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Algebraic Morse theory and the weak factorization theorem

Jaroslaw Wlodarczyk
Purdue University
Department of Mathematics
West Lafayette IN 47907-2067
United Sates of America

Strada Costiera 11, 34014 Trieste, Italy - Tel. +39 040 2240 111; Fax +39 040 224 163 - sci_info@ictp.it, www.ictp.it

Algebraic Morse theory and the weak factorization theorem

Jarosław Włodarczyk*

Abstract. We develop a Morse-like theory for complex algebraic varieties. In this theory a Morse function is replaced by a $\mathbb{C}^*$-action. The critical points of the Morse function correspond to connected fixed point components. "Passing through the fixed points" induces some simple birational transformations called blow-ups, blow-downs and flips which are analogous to spherical modifications.

In classical Morse theory by means of a Morse function we can decompose the manifold into elementary pieces – "handles". In algebraic Morse theory we decompose a birational map between two smooth complex algebraic varieties into a sequence of blow-ups and blow-downs with smooth centers.

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1. Introduction

We shall work over an algebraically closed field K of characteristic zero. We denote by K^* the multiplicative group of K.

In this paper we outline our proof of the following theorem:

Theorem 1.1 (The Weak Factorization Theorem). 1. *Let $f: X \dashrightarrow Y$ be a birational map of smooth complete varieties over a field of characteristic zero, which is an isomorphism over an open set U. Then f can be factored as*

$$X = X_0 \overset{f_0}{\dashrightarrow} X_1 \overset{f_1}{\dashrightarrow} \cdots \overset{f_{n-1}}{\dashrightarrow} X_n = Y,$$

where each X_i is a smooth complete variety and f_i is a blow-up or blow-down at a smooth center which is an isomorphism over U.

2. *Moreover, if $X \setminus U$ and $Y \setminus U$ are divisors with simple normal crossings, then each $D_i := X_i \setminus U$ is a divisor with simple normal crossings and f_i is a blow-up or blow-down at a smooth center which has normal crossings with components of D_i.*

3. *There is an index $1 \le r \le n$ such that for all $i \le r$ the induced birational map $X_i \overset{f_0}{\dashrightarrow} X$ is a projective morphism and for all $r \le i \le n$ the map $X_i \overset{f_0}{\dashrightarrow} Y$ is projective morphism.*

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4. *The above factorization commutes with any automorphisms ϕ_X of X, and ϕ_Y of Y such that $f \circ \phi_X = \phi_Y \circ f$.*

The theorem was proven in [38] and in [3] in a more general version. The above formulation essentially reflects the statement of the theorem in [3].

The weak factorization theorem extends a theorem of Zariski, which states that any birational map between two smooth complete surfaces can be factored into a succession of blow-ups at points followed by a succession of blow-downs at points. A stronger version of the above theorem, called the strong factorization conjecture, remains open.

Conjecture 1.2 (Strong Factorization Conjecture). Any birational map $f\colon X \dashrightarrow Y$ of smooth complete varieties can be factored into a succession of blow-ups at smooth centers followed by a succession of blow-downs at smooth centers.

Note that both statements are equivalent in dimension 2. One can find the formulation of the relevant conjectures in many papers. Hironaka [17] formulated the strong factorization conjecture. The weak factorization problem was stated by Miyake and Oda [30]. The toric versions of the strong and weak factorizations were also conjectured by Miyake and Oda [30] and are called the strong and weak Oda conjectures. The 3-dimensional toric version of the weak form was established by Danilov [12] (see also Ewald [14]). The weak toric conjecture in arbitrary dimensions was proved in [36] and later independently by Morelli [27], who also claimed to have a proof of the strong factorization conjecture (see also Morelli [28]). Morelli's proof of the weak Oda conjecture was completed, revised and generalized to the toroidal case by Abramovich, Matsuki and Rashid in [4]. A gap in Morelli's proof of the strong Oda conjecture, which went unnoticed in [4], was later found by K. Karu.

The local version of the strong factorization problem was posed by Abhyankar in dimension 2 and by Christensen in general; Christensen has solved it for 3-dimensional toric varieties [8]. The local version of the weak factorization problem (in characteristic 0) was solved by Cutkosky [9], who also showed that Oda's strong conjecture implies the local version of the strong conjecture for proper birational morphisms [10] and proved the local strong factorization conjecture in dimension 3 ([10]) via Christensen's theorem. Finally Karu generalized Christensen's result to any dimension and completed the argument for the local strong factorization ([22]).

The proofs in [38] and [3] are both build upon the idea of cobordism which was developed in [37] and was inspired by Morelli's theory of polyhedral cobordisms [27]. The main idea of [37] is to construct a space with a K^*-action for a given birational map. The space called a birational cobordism resembles the idea of Morse cobordism and determines a decomposition of the birational map into elementary transformations (see Remark 2.6). This gives a factorization into a sequence of weighted blow-ups and blow-downs. One can view the birational maps determined by cobordisms also in terms of VGIT developed in papers of Thaddeus ([34]) and Dolgachev–Hu ([13]). As shown in [37] the weighted blow-ups which occur in the factorization have locally a natural toric and combinatorial description which is crucial for their further regularization.

The two existing methods of regularizing centers of this factorization are π-desingularization of cobordisms as in [38] and local torification of the action as in [3].

The present proof is essentially the same as in [38]. Instead of working in full generality and developing the suitable language for toroidal varieties we focus on applying the general ideas to a particular construction of a smooth cobordism. The reader can find also a more general and extended version of this proof in [39]. The π-desingularization is a desingularization of geometric quotients of a K^*-action. This can be done locally and the procedure can be globalized in the functorial and even canonical way. The π-desingularization makes all the intermediate varieties (which are geometric quotients) smooth, and also the connecting blow-ups have smooth centers.

The proof of Abramovich, Karu, Matsuki and the author [3] relies on a subtle analysis of differences between locally toric and toroidal structures defined by the action of K^*. The Abramovich–de Jong idea of torification is roughly speaking to construct the ideal sheaves whose blow-ups (or principalizations) introduce the structure of toroidal varieties in neighborhoods of fixed points of the action. This allows one to pass from birational maps between intermediate varieties in the neighborhood of fixed points to birational toroidal maps. The latter can be factored into a sequence of smooth blow-ups by using the same combinatorial methods as for toric varieties. Combining all the local factorizations together we get a global factorization.

For simplicity we restrict our considerations to projective varieties. We will not discuss here compatibility with divisors and functorial properties.

2. Birational cobordisms

2.1. Definition of a birational cobordism. Recall some basic definitions from Mumford's GIT theory.

Definition 2.1. Let K^* act on X. By a *good quotient* we mean a variety $Y = X//K^*$ together with a morphism $\pi : X \to Y$ which is constant on G-orbits such that for any affine open subset $U \subset Y$ the inverse image $\pi^{-1}(U)$ is affine and $\pi^* \colon O_Y(U) \to O_X(\pi^{-1}(U))^{K^*}$ is an isomorphism. If additionally for any closed point $y \in Y$ its inverse limit $\pi^{-1}(x)$ is a single orbit we call $Y := X/K^*$ together with $\pi : X \to Y$ a *geometric quotient*.

Remark 2.2. A geometric quotient is a space of orbits while a good quotient is a space of equivalence classes of orbits generated by the relation that two orbits are equivalent if their closures intersect.

Definition 2.3. Let K^* act on X. We say that $\lim_{t\to 0} tx$ exists (respectively $\lim_{t\to\infty} tx$ exists) if the morphism $\mathrm{Spec}(K) \to X$ given by $t \mapsto tx$ extends to $\mathrm{Spec}(K^*) \subset \mathbb{A}^1 \to X$ (or respectively $\mathrm{Spec}(K^*) \subset \mathbb{P}^1 \setminus \{0\} \to X$).

Definition 2.4 ([37]). Let X_1 and X_2 be two birationally equivalent normal varieties. A *birational cobordism* or simply a *cobordism* $B := B(X_1, X_2)$ between them is a normal variety B with an algebraic action of K^* such that the sets

$$B_- := \{x \in B \mid \lim_{t\to 0} tx \text{ does not exist}$$

and

$$B_+ := \{x \in B \mid \lim_{t\to \infty} tx \text{ does not exist}\}$$

are nonempty and open and there exist geometric quotients B_-/K^* and B_+/K^* such that $B_+/K^* \simeq X_1$ and $B_-/K^* \simeq X_2$ and the birational map $X_1 \dashrightarrow X_2$ is given by the above isomorphisms and the open embeddings of $B_+ \cap B_-/K^*$ into B_+/K^* and B_-/K^* respectively.

Remark 2.5. An analogous notion of cobordism of fans of toric varieties was introduced by Morelli in [27].

Remark 2.6. The above definition can also be considered as an analog of the notion of cobordism in Morse theory. Let W be a cobordism in Morse theory of two differentiable manifolds X and X' and $f\colon W \to [a,b] \subset \mathbb{R}$ be a Morse function such that $f^{-1}(a) = X$ and $f^{-1}(b) = X'$. Then X and X' have open neighborhoods $X \subseteq V \subseteq W$ and $X' \subseteq V' \subseteq W'$ such that $V \simeq X \times [a, a+\varepsilon)$ and $V' \simeq X' \times (b-\varepsilon, b]$ for which $f_{|V}\colon V \simeq X \times [a, a+\varepsilon) \to [a,b]$ and $f_{|V'}\colon V' \simeq X' \times (b-\varepsilon, b] \to [a,b]$ are the natural projections on the second coordinate. Let $W' := W \cup_V X \times (-\infty, a+\varepsilon) \cup_{V'} X' \times (b-\varepsilon, +\infty)$. One can easily see that W' is isomorphic to $W \setminus X \setminus X' = \{x \in W \mid a < f(x) < b\}$. Let $f'\colon W' \to \mathbb{R}$ be the map defined by glueing the function f and the natural projection on the second coordinate. Then $\mathrm{grad}(f')$ defines an action on W' of a 1-parameter group $T \simeq \mathbb{R} \simeq \mathbb{R}^*_{>0}$ of diffeomorphisms. The last group isomorphism is given by the exponential.

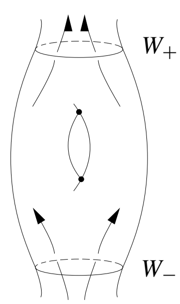

Figure 1. Cobordism in Morse theory.

Then one can see that $W'_- := \{x \in W' \mid \lim_{t\to 0} tx \text{ does not exist}\}$ and $W'_+ := \{x \in W' \mid \lim_{t\to\infty} tx \text{ does not exist}\}$ are open and X and X' can be considered as quotients of these sets by T. The critical points of the Morse function are T-fixed points. "Passing through the fixed points" of the action induces a simple birational transformation similar to spherical modification in Morse theory (see Example 2.7).

Example 2.7. Let K^* act on $B := \mathbb{A}_K^{l+m+r}$ by

$$t(x_1, \dots, x_l, y_1, \dots, y_m, z_1, \dots, z_r)$$
$$= (t^{a_1} \cdot x_1, \dots, t^{a_l} \cdot x_l, t^{-b_1} \cdot y_1, \dots, t^{-b_m} \cdot y_m, z_1, \dots, z_r),$$

where $a_1, \dots, a_l, b_1, \dots, b_m > 0$. Set $\bar{x} = (x_1, \dots, x_l)$, $\bar{y} = (y_1, \dots, y_m)$, $\bar{z} = (z_1, \dots, z_r)$. Then

$$B_- = \{p = (\bar{x}, \bar{y}, \bar{z}) \in \mathbb{A}_K^{l+m+r} \mid \bar{y} \neq 0\},$$
$$B_+ = \{p = (\bar{x}, \bar{y}, \bar{z}) \in \mathbb{A}_K^{l+m+r} \mid \bar{x} \neq 0\}.$$

Case 1. $a_i = b_i = 1$, $r = 0$ (Atiyah, Reid). One can easily see that $B//K^*$ is the affine cone over the Segre embedding $\mathbb{P}^{l-1} \times \mathbb{P}^{m-1} \to \mathbb{P}^{l \cdot m-1}$, and B_+/K^* and B_-/K^* are smooth.

The relevant birational map $\phi\colon B_-/K^* \dashrightarrow B_+/K^*$ is a flip for $l, m \geq 2$ replacing $\mathbb{P}^{l-1} \subset B_-/K^*$ with $\mathbb{P}^{m-1} \subset B_+/K^*$. For $l = 1$, $m \geq 2$, ϕ is a blow-down, and for $l \geq 2$, $m = 1$ it is a blow-up. If $l = m = 1$ then ϕ is the identity. One can show that $\phi\colon B_-/K^* \dashrightarrow B_+/K^*$ factors into the blow-up of $\mathbb{P}^{l-1} \subset B_-/K^*$ followed by the blow-down of $\mathbb{P}^{m-1} \subset B_+/K^*$.

Case 2. General case. For $l = 1$, $m \geq 2$, ϕ is a toric blow-up whose exceptional fibers are weighted projective spaces. For $l \geq 2$, $m = 1$, ϕ is a toric blow-down. If $l = m = 1$ then ϕ is the identity. The birational map $\phi\colon B_-/K^* \dashrightarrow B_+/K^*$ factors into a weighted blow-up and a weighted blow-down.

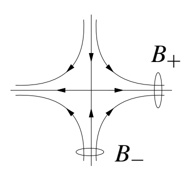

Figure 2. Affine Cobordism.

Remark 2.8. In Morse theory we have an analogous situation. In cobordisms with one critical point we replace S^{l-1} by S^{m-1}.

2.2. Fixed points of the action. Let X be a variety with an action of K^*. Denote by X^{K^*} the set of fixed points of the action and by $\mathcal{C}(X^{K^*})$ the set of its irreducible fixed components. For any $F \in \mathcal{C}(X^{K^*})$ set

$$F^+(X) = F^+ = \{x \in X \mid \lim_{t\to 0} tx \in F\}, \quad F^-(X) = F^- = \{x \in X \mid \lim_{t\to\infty} tx \in F\}.$$

Example 2.9. In Example 2.7,

$$F = \{p \in B \mid \bar{x} = \bar{y} = 0\}, \quad F^- = \{p \in B \mid \bar{x} = 0\}, \quad F^+ = \{p \in B \mid \bar{y} = 0\}.$$

Lemma 2.10. *If F is the fixed point set of an affine variety U then F, F^+ and F^- are closed in U. Moreover the ideals $I_{F^+}, I_{F^-} \subset K[V]$ are generated by all semiinvariant functions with positive (respectively negative) weights.*

Proof. Embed U equivariantly into affine space $\mathbb{A}^n$ with linear action and use the example above. □

2.3. Existence of a smooth birational cobordism

Proposition 2.11. *Let $\phi\colon X \dashrightarrow Y$ be a birational map between smooth projective varieties. Then ϕ factors as $X \leftarrow Z \to Y$, where $Z \to X$ and $Z \to Y$ are birational morphisms from a smooth projective variety Z.*

Proof. Let $\Gamma(X, Y) \subset X \times Y$ be the graph of ϕ and Z be its canonical resolution of singularities [17]. □

It suffices to construct the cobordism and factorization for the projective morphism $Y \to X$.

Proposition 2.12 ([37]). *Let $\varphi\colon Y \to X$ be a birational morphism of smooth projective varieties with the exceptional divisor D. Let $U \subset X, Y$ be an open subset where φ is an isomorphism. There exists a smooth projective variety $\overline{B}$ with a K^*-action, which contains fixed point components isomorphic to X and Y such that*

- *$B = B(X, Y) := \overline{B} \setminus (X \cup Y)$ is a cobordism between X and Y;*
- *$U \times K^* \subset B_- \cap B_+ \subset B$;*
- *there are K^*-equivariant isomorphisms $X^- \simeq X \times (\mathbb{P}^1 \setminus \{0\})$ and $Y^+ \simeq \mathcal{O}_Y(D)$;*
- *$X^- \setminus X = B_+$ and $Y^+ \setminus Y = B_-$*

In further considerations we shall refer to $\overline{B}$ as a *compactified cobordism.*

Proof. We follow here the Abramovich construction of cobordism. Let $\mathcal{I} \subset \mathcal{O}_X$ be a sheaf of ideals such that $Y = \mathrm{Bl}_{\mathcal{I}} X$ is obtained from X by blowing up of $\mathcal{I}$. Let z denote the standard coordinate on $\mathbb{P}^1$ and let $\mathcal{I}_0$ be the ideal of the point $z = 0$ on $\mathbb{P}^1$. Set $W := X \times \mathbb{P}^1$ and denote by $\pi_1\colon W \to X, \pi_2\colon W \to \mathbb{P}^1$ the standard projections. Then $\mathcal{J} := \pi_1{}^*(\mathcal{I}) + \pi_2{}^*(\mathcal{I}_0)$ is an ideal supported on $X \times \{0\}$. Set $W' := \mathrm{Bl}_{\mathcal{J}} W$. The proper transform of $X \times \{0\}$ is isomorphic to Y and we identify it with Y. Let us describe Y locally. Let $f_1, \ldots, f_k$ generate the ideal $\mathcal{I}$ on some open affine set $U \subset X$. Then after the blow-up $Y \to X$ at $\mathcal{I}$ the inverse image of U is a union of open charts $U_i \subset Y$, where

$$K[U_i] = K[U][f_1/f_i, \ldots, f_k/f_i].$$

Now the functions $f_1, \dots, f_k, z$ generate the ideal $\mathcal{J}$ on $U \times \mathbb{A}^1 \subset W$. After the blow-up $W' \to W$ at $\mathcal{J}$, the inverse image of $U \times \mathbb{A}^1$ is a union of open charts $V_i \supset Y$, where

$$K[V_i] = K[U][f_1/f_i, \dots, f_k/f_i, z/f_i] = K[U_i][z/f_i]$$

and the relevant V_z which does not intersect Y. Then $V_i = U_i^+ \simeq U_i \times \mathbb{A}^1$ where $z' := z/f_i$ is the standard coordinate on $\mathbb{A}^1$. The action of K^* on the factor U is trivial while on $\mathbb{A}^1$ it is standard given by $t(z') = tz$. Thus the open subset $Y^+ = \bigcup U_i^+ = \bigcup V_i \subset W'$ is a line bundle over Y with the standard action of K^*. On the other hand the neighborhood $X^- := X \times (\mathbb{P}^1 \setminus \{0\})$ of $X \subset W$ remains unchanged after the blow-up of $\mathcal{J}$. We identify X with $X \times \{\infty\}$. We define $\overline{B}$ to be the canonical desingularization of W. Then $B := \overline{B} \setminus X \setminus Y$. We get $B_-/K^* = (Y^+ \setminus Y)/K^* = Y$, while $B_+/K^* = (X^+ \setminus X)/K^* = X$. □

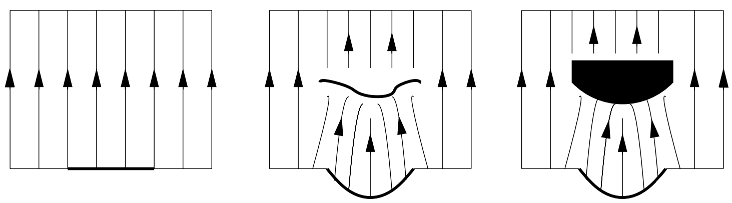

Figure 3. Compactified cobordism.

Remark 2.13. The Abramovich construction can be considered as a generalization of the Fulton–Macpherson construction of the deformation to the normal cone. If we let $\mathcal{I} = \mathcal{I}_C$ be the ideal sheaf of the smooth center then the relevant blow-up is already smooth. On the other hand this is a particular case of the very first example in [37] of a cobordism which is a K^*-equivariant completion of the space

$$L(Y, D; X, 0) := O_Y(D) \cup_{U \times K^*} X \times (\mathbb{P}^1 \setminus \{0\}).$$

Another variant of our construction is given by Hu and Keel in [21].

2.4. Collapsibility

Definition 2.14 ([37]). Let X be a cobordism or any variety with a K^*-action.

1. We say that $F \in \mathcal{C}(X^{K^*})$ is an *immediate predecessor* of $F' \in \mathcal{C}(X^{K^*})$ if there exists a nonfixed point x such that $\lim_{t\to 0} tx \in F$ and $\lim_{t\to\infty} tx \in F'$.

2. We say that F *precedes* F' and write $F < F'$ if there exists a sequence of connected fixed point set components $F_0 = F, F_1, \dots, F_l = F'$ such that F_{i-1} is an immediate predecessor of F_i (see [5]).

3. We call a cobordism (or a variety with a K^*-action) *collapsible* (see also Morelli [27]) if the relation $<$ on its set of connected components of the fixed point set is an order. (Here an order is just required to be transitive.)

Definition 2.15 ([3], [37]). A function $\chi\colon \mathcal{C}(X^{K^*}) \to \mathbb{Z}$ is *strictly increasing* if $\chi(F) < \chi(F')$ whenever $F < F'$.

2.5. Existence of a strictly increasing function for $\mathbb{P}^k$ and $\bar{B}$. The space $\mathbb{P}^k = \mathbb{P}(\mathbb{A}^{k+1})$ splits according to the weights as

$$\mathbb{P}^k = \mathbb{P}(\mathbb{A}^{k+1}) = \mathbb{P}(\mathbb{A}_{a_1} \oplus \cdots \oplus \mathbb{A}_{a_r})$$

where K^* acts on $\mathbb{A}_{a_i}$ with the weight a_i. Assume that $a_1 < \cdots < a_r$. Let $\bar{x}_{a_i} = [x_{i,1}, \ldots, x_{i,r_i}]$ be the coordinates on $\mathbb{A}_{a_i}$. The action of K^* is given by

$$t[\bar{x}_{a_1}, \ldots, \bar{x}_{a_r}] = [t^{a_1}\bar{x}_{a_1}, \ldots, t^{a_r}\bar{x}_{a_r}].$$

It follows that the fixed point components of $(\mathbb{P}^k)^{K^*}$ are $\mathbb{P}(\mathbb{A}_{a_i})$. We define a strictly increasing function $\chi_{\mathbb{P}}\colon \mathcal{C}(\mathbb{P}^{K^*}) \to \mathbb{Z}$ by

$$\chi_{\mathbb{P}}(\mathbb{P}(\mathbb{A}_{a_i})) = a_i.$$

We see that for $\bar{x} = [\bar{x}_{a_0}, \ldots, \bar{x}_{a_r}]$, $\lim_{t\to 0} tx \in \mathbb{P}(\mathbb{A}_{a_{\min}})$, $\lim_{t\to\infty} tx \in \mathbb{P}(\mathbb{A}_{a_{\max}})$, where

$$a_{\max} = \max\{a \mid \bar{x}_a \neq 0\}, \quad a_{\min} = \min\{a \mid \bar{x}_a \neq 0\}.$$

Then $\mathbb{P}(\mathbb{A}_{a_i}) < \mathbb{P}(\mathbb{A}_{a_j})$ iff $a_i < a_j$.

By the Sumihiro theorem ([33]), we embed $\bar{B}$ equivariantly into a projective space $\mathbb{P}^k$. Then every fixed point component F in $\mathcal{C}(\bar{B}^{K^*})$ is contained in $\mathbb{P}(\mathbb{A}_a) \in \mathcal{C}((\mathbb{P}^k)^{K^*})$ and we put $\chi_B(F) = \chi_{\mathbb{P}}(\mathbb{P}(\mathbb{A}_a)) = a_i$. The function $\chi_{\mathbb{P}}$ is strictly increasing on $\mathcal{C}((\mathbb{P}^k)^{K^*})$ and the function χ_B is strictly increasing on $\mathcal{C}(\bar{B}^{K^*})$. This implies

Lemma 2.16. *A compactified cobordism $\bar{B}$ is collapsible.*

2.6. Decomposition of a birational cobordism

Definition 2.17 ([3], [37]). A cobordism B is *elementary* if any $F \neq F' \in \mathcal{C}(B^{K^*})$ are incomparable with respect to $>$.

The function χ_F defines a decomposition of $\mathcal{C}(B^{K^*})$ into elementary cobordisms

$$B_{a_i} := B \setminus \Big(\bigcup_{\chi_B(F)<a_i} F^- \cup \bigcup_{\chi_B(F)>a_i} F^+ \Big),$$

where $a_1 < \cdots < a_r$ are the values of χ_B. This yields

Lemma 2.18. 1. $(B_{a_1})_- = B_-$, $(B_{a_r})_+ = B_+$.

2. $(B_{a_{i+1}})_- = (B_{a_i})_+ = B \setminus \big(\bigcup_{\chi_B(F)\leq a_i} F^- \cup \bigcup_{\chi_B(F)\geq a_{i+1}} F^+\big)$.

3. $\chi(F) = a_i$ *for any* $F \in \mathcal{C}(B_{a_i})$.

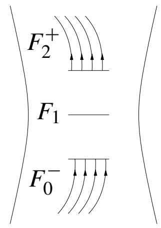

Figure 4. Elementary birational cobordism.

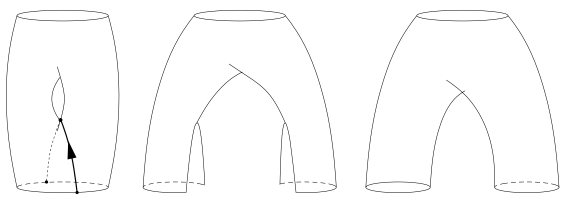

Figure 5. “Handle”-elementary cobordism in Morse Theory.

2.7. Decomposition of $\mathbb{P}^k$. Set $\mathbb{A}_{\geq a_i} := \mathbb{A}_{a_i} \oplus \cdots \oplus \mathbb{A}_{a_r}$, $\mathbb{A}_{>a_i} := \mathbb{A}_{a_{i+1}} \oplus \cdots \oplus \mathbb{A}_{a_r}$, and define $\mathbb{A}_{<a_i}$, $\mathbb{A}_{\leq a_i}$ analogously.

Lemma 2.19. $\mathbb{P}(\mathbb{A}_{a_i})^+ = \mathbb{P}(\mathbb{A}_{\geq a_i})$ *and* $\mathbb{P}(\mathbb{A}_{a_i})^- = \mathbb{P}(\mathbb{A}_{\leq a_i})$.

Lemma 2.20. *Set* $\mathbb{P}_{a_i} := \mathbb{P}^k \setminus \big(\bigcup_{\chi_{\mathbb{P}}(F)<a_i} F^- \cup \bigcup_{\chi_{\mathbb{P}}(F)>a_i} F^+\big)$. *Then*

$$\mathbb{P}_{a_i} = \mathbb{P}^k \setminus \mathbb{P}(\mathbb{A}_{>a_i}) \setminus \mathbb{P}(\mathbb{A}_{<a_i}), \quad (\mathbb{P}_{a_i})_+ = \mathbb{P}^k \setminus \mathbb{P}(\mathbb{A}_{\geq a_i}) \setminus \mathbb{P}(\mathbb{A}_{<a_i})$$
$$(\mathbb{P}_{a_i})_- = \mathbb{P}^k \setminus \mathbb{P}(\mathbb{A}_{>a_i}) \setminus \mathbb{P}(\mathbb{A}_{\leq a_i}).$$

Lemma 2.21. $B_{a_i} = \overline{B} \cap \mathbb{P}_{a_i}$, $(B_{a_i})_- = \overline{B} \cap (\mathbb{P}_{a_i})_-$, $(B_{a_i})_+ = \overline{B} \cap (\mathbb{P}_{a_i})_+$.

2.8. GIT and existence of quotients for $\mathbb{P}^k$. The sets $\mathbb{P}_{a_i}$ can be interpreted in terms of Mumford’s GIT theory. Any lifting of the action of K^* on $\mathbb{P}^k = \mathbb{P}(\mathbb{A}^{k+1})$ to $\mathbb{A}^{k+1}$ is called a *linearization*. Consider the twisted action on $\mathbb{A}^{k+1}$,

$$t_r(x) = t^{-r} \cdot t(x).$$

The twisting does not change the action on $\mathbb{P}(\mathbb{A}^{k+1})$ and defines different linearizations. If we compose the action with a group monomorphism $t \mapsto t^k$ the weights of the new action $t^k(x)$ will be multiplied by k. The good and geometric quotients for $t(x)$ and $t^k(x)$ are the same. Keeping this in mind it is convenient to allow linearizations with rational weights.

Definition 2.22. A point $x \in \mathbb{P}^k$ is *semistable* with respect to t_r, written $x \in (\mathbb{P}^k, t_r)^{\mathrm{ss}}$, if there exists an invariant section $s \in \Gamma(\mathcal{O}_{\mathbb{P}^{k+1}}(n)^{t_r})$, for some $n \in \mathbb{N}$ such that $s(x) \neq 0$.

Lemma 2.23 ([3]). $\mathbb{P}_{a_i} = (\mathbb{P}^k, t_{a_i})^{ss}$, $(\mathbb{P}_{a_i})_- = (\mathbb{P}^k, t_{a_i - \frac{1}{2}})^{ss}$, $(\mathbb{P}_{a_i})_+ = (\mathbb{P}^k, t_{a_i + \frac{1}{2}})^{ss}$.

Proof. $x \in \mathbb{P}_{a_i}$ iff either $\bar{x}_{a_i} \neq 0$ or $\bar{x}_{a_{j_1}} \neq 0$ and $\bar{x}_{a_{j_2}} \neq 0$ for $a_{j_1} < r = a_i < a_{j_2}$. In both situations we find a nonzero t_r-invariant section $s_i = x_i$ or $s_{j_1 j_2} = x_{j_1}^{b_1} x_{j_2}^{b_2}$ for suitable coprime b_1 and b_2.

$x \in (\mathbb{P}_{a_i})_-$ iff $\bar{x}_{a_{j_1}} \neq 0$ and $\bar{x}_{a_{j_2}} \neq 0$ for $a_{j_1} < a_i \leq a_{j_2}$ (or equivalently $a_{j_1} < r = a_i - 1/2 < a_{j_2}$). As before there is a nonzero t_r-invariant section $x_{j_1}^{b_1} x_{j_2}^{b_2}$ for suitable coprime b_1 and b_2. □

It follows from GIT theory that $(\mathbb{P}^k, t^r)^{ss} /\!/ K^*$ exists and it is a projective variety. By Lemma 2.21 and the above we get

Corollary 2.24. *There exist quotients* $\pi_{a_i} : B_{a_i} \to B_{a_i} /\!/ K^*$ *and* $\pi_{a_{i-}} = B_{a_{i-}} \to (B_{a_i})_- / K^*$, $\pi_{a_{i+}} = (B_{a_i})_+ \to (B_{a_i})_+ / K^*$.

2.9. Local description

Proposition 2.25 ([37]). *Let* B_a *be a smooth elementary cobordism. Then for any* $x \in F_0$ *there exists an invariant neighborhood* V_x *of* x *and a* K^**-equivariant étale morphism* (*i.e. locally analytic isomorphism*) $\phi : V_x \to \mathrm{Tan}_x$, *where* $\mathrm{Tan}_x \simeq \mathbb{A}^n_K$ *is the tangent space with the induced linear* K^**-action, such that in the diagram*

$$\begin{array}{ccccccc}
(B_a)_-/K^* & \supset & V_x/\!/K^* \times_{\mathrm{Tan}_x/\!/K^*} \mathrm{Tan}_{x-}/K^* & \simeq & V_{x-}/K^* & \to & \mathrm{Tan}_{x-}/K^* \\
\downarrow & & & & \downarrow & & \downarrow \\
B_a/\!/K^* & & \supset & & V_x/\!/K^* & \to & \mathrm{Tan}_x/\!/K^* \\
\uparrow & & & & \uparrow & & \uparrow \\
(B_a)_+/K^* & \supset & V_x/\!/K^* \times_{\mathrm{Tan}_x/\!/K^*} \mathrm{Tan}_{x+}/K^* & \simeq & V_{x+}/K^* & \to & \mathrm{Tan}_{x+}/K^*
\end{array}$$

the vertical arrows are defined by open embeddings and the horizontal morphisms are defined by ϕ *and are étale.*

Proof. By taking local semiinvariant parameters at the point $x \in F_0$ one can construct an equivariant morphism $\phi : U_x \to \mathrm{Tan}_x \simeq \mathbb{A}^n_K$ from some open affine invariant neighborhood U_x such that ϕ is étale at x. By Luna's Lemma (see [Lu], Lemme 3 (Lemme Fondamental)) there exists an invariant affine neighborhood $V_x \subseteq U_x$ of the point x such that $\phi_{|V_x}$ is étale, the induced map $\phi_{|V_x/K^*} : V_x /\!/ K^* \to \mathrm{Tan}_x /\!/ K^*$ is étale and $V_x \simeq V_x /\!/ K^* \times_{\mathrm{Tan}_x /\!/ K^*} \mathrm{Tan}_x$. This defines the isomorphisms $V_x /\!/ K^* \times_{\mathrm{Tan}_x /\!/ K^*} \mathrm{Tan}_{x-}/K^* \simeq V_{x-}/K^*$. Note that $(B_a)_- = B_a \setminus \bigcup F^+_{F \in \mathcal{C}(B^{K^*})}$ and $V_x \cap F^+ = (V_x \cap F)^+$. (Both sets are closed and irreducible.) Thus $(V_x)_- = V_x \cap (B_a)_-$ and we get the horizontal inclusions. □

Proposition 2.26 ([37]). *There is a factorization of the morphism* $\phi : Y \to X$ *given by* $Y = (B_{a_1})_-/K^* \dashrightarrow (B_{a_1})_+/K^* = (B_{a_2})_-/K^* \dashrightarrow \cdots \dashrightarrow (B_{a_{k-1}})_+/K^* = (B_{a_k})_-/K^* \dashrightarrow (B_{a_k})_+/K^* = X$.

Remark 2.27. The birational maps $(B_a)_-/K^* \dashrightarrow (B_a)_+/K^*$ are locally described by Example 2.7. Both spaces have cyclic singularities and differ by the composite of a weighted blow-up and a weighted blow-down. To achieve the factorization we need to desingularize quotients as in for instance case 1 of the example. It is hopeless to modify weights by birational modification of smooth varieties. Instead we want to view Example 2.7 from the perspective of toric varieties.

3. Toric varieties

3.1. Fans and toric varieties. Let $N \simeq \mathbb{Z}^k$ be a lattice contained in the vector space $N^{\mathbb{Q}} := N \otimes \mathbb{Q} \supset N$.

Definition 3.1 ([11], [31]). By a *fan* Σ in $N^{\mathbb{Q}}$ we mean a finite collection of finitely generated strictly convex cones σ in $N^{\mathbb{Q}}$ such that

- any face of a cone in Σ belongs to Σ,
- any two cones of Σ intersect in a common face.

If σ is a face of σ' we shall write $\sigma \preceq \sigma'$.

We say that a cone σ in $N^{\mathbb{Q}}$ is *regular* if it is generated by a part of a basis of the lattice $e_1, \ldots, e_k \in N$, written $\sigma = \langle e_1, \ldots, e_k \rangle$. A cone σ is *simplicial* if it is generated over $\mathbb{Q}$ by linearly independent integral vectors $v_1, \ldots, v_k$, written $\sigma = \langle v_1, \ldots, v_k \rangle$

Definition 3.2. Let Σ be a fan and $\tau \in \Sigma$. The *star* of the cone τ and the *closed star* of τ are defined as follows:

$$\mathrm{Star}(\tau, \Sigma) := \{\sigma \in \Sigma \mid \tau \preceq \sigma\},$$

$$\overline{\mathrm{Star}}(\tau, \Sigma) := \{\sigma \in \Sigma \mid \sigma' \preceq \sigma \text{ for some } \sigma' \in \mathrm{Star}(\tau, \Sigma)\}.$$

To a fan Σ there is associated a toric variety $X_\Sigma \supset T$, i.e. a normal variety on which a torus T acts effectively with an open dense orbit (see [23], [12], [31], [15]). To each cone $\sigma \in \Sigma$ corresponds an open affine invariant subset X_σ and its unique closed orbit O_σ. The orbits in the closure of the orbit O_σ correspond to the cones of $\mathrm{Star}(\sigma, \Sigma)$. In particular, $\tau \preceq \sigma$ iff $\overline{O}_\tau \supset O_\sigma$.

The fan Σ is *nonsingular* (resp. *simplicial*) if all its cones are nonsingular (resp. simplicial). Nonsingular fans correspond to nonsingular varieties.

Denote by

$$M := \mathrm{Hom}_{\mathrm{alg.gr.}}(T, K^*)$$

the lattice of group homomorphisms to K^*, i.e. characters of T. The dual lattice $\mathrm{Hom}_{\mathrm{alg.gr.}}(K^*, T)$ of 1-parameter subgroups of T can be identified with the lattice N. Then the vector space $M^{\mathbb{Q}} := M \otimes \mathbb{Q}$ is dual to $N^{\mathbb{Q}} = N \otimes \mathbb{Q}$.

The elements $F \in M = N^*$ are functionals on N and integral functionals on $N^{\mathbb{Q}}$. For any $\sigma \subset N^{\mathbb{Q}}$ we denote by

$$\sigma^{\vee} := \{F \in M \mid F(v) \geq 0 \text{ for any } v \in \sigma\}$$

the set of integral vectors of the dual cone to σ. Then the ring of regular functions $K[X_\sigma]$ is $K[\sigma^\vee]$.

We call a vector $v \in N$ *primitive* if it generates the sublattice $\mathbb{Q}_{\geq 0} v \cap N$. Primitive vectors correspond to 1-parameter monomorphisms.

For any $\sigma \subset N^{\mathbb{Q}}$ set

$$\sigma^{\perp} := \{m \in M \mid (v, m) = 0 \text{ for any } v \in \sigma\}.$$

The latter set represents all invertible characters on X_σ. All noninvertible characters are in $\sigma^\vee \setminus \sigma^\perp$ and vanish on O_σ. The ring of regular functions on $O_\sigma \subset X_\sigma$ can be written as $K[O_\sigma] = K[\sigma^\perp] \subset K[\sigma^\vee]$.

3.2. Star subdivisions and blow-ups

Definition 3.3 ([23], [31], [12], [15]). A *birational toric morphism* or simply a *toric morphism* of toric varieties $X_\Sigma \to X_{\Sigma'}$ is a morphism identical on $T \subset X_\Sigma, X_{\Sigma'}$.

By the *support* of a fan Σ we mean the union of all its faces, $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma$.

Definition 3.4 ([23], [31], [12], [15]). A *subdivision* of a fan Σ is a fan Δ such that $|\Delta| = |\Sigma|$ and any cone $\sigma \in \Sigma$ is a union of cones $\delta \in \Delta$.

Definition 3.5. Let Σ be a fan and ϱ be a ray passing in the relative interior of $\tau \in \Sigma$. Then the *star subdivision* $\varrho \cdot \Sigma$ of Σ with respect to ϱ is defined to be

$$\varrho \cdot \Sigma = (\Sigma \setminus \mathrm{Star}(\tau, \Sigma)) \cup \{\varrho + \sigma \mid \sigma \in \overline{\mathrm{Star}}(\tau, \Sigma) \setminus \mathrm{Star}(\tau, \Sigma)\}.$$

If Σ is nonsingular, i.e. all its cones are nonsingular, $\tau = \langle v_1, \dots, v_l \rangle$ and $\varrho = \langle v_1 + \cdots + v_l \rangle$ then we call the star subdivision $\varrho \cdot \Sigma$ *nonsingular*.

Proposition 3.6 ([23], [12], [31], [15]). *Let X_Σ be a toric variety. There is a* 1-1 *correspondence between subdivisions of the fan Σ and proper toric morphisms $X_{\Sigma'} \to X_\Sigma$.*

Remark 3.7. Nonsingular star subdivisions from 3.5 correspond to blow-ups of smooth varieties at closures of orbits ([31], [15]). Arbitrary star subdivisions correspond to blow-ups of some ideals associated to valuations (see Lemma 5.20).

4. Polyhedral cobordisms of Morelli

4.1. Preliminaries. By $N^{\mathbb{Q}+}$ we shall denote a vector space $N^{\mathbb{Q}+} \approx \mathbb{Q}^k$ containing a lattice $N^+ \simeq \mathbb{Z}^k$, together with a primitive vector $v_0 \in N^+$ and the canonical projection

$$\pi : N^{\mathbb{Q}+} \to N^{\mathbb{Q}} \simeq N^{\mathbb{Q}+}/\mathbb{Q} \cdot v_0.$$

Definition 4.1 ([27]). A cone $\sigma \subset N^{\mathbb{Q}+}$ is *π-strictly convex* if $\pi(\sigma)$ is strictly convex (contains no line). A fan Σ is π-strictly convex if it consists of π-strictly convex cones.

In the following all the cones in $N^{\mathbb{Q}+}$ are assumed to be π-strictly convex and simplicial. The π-strictly convex cones σ in $N^{\mathbb{Q}+}$ split into two categories.

Definition 4.2. A cone $\sigma \subset N^{\mathbb{Q}+}$ is called *independent* if the restriction of π to σ is a linear isomorphism (equivalently $v_0 \notin \operatorname{span}(\sigma)$). A cone $\sigma \subset N^{\mathbb{Q}+}$ is called *dependent* if the restriction of π to σ is a lattice submersion which is not an isomorphism (equivalently $v_0 \in \operatorname{span}(\sigma)$).

A dependent cone is called a *circuit* if all its proper faces are independent.

Lemma 4.3. *Any dependent cone σ contains a unique circuit δ.*

4.2. K^*-actions and $N^{\mathbb{Q}+}$.

The vector $v_0 = (a_1, \ldots, a_k) \in N^{\mathbb{Q}+}$ defines a 1-parameter subgroup $t^{v_0} := t_1^{a_1} \ldots t_k^{a_k}$ acting on T and all toric varieties $X \supset T$. Denote by M^+ the lattice dual to N^+. Then the lattice $N := N^+/\mathbb{Z} \cdot v_0$ is dual to the lattice $M := \{a \in M^+ \mid (a, v_0) = 0\}$ of all the characters invariant with respect to the group action. The natural projection of cones $\pi : \sigma \to \sigma^\Gamma$ defines the good quotient morphism

$$X_\sigma = \operatorname{Spec} K[\sigma^\vee] \to X_\sigma /\!/ K^* = \operatorname{Spec} K[\sigma^\vee \cap M] = \operatorname{Spec} K[(\sigma^\Gamma)^\vee] = X_{\sigma^\Gamma}.$$

Lemma 4.4. *A cone σ is independent iff the geometric quotient $X_\sigma \to X_\sigma/K^*$ exists or alternatively if X_σ contains no fixed points. The cone σ is dependent if O_σ is a fixed point set.*

Proof. Note that the set $X_\sigma^{K^*}$ is closed and if it is nonempty then it contains O_σ. Then a point $p \in O_\sigma$ is fixed, i.e. $t^{v_0} p = p$, iff for all functionals $F \in \sigma^\perp$ (i.e. $x^F(p) \neq 0$) we have $x^F(p) = x^F(t^{v_0} p) = t^{F(v_0)} x^F(p)$.

Then for all $F \in \sigma^\perp \subset \operatorname{span}(\sigma)^\perp$ we have $F(v_0) = 0$ so $v_0 \in \operatorname{span}(\sigma)$. □

Corollary 4.5. *A cone $\delta \in \Sigma$ is a circuit if and only if O_δ is the generic orbit of some $F \in \mathcal{C}(X_\Sigma^{K^*})$.*

Proof. O_σ is fixed with respect to the action of K^* if σ is dependent. Thus $O_\sigma \subset \overline{O}_\delta$ where δ is the unique circuit in σ (Lemma 4.3). □

4.3. Morelli cobordisms

Definition 4.6 (Morelli [27], [4]). A fan Σ in $N^{\mathbb{Q}+} \supset N^+$ is called a *polyhedral cobordism* or simply a cobordism if the sets of cones

$$\partial_-(\Sigma) := \{\sigma \in \Sigma \mid \text{there is } p \in \operatorname{int}(\sigma) \text{ so that } p - \varepsilon \cdot v_0 \notin |\Sigma| \text{ for all small } \varepsilon > 0\},$$

$$\partial_+(\Sigma) := \{\sigma \in \Sigma \mid \text{there is } p \in \operatorname{int}(\sigma) \text{ so that } p + \varepsilon \cdot v_0 \notin |\Sigma| \text{ for all small } \varepsilon > 0\}$$

are subfans of Σ and $\pi(\partial_-(\Sigma)) := \{\pi(\tau) \mid \tau \in \partial_-(\Sigma)\}$ and $\pi(\partial_+(\Sigma)) := \{\pi(\tau) \mid \tau \in \partial_+(\Sigma)\}$ are fans in $N^{\mathbb{Q}}$.

4.4. Dependence relation. Let $\sigma = \langle v_1, \dots, v_k\rangle$ be a dependent (simplicial) cone. Then, by definition $v_0 \in \mathrm{span}(v_1, \dots, v_k)$ where $v_1, \dots, v_k$ are linearly independent. There exists a unique up to rescaling integral relation

$$r_1 v_1 + \cdots + r_k v_k = a v_0, \quad \text{where } a > 0. \qquad (*)$$

Definition 4.7 ([27]). The rays of σ are called *positive, negative* and *null* vectors, according to the sign of the coefficient in the defining relation.

Remark 4.8. Note that the relation $(*)$ defines a unique relation

$$r'_1 w_1 + \cdots + r'_k w_k = 0 \qquad (**)$$

where w_i are generating vectors in the rays $\pi(\langle v_i\rangle)$, $r'_i w_i = r_i \pi(v_i)$. In particular $r'_i / r_i > 0$.

Lemma 4.9. *Let $\sigma = \langle v_1, \dots, v_k\rangle$ be a dependent cone. Then an independent face τ is in $\partial_+(\sigma)$ (resp. $\tau \in \partial_+(\sigma)$) if τ is a face of $\langle v_1, \dots, \check{v_i}, \dots, v_k\rangle$ for some index i such that $r_i < 0$ (resp. $r_i > 0$).*

Proof. By definition $\tau \in \partial_+(\sigma)$ there exists $p \in \mathrm{int}(\tau)$ such that for any sufficiently small $\varepsilon > 0$, $p + \varepsilon v_0 \notin \sigma$. Write $p = \sum \alpha_i v_i = \sum_{r_i>0} \alpha_i v_i + \sum_{r_i<0} \alpha_i v_i + \sum_{r_i=0} \alpha_i v_i$, where $\alpha_i \geq 0$. Then one of the coefficients in

$$p + \varepsilon v_0 = \sum_{r_i>0} (\alpha_i + r_i\varepsilon) v_i + \sum_{r_i<0} (\alpha_i + r_i\varepsilon) v_i + \sum_{r_i=0} (\alpha_i + r_i\varepsilon) v_i$$

is negative for small $\varepsilon > 0$. This is possible if $\alpha_i = 0$ for some index i with $r_i < 0$. □

Lemma 4.10. *A cone τ is in $\partial_+(\sigma)$ iff there exists $F \in \sigma^\vee \cap \tau^\perp$ such that $F(v_0) < 0$.*

Proof. If $\tau \in \partial_+(\sigma)$ then there exists $p \in \mathrm{int}(\tau)$ for which $p + \varepsilon v_0 \notin \sigma$. Hence there exists $F \in \sigma^\vee$ such that $F(p + \varepsilon v_0) < 0$ for small $\varepsilon > 0$. Then $F(p) = 0$ and $F(v_0) < 0$. Since $p \in \mathrm{int}(\tau)$ we have $F_{|\tau} = 0$. □

Corollary 4.11. $\partial_+(\sigma)$ *(resp.* $\partial_-(\sigma)$*) is a fan.*

Proof. By the lemma above, if $\tau \in \sigma^+$ then every face τ' of τ is in σ^+. □

Lemma 4.12. *Let σ be a dependent cone in $N^{\mathbb{Q}+}$. Then $B := X_\sigma$ is a birational cobordism such that*

- $(X_\sigma)_+ = X_{\partial_-(\sigma)}$, $(X_\sigma)_- = X_{\partial_+(\sigma)}$.
- $(X_\sigma)_+/K^* \cong X_{\pi(\partial_-(\sigma))}$, $(X_\sigma)_-/K^* \cong X_{\pi(\partial_+(\sigma))}$.
- $\pi(\partial_-(\sigma))$ *and* $\pi(\partial_+(\sigma))$ *are both decompositions of* $\pi(\sigma)$.
- *There is a factorization into a sequence of proper morphisms* $(X_\sigma)_+/K^* \to (X_\sigma)/\!/K^* \leftarrow (X_\sigma)_-/K^*$.

Proof. We have $p \in O_\tau$ where $O_\tau \subset (X_\sigma)_-$ iff $\lim t^{v_0} p \notin X_\sigma$. This is equivalent to existence of a functional $F \in \sigma^\vee$ for which $x^F(t^{v_0}p) = t^{F(v_0)}x^F(p)$ has a pole at $t = 0$. This means exactly that $x^F(p) \neq 0$ and $F(v_0) < 0$. The last condition says $F_{|\tau} = 0$ and $F(v_0) < 0$, which is equivalent to $\tau \in \partial_+(\sigma)$.

Suppose that $x \in \pi(\sigma)$. Then $\pi^{-1}(x) \cap \sigma$ is a line segment or a point. Let $y = \sup\{\pi^{-1}(x) \cap \sigma\}$. Then $y \in \mathrm{int}(\tau)$, where $\tau \prec \sigma$ and $y + \varepsilon v_0 \notin \sigma$, which implies that $\tau \in \partial_+(\sigma)$. Thus every point in $\pi(\sigma)$ belongs to a relative interior of a unique cone $\pi(\tau) \in \pi(\partial_+(\sigma))$. Since $\pi_{|\tau}$ is a linear isomorphism and $\partial_+(\sigma)$ is a fan, all faces of $\pi(\tau)$ are in $\pi(\partial_-(\sigma))$. Finally, $\pi(\partial_+(\sigma))$ and $\pi(\partial_-(\sigma))$ are both decompositions of $\pi(\sigma)$ corresponding to toric varieties $(X_\sigma)_-/K^* = X_{\pi(\partial_+(\sigma))}$ and $(X_\sigma)_+/K^* = X_{\pi(\partial_-(\sigma))}$. □

The above yields

Lemma 4.13. *$B = X_\sigma$ is an elementary cobordism with a single fixed point component $F := \overline{O}_\delta$, where $\delta = \langle v_i \mid r_i \neq 0\rangle$ is a circuit. Moreover $(X_\sigma)_+ = X_{\partial_-(\sigma)} = X_\sigma \setminus \overline{O}_{\sigma_+}$, where*

$$\sigma_+ := \langle v_i \mid r_i > 0\rangle, \quad \sigma_- := \langle v_i \mid r_i < 0\rangle.$$

In particular $F^+ = (\overline{O}_\delta)^+ = \overline{O}_{\sigma_+}$ and $F^- = (\overline{O}_\delta)^- = \overline{O}_{\sigma_-}$.

4.5. Example 2.7 revisited. The cobordism X_σ from the lemma generalizes the cobordism $B = \mathbb{A}_K^{l+m+r} \supset T = (K^*)^{l+m+r}$ from Example 2.7. The action of K^* determines a 1-parameter subgroup of T which corresponds to a vector $v_0 = [a_1, \dots, a_l, -b_1, \dots, -b_m, 0, \dots, 0]$. The cobordism B is associated with a nonsingular cone $\Delta \subset N_{\mathbb{Q}}$, while B_- and B_+ correspond to the fans $\partial_+(\Delta)$ and $\partial_-(\Delta)$ consisting of the faces of Δ visible from above and below respectively.

The quotients B_+/K^* , B_-/K^* and $B//K^*$ are toric varieties corresponding to the fans $\pi(\partial_+(\Delta)) = \{\pi(\sigma) \mid \sigma \in \partial_+(\Delta)\}, \pi(\partial_-(\Delta)) = \{\pi(\sigma) \mid \sigma \in \Delta^-\}$ and $\pi(\Delta)$ respectively, where π is the projection defined by v_0.

The relevant birational map $\phi\colon B_-/K^* \dashrightarrow B_+/K^*$ for $l, m \geq 2$ is a toric flip associated with a bistellar operation replacing the triangulation $\pi(\partial_-(\Delta))$ of the cone $\pi(\Delta)$ with $\pi(\partial_+(\Delta))$.

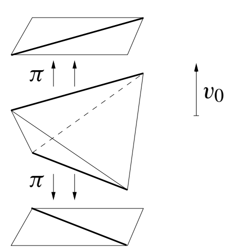

Figure 6. Morelli cobordism

4.6. π-nonsingular cones

Definition 4.14 (Morelli). An independent cone τ is *π-nonsingular* if $\pi(\tau)$ is nonsingular. A fan Σ is *π-nonsingular* if all independent cones in Σ are π-nonsingular. In particular a dependent cone σ is *π-nonsingular* if all its independent faces are π-nonsingular.

Lemma 4.15. *Let $\sigma = \langle v_1, \dots, v_k\rangle$ be a dependent cone and w_i be primitive generators of the rays $\pi(v_i)$. Let $\sum r_i' w_i = 0$ be the unique relation $(**)$ between vectors w_i. Then the ray $\varrho := \pi(\sigma_+) \cap \pi(\sigma_-)$ is generated by the vector $\sum_{r_i'>0} r_i' w_i = \sum_{r_i'<0} -r_i' w_i$ and $\varrho\cdot\pi(\partial_+(\sigma)) = \varrho\cdot\pi(\partial_-(\sigma))$. If σ is a π-nonsingular dependent cone then the ray ϱ defines regular star subdivisions of $\pi(\partial_+(\sigma))$ and $\pi(\partial_-(\sigma))$.*

Proof. Note that $\pi(\partial_+(\sigma)) \setminus \pi(\partial_-(\sigma))$ are exactly the cones containing $\pi(\sigma_+)$. That is, $\pi(\partial_+(\sigma)) \setminus \pi(\partial_-(\sigma)) = \mathrm{Star}(\pi(\sigma_+), \pi(\partial_+(\sigma)))$. This gives $\varrho \cdot \pi(\partial_+(\sigma)) = (\pi(\sigma_+) \cap \pi(\sigma_-)) \cup \{\varrho + \tau \mid \tau \in \pi(\sigma_+) \cap \pi(\sigma_-)\} = \varrho \cdot \pi(\sigma_-)$. Assume now that σ is π-nonsingular and all the coefficients r_i' are coprime. By Lemma 4.9 and the π-nonsingularity the set of vectors $w_1, \dots, \check{w}_i, \dots, w_k$ where $r_i' \neq 0$ is a basis of the lattice $\pi(\sigma) \cap N$. Thus every vector w_i, where $r_i' \neq 0$, can be written as an integral combination of others. Since the relation $(**)$ is unique it follows that the coefficient r_i' is equal to ± 1. Thus ϱ is generated by the vector $\sum_{r_i'>0} w_i = \sum_{r_i'<0} w_i$ and determines regular star subdivisions. □

Corollary 4.16. *If σ is dependent then there exists a factorization*

$$(X_\sigma)_-/K^* \xleftarrow{\phi_-} \Gamma((X_\sigma)_-/K^*, (X_\sigma)_+/K^*) \xrightarrow{\phi_+} (X_\sigma)_+/K^*,$$

where $\Gamma((X_\sigma)_-/K^*, (X_\sigma)_+/K^*)$ is the normalization of the graph of $(X_\sigma)_-/K^* \to (X_\sigma)_+/K^*$. If σ is π-nonsingular the morphisms ϕ_-, ϕ_+ are blow-ups of smooth centers.

Proof. By definition $\Gamma((X_\sigma)_-/K^*, (X_\sigma)_+/K^*)$ is a toric variety. By the universal property of the graph (dominating component of the fiber product) it corresponds to the coarsest simultaneous subdivision of both $\pi(\sigma_-)$ and $\pi(\sigma_+)$, that is, to the fan $\{\tau_1 \cap \tau_2 \mid \tau_1 \in \pi(\sigma_-), \tau_2 \in \pi(\sigma_+)\} = \varrho \cdot \pi(\sigma_-) = \varrho \cdot \pi(\sigma_+)$. □

4.7. The π-desingularization lemma of Morelli and centers of blow-ups. For any simplicial cone $\sigma = \langle v_1, \dots, v_k\rangle$ in N set

$$\mathrm{par}(\sigma) := \{v \in \sigma \cap N_\sigma \mid v = \alpha_1 v_1 + \cdots + \alpha_k v_k, \text{ where } 0 \leq \alpha_i < 1\},$$

$$\overline{\mathrm{par}(\sigma)} := \{v \in \sigma \cap N_\sigma \mid v = \alpha_1 v_1 + \cdots + \alpha_k v_k, \text{ where } 0 \leq \alpha_i \leq 1\}.$$

We associate with a dependent cone σ and an integral vector $v \in \pi(\sigma)$ a vector $\mathrm{Mid}(v, \sigma) := \pi^{-1}_{|\partial_-(\sigma)}(v) + \pi^{-1}_{|\partial_+(\sigma)}(v) \in \sigma$ ([27]), where $\pi_{|\partial_-(\sigma)}$ and $\pi_{|\partial_+(\sigma)}$ are the restrictions of π to $\partial_-(\sigma)$ and $\partial_+(\sigma)$.

We also set $\mathrm{Ctr}_-(\sigma) := \sum_{r_i<0} w_i$, $\mathrm{Ctr}_+(\sigma) := \sum_{r_i>0} w_i$.

Lemma 4.17 (Morelli [27], [28], [4]). *Let Σ be a simplicial cobordism in N^+. Then there exists a simplicial cobordism Δ obtained from Σ by a sequence of star subdivisions such that Δ is π-nonsingular. Moreover, the sequence can be taken so that any independent and already π-nonsingular face of Σ remains unaffected during the process. All the centers of the star subdivisions are of the form $\pi_{|\tau}^{-1}(\mathrm{par}(\pi(\tau)))$ where τ is independent, and $\mathrm{Mid}(\mathrm{Ctr}_{\pm}(\sigma),\sigma)$, where σ is dependent.*

Remark 4.18. It follows from Lemma 4.17 that π-desingularization can be done for an open affine neighborhood of a point x of $F \in \mathcal{C}(B^{K^*})$ on the smooth cobordism B which is étale isomorphic with the tangent space Tan_x. We need to show how to globalize this procedure in a coherent and possibly canonical way. This will replace the tangent space Tan_x in the local description of flips defined by elementary cobordisms (as in Proposition 2.25) with π-nonsingular X_σ.

By Corollary 4.16 we get a factorization into a blow-up and a blow-down at smooth centers: $(B_a)_-/K^* \overset{\phi_-}{\longleftarrow} \Gamma((B_a)_-/K^*, (B_a)_+/K^*) \overset{\phi_+}{\longrightarrow} (B_a)_+/K^*$.

5. π-desingularization of birational cobordisms

5.1. Stratification by isotropy groups on a smooth cobordism. Let B be a smooth cobordism of dimension n. Denote by Γ_x the isotropy group of a point $x \in B$. Define the stratum $s = s_x$ through x to be an irreducible component of the set $\{p \in B \mid \Gamma_x = \Gamma_p\}$.

We can find Γ_x-semiinvariant parameters in the affine open neighborhood U of x such that Γ_x acts nontrivially on $u_1, \dots, u_k$ and trivially on $u_{k+1}, \dots, u_n$.

After suitable shrinking of U the parameters define an étale Γ_x-equivariant morphism $\varphi\colon U \to \mathrm{Tan}_x = \mathbb{A}^n$. By definition the stratum s is locally described by $u_1 = \cdots = u_k = 0$. The parameters $u_1, \dots, u_k$ determine a Γ_x-equivariant smooth morphism

$$\psi\colon U \to \mathrm{Tan}_{B,x}/\mathrm{Tan}_{s,x} = \mathbb{A}^k.$$

We shall view $\mathbb{A}^k = X_\sigma$ as a toric variety with a torus T_σ and refer to ψ as a *toric chart*. This assigns to a stratum s the cone σ and the relevant group Γ_σ acting on X_σ. Then Luna's [24] fundamental lemma implies that the morphisms ϕ and ψ preserve stabilizers, the induced morphism $\psi_\Gamma\colon U/\!/\Gamma_x \to X_\sigma/\!/\Gamma_\sigma$ is smooth and $U \simeq U/\!/\Gamma_x \times_{\mathbb{A}^k/\!/\Gamma_x} \mathbb{A}^k$.

The invariant Γ_x can be defined for $X_\sigma = \mathbb{A}^k$ and determine the relevant T_σ-invariant stratification S_σ on X_σ. By shrinking U we may assume that the strata on U are inverse images of the strata on X_σ. Any stratum s_y on U through y after a suitable rearrangement of $u_1, \dots, u_k$ is described in the neighborhood $U' \subset U$ of y by $u_1 = \cdots = u_\ell = 0$, where $\Gamma_y \le \Gamma_x$ acts nontrivially on $u_1, \dots, u_\ell$ and trivially on $u_{\ell+1} \dots, u_k, u_{k+1}, \dots, u_n$. The remaining Γ_y-invariant parameters at y are $u_{\ell+1} - u_{\ell+1}(y), \dots, u_n - u_n(y)$. Then the closure of $\bar{s}_y$ is described on U by $u_1 = \cdots = u_\ell = 0$ and contains s_x. This shows

Lemma 5.1. *The closure of any stratum is a union of strata.*

We can introduce an order on the strata by setting

$$s' \leq s \quad \text{iff} \quad \bar{s}' \subseteq s.$$

Lemma 5.2. *If $s' \leq s$ then there exists an inclusion $i_{\sigma'\sigma}\colon \sigma' \hookrightarrow \sigma$ onto a face of σ. The inclusion $i_{\sigma'\sigma}$ defines a $\Gamma_{\sigma'}$-equivariant morphism of toric varieties $X_{\sigma'} \to X_{\sigma'} \times 1 \hookrightarrow X_{\sigma'} \times T \subset X_\sigma$, where $T_{\sigma'} \times T = T_\sigma$ and $\Gamma_{\sigma'} \subset T_{\sigma'}$. Moreover we can write $X_\sigma \cong X_{\sigma'} \times \mathbb{A}^r$ where $\Gamma_{\sigma'}$ acts trivially on $\mathbb{A}^r$ and nontrivially on all coordinates of $X_{\sigma'} \simeq \mathbb{A}^\ell$.*

In the above situation we shall write

$$\sigma' \leq \sigma.$$

The lemma above immediately implies

Lemma 5.3. *If $\tau < \sigma$ (that is, $\tau \leq \sigma$, $\tau \neq \sigma$) then $\Gamma_\tau \subsetneq \Gamma_\sigma$.*

Consider the stratification S_σ on X_σ. Every stratum $s_\tau \in S_\sigma$, where $\tau \leq \sigma$, is a union of orbits $O_{\tau'}$. Set

$$\bar{\tau} := \{\tau' \mid O_{\tau'} \subset s_\tau\}.$$

Lemma 5.4. *Any cone from the set $\tau' \in \bar{\tau}$ can be expressed as $\tau' \simeq \tau \times \langle e_1, \ldots, e_r\rangle \subset \sigma$, and $X_{\tau'} = X_\tau \times \mathbb{A}^s \times T^{r-s}$ where Γ_τ acts trivially on $\mathbb{A}^r \times T^{r-s}$.*

Lemma 5.5. *We have $\Gamma_\tau = \Gamma_{\tau'} := \{g \in \Gamma_\sigma \mid$ for all $x \in O_{\sigma'}$, $gx = x\}$ for any $\tau' \in \bar{\tau}$.*

5.2. Local projections

Definition 5.6. A cone σ in $N^{\mathbb{Q}}$ is *of maximal dimension* if $\dim \sigma = \dim N^{\mathbb{Q}}$.

Every cone σ in $N^{\mathbb{Q}}$ defines a cone of maximal dimension in $N^{\mathbb{Q}} \cap \operatorname{span}\{\sigma\}$ with lattice $N \cap \operatorname{span}\{\sigma\}$. We denote it by $\underline{\sigma}$. There is a noncanonical isomorphism

$$X_\sigma = X_{\underline{\sigma}} \times O_\sigma.$$

The vector space $\operatorname{span}\{\sigma\} \subset N^{\mathbb{Q}}$ corresponds to a subtorus $T_{\underline{\sigma}} \subset T_\sigma$ defined as $T_{\underline{\sigma}} := \{t \in T_\sigma \mid tx = x \text{ for } x \in O_\sigma\}$. Then O_σ is isomorphic to the torus $T_\sigma / T_{\underline{\sigma}}$ with dual lattice $\sigma^\perp \subset M^Q$.

Lemma 5.7. *If $\Gamma \subset T_\sigma$ acts freely on $X_\sigma = X_{\underline{\sigma}} \times O_\sigma$ then*

$$X_\sigma / \Gamma = X_{\underline{\sigma}} \times O_\sigma / \Gamma,$$

where $O_\sigma \simeq O_\sigma / \Gamma$ if Γ is finite, while O_σ / Γ is isomorphic to a torus of dimension $\dim O_\sigma - 1$ *if* $\Gamma = K^*$.

Proof. By assumption $\Gamma \cap T_{\underline{\sigma}}$ is trivial. Hence Γ acts trivially on $X_{\underline{\sigma}}$ and $X_\sigma/\Gamma = X_{\underline{\sigma}} \times O_\sigma/\Gamma$. □

Let $\pi_\sigma : (\sigma, N_\sigma) \to (\sigma^\Gamma, N_\sigma^\Gamma)$ denote the projection corresponding to the quotient map $X_\sigma \to X_\sigma//\Gamma_\sigma$.

Lemma 5.8. *If $\tau \leq \sigma$ then $\pi_\tau(\tau) \simeq \pi_\sigma(\tau)$.*

Proof. $X_{\underline{\tau}} \times O_\tau$ is an open subvariety in X_σ and Γ_τ acts trivially on O_τ. We have

$$(X_{\underline{\tau}} \times O_\tau)/\Gamma_\tau = X_{\underline{\tau}}/\Gamma_\tau \times O_\tau = X_{\pi_\tau(\tau)} \times O_\tau.$$

$\Gamma_\sigma/\Gamma_\tau$ acts freely on $(X_{\underline{\tau}} \times O_\tau)/\Gamma_\tau = X_{\pi_\tau(\tau)} \times O_\tau$. Thus by the previous lemma $X_{\pi_\sigma(\tau)} \cong X_{\pi_\tau(\tau)} \times O_\tau/\Gamma_\sigma$. □

Lemma 5.9. *Let Γ be a subgroup of Γ_σ, and $\pi_\Gamma : \sigma \to \sigma^\Gamma$ be the projection corresponding to the quotient $X_\sigma \to X_\sigma/\Gamma$. For any $\tau \leq \sigma$ and $\tau' \in \bar{\tau}$ we have $\tau' = \tau \oplus \langle e_1, \ldots, e_k\rangle$ where $\langle e_1, \ldots, e_k\rangle$ is regular and $\pi_\Gamma(\tau') = \pi_\Gamma(\tau) \oplus \langle e_1, \ldots, e_k\rangle$.*

Proof. $X_{\tau'} = X_\tau \times \mathbb{A}^k \times O_{\tau'}$ where the action of $\Gamma_\tau \cap \Gamma$ on $\mathbb{A}^k \times O_\tau$ is trivial. Thus $X_{\tau'}/\Gamma_\tau = X_\tau/\Gamma_\tau \times \mathbb{A}^k \times O_{\tau'}$. Now $\Gamma/(\Gamma_\tau \cap \Gamma)$ acts freely on $O_{\tau'} \subset s_\tau$ and we use Lemma 5.7. □

5.3. Independent and dependent cones. By Lemma 5.8 there exists a lattice isomorphism $j_{\tau\sigma} : \pi_\tau(\tau) \to \pi_\sigma(\tau)$, where $\tau \leq \sigma$. Thus the projections π_τ and π_σ are coherent and related: $j_{\tau\sigma}\pi_\tau = \pi_\sigma$.

Case 1: $\Gamma_\sigma = K^*$. The action of K^* on X_σ corresponds to a primitive vector $v_\sigma \in N_\sigma$. The invariant characters $M_\sigma^\Gamma \subset M_\sigma$ are precisely those $F \in M_\sigma^\Gamma$ such that $F(v_\sigma) = 0$. The dual morphism is a projection $\pi_\sigma : N_\sigma \to N_\sigma/\mathbb{Z} \cdot v_\sigma = N_\sigma^\Gamma$.

The quotient morphism of toric varieties $X_\sigma \to X_\sigma/\Gamma_\sigma$ corresponds to the projection $\sigma \to \pi_\sigma(\sigma)$.

Case 2: $\Gamma_\sigma \cong \mathbb{Z}_n$. The invariant characters $M_\sigma^\Gamma \subset M_\sigma$ form a sublattice of dimension $\dim(M_\sigma^\Gamma) = \dim(M_\sigma)$, where $M_\sigma/M_\sigma^\Gamma \simeq \mathbb{Z}_n$. The dual morphism defines an inclusion $\pi : N_\sigma \hookrightarrow N_\sigma^\Gamma$. The projection $\sigma \to \pi_\sigma(\sigma)$ is a linear isomorphism which does not preserve lattices. This gives

Lemma 5.10. *X_τ is independent iff Γ_τ is finite. X_σ is dependent iff $\Gamma_\sigma = K^*$.*

Definition 5.11. Let Δ^σ be a decomposition of a cone $\sigma \in \Sigma$. A cone $\tau \in \Delta^\sigma$ is *independent* if $\pi_{\sigma|\tau}$ is a linear isomorphism. A cone τ is *dependent* if $\pi_{\sigma|\tau}$ is not a linear isomorphism.

5.4. Semicomplexes and birational modification of cobordisms. By glueing cones σ corresponding to strata along their faces we construct a *semicomplex* Σ, that is, a partially ordered set of cones such that for $\sigma \le \sigma'$ there exists a face inclusion $i_{\sigma\sigma'}\colon \sigma \to \sigma'$.

Remark 5.12. The glueing need not be transitive: for $\sigma \le \sigma' \le \sigma''$ we have $i_{\sigma'\sigma''}i_{\sigma\sigma'} \neq i_{\sigma\sigma''}$. Instead, there exists an automorphism α_σ of σ such that $i_{\sigma'\sigma''}i_{\sigma\sigma'} = i_{\sigma\sigma''}\alpha_\sigma$.

For any fan Σ denote by $\mathrm{Vert}(\Sigma)$ the set of all 1-dimensional faces (rays) in Σ. Denote by $\mathrm{Aut}(\sigma)$ the automorphisms of σ inducing Γ_σ-equivariant automorphisms.

Definition 5.13. By a *subdivision* of Σ we mean a collection $\Delta = \{\Delta^\sigma \mid \sigma \in \Sigma\}$ of subdivisions Δ^σ of σ such that:

1. If $\tau \le \sigma$ then the restriction $\Delta^\sigma_{|\tau}$ of Δ^σ to τ is equal to Δ^τ.
2. All rays in $\mathrm{Vert}(\Delta^\sigma) \setminus \mathrm{Vert}(\sigma)$ are contained in $\bigcup_{\tau\le\sigma} \mathrm{int}(\tau)$.
3. Δ^σ is $\mathrm{Aut}(\sigma)$-invariant.

Remark 5.14. Condition 3 is replaced with a stronger one in the following proposition.

Lemma 5.15. *If $\tau' \in \bar{\tau}$, $\tau' \prec \sigma \in \Sigma$ then $\mathrm{Vert}(\Delta^\sigma_{|\tau'}) \setminus \mathrm{Vert}(\tau') \subset \tau$ and*

$$\Delta^\sigma_{|\tau'} = \Delta^\sigma_{|\tau} \oplus \langle e_1, \dots, e_k\rangle = \Delta^\tau \times \langle e_1, \dots, e_k\rangle.$$

Lemma 5.16. *For every point $x \in B \setminus (B_+ \cap B_-)$, $x \in s'$ there exists a toric chart $x \in U_\sigma \to X_\sigma$, with $\Gamma_\sigma = K^*$, corresponding to a stratum $s \subset \bar{s}'$. In particular the maximal cones of Σ are circuits.*

Proof. Let τ correspond to a stratum $s' \ni x$. By definition of cobordism $\lim_{t\to 0} tx = x_0$ or $\lim_{t\to\infty} tx = x_0$ exists. The point x_0 is K^*-fixed and belongs to a stratum s, with $\Gamma_s = \Gamma_\sigma = K^*$. Since U is a K^*-invariant neighborhood of x_0 it contains an orbit $K^* \cdot x$ and the point x. Moreover $\bar{s}' \supset s$ and $\tau \le \sigma$. □

Lemma 5.17. *Let σ be the cone corresponding to a stratum s on B and $x \in s$. Then $\widehat{X}_x = \mathrm{Spec}\, \widehat{O}_{x,B} \simeq (X_\sigma \times \mathbb{A}^{\dim(s)})^\wedge \cong \mathrm{Spec}\, K[[x_1, \dots, x_k, \dots, x_n]]$.*

Set $\widetilde{X}_\sigma := (X_\sigma \times \mathbb{A}^{\dim(s)})^\wedge$ and let G_σ denote the group of all Γ_σ-equivariant autorphisms of $\widetilde{X}_\sigma$.

The subdivision Δ^σ of σ defines a toric morphism and induces a proper birational Γ_σ-equivariant morphism

$$\widetilde{X}_{\Delta^\sigma} := X_{\Delta^\sigma} \times_{X_\sigma} \widetilde{X}_\sigma \to \widetilde{X}_\sigma.$$

Proposition 5.18. *Let $\Delta = \{\Delta^\sigma \mid \sigma \in \Sigma\}$ be a subdivision of Σ such that:*

$$\text{For every } \sigma \in \Sigma \text{ the morphism } \widetilde{X}_{\Delta^\sigma} \to \widetilde{X}_\sigma \text{ is } G_\sigma\text{-equivariant.} \tag{1}$$

Then Δ defines a K^*-equivariant birational modification $f : B' \to B$ such that for every toric chart $\varphi_\sigma : U \to X_\sigma$ there exists a Γ_σ-equivariant fiber square

$$\begin{array}{ccc} U_\sigma \times_{X_\sigma} X_{\Delta^\sigma} \simeq f^{-1}(U_\sigma) & \to & X_{\Delta^\sigma} \\ \downarrow f & & \downarrow \\ U_\sigma & \to & X_\sigma . \end{array} \tag{2}$$

Definition 5.19. A decomposition Δ of Σ is *canonical* if it satisfies condition (1).

Proof. The above diagrams define open subsets $f_\sigma^{-1}(U_\sigma)$ together with proper birational Γ_σ-equivariant morphisms $f_\sigma^{-1}(U_\sigma) \to U_\sigma$. Let $s' \le s$ be a stratum corresponding to the cone $\tau \le \sigma$. By Lemma 5.15, the restriction of the diagram (2) defined by $U_\sigma \to X_\sigma$ to a neighborhood U_τ of $y \in s'$ determines a diagram defined by the induced toric chart $U_\tau \to X_\tau$ and the decomposition Δ^τ of τ. In order to show that the $f_\sigma^{-1}(U)$ glue together we need to prove that for $x \in s$ and two different charts of the form $\varphi_{1,\sigma} : U_{1,\sigma} \to X_\sigma$ and $\varphi_{2,\sigma} : U_{2,\sigma} \to X_\sigma$ where $x \in U_{1,\sigma}, U_{2,\sigma}$ the induced varieties $V_1 := f_{1,\sigma}^{-1}(U_{1,\sigma})$ and $V_2 := f_{2,\sigma}^{-1}(U_{2,\sigma})$ are isomorphic over $U_{1,\sigma} \cap U_{2,\sigma}$. For simplicity assume that $U_{1,\sigma} = U_{2,\sigma} = U$ by shrinking $U_{1,\sigma}$ and $U_{2,\sigma}$ if necessary. The charts $\varphi_{1,\sigma}, \varphi_{2,\sigma} : U \to X_\sigma$ are defined by the two sets of semiinvariant parameters, $u_1^1, \dots, u_k^1$ and $u_1^2, \dots, u_k^2$ with a nontrivial action of Γ_σ. These sets can be extended to full sets of parameters $u_1^1, \dots, u_k^1, u_{k+1}, \dots, u_n$ and $u_1^2, \dots, u_k^2, u_{k+1}, \dots, u_n$ where Γ_σ acts trivially on $u_{k+1}, \dots, u_n$, and $u_{k+1} \dots, u_n$ define parameters on the stratum s at x. These two sets of parameters define étale morphisms $\varphi_{1,\sigma}, \varphi_{2,\sigma} : U \to X_\sigma \times \mathbb{A}^{n-k}$ and fiber squares

$$\begin{array}{cccc} \bar{\varphi}_{i,\sigma} : & V_i & \to & X_{\Delta^\sigma} \times \mathbb{A}^{n-k} \\ & \downarrow & & \downarrow \\ \varphi_{i,\sigma} : & U & \to & X_\sigma \times \mathbb{A}^{n-k}. \end{array}$$

Suppose the induced Γ-equivariant birational map $f : V_1 \dashrightarrow V_2$ is not an isomorphism over U.

Let V be the graph of f which is a dominating component of the fiber product $V_1 \times_U V_2$. Then either $V \to V_1$ or $V \to V_2$ is not an isomorphism (i.e. collapses a curve to a point) over some $x \in s \cap U$. Consider an étale Γ_σ-equivariant morphism $e : \widehat{X}_x \to U$. Pull-backs of the morphisms $V_i \to U$ via e define two different nonisomorphic Γ_σ-equivariant liftings $Y_i \to \widehat{X}_x$, since the graph Y of $Y_1 \dashrightarrow Y_2$ (which is a pull-back of V) is not isomorphic to at least one Y_i. But these two liftings are defined by two isomorphisms $\widehat{\varphi}_1, \widehat{\varphi}_2 : \widehat{X}_x \simeq \widetilde{X}_\sigma$. These isomorphisms differ by some automorphism $g \in G_\sigma$, so we have $\widehat{\varphi}_1 = g \circ \widehat{\varphi}_2$. Since g lifts to the automorphism of $\widetilde{X}_{\Delta^\sigma}$ we get $Y_1 \simeq Y_2 \simeq \widetilde{X}_{\Delta^\sigma}$, which contradicts the choice of Y_i.

Thus V_1 and V_2 are isomorphic over any $x \in s$ and B' is well defined by glueing pieces $f_\sigma^{-1}(U)$ together. We need to show that the action of K^* on B lifts to the action of K^* on B'.

Note that B' is isomorphic to B over the open generic stratum $U \supset B_+ \cup B_-$ of points x with $\Gamma_x = \{e\}$. By Lemma 5.16 every point $x \in B \setminus (B_+ \cap B_-)$ is in U_σ, with $\Gamma_\sigma = K^*$. Then the diagram (2) defines the action of K^* on $f^{-1}(U_\sigma)$. □

5.5. Basic properties of valuations. Let $K(X)$ be the field of rational functions on an algebraic variety or an integral scheme X. A *valuation* on $K(X)$ is a group homomorphism $\mu\colon K(X)^* \to G$ from the multiplicative group $K(X)^*$ to a totally ordered group G such that $\mu(a+b) \geq \min(\mu(a), \mu(b))$. By the *center* of a valuation μ on X we mean an irreducible closed subvariety $Z(\mu) \subset X$ such that for any open affine $V \subset X$, intersecting $Z(\mu)$, the ideal $I_{Z(\mu)\cap V} \subset K[V]$ is generated by all $f \in K[V]$ such that $\mu(f) > 0$ and for any $f \in K[V]$, we have $\mu(f) \geq 0$. Each vector $v \in N^{\mathbb{Q}}$ defines a linear function on M which determines a valuation $\mathrm{val}(v)$ on a toric variety $X_\Sigma \supset T$.

For any regular function $f = \sum_{w\in M} a_w x^w \in K[T]$ set

$$\mathrm{val}(v)(f) := \min\{(v, w) \mid a_w \neq 0\}.$$

If $v \in \mathrm{int}(\sigma)$, where $\sigma \in \Delta$, then $\mathrm{val}(v)$ is positive for all x^F, where $F \in \sigma^\vee \setminus \sigma^\perp$. In particular we get

$$Z(\mathrm{val}(v)) = \overline{O}_\sigma \quad \text{iff} \quad v \in \mathrm{int}\,\sigma.$$

If $v \in \sigma$ then $\mathrm{val}(v)$ is a valuation on $R = K[X_\sigma] = K[\sigma^\vee]$, that is, $\mathrm{val}(v)(f) \geq 0$ for all $f \in K[\sigma^\vee] \setminus \{0\}$. We construct ideals for all $a \in \mathbb{N}$ which uniquely determine $\mathrm{val}(v)$:

$$I_{\mathrm{val}(v),a} = \{f \in R \mid \mathrm{val}(v)(f) \geq a\} = (x^F \mid F \in \sigma^\vee, F(v) \geq a) \subset R.$$

By glueing $I_{\mathrm{val}(v),a}$ for all $v \in \sigma$ and putting $\mathcal{I}_{\mathrm{val}(v),a|X_\sigma} = \mathcal{O}_{X_\sigma}$ if $v \notin \sigma$ we construct a coherent sheaf of ideals $\mathcal{I}_{\mathrm{val}(v),a}$ on X_Δ.

Lemma 5.20 ([23]). *The star subdivision $\langle v\rangle \cdot \Sigma$ corresponds to the normalized blow-up of $\mathcal{I}_{\mathrm{val}(v),a}$ on X_Σ for a sufficiently divisible $a \in \mathbb{N}$.*

5.6. Stable vectors. Let $g\colon X \to Y$ be any dominant morphism of integral schemes (that is, $\overline{g(X)} = Y$) and μ be a valuation of $K(X)$. Then g induces a valuation $g_*(\mu)$ on $K(Y) \simeq g(K(Y)) \subset K(X)$: $g_*\mu(f) = \mu(f \circ g)$.

Definition 5.21. Let Σ be the semicomplex defined for the cobordism B. A vector $v \in \mathrm{int}(\sigma)$, where $\sigma \in \Sigma$, is called *stable* if for every $\sigma \leq \sigma'$, $\mathrm{val}(v)$ is $G_{\sigma'}$-invariant on $\widetilde{X}_{\sigma'}$.

Lemma 5.22. *If $\widetilde{X}_{\Delta^\sigma} \to \widetilde{X}_\sigma$ is G_σ-equivariant and* $\mathrm{val}(v)$ *is G_σ-invariant then $\widetilde{X}_{\langle v\rangle\cdot\Delta^\sigma} \to \widetilde{X}_\sigma$ is G_σ-equivariant.*

Proof. The morphism $\widetilde{X}_{\langle v\rangle\cdot\Delta^\sigma} \to \widetilde{X}_{\Delta^\sigma}$ is a pull-back of the morphism $X_{\langle v\rangle\cdot\Delta^\sigma} \to X_{\Delta^\sigma}$. Thus, by Lemma 5.20, $\widetilde{X}_{\langle v\rangle\cdot\Delta^\sigma} \to \widetilde{X}_{\Delta^\sigma}$ is a normalized blow-up of $\mathcal{I}_{\mathrm{val}(v),a}$ on $\widetilde{X}_{\Delta^\sigma}$. But the latter sheaf is G_σ-invariant. □

Proposition 5.23. *Let $\Delta = \{\Delta^\sigma \mid \sigma \in \Sigma\}$ be a canonical subdivision of Σ and v be a stable on Σ. Then $\langle v\rangle \cdot \Delta := \{\langle v\rangle \cdot \Delta^\sigma \mid \sigma \in \Sigma\}$ is a canonical subdivision of Σ.*

5.7. Convexity

Lemma 5.24. *Let* $\mathrm{val}(v_1)$ *and* $\mathrm{val}(v_2)$ *be G_σ-invariant valuations on X_σ. Then all valuations* $\mathrm{val}(v)$*, where $v = av_1 + bv_2$, $a, b \geq 0$, $a, b \in \mathbb{Q}$, are G_σ-invariant.*

Proof. Let $\Delta = \langle v_1\rangle \cdot \langle v_2\rangle \cdot \sigma$ be a subdivision of σ. Then by Lemma 5.22, the morphism $\widetilde{X}_\Delta \to \widetilde{X}_\sigma$ is G_σ-equivariant. The exceptional divisors D_1 and D_2 of the morphism are G_σ-invariant and correspond to one-dimensional cones (rays) $\langle v_1\rangle, \langle v_2\rangle \in \Delta$. The cone $\tau = \langle v_1, v_2\rangle \in D$ corresponds to the orbit O_τ whose closure is $D_1 \cap D_2$ and thus the generic point is G_σ-invariant. The action of G_σ on $\widetilde{X}_\sigma$ induces an action on the local ring $\widetilde{X}_{\Delta,O_\tau}$ at the generic point of O_τ and on its completion $K(O_\tau)[[\underline{\tau}^\vee]]$. Note that for any $v \in \tau$, $\mathrm{val}(v)_{|K(O_\tau)} = 0$. For any $F \in \underline{\tau}^\vee = \frac{\tau^\vee}{\tau^\perp}$ the divisor (x^F) of the character x^F on $\widehat{X}_\tau := \operatorname{Spec} K(O_\tau)[[\underline{\tau}^\vee]]$ is a combination $n_1D_1 + n_2D_2$ for $n_1n_2 \in \mathbb{Z}$. Since D_1 and D_2 are G_σ-invariant, the divisor $(x^F) = n_1D_1 + n_2D_2$ is G_σ-invariant, that is, for any $g \in G$, we have $gx^F = u_{g,F} \cdot x^F$ where $u_{g,F}$ is invertible on $K(O_\tau)[[\underline{\tau}^\vee]]$. Thus for every $v \in \tau$ and $g \in G$ we have

$$\begin{aligned} g^*(I_{\mathrm{val}(v),a}) &= g^*(x^F | F \in \underline{\sigma}^\vee, F(v) \geq a) \\ &= (u_{g,F}x^F | F \in \sigma^\vee, F(v) \geq a) = I_{\mathrm{val}(v),a}. \end{aligned}$$

Thus $\mathrm{val}(v)$ is G_σ-invariant on $K(O_\tau)[[\underline{\tau}^\vee]]$ and on its subring $\mathcal{O}_{\widetilde{X}_{\Delta,O_\tau}}$. The latter ring has the same quotient field as $\widetilde{X}_\sigma$ so $\mathrm{val}(v)$ is G_σ-invariant on $\widetilde{X}_\sigma$. □

Lemma 5.25. *Let $\sigma \in \Sigma$ and $v_1, v_2 \in \sigma$ be stable vectors. Then all vectors $v = av_1 + bv_2 \in \sigma$, where $a, b \in \mathbb{Q}_{>0}$, are stable.*

5.8. Basic properties of stable vectors

Lemma 5.26. *Let* $\mathrm{Tan}_0 = \mathbb{A}^n = \mathrm{Tan}_0^{a_0} \oplus \mathrm{Tan}_0^{a_1} \oplus \cdots \oplus \mathrm{Tan}_0^{a_k}$ *denote the tangent space of* $\widetilde{X}_\sigma = \operatorname{Spec} K[[u_1, \ldots, u_n]]$ *at* 0 *and its decomposition according to the weight distribution. Let $d : G_\sigma \to GL(\mathrm{Tan}_0)$ be the differential morphism defined as $g \mapsto dg$. Then $d(G_\sigma) = GL(\mathrm{Tan}_0^{a_1}) \times \cdots \times GL(\mathrm{Tan}_0^{a_k})$.*

Lemma 5.27. *Let $v \in \sigma$, where $\sigma \in \Sigma$, be an integral vector such that for any $g \in G_\sigma$, there exists an integral vector $v_g \in \sigma$ such that $g_*(\mathrm{val}(v)) = \mathrm{val}(v_g)$. Then* $\mathrm{val}(v)$ *is G_σ-invariant on $\widetilde{X}_\sigma$.*

Proof. Set $W = \{v_g \mid g \in G\}$. For any natural number n, the ideals $I_{\mathrm{val}(v_g),a}$ are generated by monomials. They define the same Hilbert–Samuel function $k \mapsto \dim_K(K[\widetilde{X}_\sigma]/(I_{\mathrm{val}(v_g),a} + m^k))$, where $m \subset K[\widetilde{X}_\sigma]$ denotes the maximal ideal. It follows that the set W is finite. On the other hand since $I_{\mathrm{val}(v_g),a}$ are generated by monomials they are uniquely determined by the ideals $\mathrm{gr}(I_{\mathrm{val}(v_g),a})$ in the graded ring

$$\mathrm{gr}(O_{\widetilde{X}_\sigma}) = O_{\widetilde{X}_\sigma}/m \oplus m/m^2 \oplus \cdots .$$

The connected group $d(G_\sigma)$ acts algebraically on $\mathrm{gr}(O_{\widetilde{X}_\sigma})$ and on the connected component of the Hilbert scheme with fixed Hilbert polynomial. In particular it acts trivially on its finite subset W and consequently $d(G_\sigma)$ preserves $\mathrm{gr}(I_{\mathrm{val}(v_g),a})$ and G_σ preserves $I_{\mathrm{val}(v_g),a}$. $\square$

Let $R \subset K$ be a ring contained in the field. We can order valuations by writing

$$\mu_1 > \mu_2 \quad \text{if} \quad \mu_1(a) \geq \mu_2(a) \text{ for all } a \in R \text{ and } \mu_1 \neq \mu_2.$$

A cone σ defines a partial ordering: $v_1 > v_2$ if $v_1 - v_2 \in \sigma$. Both orders coincide for $K[X_\sigma] \subset K(X_\sigma)$: $v_1 > v_2$ iff $\mathrm{val}(v_1) > \mathrm{val}(v_2)$.

Lemma 5.28. *Let σ be a cone in $N_\sigma^{\mathbb{Q}}$ with the lattice of* 1*-parameter subgroups $N_\sigma \subset N_\sigma^Q$ and the dual lattice of characters M_σ. Let μ be any integral* (*or rational*) *valuation centered on $\overline{O}_\tau$, where $\tau \preceq \sigma$. Then the restriction of μ to $M_\sigma \subset K(\widetilde{X}_\sigma)^*$ defines a functional on $\tau^\vee \subseteq M_\sigma^Q$ corresponding to a vector $v_\mu \in \mathrm{int}\,\tau$ such that $F(v_\mu) = \mu(x^F)$ for $F \in M_\sigma$ and $\mu \geq \mathrm{val}(v_\mu)$ on $\widetilde{X}_\sigma$.*

Proof. $I_{\mu,a} \supseteq (x^F \mid \mu(x^F) \geq a) = (x^F \mid F(v_\mu) \geq a) = I_{\mathrm{val}(v_\mu),a}$. $\square$

Lemma 5.29. *Let $\Gamma \subset \Gamma_\sigma$ be a finite group acting on $\widetilde{X}_\sigma$. Let $\pi : N^{\mathbb{Q}} \to (N^\Gamma)^{\mathbb{Q}}$ denote the projection corresponding to the geometric quotient $\widetilde{X}_\sigma \to \widetilde{X}_{\pi(\sigma)} = \widetilde{X}_\sigma/\Gamma$. Then* $\mathrm{val}(v)$ *is G_σ-invariant on $\widetilde{X}_\sigma$ iff* $\mathrm{val}(\pi(v))$ *is G_σ-invariant on $\widetilde{X}_{\pi(\sigma)}$.*

Proof. ($\Rightarrow$) $\mathrm{val}(v)$ is G_σ-invariant on $K[\widetilde{X}_\sigma]$ and it is invariant on $K[\widetilde{X}_\sigma]^\Gamma$.

($\Leftarrow$) Note that π defines an inclusion of same dimension lattices $N \hookrightarrow N^\Gamma$ and $M^\Gamma \hookrightarrow M$.

Assume that $\mathrm{val}(\pi(v))$ is G_σ-invariant. It defines a functional on the lattice M^Γ and its unique extension to $M \supset M^\Gamma$ corresponding to $\mathrm{val}(v)$. Since $g_*(\mathrm{val}(\pi(v))) = \mathrm{val}(\pi(v))$, we have $g_*(\mathrm{val}(v))_{|M^\Gamma} = \mathrm{val}(v)_{|M^\Gamma}$ and consequently $g_*(\mathrm{val}(v))_{|M} = \mathrm{val}(v)_{|M}$. By Lemma 5.28, $g_*(\mathrm{val}(v)) \geq \mathrm{val}(v)$ for all $g \in G_\sigma$. Thus $\mathrm{val}(v) \geq g_*^{-1}(\mathrm{val}(v))$ for all $g^{-1} \in G_\sigma$. Finally $g_*(\mathrm{val}(v)) = \mathrm{val}(v)$. $\square$

5.9. Stability of centers from par($\pi(\tau)$). In the following let Δ^σ be a decomposition of $\sigma \in \Sigma$ such that $\widetilde{X}_{\Delta^\sigma} \to \widetilde{X}_\sigma$ is G_σ-equivariant, $\tau \in \Delta^\sigma$ be its face and Γ be a finite subgroup of Γ_σ. Denote by $\pi : (\sigma, N_\sigma) \to (\sigma^\Gamma, N_\sigma^\Gamma)$ the linear isomorphism and the lattice inclusion corresponding to the quotient $X_\sigma \to X_\sigma/\Gamma = X_{\pi(\sigma)}$.

Lemma 5.30. *Assume that for any $g \in G_\sigma$, there exists a cone $\tau_g \in \Delta^\sigma$ such that $g \cdot (\overline{O}_\tau) = \overline{O}_{\tau_g}$. Let $v \in \operatorname{int}(\pi(\tau)) \cap N^\Gamma_\sigma$ be an integral vector such that* $\operatorname{val}(v)$ *is not G_σ-invariant on $\widetilde{X}_\sigma/\Gamma$. Then there exist integral vectors $v_1 \in \operatorname{int}(\pi(\tau))$ and $v_2 \in \pi(\tau)$ such that*

$$v = v_1 + v_2.$$

Moreover if there exists $v_0 \in \pi(\sigma)$ (not necessarily integral) such that $\operatorname{val}(v_0)$ *is G_σ-invariant and $v > v_0$ on $\pi(\sigma)$ then $v_1 > v_0$ on $\pi(\sigma)$.*

Proof. If val(v) is not G_σ-invariant on $\widetilde{X}_\sigma/\Gamma$ then by Lemma 5.27 there exists an element $g \in G_\sigma$ such that $\mu_g := g_*(\operatorname{val}(v))$ is not a toric valuation. By the assumption μ_g is centered on $\overline{O}_{\pi(\tau_g)}$. Then by Lemma 5.28 it defines $v_g \in \operatorname{int}\pi(\tau_g)$ such that $\mu_g(x^F) = F(v_g)$ for $F \in \sigma^\vee$. Moreover $\mu_g > \operatorname{val}(v_g)$. Then the valuation $g_*^{-1}(\operatorname{val}(v_g))$ is centered on $\overline{O}_{\pi(\tau)}$. Thus it defines an integral $v_1 \in \operatorname{int}(\pi(\tau))$ such that $v \geq v_1$ on $\pi(\tau)$ and $v_2 := v - v_1$. Then

$$\operatorname{val}(v) = g_*^{-1}(\mu_g) > g_*^{-1}(\operatorname{val}(v_g)) \geq \operatorname{val}(v_1).$$

Note also that if $v \geq v_0$ then $\mu_g = g_*(\operatorname{val}(v)) \geq \operatorname{val}(v_0)$ and $\operatorname{val}(v_g) \geq \operatorname{val}(v_0)$. Thus also $\operatorname{val}(v_1) \geq \operatorname{val}(v_0)$. □

Lemma 5.31. *All valuations* $\operatorname{val}(v)$*, where $v \in \varrho$, $\varrho \in \operatorname{Vert}(\Delta^\sigma) \setminus \operatorname{Vert}(\sigma)$, are G_σ-invariant.*

Proof. Let v_ϱ be the primitive generator of $\varrho \in \operatorname{Vert}(\Delta^\sigma) \setminus \operatorname{Vert}(\sigma)$. The ray ϱ corresponds to an exceptional divisor D_ϱ. By the definition there is no decomposition $v_\varrho = v_1 + v$. Thus by the previous lemma (for $\Gamma = \{e\}$), $\operatorname{val}(v)$ is G_σ-invariant. □

Lemma 5.32. *For any $\tau \leq \sigma$, the closure of the orbit $\overline{O}_\tau \subset \widetilde{X}_\sigma$ is G_σ-invariant.*

Proof. By Lemma 5.2, the ideal of $\overline{O}_\tau \subset \widetilde{X}_\sigma$ is generated by all functions with nontrivial Γ_σ-weights. □

Lemma 5.33. *The valuations* $\operatorname{val}(v)$*, where $v \in \operatorname{par}(\pi(\tau))$, are G_σ-invariant on $\widetilde{X}_{\Delta^\sigma}$. Moreover $v \in \operatorname{int}(\pi(\sigma_0))$, for some $\sigma_0 \leq \sigma$.*

Proof. Let $v \in \operatorname{par}(\pi(\tau))$, where $\pi(\tau) \in \pi(\Delta)$ is a minimal integral vector such that $\operatorname{val}(v)$ is not G_σ-invariant. We may assume that $v \in \operatorname{int}(\pi(\tau))$ passing to its face if necessary. Let $\sigma' \in \overline{\sigma}_0$ be a face of σ such that $v \in \operatorname{int}\pi(\sigma')$. In particular $\pi(\sigma') \subset \pi(\tau)$. Then $\pi(\Delta^\sigma)_{|\pi(\sigma')} = \pi(\Delta^\sigma)_{|\pi(\sigma_0)} \oplus \langle e_1, \ldots, e_k\rangle$ by Lemmas 5.9 and 5.15 and $v \in \operatorname{par}(\pi(\tau)) \subset \pi(\sigma_0)$. Thus $\sigma' = \sigma_0$ and $v \in \operatorname{int}(\pi(\sigma_0))$. Let

$$\pi(\tau) = \langle v_1, \ldots, v_k, w_1, \ldots, w_\ell\rangle,$$

where $v_1, \ldots, v_k \in \operatorname{Vert}(\pi(\tau))$ and $w_1, \ldots, w_\ell \in \operatorname{Vert}(\pi(\Delta)) \setminus \operatorname{Vert}(\pi(\sigma))$. By Lemma 5.31, $\operatorname{val}(w_1), \ldots, \operatorname{val}(w_\ell)$ are G_σ-invariant. Write

$$v = \alpha_1 v_1 + \cdots + \alpha_k v_k + \alpha_{k+1} w_1 + \cdots + \alpha_{k+\ell} w_\ell,$$

where $0 < \alpha_i < 1$. Note that

$$v \geq v_0 = \alpha_{k+1} w_1 + \cdots + \alpha_{k+\ell} w_\ell$$

and $\overline{O}_{\pi(\sigma_0)} \subset \widetilde{X}_{\pi(\sigma)}$ is G_σ-invariant. By Lemma 5.30 for $v \in \pi(\sigma_0) \leq \pi(\sigma)$ and $v > v_0$ we can find integral vectors $v', v'' \in \pi(\sigma)$ such that $v = v' + v''$, $v' \geq v_0$. Then

$$v'' := v - v' \leq v - v_0 = \alpha_1 v_1 + \cdots + \alpha_k v_k.$$

Thus $v'' \in \mathrm{par}\langle v_1, \ldots, v_k \rangle \subseteq \mathrm{par}(\pi)(\tau)$. Write $v'' := \beta_1 v_1 + \cdots + \beta_k v_k$, where $\beta_i \leq \alpha_i$. Then

$$v' = v - v'' = (\alpha_1 - \beta_1) v_1 + \cdots + (\alpha_k - \beta_k) v_k + \alpha_{k+1} w_1 + \cdots + \alpha_{k+\ell} \in \mathrm{par}(\pi(\tau)).$$

By the minimality assumption, $\mathrm{val}(v')$ and $\mathrm{val}(v'')$ are G_σ-invariant and it follows from Lemma 5.24 that $\mathrm{val}(v) = \mathrm{val}(v' + v'')$ is G_σ-invariant. □

Corollary 5.34. *Let $\Delta = \{\Delta^\sigma \in \Sigma\}$ be a decomposition of Σ. Let $\tau \in \Delta^\sigma$ be an independent face. Then the vectors in $(\pi_{\sigma_{|\tau}})^{-1}(\mathrm{par}(\pi_\sigma(\tau)))$ are stable.*

Proof. Put $\Gamma = \Gamma_\tau$. Let $\pi\colon (\sigma, N_\sigma) \to (\sigma, N_\sigma^\Gamma)$ be the linear isomorphism and a lattice inclusion corresponding to the quotient $X_\sigma \to X_\sigma / \Gamma$. Then by Lemma 5.8, $\pi(\tau) \simeq \pi_\tau(\tau) \simeq \pi_\sigma(\tau)$ and by Lemma 5.33 vectors in $(\pi_{\sigma_{|\tau}})^{-1}(\mathrm{par}(\pi_\sigma(\tau))) = \pi^{-1}(\mathrm{par}(\pi(\tau)))$ are stable. □

Corollary 5.35. 1. *Assume that for any $g \in G_\sigma$, there exists $\tau_g \in \Delta^\sigma$ such that $g(\overline{O}_\tau) = \overline{O}_{\tau_g}$. Then $\overline{O}_\tau$ is G_σ-invariant. Moreover all valuations* $\mathrm{val}(v)$, *where $v \in \overline{\mathrm{par}}\,(\tau) \cap \mathrm{int}(\tau)$, are G_σ-invariant.*

2. *Let $\tau \in \Delta^\sigma$ be an independent cone such that $\overline{O}_\tau$ is G_σ-invariant. Then for any $v \in \pi_\sigma^{-1}(\overline{\mathrm{par}}\,(\pi(\tau)) \cap \mathrm{int}(\pi(\tau)))$ the valuation* $\mathrm{val}(v)$ *is G_σ-invariant.*

Proof. 1. Let $\tau = \langle v_1, \ldots, v_k \rangle$ and $v = \alpha_1 v_1 + \cdots + \alpha_k v_k$, where $0 < \alpha_i \leq 1$, be a minimal vector in $\mathrm{int}(\tau) \cap \overline{\mathrm{par}}\,(\tau)$ such that $\mathrm{val}(v)$ is not G_σ-invariant. Then by Lemma 5.30, the vector v can be written as $v = v' + v''$, where $v', v'' < v$, $v' \in \mathrm{int}(\tau)$, $v'' \in \tau$. Thus $v' = \alpha'_1 v_1 + \cdots + \alpha'_k v_k$ where $0 < \alpha'_i \leq \alpha_i \leq 1$ and $v'' = \alpha''_1 v_1 + \cdots + \alpha''_k v_k$, where $0 \leq \alpha''_i = \alpha_i - \alpha'_i < 1$. Then $v' \in \mathrm{int}(\tau) \cap \overline{\mathrm{par}}\,(\tau)$ and $v'' \in \mathrm{par}(\tau)$. By Corollary 5.34, $\mathrm{val}(v'')$ is G_σ-invariant on $\widetilde{X}_\sigma$. By the minimality assumption $\mathrm{val}(v')$ is G_σ-invariant. Since $v = v' + v''$, the valuation $\mathrm{val}(v)$ is G_σ-invariant on $\widetilde{X}_\sigma$ and its center $Z(\mathrm{val}(v))$ equals $\overline{O}_\tau$.

2. Let $\pi\colon N \to N^\Gamma$ be the projection corresponding to the quotient $X_\sigma \to X_\sigma / \Gamma_\tau$. Then $\pi(\tau) \simeq \pi_\sigma(\tau)$. The proof is now exactly the same as the proof in 1 except that we replace $\widetilde{X}_{\Delta^\sigma}$ with $\widetilde{X}_{\Delta^\sigma} / \Gamma_\tau$. □

Corollary 5.36. *Let $\delta \in \Delta^\sigma$ be a circuit. Then $\overline{O}_\delta$ is G_σ-invariant.*

Proof. By Corollary 4.5, $\overline{O}_\delta$ is an irreducible component of a G_σ-invariant closed subscheme $\widetilde{X}_{\Delta^\sigma}^{K^*}$. Thus by the previous corollary it is G_σ-invariant. □

5.10. Stability of $\mathrm{Ctr}_+(\sigma)$. In the sequel $\delta = \langle v_1, \ldots, v_k\rangle \in \Delta^\sigma$ is a circuit. Let $\Gamma \subset \Gamma_\sigma = K^*$ be a finite group. Denote by π (resp. π_Γ) the projection corresponding to the quotient $X_\delta \to X_\delta//K^*$ (resp. $X_\delta \to X_\delta/\Gamma$). Write $\pi(\delta) = \langle w_1, \ldots, w_k\rangle$ and let $\sum_{r_i'>0} r_i' w_i = 0$ be the unique relation between vectors (∗∗) as in Section 4.4. Set $\mathrm{Ctr}_+(\delta) = \sum_{r_i'>0} w_i \in \overline{\mathrm{par}}(\pi(\delta_+)) \cap \mathrm{int}(\pi(\delta_+))$, where $\delta_+ = \langle v_i \mid r_i > 0\rangle$.

Denote by $\widehat{X}_\delta$ the completion of $\widetilde{X}_{\Delta^\sigma}$ at O_δ. By Corollary 5.36, the generic point $O_\delta \in \widetilde{X}_{\Delta^\sigma}$ is G_σ-invariant and thus G_σ acts on $\widehat{X}_\delta$. Moreover $K[\widehat{X}_\delta] = K(O_\delta)[[\delta^\vee]]$ is faithfully flat over a $\mathcal{O}_{\widetilde{X}_{\Delta^\sigma}, O_\delta}$. Also, $\widehat{\mathcal{O}}_{X_{\pi_\Gamma(\Delta)}, \widetilde{O}_{\pi_\Gamma(\delta)}} = K(\widetilde{O}_{\pi_\Gamma(\delta)})[[\underline{\pi_\Gamma(\delta)}^\vee]]$ is faithfully flat over $\mathcal{O}_{X_{\pi_\Gamma(\Delta)}, \widetilde{O}_{\pi_\Gamma(\delta)}}$ and we get

Lemma 5.37. *The valuation* $\mathrm{val}(v)$*, where* $v \in \pi_\Gamma(\delta)$*, is* G_σ*-invariant on* $\widehat{X}_\delta/\Gamma$ *iff it is* G_σ*-invariant on* $\widetilde{X}_{\Delta^\sigma}/\Gamma$.

Lemma 5.38. $\overline{O}_{\delta_-}, \overline{O}_{\delta_+} \subset \widehat{X}_\delta$ *and* $\overline{O}_{\delta_-}, \overline{O}_{\delta_+} \subset \widetilde{X}_{\Delta^\sigma}$ *are* G_σ*-invariant.*

Proof. By Lemmas 4.13 and 2.10, the ideal $I_{\overline{O}_{\delta_+}} \subset K[\widehat{X}_\sigma]$ of $\overline{O}_{\delta_+} = (O_\delta)^+$ is generated by functions with positive weights. □

Proposition 4.12, Lemma 4.13 and the above imply:

Corollary 5.39. *The morphisms* $\widehat{\phi}_- \colon (\widehat{X}_\delta)_-/K^* \to \widehat{X}_\delta//K^*$ *and* $\widehat{\phi}_+ \colon (\widehat{X}_\delta)_+/K^* \to \widehat{X}_\delta//K^*$ *are* G_σ*-equivariant, proper and birational.*

Lemma 5.40. *The vector* $v := \mathrm{Mid}\,(\mathrm{Ctr}_+(\delta), \delta) = \pi^{-1}_{|\partial_-(\delta)}(\mathrm{Ctr}_+(\delta)) + \pi^{-1}_{|\partial_+(\delta)}(\mathrm{Ctr}_+(\delta))$ *is stable.*

Proof. Set $v_- := \pi^{-1}_{|\partial_-(\delta)}(\mathrm{Ctr}_+(\delta))$ and $v_+ := \pi^{-1}_{|\partial_+(\delta)}(\mathrm{Ctr}_+(\delta))$. By Lemma 5.38, $\overline{O}_{\delta_+} \subset \widetilde{X}_{\Delta^\sigma}$ is G_σ-invariant and, by Corollary 5.35(2) and Lemma 5.37, $\mathrm{val}(v_+)$ is G_σ-invariant on $\widetilde{X}_\sigma$ and on $\widehat{X}_\delta$. Hence the valuation $\mathrm{val}(v_+)$ descends to a G_σ-invariant valuation $\mathrm{val}(\pi(v_+))$ on $\widehat{X}_\delta//K^* = K(O_\delta)[[\delta^\vee]]^{K^*}$. By Corollary 5.39, $\mathrm{val}(\pi(v_-)) = \mathrm{val}(\pi(v_+))$ is G_σ-invariant on $(\widehat{X}_\delta)_+/K^* = \widehat{X}_{\partial_-(\delta)}/K^* = \widehat{X}_{\pi(\partial_-(\delta))}$. Let $\Gamma \subset K^*$ be the subgroup generated by all subgroups $\Gamma_\tau \subset K^*$, where $\tau \in \partial_-(\delta)$. Then K^*/Γ acts freely on $X_{\partial_-(\delta)}/\Gamma = (X_\delta)_+/\Gamma$. Let $j \colon (X_\delta)_+/\Gamma \to (X_\delta)_+/K^*$ be the natural morphism. Let $\pi_\Gamma \colon \delta \to \pi_\Gamma(\delta)$ be the projection corresponding to the quotient $X_\delta \to X_\delta/\Gamma$. By Lemma 5.7, for any $\tau \in \partial_-(\sigma)$, the restriction of j to $X_\tau/\Gamma \subset (X_\delta)_+/\Gamma$ is given by $j \colon X_\tau/\Gamma = X_{\underline{\tau}}/\Gamma \times O_\tau/\Gamma \to X_\tau/K^* = X_{\underline{\tau}}/\Gamma \times O_\tau/K^*$. Thus $\mathcal{I}_{\mathrm{val}(\pi_\Gamma(v_-)),a} = \hat{j}^*(\mathcal{I}_{\mathrm{val}(\pi(v_-)),a})$, where $\hat{j} \colon (\hat{X}_\delta)_+/\Gamma \to (\hat{X}_\delta)_+/K^*$ is the natural morphism induced by j. Since the morphism $\hat{j}$ is G_σ-equivariant it follows that $\mathrm{val}(\pi_\Gamma(v_-))$ is G_σ-equivariant on $(\widehat{X}_\delta)_+/\Gamma$. Since $(\widehat{X}_\delta)_+ \subset \widehat{X}_\delta$ is an open G_σ-equivariant inclusion and Γ is finite we get that the morphism $(\widehat{X}_\delta)_+/\Gamma \subset (\widehat{X}_\delta)/\Gamma$ is an open G_σ-equivariant inclusion. Thus the valuation $\mathrm{val}(\pi(v_-))$ is G_σ-equivariant on $\widehat{X}_\delta/\Gamma$ and on $\widetilde{X}_{\Delta^\sigma}/\Gamma$ (Lemma 5.37). Finally, by Lemma 5.29, $\mathrm{val}(v_-)$ it is G_σ-equivariant on $\widetilde{X}_{\Delta^\sigma}$. Thus by the convexity $\mathrm{val}(v) = \mathrm{val}(v_+ + v_-)$ is G_σ-equivariant on $\widetilde{X}_{\Delta^\sigma}$. □

5.11. π-desingularization of cobordisms. Let $\delta_1, \ldots, \delta_k \in \Sigma$ be the circuits in Σ. Note that common faces of distinct circuits are independent. Also, every independent $\tau \in \Sigma$ is a face of some circuit $\tau < \delta$. Thus π-desingularization of circuits δ_i will determine π-desingularization of all faces in Σ. Apply Morelli π-desingularization to δ_1 to get $\Delta_1^{\sigma_1} := \langle v_{r_1}\rangle \ldots \langle v_1\rangle \cdot \delta_1$. This defines a canonical subdivision Δ_1 of Σ, where $\Delta_1 := \langle v_{r_1}\rangle \ldots \langle v_1\rangle \cdot \Sigma$. Next apply the π-desingularization to the subdivision $\Delta_1^{\sigma_2}$ of σ_2 to get $\Delta_2^{\sigma_2} := \langle v_{r_2}\rangle \ldots \langle v_{r_1+1}\rangle \cdot \Delta_1^{\sigma_2}$ and $\Delta_2 = \langle v_{r_2}\rangle \ldots \langle v_{r_1+1}\rangle \cdot \Delta_1$. Continue the process for other circuits to get the π-nonsingular subdivision $\Delta_k = \langle v_{r_k}\rangle \ldots \langle v_1\rangle \cdot \Sigma$ of Σ.

5.12. Proof of the Weak Factorization Theorem. The decomposition Δ of Σ is obtained by a sequence of star subdivisions at stable centers (Lemmas 5.40, 5.34). By Propositions 5.23 and 5.18, Δ defines a birational projective modification $f\colon B^\pi \to B$. The modification does not affect points with trivial stabilizers $B_- = X^- \setminus X$ and $Y^+ \setminus Y$ (see Proposition 2.12). This means that $(B^\pi)_- = B_-$ and $(B^\pi)_+ = B_+$ and B^π is a cobordism between X and Y. Moreover B^π admits a projective compactification $\overline{B^\pi} = B^\pi \cup X \cup Y$. The cobordism $B^\pi \subset \overline{B^\pi}$ admits a decomposition into elementary cobordisms B^π_a, defined by the strictly increasing function χ_{B^π}. Let $F \in \mathcal{C}((B^\pi_a)^{K^*})$ be a fixed point component and $x \in F$ be a point. By Proposition 5.18 the modification $f\colon B^\pi \to B$ is locally described for a toric chart $\phi_\sigma\colon U \to X_\sigma$ by a smooth Γ_σ-equivariant morphism $\phi_{\Delta^\sigma}\colon f^{-1}(U) \to X_{\Delta^\sigma}$. Then by Lemma 4.4, $\phi_{\Delta^\sigma}(x)$ is in O_δ, where $\delta \in \Delta^\sigma$ is dependent and π-nonsingular. In particular the cone $\sigma \in \Sigma$ is also dependent and $\Gamma_\sigma = K^*$. So we locally have a smooth K^*-equivariant morphism

$$\phi_\delta\colon V_x \to X_\delta,$$

where $V_x \subset \phi^{-1}_{\Delta^\sigma}(X_\delta)$ is an affine K^*-invariant subset of B^π_a. This gives a diagram

$$\begin{array}{ccccc}
(B^\pi_a)_-/K^* & \supset & V_{x-}/K^* & \to & X_{\delta-}/K^* \\
\uparrow{\scriptstyle\psi_-} & & \uparrow & & \uparrow{\scriptstyle\phi_-} \\
\Gamma((B^\pi_a)_-/K^*, (B^\pi_a)_+/K^*) & \supset & \Gamma(V_{x-}/K^*, V_{x+}/K^*) & \to & \Gamma(X_{\delta-}/K^*, X_{\delta+}/K^*) \\
\downarrow{\scriptstyle\psi_+} & & \downarrow & & \downarrow{\scriptstyle\phi_+} \\
(B^\pi_a)_+/K^* & \supset & V_{x+}/K^* & \to & X_{\delta+}/K^*
\end{array}$$

with horizontal arrows smooth. Here $\Gamma(X_-/K^*, X_+/K^*)$ denotes the normalization of the graph of a birational map $X_-/K^* \dashrightarrow X_+/K^*$ for a relevant cobordism X. We use functoriality of the graph (a dominated component of the fiber product $X_-/K^* \times_{X//K^*} X_+/K^*$). By Corollary 4.16 the morphisms ϕ_- and ϕ_+ are blow-ups at smooth centers. Thus ψ_- and ψ_+ are locally blow-ups at smooth centers so they are globally blow-ups at smooth centers.

References

[1] Abhyankar, S., On the valuations centered in a local domain. *Amer. J. Math.* **78** (1956), 321–348.

[2] Abramovich, D., and de Jong, A. J., Smoothness, semistability, and toroidal geometry. *J. Algebraic Geom.* **6** (1997), 789–801.

[3] Abramovich, D., Karu, K., K. Matsuki, J. Włodarczyk, Torification and factorization of birational maps. *J. Amer. Math. Soc.* **15** (2002), 531–572.

[4] Abramovich, D., Matsuki, K., and Rashid, S., A note on the factorization theorem of toric birational maps after Morelli and its toroidal extension. *Tohoku Math. J.* (2) **51** (1999), 489–537.

[5] Białynicki-Birula, A., Święcicka, J., Complete quotients by algebraic torus actions. In *Group actions and vector fields*, Lecture Notes in Math. 956, Springer-Verlag, Berlin 1982, 10–21.

[6] Bierstone, E., and Milman, D., Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant. *Invent. Math.* **128** (1997), 207–302.

[7] M. Brion and C. Procesi, Action d'un tore dans une variété projective. in *Operator algebras, unitary representations, enveloping algebras, and invariant theory* (Paris, 1989), Progr. Math. 92, Birkhäuser, Boston 1990, 509–539.

[8] Christensen, C., Strong domination/weak factorization of three dimensional regular local rings. *J. Indian Math. Soc.* **45** (1981), 21–47.

[9] Cutkosky, S. D., Local factorization of birational maps. *Adv. in Math.* **132** (1997), 167–315.

[10] Cutkosky, S. D., Local factorization and monomialization of morphisms. *Astérisque* **260** (1999).

[11] Danilov, V. I., The geometry of toric varieties. *Russian Math. Surveys* **33** (1978), 97–154.

[12] Danilov, V. I., Birational geometry of toric 3-folds. *Math. USSR-Izv.* **21** (1983), 269–280.

[13] Dolgachev, I. V., and Hu, Y., Variation of geometric invariant theory quotients. *Inst. Hautes Études Sci. Publ.* Math. **87** (1998), 5–56.

[14] Ewald, G., Blow-ups of smooth toric 3-varieties. *Abh. Math. Sem. Univ. Hamburg* **57** (1987), 193–201.

[15] Fulton, W., *Introduction to Toric Varieties*. Ann. of Math. Stud. 131, Princeton University Press, Princeton 1993.

[16] Hartshorne, R., *Algebraic Geometry*. Grad. Texts in Math. 52, Springer-Verlag, New York, Heidelberg 1977.

[17] Hironaka, H., On the theory of birational blowing-up. Harvard University Ph.D. Thesis, 1960.

[18] Hironaka, H., Resolution of singularities of an algebraic variety over a field of characteristic zero. *Ann. of Math.* **79** (1964), 109–326.

[19] Hu, Y., The geometry and topology of quotient varieties of torus actions. *Duke Math. J.* **68** (1992), 151–184; Erratum ibid. **68** (1992), 609.

[20] Hu, Y., Relative geometric invariant theory and universal moduli spaces. *Internat. J. Math.* **7** (1996), 151–181.

[21] Hu, Y., and Keel, Ś., A GIT proof of Włodarczyk's weighted factorization theorem. Preprint; math.AG/9904146.

[22] Karu, K., Local strong factorization of toric birational maps. *J. Algebraic Geom.* **14** (2005), 165–175.

[23] Kempf, G., Knudsen, F., Mumford, D., and Saint-Donat, B., *Toroidal embeddings I.* Lecture Notes in Math. 339, Springer-Verlag, Berlin 1973.

[24] Luna, D., Slices étales. *Sur les groupes algébriques*, *Bull. Soc. Math. France* **101** (33) (1973), 81–105.

[25] Matsuki, K., Lectures on factorization of birational maps. *Publ. Res. Inst. Math. Sci.*, preprint, 1999.

[26] Milnor, J., *Morse Theory*. Ann. of Math. Stud. 51, Princeton University Press, Princeton 1963.

[27] Morelli, R., The birational geometry of toric varieties. *J. Algebraic Geom.* **5** (1996), 751–782.

[28] Morelli, R., Correction to "The birational geometry of toric varieties". 2000; http://www.math.utah.edu/~morelli/Math/math.html.

[29] Mumford, D., Fogarty, J., and Kirwan, F., *Geometric Invariant Theory*. Third enlarged edition, Ergeb. Math. Grenzgeb. 34, Springer-Verlag, Berlin 1994.

[30] Oda, T., *Torus embeddings and applications*. Based on joint work with Katsuya Miyake, Tata Inst. Fund. Res., Bombay, 1978.

[31] Oda, T., *Convex Bodies and Algebraic Geometry*. Ergeb. Math. Grenzgeb. 15, Springer-Verlag, Berlin 1988.

[32] Reid, M., Decomposition of Toric Morphisms. In *Arithmetic and Geometry* (ed. by M. Artin and J. Tate), Progr. Math. 36, Birkhäuser, Boston 1983, 395–418.

[33] Sumihiro, H., Equivariant Completion I, II. *J. Math. Kyoto Univ.* **14** (1974), 1–28; **15** (1975) 573–605.

[34] Thaddeus, M., Geometric invariant theory and flips. *J. Amer. Math. Soc.* **9** (1996), 691–723.

[35] Villamayor, O., Constructiveness of Hironaka's resolution. *Ann. Sci. École Norm. Sup.* (4) **22** (1) (1989), 1–32.

[36] Włodarczyk, J., Decomposition of birational toric maps in blow-ups and blow-downs. A proof of the Weak Oda Conjecture. *Trans. Amer. Math. Soc.* **349** (1997), 373–411.

[37] Włodarczyk, J., Birational cobordisms and factorization of birational maps. *J. Algebraic Geom.* **9** (2000), 425–449.

[38] Włodarczyk, J., Toroidal varieties and the weak factorization theorem. *Invent. Math.* 154 (2003), 223–231.

[39] Włodarczyk, J., Simple constructive weak factorization. Preprint; AG/0601649.

[40] Włodarczyk, J., Simple Hironaka resolution in characteristic zero. *J. Amer. Math. Soc.* **18** (2005), 779–822.

[41] Zariski, O., *Algebraic Surfaces*. Ergeb. Math. Grenzgeb. 3, Springer-Verlag, Berlin 1935.

Department of Mathematics, Purdue University, West Lafayette, IN 47907, U.S.A.
E-mail: wlodar@math.purdue.edu