Primer in Special Relativity

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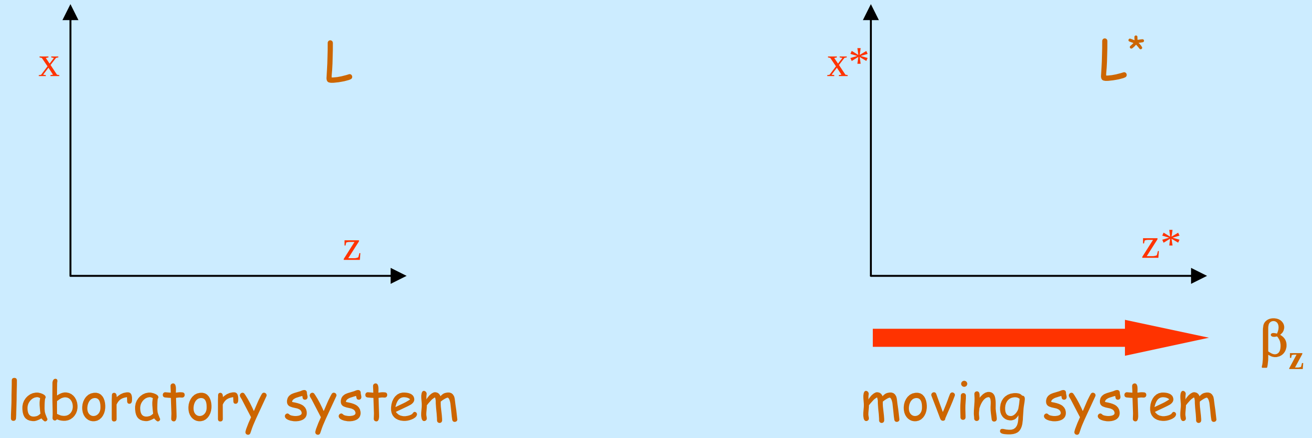
Lorentz transformation:



$$\begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & \beta\gamma \\ 0 & 0 & \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x^* \\ y^* \\ z^* \\ ct^* \end{pmatrix}$$

$$\beta = \frac{v}{c} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \beta = \sqrt{1 - \frac{1}{\gamma^2}}$$

Contraction-dilatation

Lorentz contraction:

consider rod in lab system of length $\Delta z = z_2 - z_1$

$$\Delta z = \gamma\,(z_2^* + v_z t^*) - \gamma\,(z_1^* + v_z t^*) = \gamma\,\Delta z^*$$

Time dilatation:

consider two events happening at same place

$$\Delta t = t_2 - t_1 = \gamma\left(t_2^* + \frac{\beta z_2^*}{c}\right) - \gamma\left(t_1^* + \frac{\beta z_1^*}{c}\right) = \gamma\,\Delta t^*$$

Lorentz transformations of fields

$$\begin{pmatrix} E_x^* \\ E_y^* \\ E_z^* \\ cB_x^* \\ cB_y^* \\ cB_z^* \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & 0 & -\beta\gamma & 0 \\ 0 & \gamma & 0 & \beta\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \beta\gamma & 0 & \gamma & 0 & 0 \\ -\beta\gamma & 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \\ cB_x \\ cB_y \\ cB_z \end{pmatrix}$$

magnetostatic field in lab system → EM field in particle system

$$\begin{aligned} E_x^* &= -\beta\gamma c B_y \\ E_y^* &= 0 \\ E_z^* &= 0 \end{aligned}$$

we don't really know velocities ! can we express β,γ differently?

space - time

imagine a light flash to appear at time t=0 from the origin of the lab coordinate system

at time t edge of light pulse has expanded to $x^2 + y^2 + z^2 = c^2 t^2$

observing from L* - system, we get from Lorentz transformations

$$x^{*2} + y^{*2} + z^{*2} = c^2 t^{*2} \quad !$$

velocity of light is Lorentz invariant !

4 - vectors

Minkowski combined space-time to form a 4-dimensional coordinate system:

space - time 4-vector $\tilde{\mathbf{s}} = (x^0, x^1, x^2, x^3) = (\mathrm{i}ct, x, y, z)$ world point

all world points = world

variation of world point = world line

world time is defined by $c\tau = \sqrt{-\tilde{\mathbf{s}}^2}$ which is Lorentz invariant

$$\left.\begin{aligned} c\,\mathrm{d}\tau &= \sqrt{c^2(\mathrm{d}t)^2 - (\mathrm{d}x)^2 - (\mathrm{d}y)^2 - (\mathrm{d}z)^2} \\ &= \sqrt{c^2 - (v_x^2 + v_y^2 + v_z^2)}\,\mathrm{d}t \\ &= \sqrt{c^2 - v^2}\,\mathrm{d}t = \sqrt{1 - \beta^2}\,c\,\mathrm{d}t, \end{aligned}\right\} \quad \mathrm{d}\tau = \frac{1}{\gamma}\,\mathrm{d}t$$

γ: relativistic factor

length of 4 - vectors

length of a 4-vector is Lorentz invariant

examples $\tilde{\mathbf{s}}^2 = -c^2\tau^2$

actually product of any two 4-vectors is Lorentz invariant

how do we know a vector is a 4-vector?

if the length of a vector is Lorentz invariant, it's a 4-vector

invariance of 4-vectors

$$
\begin{aligned}
\tilde{s}^{*2} &= x^{*2} + y^{*2} + z^{*2} - c^2 t^{*2} \\
&= x^2 + y^2 + (\gamma z - \beta\gamma\, ct)^2 - (-\beta\gamma\, z + \gamma\, ct)^2 \\
&= x^2 + y^2 + z^2 - c^2 t^2 \\
&= \tilde{s}^2
\end{aligned}
$$

any product of two 4-vectors is Lorentz invariant

$$\tilde{a}^* \tilde{b}^* = \tilde{a}\tilde{b}$$

homework?

4-velocity

4-velocity: $\tilde{\mathbf{v}} = \frac{\mathrm{d}\tilde{\mathbf{s}}}{\mathrm{d}\tau} = \gamma \frac{\mathrm{d}\tilde{\mathbf{s}}}{\mathrm{d}t} = \gamma(\mathrm{i}c, \dot{x}, \dot{y}, \dot{z})$

in moving system $(\gamma = 1; \dot{x} = \dot{y} = \dot{z} = 0)$: $\tilde{\mathbf{v}}^2 = -c^2 = \mathrm{const}$

velocity of light is Lorentz invariant !

$$c = 299,792,458\ m/s$$

4-acceleration

$$\tilde{a} = \frac{d\tilde{v}}{d\tau} = \gamma \frac{d}{dt}\left(\gamma \frac{d\tilde{s}}{dt}\right)$$

$$\tilde{a}^2 = \gamma^6 \left\{\mathbf{a}^2 - [\boldsymbol{\beta} \times \mathbf{a}]^2\right\} = \tilde{a}^{*2}$$

prove this as an exercise !

4-momentum

energy - momentum 4-vector: $c\tilde{\mathbf{p}} = (\mathrm{i}E, cp_x, cp_y, cp_z)$

with $E_0 = Amc^2$

$$c^2\tilde{\mathbf{p}}^2 = -E^2 + c^2p_x^2 + c^2p_y^2 + c^2p_z^2$$

total energy

$$E^2 = c^2p^2 + A^2m^2c^4$$

Relativistic factor depends on particle velocity, but generally we don't know velocities.

look for different expression

relativistic factor

$(\mathrm{i}E, c\mathbf{p})$ → $E^2 = c^2p^2 + A^2m^2c^4$ (1)

$\gamma(\mathrm{i}c, \dot{\mathbf{r}})$ → $-c\gamma E + c\gamma\dot{\mathbf{r}}\mathbf{p} = -cAmc^2$

$$-\gamma E + c\gamma\boldsymbol{\beta}\mathbf{p} = -Amc^2 \rightarrow cp = \frac{\gamma E - Amc^2}{\gamma\beta}$$ since $\mathbf{p}\parallel\boldsymbol{\beta}$

insert into (1) $E^2 = \left(\frac{\gamma E - Amc^2}{\gamma\beta}\right)^2 + (Amc^2)^2$

with $\beta^2\gamma^2 = \gamma^2 - 1$ we get $E - \gamma Amc^2 = 0$ or

relativistic factor

$$\gamma = \frac{E}{Amc^2}$$

Conventions

particles: **electrons, protons, ions**

energy: **eV, keV, MeV, GeV**

1 eV = kin.energy gained while traveling through potential difference of 1 Volt

momentum: **eV/c, keV/c, MeV/c, GeV/c**

mostly we use c**p** for the momentum

proton rest mass: $m_p c^2 = 938.272$ MeV
electron rest mass: $m_e c^2 = 0.510999$ MeV

Momentum

$$cp \approx \sqrt{2Amc^2E_{\text{kin}}} = Amc^2\beta \approx cAmv$$

nonrelativistic case

examples

$$20\ \text{keV A}^+ : A = 40$$
$$Amc^2 = 37531\,\text{MeV} \gg 0.020\,\text{MeV}$$
$$v = c\sqrt{\frac{2 \cdot 0.02}{37531}} = 0.00103c$$

non relativistic

$$400\ \text{keV He}^+ : A = 2$$
$$Amc^2 = 1876.56\,\text{MeV} \gg 0.4\,\text{MeV}$$
$$v = c\sqrt{\frac{2 \cdot 0.4}{1876.56}} = 0.02065c$$

starting to become relativistic

$$20\ \text{MeV electrons}: A = 1$$
$$mc^2 = 0.511\ \text{MeV} \ll 20\ \text{MeV}$$
$$v = c\sqrt{1 - \frac{0.511^2}{20^2}} = 0.99967c$$

highly relativistic

summary of formulas

relativistic factor $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ or $\gamma = \frac{E}{Amc^2} = 1 + \frac{E_{\text{kin}}}{Amc^2}$

total energy $E^2 = c^2p^2 + A^2m^2c^4$

momentum $cp = \beta E$

velocity $\beta = \sqrt{1-\gamma^{-2}} = \sqrt{1 - \frac{1}{\left(1+\frac{E_{\text{kin}}}{Amc^2}\right)}} = \frac{\sqrt{E_{\text{kin}}^2 + 2Amc^2E_{\text{kin}}}}{E_{\text{kin}} + Amc^2}$

Emission of Radiation, spectral and spatial distribution

consider EM wave in particle system $E^* = E_0^* e^{i\Phi^*}$

phase of the wave is: $\Phi^* = \omega^*[t^* - \frac{1}{c}(n_x^* x^* + n_y^* y^* + n_z^* z^*)]$

Phase is product of two 4-vectors! $(i\frac{1}{c}E, \mathbf{p})(ict, \mathbf{r}) = -tE + \mathbf{rp}$

with $E = \hbar\omega$ and $\mathbf{p} = \hbar\mathbf{k} = \hbar k\mathbf{n}$

$$(i\frac{1}{c}E, \mathbf{p})(ict, \mathbf{r}) = -tE + \mathbf{rp} = -t\hbar\omega + \hbar k\mathbf{nr} = -t\hbar\omega + \hbar\frac{\omega}{c}\mathbf{nr}$$

Radiation phase is Lorentz invariant:

$$\omega^*[ct^* - n_x^* x^* - n_y^* y^* - n_z^* z^*] = \omega[ct - n_x x - n_y y - n_z z]$$

Now apply Lorentz transformation and collect coefficients of (t,x,y,z)

Relativistic Doppler Effect

$$\omega^*[(-\beta\gamma z + \gamma ct) - n_x^* x - n_y^* y - n_z^*(\gamma z - \beta\gamma ct)] = \omega[ct - n_x x - n_y y - n_z z]$$

coefficients must be zero !

example ct-term: $\omega^* \gamma(1 + n_z^* \beta) = \omega$

$$\omega = \gamma(1 + n_z^* \beta)\omega^*$$

relativistic Doppler effect

example: Undulator Radiation

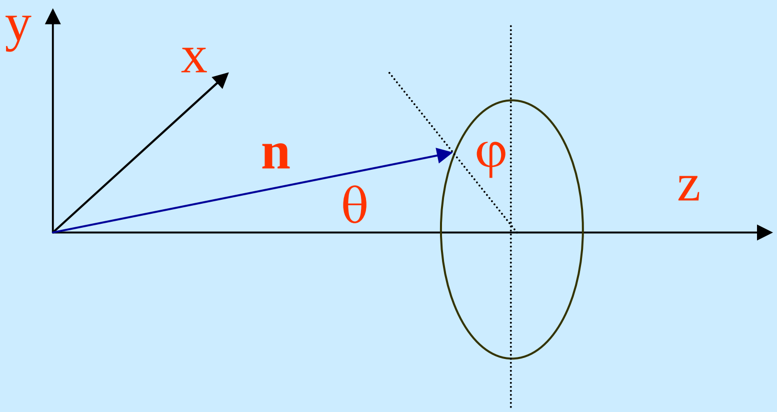
$$\lambda = \frac{\lambda_{\mathrm{u}}}{2\gamma^2}\left(1 + \tfrac{1}{2}\gamma^2\theta^2\right)$$

Collimation

from coefficients of spatial terms, we get:

$$n_{x,y} = \frac{n^*_{x,y}}{\gamma(1+n^*_z\beta)} \quad \text{and} \quad n_z = \frac{\beta+n^*_z}{(1+n^*_z\beta)}$$

n is a unit vector and therefore:

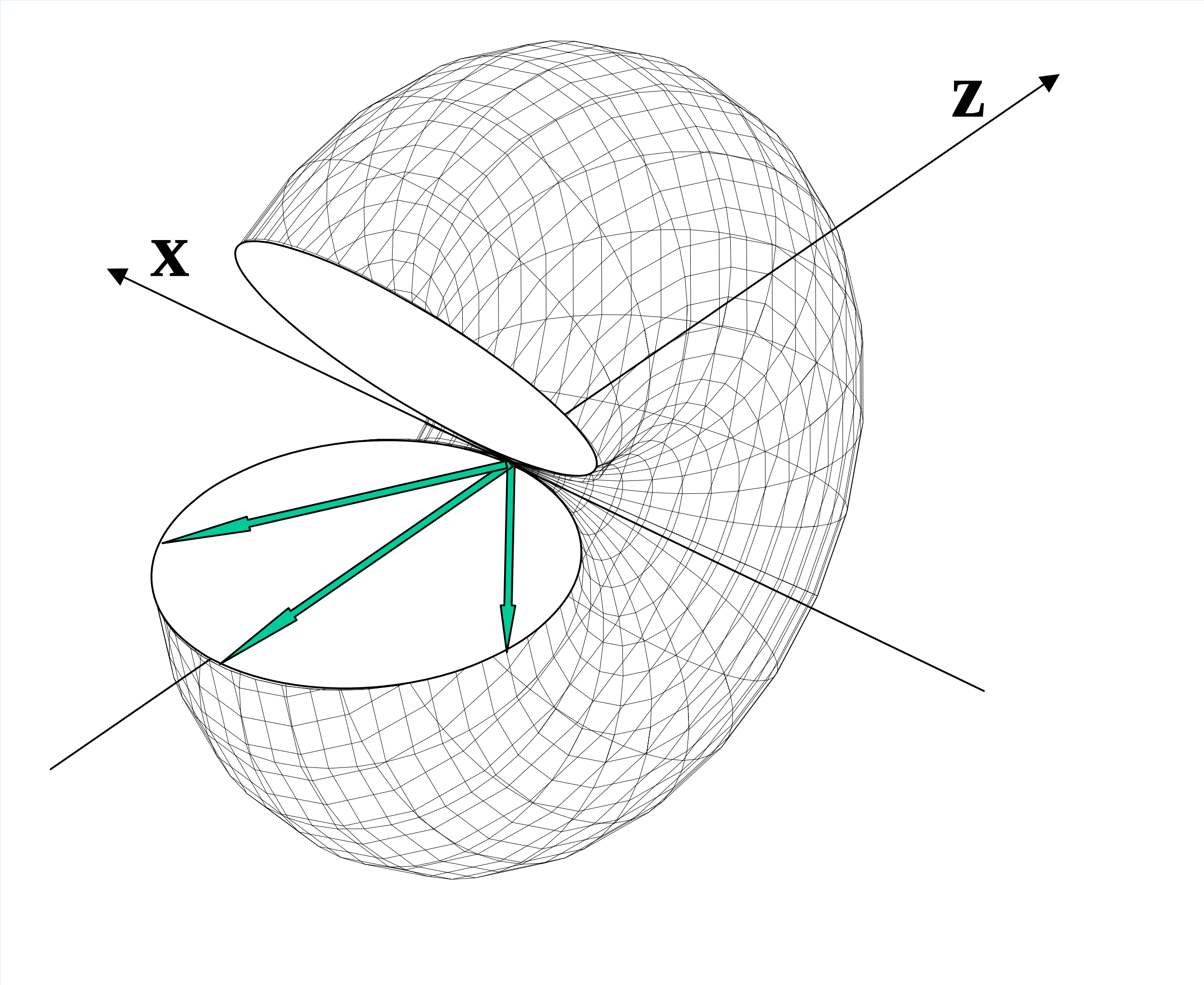


$$\begin{aligned} n_x &= \sin\theta\sin\varphi \\ n_y &= \sin\theta\cos\varphi \\ n_z &= \cos\theta \end{aligned}$$

$\sin\theta \approx \theta \approx \frac{\sin\theta^*}{\gamma(1+\beta\cos\theta^*)}$ or for $-\pi/2 < \theta^* < \pi/2$

$$|\theta| \le \pm\frac{1}{\gamma}$$

radiation emission in particle system



radiation emission in laboratory system