j}}{x^-_{4,j}}$$

$$1 = \prod_{\substack{j=1\\ j\neq k}}^{K_2} \frac{u_{2,k}-u_{2,j}-i}{u_{2,k}-u_{2,j}+i} \prod_{j=1}^{K_3} \frac{u_{2,k}-u_{3,j}+\frac{i}{2}}{u_{2,k}-u_{3,j}-\frac{i}{2}}$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{3,k}-u_{2,j}+\frac{i}{2}}{u_{3,k}-u_{2,j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{3,k}-x^+_{4,j}}{x_{3,k}-x^-_{4,j}}$$

$$1 = \left(\frac{x^-_{4,k}}{x^+_{4,k}}\right)^{K_0} \prod_{\substack{j=1\\ j\neq k}}^{K_4} \left(\frac{u_{4,k}-u_{4,j}+i}{u_{4,k}-u_{4,j}-i}\,\sigma^2(x_{4,k},x_{4,j})\right) \prod_{j=1}^{K_3} \frac{x^-_{4,k}-x_{3,j}}{x^+_{4,k}-x_{3,j}} \prod_{j=1}^{K_5} \frac{x^-_{4,k}-x_{5,j}}{x^+_{4,k}-x_{5,j}}$$

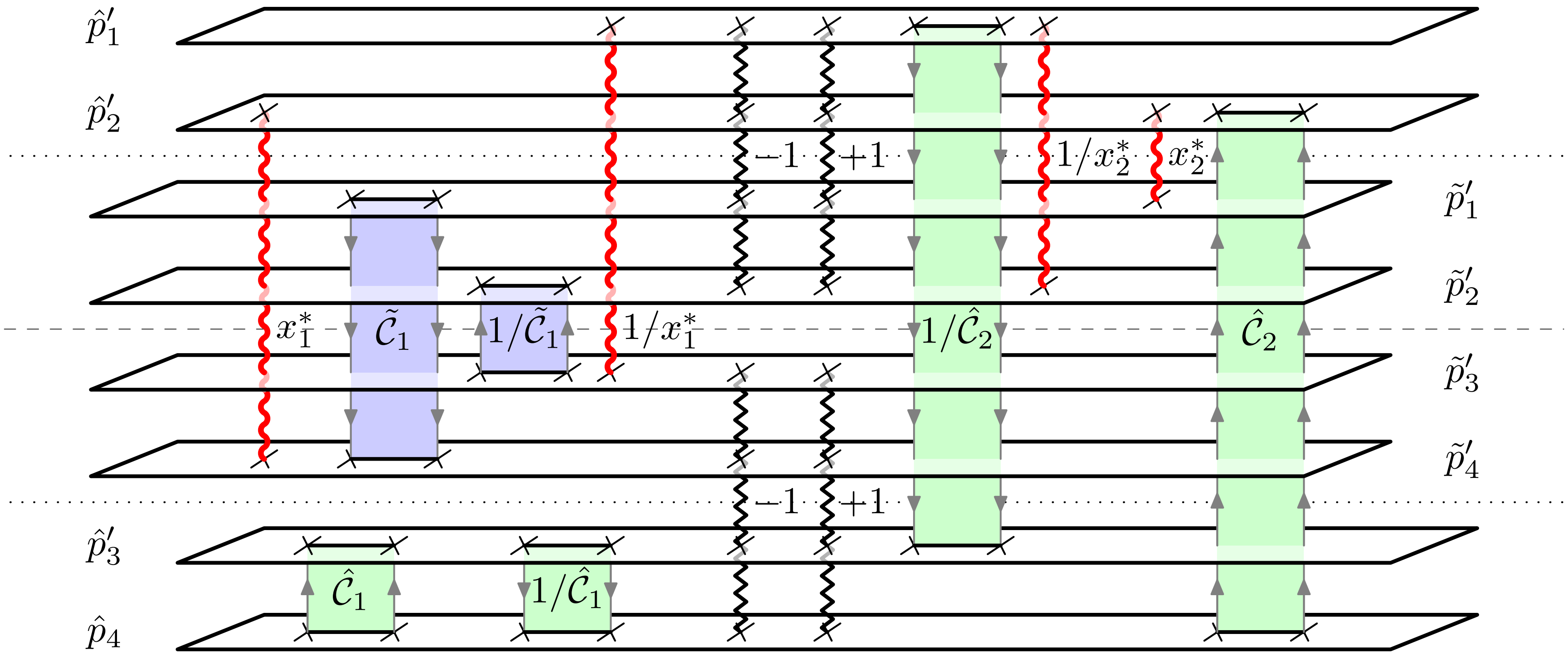
$$1 = \prod_{j=1}^{K_6} \frac{u_{5,k}-u_{6,j}+\frac{i}{2}}{u_{5,k}-u_{6,j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{5,k}-x^+_{4,j}}{x_{5,k}-x^-_{4,j}}$$

$$1 = \prod_{\substack{j=1\\ j\neq k}}^{K_6} \frac{u_{6,k}-u_{6,j}-i}{u_{6,k}-u_{6,j}+i} \prod_{j=1}^{K_5} \frac{u_{6,k}-u_{5,j}+\frac{i}{2}}{u_{6,k}-u_{5,j}-\frac{i}{2}}$$

some free parameters in $\sigma(x_1,x_2)$, for $\mathcal{N}=4$ SYM: $\sigma=1$

Should work asymptotically to $\mathcal{O}(g^{2K_0})$. Tested at three loops.
Much better than by field theory and Feynman diagrams.

Complete Algebraic Curve



- $p'(z)$ is a curve of degree $4+4$. [NB, Kazakov Sakai, Zarembo]
- Bosonic modes: Square roots, branch cuts (Bose condensates).
- Fermionic excitations: Poles (Pauli principle).
- Branch cuts/poles have associated mode number n (position in $\mathbb{C}$).
- Branch cuts have associated amplitude (length).

Bethe Equations for Quantum Strings

Conjectured Bethe equations for quantum strings

[Arutyunov Frolov Staudacher][NB, Staudacher hep-th/0504190]

coupling constant

$$g^2 = \frac{\lambda}{8\pi^2}$$

transformation between u and x

$$x(u) = \tfrac{1}{2}u + \tfrac{1}{2}u\sqrt{1 - 2g^2/u^2}\,, \qquad u(x) = x + \frac{g^2}{2x}$$

$x^\pm$ parameters

$$x^\pm = x(u \pm \tfrac{i}{2})$$

spin chain momentum

$$e^{ip} = \frac{x^+}{x^-}$$

local charges

$$q_r(x) = \frac{1}{r-1}\left(\frac{i}{(x^+)^{r-1}} - \frac{i}{(x^-)^{r-1}}\right), \qquad Q_r = \sum_{j=1}^{K_4} q_r(x_{4,j})$$

anomalous dimension

$$\delta D = g^2 Q_2 = g^2 \sum_{j=1}^{K_4}\left(\frac{i}{x^+_{4,j}} - \frac{i}{x^-_{4,j}}\right)$$

Bethe equations

$$1 = \prod_{j=1}^{K_4} \frac{x^+_{4,j}}{x^-_{4,j}}$$

$$1 = \prod_{\substack{j=1\\ j\neq k}}^{K_2} \frac{u_{2,k} - u_{2,j} - i}{u_{2,k} - u_{2,j} + i} \prod_{j=1}^{K_3} \frac{u_{2,k} - u_{3,j} + \frac{i}{2}}{u_{2,k} - u_{3,j} - \frac{i}{2}}$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{3,k} - u_{2,j} + \frac{i}{2}}{u_{3,k} - u_{2,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{3,k} - x^+_{4,j}}{x_{3,k} - x^-_{4,j}}$$

$$1 = \left(\frac{x^-_{4,k}}{x^+_{4,k}}\right)^{K_0} \prod_{\substack{j=1\\ j\neq k}}^{K_4} \left(\frac{u_{4,k} - u_{4,j} + i}{u_{4,k} - u_{4,j} - i}\, \sigma^2(x_{4,k}, x_{4,j})\right) \prod_{j=1}^{K_3} \frac{x^-_{4,k} - x_{3,j}}{x^+_{4,k} - x_{3,j}} \prod_{j=1}^{K_5} \frac{x^-_{4,k} - x_{5,j}}{x^+_{4,k} - x_{5,j}}$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{5,k} - u_{6,j} + \frac{i}{2}}{u_{5,k} - u_{6,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{5,k} - x^+_{4,j}}{x_{5,k} - x^-_{4,j}}$$

$$1 = \prod_{\substack{j=1\\ j\neq k}}^{K_6} \frac{u_{6,k} - u_{6,j} - i}{u_{6,k} - u_{6,j} + i} \prod_{j=1}^{K_5} \frac{u_{6,k} - u_{5,j} + \frac{i}{2}}{u_{6,k} - u_{5,j} - \frac{i}{2}}$$

function $\sigma(x_1, x_2)$ for quantum strings, coefficients: $c_{r,s} = \delta_{r+1,s} + \mathcal{O}(1/g)$

$$\sigma(x_1, x_2) = \exp\left(i \sum_{r<s=2}^{\infty} (\tfrac{1}{2}g^2)^{(r+s-1)/2}\, c_{r,s}(g)\, \big(q_r(x_1)\, q_s(x_2) - q_r(x_2)\, q_s(x_1)\big)\right)$$

Various $\mathcal{O}(1/L)$ tests

[NB, Tseytlin Zarembo][Hernández, López Periáñez, Sierra][NB, Freyhult hep-th/0506243][Schäfer-Nameki Zamaklar, Zarembo][NB, Tseytlin hep-th/0509084]