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A Program for Resolution of Singularities, in all Characteristics $p > 0$ and in all Dimensions

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A PROGRAM FOR RESOLUTION OF SINGULARITIES, IN ALL CHARACTERISTICS $p > 0$ AND IN ALL DIMENSIONS

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1. Review on Idealistic Exponents and Infinitely Near Singularities

We will be working on singular objects given in a smooth irreducible algebraic scheme Z of dimension $n \geq 2$ over a perfect base field $\mathbb{K}$. An *idealistic exponent* $E = (J, b)$ will be nothing but a pair of a coherent ideal sheaf $J \subset \mathcal{O}_Z$ and a positive integer b. However we will later add far deeper meanings to it which are *algebraic* on one hand and *geometric* on the other.

Example 1.1. An $f \in \mathbb{K}[x], \neq 0$, with a system of variables $x = (x_1, \cdots, x_n)$ defines a hypersurface $F = Spec(\mathbb{K}[x]/f\mathbb{K}[x])$ in an affine space $Z = Spec(\mathbb{K}[x])$ of dimension n. For a positive integer b, a pair $E = (J, b)$ with $J = f\mathbb{K}[x]$ will be associated with problems on those singular points of multiplicities $\geq b$ of F, especially with that of eliminating all such singular points by means of sequences of suitable blow-ups over the ambient space Z. Naturally the most interesting case is when b is the maximum of the *multiplicities* of F at the points ξ of Z, which are

$$mult_\xi(F) \;=\; ord_\xi(f), \xi \in Z.$$

But for some technical reason, we need to consider the cases of arbitrary $b > 0$. To be more precise but only primarily so, we will view E as being the totality of all those *infinitely near singularities* of multiuplicities $\geq b$ which mean those points of multiplicities $\geq b$ of the transforms of F by *permissible* blow-ups. A blow-up with center D will be said to be *permissible* for E if D is smooth and $ord_\xi(f) \geq b$ for all $\xi \in D$. To be furthermore precise, we will need to consider the same of $E[t] = (J[t], b)$ on $Z[t]$ where t is any finite system of additional indeterminates and $Z[t] = Spec(\mathbb{K}[x,t])$ and $J[t] = J\mathcal{O}_{Z[t]}$. The reason for the needs will be made apparent by the key technical theorems, especially the theorem of *Numerical Exponents*, [11], [12],[13].

Definition 1.1. For $E = (J, b)$, we define

$$ord_\xi(E) = ord_\xi(J)/b \;\; and \;\; Sing(E) = \{\xi \in Z \mid ord_M(E) \geq 1\},$$

which is called the singular locus of E.

Now, back to the general case of $E = (J, b)$ on Z, we set out some basic definitions:

Definition 1.2. A blow-up $\pi : Z' \to Z$ with center $D \subset Z$ is said to be *permissible* for E if

(1) D is a smooth irreducible closed subscheme of Z, and
(2) D is contained in the singular locus $Sing(E)$ of Def.(1.1).

Definition 1.3. The *transform* $E' = (J', b)$ of $E = (J, b)$ by a permissible blow-up $\pi : Z' \to Z$ with center $D \subset Z$ is defined by letting

$$J\mathcal{O}_{Z'} \;=\; J'\Big(I(D,Z)\mathcal{O}_{Z'}\Big)^{-b}$$

where $I(D,Z)$ denotes the ideal sheaf defining $D \subset Z$. It should be noted that, assuming $D \neq Z$, the ideal $I(D,Z)\mathcal{O}_{Z'}$ is locally everywhere nonzero principal and divides $J\mathcal{O}_{Z'}$ because of the *pemissibility condition* of Def.(1.2). Hence $J' \subset \mathcal{O}_Z$. If $D = Z$, then we define Z' to be empty so that the above definition is still logically valid.

Remark 1.1. Assume that a proper blow-up $\pi : Z' \to Z$ is permissible for both $E_i = (J_i, b_i), i = 1, 2$. Let ${E_i}'$ be the transform of E_i by π. Then ${E_1}' \cap {E_2}'$ is equivalent to the transform of $E_1 \cap E_2$ by π.

Having the definition of permissibility and transform, we can naturally extend the notion of permissibility to successive blow-ups for E. We can thus speak of permissibility of LSB's which are defined below.

Definition 1.4. An LSB *over* Z, a short form of a *sequence of local smooth blow-ups*, is defined to be the following diagram:

$$\begin{array}{ccccc} Z_r & \overset{\pi_{r-1}}{\longrightarrow} & U_{r-1} \subset Z_{r-1} & \overset{\pi_{r-2}}{\longrightarrow} & \\ & & \cup & & \\ & & D_{r-1} & & \\ & & & \cdots & \end{array}$$

$$\begin{array}{ccccc} \overset{\pi_1}{\longrightarrow} & U_1 \subset Z_1 & \overset{\pi_0}{\longrightarrow} & U_0 \subset Z_0 = Z \\ & \cup & & \cup \\ & D_1 & & D_0 \end{array}$$

where U_i is an open subscheme of Z_i, D_i is a regular closed subscheme of U_i and the arrows mean: $\pi_i : Z_{i+1} \to U_i$ is the blowing-up with center D_i.

Definition 1.5. Pick any finite system of indeterminates $t = (t_1, \cdots, t_l)$. Let $Z[t] = Spec(\mathbb{K}[t]) \times_{\mathbb{K}} Z$ and $E[t] = (J[t], b)$ with $J[t] = J\mathcal{O}_{Z[t]}$ with respect to the canonical projection. We then define the following t-indexed disjoint union:

$$\mathfrak{S}(E) \;=\; \bigcup_t \textit{ the set } \Big\{\textit{all those LSBs over } Z[t] \textit{ which are permissible for } E[t]\Big\}$$

which will be called the totality of *infinitely near singularities* of E on Z. We then define the inclusion relations for any two $E_j, j = 1, 2$, as follows:

$$E_1 \subset E_2 \ (\textit{symbolically})$$
$$\Longleftrightarrow$$
$$\mathfrak{S}(E_1) \subset \mathfrak{S}(E_2) \ (\textit{set-theoretically})$$
$$\Longleftrightarrow$$
$$\Big(\forall t, \textit{ an } LSB \textit{ is permissible for } E_1[t] \Longrightarrow \textit{ it is permissible for } E_2[t]\Big)$$

We then define *equivalence relation* by saying that

$$E_1 \sim E_2 \Longleftrightarrow E_1 \subset E_2 \text{ and } E_1 \supset E_2.$$

Finally we shall use the notation $\cap_\alpha E_\alpha$ for any number of idealistic exponents E_α, meaning that we have an idealistic exponent F such that

$$\cap_\alpha \mathfrak{S}(E_\alpha) = \mathfrak{S}(F)$$

in the set-theoretical sense (for every t indivisually as above).

What follows are most of the elementary but basic *facts* on relations among idealistic exponents, whose proofs are more or less straight forward.

[Fact 1] $(J^e, eb) \sim (J, b)$ for every positive integer e.

[Fact 2] For every common multiple m of b_1 and b_2, we have

$$(J_1, b_1) \cap (J_2, b_2) \sim (J_1^{\frac{m}{b_1}} + J_2^{\frac{m}{b_2}}, m)$$

In particular if $b_1 = b_2 = b$ $(= m)$ and $J_1 \subset J_2$ then we have $(J_1, b) \supset (J_2, b)$. It also follows that the intersection of any finite number of idealisitic exponents is equivalent to an idealistic exponent.

[Fact 3] We always have

$$(J_1J_2, b_1 + b_2) \supset (J_1, b_1) \cap (J_2, b_2)$$

The reversed inclusion does not hold in general. However, if $Sing(J_i, b_i + 1)$ are both *empty* for $i = 1, 2$, then the left hand side becomes *equivalent* to the right hand side. Moreover, we always have

$$(J, b) \subset (J_k, b_k), 1 \le k \le r, \Rightarrow (J, b) \subset (\prod_{1\le k\le r} J_k \ , \sum_{1\le k\le r} b_k)$$

[Fat 4] Let us compare two idealistic exponents having the same ideal but different b's, say $F_1 = (J, b_1)$ and $F_2 = (J, b_2)$ with $b_1 > b_2$. Then we have

1) $F_1 \subset F_2$.

2) For any LSB permissble for F_1, and hence so for F_2, their final transforms differ only by a locally principal non-zero factor supported by the union of the exceptional divisors.

> To be precise, their final transforms being denoted by $F_1^* = (J_1^*, b_1)$ and $F_2^* = (J_2^*, b_2)$, we have $J_2^* = MJ_1^*$ where M is a positive power product of the ideals of the strict transforms of the exceptional divisors created by the blowing-ups belonging to the LBS.

Remark 1.2. Incidentally, changing the number b turns out to be a useful technique in connection with the problem of transforming singular data into normal crossing data which appears in a process of desingularization.

[Fact 5] We have $(J_1, b) \supset (J_2, b)$ if J_1 is contained in the *integral closure* of J_2 in the sense of *integral dependence* (after Oscar Zariski) defined in the theory of *ideals*. Recall the definition:

> For ideals $H_i, i = 1, 2$, in a commutative ring R, H_1 is *integral* over H_2 in the sense of the *ideal theory* if and only if $\sum_{a\ge 0} H_1^a T^a$ is *integral* over $\sum_{a\ge 0} H_2^a T^a$ in the sense of the *ring theory*, where T is an indeterminate over R. In our case, since Z is regular and hence normal, if $\pi : \tilde{Z} \longrightarrow Z$ is any proper birational morphism

such that $\tilde{Z}$ is normal and $J_2\mathcal{O}_{\tilde{Z}}$ is locally non-zero principal, then the direct image $\pi_*(J_2\mathcal{O}_{\tilde{Z}})$ is equal to the *integral closure* of J_2. As an example of such π, we could take the *normalized blowing-up* of J_2, i.e., the blowing-up of J_2 followed by normalization.

Let us now recall what we called the *Three Key Theorems* in my paper [12] which are technically useful in the general theory of infinitely near sigularities. See also [11] and [13] for the details with proofs.

We let $Diff_Z^{(i)} = Diff_{Z/\mathbb{K}}^{(i)}$, which denotes the $\mathcal{O}_Z$-module of all those differential operators of orders $\leq i$ from $\mathcal{O}_Z$ into itself, which are acting trivially in $\mathbb{K}$.

The first Key Tech:

Theorem 1.1 (Differentiation Theorem, or Diff theorem). *For every $\mathcal{O}_Z$-submodule $\mathcal{D}$ of $Diff_Z^{(i)}$, we have the following inclusion in the sense of infinitely near singularities*

$$(\mathcal{D}J, b-i) \supset (J,b) \quad meaning \quad \mathfrak{S}(\mathcal{D}J, b-i) \supset \mathfrak{S}(J,b)$$

which is equivalent to saying that

$$\mathfrak{S}(J,b) \bigcap \mathfrak{S}(\mathcal{D}J, b-i) \quad = \quad \mathfrak{S}(J,b).$$

Incidentally the last equality will often be expressed symbolically as

$$(J,b) \bigcap (\mathcal{D}J, b-i) \quad \sim \quad (J,b).$$

Quite generally, let $W \subset Z$ be a *regular* irreducible closed subscheme W of an *excellent* scheme Z. In this paper Z is assumed to be of finite type over a field $\mathbb{K}$ and hence is *excellent* and *regularity* of a subscheme is equivalent to its *smoothness* because $\mathbb{K}$ is perfect. Just for the sake of generality, we make the following definition for a general $W \subset Z$ as above, thinking of such a case in which Z may be an *arithmetic scheme* which is of finite type over the ring of integers $\mathbb{Z}$.

Definition 1.6. Given idealistic exponents $E = (J,b)$ on Z and $F = (H,a)$ on W, F is called an *ambient reduction* of E from Z to W if the following condition is satisfied:

> *Pick any finite system of indeterminates $t = (t_1, \cdots, t_r)$ and any LSB over $Z[t]$, subject to one condition that all the centers of the LSB are contained in the respective strict transforms of $W[t]$, which therefore induces an LSB over $W[t]$. The condition is that the LSB on $Z[t]$ is permissible for $E[t]$ if and only if the induced LSB on $W[t]$ is permissible for $F[t]$.*

It should be noted here that every LSB over $W[t]$ extends to an LSB over $Z[t]$ (with only non-uniqueness of open restrictions in the latter) and it is hence an induced LSB from $Z[t]$ to $W[t]$.

Remark 1.3. The author does not know the existence of an ambient reduction in the *arithmetic case in general* according to the above definition, while in the *algebraic case* its existence is proven constructively and universally as in the second technical therem given below.

The second Key Tech:

Theorem 1.2 (Ambient Reduction Theorem). *Back to the case in which Z is of finite type over a perfect base field $\mathbb{K}$ of any characteristic, we consider an idealistic exponent $E = (J, b)$ in Z. We let $b^\sharp = b!$ and let*

$$J^\sharp = \sum_{j=0}^{b-1} \left(Diff_Z^{(j)} J \right)^{\frac{b!}{b-i}}.$$

Then, for every smooth irreducible closed subscheme $W \subset Z$, $F = (J^\sharp \mathcal{O}_W, b^\sharp)$ is an ambient reduction *of E from Z to W.*

Definition 1.7. We say that an ambient reduction F of E from Z to W is an *ambient equi-reduction*, or *ambient equivalent reduction* if the following condition is satisfied:

> *Pick any finite system of indeterminates $t = (t_1, \cdots, t_r)$ and any LSB over $Z[t]$. If it is permissible for E, then all the centers of the LSB are necessarily contained in the respective strict transforms of $W[t]$, which therefore induces an LSB over $W[t]$. Moreover F is an ambient reduction fron Z to W in the sense of Def.(1.6).*

Let us again go back to the case in which Z is of finite type over a perfect base field $\mathbb{K}$.

The third Key Tech:

Theorem 1.3 (Numerical Exponent Theorem). *Let $W_i, i = 1, 2$, be two smooth irreducible closed subschemes of Z having the same dimension. Let $F_i = (I_i, c_i)$, be an idealistic exponent on $W_i, i = 1, 2$. Assume that we have an idealistic exponent $E = (J, b)$ on Z of which F_i is an ambient equi-reduction from Z to W for $i = 1, 2$. Then, for every $\xi \in Z$, we have the following eequivalence:*

$$\xi \in W_1 \text{ and } ord_\xi(I_1) \geq c_1 \Longleftrightarrow \xi \in W_2 \text{ and } ord_\xi(I_2) \geq c_2$$

Moreover if so then we have

$$\frac{ord_\xi(I_1)}{c_1} = \frac{ord_\xi(I_2)}{c_2}.$$

Corollary 1.4. *Consider two idealistic exponents $E_i = (J_i, b_i), i = 1, 2$, on Z. If $\mathfrak{S}(E_1) \subset \mathfrak{S}(E_2)$, i.e., $\mathfrak{S}\Big((J_1)^{b_2} + (J_2)^{b_1}, b_1 b_2\Big) = \mathfrak{S}(E_1)$, then we have*

$$\frac{ord_\xi(J_1)}{b_1} \leq \frac{ord_\xi(J_2)}{b_2}$$

for every $\xi \in Z$ for which the first fractional number ≥ 1.

Remark 1.4. In the above theorem on *numerical exponents*, it is important that we make use of auxiliary variables t in the definition of inclusion and equivalence among idealisitic exponents. For instance, consider a plane curves $X \subset Z = Spec(\mathbb{C}[x, y])$ defined by equations $f_i = y^b - x^{c_i}$ with integers $1 < b < c_i$. Consider $E_i = (f_i \mathbb{C}[x, y], b)$ on Z. We can then prove easily that

(1) We have the equality of the integral parts

$$[\frac{c_1}{b}] = [\frac{c_2}{b}]$$

if and only if the following statement holds:

For every LSB over Z, it is permissible for E_1 if and only if it is so for E_2.

(2) We have the equality

$$\frac{c_1}{b} = \frac{c_2}{b}$$

if and only if the following statement holds:

For every t and every LSB over $Z[t]$, it is permissible for $E_1[t]$ if and only if it is so for $E_2[t]$.

The difference is apparent depending upon whether we make use of t or not.

2. Characteristic algebra of singularity

The results of this paper were all originally published in my paper [12] and later reproduced in [13].

Definition 2.1. As before let us make use of $E^\sharp = (J^\sharp, b^\sharp)$ where $b^\sharp = b!$ and

$$J^\sharp = \sum_{0 \le \mu \le b-1} \left(Diff_{Z/\mathbf{k}}^{(\mu)} J\right)^{b^\sharp/(b-\mu)}$$

Then the *characteristic algebra* of $E = (J, b)$ on Z, denoted by $\wp(E)$, is the integral closure of the subalgebra

$$\sum_{\alpha=0}^{\infty} (J^\sharp)^\alpha T^{b^\sharp \alpha} \quad \subset \quad \sum_{\alpha=0}^{\infty} \mathcal{O}_Z T^\beta \quad = \quad \mathcal{O}_Z[T] \tag{2.1}$$

where T is a dummy variable whose powers indicate degrees of the homogeneous parts. In general the integral closure of a graded subalgebra is graded and hence we can write

$$\wp(E) = \sum_{a=0}^{\infty} J_{max}(a) T^a$$

with coherent ideal sheaves $J_{max}(a) \subset \mathcal{O}_Z, \forall a$, where $J_{max}(0) = \mathcal{O}_Z$.

Incidentally, since Z is regular, $\mathcal{O}_Z[T]$ of (2.1) is clearly integrally closed in its field of fractions which is the function field of $Z[T]$ and hence we could have said that $\wp(E)$ is the integral closure of $\mathcal{O}_Z[(J^\sharp)T^{b^\sharp}]$ in the function field of $Z[T]$.

Remark 2.1. According to the above definition, for every non-negative integer a, the ideal sheaf $J_{max}(a)$ consists of exactly those $h \in \mathcal{O}_Z$ which satisfy monic equations of the form:

$$h^{mb^\sharp} + A_1 h^{(m-1)b^\sharp} + \cdots + A_m = 0, \ A_j \in (J^\sharp)^{aj}, \forall j.$$

Remark 2.2. It follows from the above definition that $\wp(E)$ is *finitely presented* as $\mathcal{O}_Z$-algebra.

Remark 2.3. The *characteristic algebra* of E defined above turns out to be a generalization of the *first characteristic exponent* for a plane curve. Just to get an idea about it, consider the case in which $\mathbb{K}$ is an algebraically closed field of characteristic zero and we are given a plane curve X in $Z = Spec(\mathbb{K}[x,y])$ defined by $f \in \mathbb{K}[x,y], \neq 0$. Let $m > 0$ be the maximum of the multiplicities of X, i.e., $m = \max\{\ ord_\eta(f) \mid \eta \in Z\ \}$. For simplicity let us assume that $ord_\xi(f) = m$ with the origin $\xi = (0,0)$ and that if we write

$$f(x,y) = \sum_{ij} c_{ij}x^i y^j \quad with \quad c_{ij} \in \mathbb{K}$$

then we have

(1) $c_{0m} \neq 0$,
(2) if $\delta = min\{\ \frac{i}{m-j} \mid i > 0, c_{ij} \neq 0\ \}$ then $\delta < \infty$ and the first Newton segmental polynomial $\sum_{ij;\delta(m-j)=i} c_{ij}x^i y^j$ is not an m-th power of any polynomial in $\mathbb{K}[x,y]$.

These conditions can be always gained by taking a suitable biregular transformation of Z except for the case in which f is an m-th power up to a unit multiple and $\delta = \infty$. The rational number δ is called the *first characteristic exponent* of the plane curve. In this case $\wp(E) = \sum_{a=0}^{\infty} J_{max}(a)T^a$ with $E = (f\mathbb{K}[x,y], m)$ is as follows:

$$J_{max}(a) \;=\; \{\ x^i y^j \mid \frac{i}{\delta} + j \geq a, i \geq 0, j \geq 0\ \}, \quad \forall a \geq 0.$$

Incidentally this assertion fails to be true in general when $char(\mathbb{K}) = p > 0$.

Theorem 2.1. *We have other characterizations of the same $\wp(E)$ for $E = (J,b)$, every one of which can be taken as its definition.*

(1) *$\wp(E)$ is the integral closure of the $\mathcal{O}_Z$-subalgebra of $\mathcal{O}_Z[T]$ generated by the set*

$$\left(Diff_Z^{(i)} J\right) T^{b-i}, 0 \leq i < b.$$

It is the same as saying that $\wp(E)$ is the integral closure of the following $\mathcal{O}_Z$-subalgebra of $\mathcal{O}_Z[T]$

$$\sum_{\alpha=(\alpha_0,\cdots,\alpha_{b-1})\in\mathbb{Z}_0^b} \Big(\prod_{i=0}^{b-1} (Diff_Z^{(i)} J\ )^{\alpha_i}\Big)\ T^{\sum_{i=0}^{b-1}\alpha_i(b-i)}$$

(2) *$\wp(E)$ is the smallest among those $\mathcal{O}_Z$-subalgebras G of $\mathcal{O}_Z[T]$ which have the following properties:*
   (a) *$\mathcal{O}_Z \subset G$ and $JT^b \subset G$.*
   (b) *For every pair of integers $c > d \geq 0$, if I is a coherent ideal sheaf in $\mathcal{O}_Z$ such that $IT^c \subset G$ then we have $\left(Diff_{Z/\mathbb{K}}^{(d)} I\right)T^{c-d} \subset G$.*
   (c) *G is integrally closed in $\mathcal{O}_Z[T]$. In particular, for every integer $m > 0$ we have*

$$fT^k \in G \Longleftrightarrow f^m T^{mk} \in G, \forall k.$$

As for the above property (b), we use the following

Lemma 2.2. *For every $c = \sum_{i=0}^{b-1}(b-i)\alpha_i$ with $\alpha \in \mathbb{Z}_0^b$ and for every $d < c$,*

$$Diff_Z^{(d)}\Big(\prod_{i=0}^{b-1}(Diff_Z^{(i)} J\ )^{\alpha_i}\Big)$$

is contained in

$$\sum_{\beta \in \mathbb{Z}_0^b, \sum_{i=0}^{b-1} \beta_i (b-i)=c-d} \left(\prod_{i=0}^{b-1} (Diff_Z^{(i)} J)^{\beta_i} \right)$$

In cotrast against the *algebraic* nature of $\wp(E)$ shown in both Def.(2.1) and Th.(2.1), we have a quite different way of defining the same $\wp(E)$, which is *geometic* in the sense of *infinitely near singularities* defined by means of of successive blow-ups, or more precisely by LSBs over $Z[t]$ with various t (see Defs. (1.4) and (1.5)).

Theorem 2.3. *For every $a \in \mathbb{Z}_0$, the ideal sheaf $J_{max}(a)$ of Def.(2.1) can be defined as follows:*

$$J_{max}(a) = \bigcup \{ \text{ ideals } I \mid \mathfrak{S}(I,a) \supset \mathfrak{S}(J,b) \ \}.$$

In fact, there always exists the maximal one among all such ideals I as above with respect to the set-theoretical inclusion relation.

Remark 2.4. This theorem, asserting that the *algebraic* definition (2.1) of $\wp(E)$ is equivalent to the *geometric* one of Th.(2.1), was proven in the paper [12], in which the geometric characterization was taken as its definition rather than the algebraic one done here. For the detail of its proof, the reader should refer to the proofs of Lemmas 2.1 - 2.2 and the equality ($\mathfrak{b}$) of page 918 of [12], included in the proof of what was called *Main Theorem* there.

3. Strategy for induction (1)

From now on we will assume that we are given a *normal crossing data*, simply called NC, in the ambient scheme Z.

Definition 3.1. An NC means a finite system of hypersurfaces $\Gamma = (\Gamma_1, \cdots, \Gamma_s)$ in Z satisfying the following conditions:

(1) For each j, Γ_j is a smooth irreducible closed hypersurface in Z,
(2) $\Gamma_j \neq \Gamma_k$ if $j \neq k$, and
(3) Γ has only *normal crossings* everywhere in Z, i.e., for every point $\eta \in Z$ we can finid a regular system of parameters $z = (z_1, \cdots, z_n)$ of the local ring $\mathcal{O}_{Z,\eta}$ such that if $\eta \in \Gamma_j$ then there exists $k, 1 \leq k \leq n$, with which the ideal of Γ_j in $\mathcal{O}_{Z,\eta}$ is generated by z_k.

Definition 3.2. For a point $\xi \in Z$, a regular system of parameters $x = (x_1, \cdots, x_n)$ of $\mathcal{O}_{Z,\xi}$ will be said Γ-*adapted* if every Γ_j passing though ξ is defined by an ideal generated by one of the x_i's. A smooth subscheme $D \subset Z$ is said to have *(only) normal crossings* with Γ at $\xi \in Z$ if there exists a Γ-*adapted* regular system of parameters x of $\mathcal{O}_{Z,\xi}$ such that the ideal of D at ξ is generated by a subsystem of x. If this is true at every point of D, then we simply say that D has *normal crossings* with Γ. An ideal I in $\mathcal{O}_{Z,\xi}$ is said to be Γ-*monomial* if there exists a Γ-adapted regular system of parameters x of $\mathcal{O}_{Z,\xi}$ in terms of which I is generated by a monomial in x.

Definition 3.3. A blow-up $\pi : Z' \to Z$ with center D is said to be *permissible* for Γ if D is smooth irreducible and have normal crossings with Γ. The *transform* Γ'

of Γ by π is defined to be

$$\Gamma' = (\Gamma'_1, \cdots, \Gamma'_s, \Gamma'_{s+1})$$

such that

(1) Γ'_i is the strict transform of Γ_i by π for every $i, 1 \le i \le s$,
(2) Γ'_{s+1} is the exceptional divisor $\pi^{-1}(D)$.

We can thus speak of whether a sequence of blow-ups over Z is (successively) *permissible* for Γ or not.

We write $|\Gamma|$ for the set-theoretical union of the $\Gamma_i, 1 \le i \le s$, which is a closed subset of Z.

Definition 3.4. A blow-up will be said to be *proper* if the center is closed. A sequence of blow-ups will be said to be *proper* if the blow-ups are all successively proper.

We then propose to give an affirmative answer to the following *Desingularization Problems*:

Problem(I.n):
Let n be the dimension of Z and let $E = (J, b)$ be an idealistic exponent with $J \ne (0)$ on Z. The question is then about the existence of a finite proper sequence of blow-ups over Z, permissible for both E and Γ, such that the final transforms $\tilde{E} = (\tilde{J}, b)$ of E is *nonsingular*, i.e., $Sing(\tilde{E}) = \emptyset$.

At this point we present one of the many steps in our process of solving the above problem. It is a problem reduction which will be repeatedly used later in the proof of desingularization by induction. It concerns with the comparison of two idealistic exponents having the same ideal but different base numbers. Keep in mind that we have a common Γ and its transforms whenever we apply permissible sequences of blow-ups. We thus have

Lemma 3.1. *Let us compare $E = (J, b)$ and $E^* = (J, m)$ where $0 < b < m$. Then the following statements are true.*

(1) *If an LSB of $Def.(1.4)$ over $Z[t]$ is permissible for $E^*[t]$ then so it is for $E[t]$ for every t. In other words, $E^* \subset E$.*
(2) *For any pair of $E_1 = (J_1, b_1)$ and $E^*_1 = (J_1, m_1)$ having $m/b = m_1/b_1$, if $E_1 \sim E$ then $E^* \sim E^*_1$ in the sense of Def.(1.5).*
(3) *After every finite sequence of blow-ups permissible for E^* and hence permissible for E, say $\pi : \tilde{Z} \longrightarrow Z$, the final transform $\tilde{E} = (\tilde{J}, b)$ of E by π differs from the final $\tilde{E}^* = (\tilde{J}^*, m)$ of E^* by a factor of the following form: There exists a $\tilde{\Gamma}$-monomial $\Delta \subset \mathcal{O}_{\tilde{Z}}$ such that $\Delta \tilde{J}^* = \tilde{J}$ everywhere in $\tilde{Z}$, where $\tilde{\Gamma}$ denotes the final tansform of Γ by π.*
(4) *If moreover $m = max\{ord_\eta(J) \mid \eta \in Z\} \ge b$ and $Sing(\tilde{E}^*) = \emptyset$, then*

$$ord_\eta(\Delta^{-1}\tilde{J}) = ord_\eta(\tilde{J}^*) < m, \ \forall \eta \in Sing(\tilde{E}).$$

Proposition 3.2. *Given any idealistic exponent $F = (I, b)$ in Z, F^* will denote the associated idealistic exponent (I, m) with $m = max\{ord_\eta(I) \mid \eta \in Z\}$.*

Suppose that the Problem($I.n$) *had been affirmatively solved for every $F^* = (I^*, m)$ in Z, then for any idealistic exponent $E = (J, b)$ on Z, there exists a finite proper sequence of blow-ups, permissible for E and Γ, such that $\tilde{J}$ of the final transform $\tilde{E} = (\tilde{J}, b)$ of E is a $\tilde{\Gamma}$-monomial at every point of $Sing(\tilde{E})$, where $\tilde{\Gamma}$ denotes the final transform of Γ.*

For a proof of this proposition, we will write F^* for $F = (I, c)$ to means $F^* = (I, c^*)$ where $c^* = max\{ord_\eta(I), \forall\eta\}$. Now, given $E = (J, b)$, we apply the supposition of the proposition to E^* in the manner of the lemma (3.1) repeatedly as follows. Write $E(0) = (J(0), b(0)$ for $E = (J, b)$ and $E(0)^* = (J(0), b(0)^*)$. Apply the lemma (3.1) to $E(0)^*$ and accordingly we find a sequence of blowups $\pi(0) : Z(1) \longrightarrow Z(0)$. We let $\tilde{E}(0)^* = (\tilde{J}(0)^*, b(0)^*)$ and $\tilde{E}(0) = (\tilde{J}(0), b(0))$ be the final transforms of $E(0)^*$ and $E(0)$ by $\pi(0)$, respectively, so that $Sing(\tilde{E}(0)^*) = \emptyset$. We also have a $\Gamma(l)$-monomial $\Delta(1)$ with $\tilde{J}(0) = \Delta(1)\tilde{J}(0)^*$ in the sense of Lem.(3.1). We let

$$m(1) \;=\; max\{\; ord_\zeta(\tilde{J}(0)^*) \mid \zeta \in Sing(\tilde{E}(0)) \subset Z(1)\}$$

so that $m(1) < m(0)$. If $m(1) = 0$, then $\tilde{J}(0) = \Delta(1)$ at every point of $Sing(\tilde{E}(0))$ and we are done. If $m(1) > 0$ then we proceed as follows: Let us define

$$E(1) \;=\; \tilde{E}(0) \cap (\; \tilde{J}(0)^*, m(1)\;) \;\sim\; \big(\tilde{J}(0)^* + \tilde{J}(0), m(1)\big)$$

where the last equivalence is because $\tilde{J}(0)_\eta \supset \tilde{J}(0)^*{}_\eta$ locally at every $\eta \in Sing\big(\tilde{E}(0)\big)$. Write $J(1) = \tilde{J}(0)^* + \tilde{J}(0)$ and $E(1) = \big(J(1), b(1)\big)$ with $b(1) = m(1)$. We then apply Lem.(3.1) to $E(1)^*$ in the same manner as above. We obtain a sequence of blowups $\pi(2) : Z(2) \longrightarrow Z(1)$, permissible for $E(1)^*$. We also obtain the transforms $\tilde{E}(1)^* = \big(\tilde{J}(1)^*, m(1)\big)$ and $\tilde{E}(1) = \big(\tilde{J}(1), b(1)\big)$ of $E(1)^*$ and $E(1)$, respectively, so that $Sing(\tilde{E}(1)^*) = \emptyset$. Similarly $m(2)$ and $E(2)$ are obtained with $m(2) < m(1)$as before. Repeat the same process again if necessary. The inequalities $m > m(1) > m2) > \cdots \geq 0$ cannot continue forever.

The usefulness of the above reduction is as follows.

Remark 3.1. Consider that J of $E = (J, b)$ is already Γ-monomial at every point of $Sing(E)$. We then have an explicit procedure which eliminates all the singularities of E, thus giving an affirmative answer to the problem Prob.(3) stated in the beginning of this section.

In fact, let us denote $m_i = ord_{\Gamma_i}(J), 1 \leq i \leq s$. Define $k > 0$ to be the smallest length among those of all subsystems S of $[1, s]$ such that

$$\bigcap_{i \in S} \Gamma_i \neq \emptyset \quad and \quad \sum_{i \in S} m_i \geq b, \quad so\ that \quad \bigcap_{i \in S} \Gamma_i \subset Sing(E).$$

Pick any such S of length k and apply the blow-up with center $\cap_{i \in S}\Gamma_i$ (or its connected components one after another) to Z. This process stops after a finite number of steps and leads to the situation in which the final transform of E has no singular points.

Remark 3.2. We could make the last process more canonical by choosing $S = (i_1, \cdots, i_k)$ of lenght k as follows:

(1) $\bigcap_{i \in S} \Gamma_i \neq \emptyset$ *and* $\sum_{i \in S} m_i \geq b$,
(2) $i_1 < i_2 < \cdots < i_k$, and
(3) $(i_1, \cdots, i_k)$ is the lexicographically smallest among those S having the above two properties.

4. Strategy for Induction (2)

From now on, we look at singular data locally at a closed point $\xi \in Z$. We let $R = \mathcal{O}_{Z,\xi}$, $M = max(R)$ and $\kappa = R/M$. For an element $f \in R$, we will use the symbol $in_M(f)$ for the M-adic *initial form* of f, which means the class of f modulo M^{k+1} where $k = ord_M(f)$. (If $k = \infty$, then $in_M(f) = 0$.) This residue field κ is also a perfect field because it is a finite algebraic extension of the perfect field $\mathbb{K}$. We assume $dim(R) = n$. We will need the following lemma which will be called *initial decomposition lemma*, in which if the characteristic of $\mathbb{K}$ is zero then we must understand $p^e = 1$ for all non-negative integers e. However our primary interest lies in the case of $char(\mathbb{K}) = p > 0$.

Lemma 4.1. *Given any non-zero ideal $J \subset R$ with $ord_M(J) = m > 0$, we can find a regular system of parameters $y = (y_1, \cdots, y_n)$ of R, an integer $r, 1 \leq r \leq n$, and a sequence of powers $q_i = p^{e_i}, e_i \geq 0, 1 \leq i \leq r$, which satisfy the following conditions: Letting $\bar{y}_i = in_M(y_i), \forall i$, and $\bar{y} = (\bar{y}_1, \cdots, \bar{y}_n)$, we have*

(1) $m \geq q_1 \geq q_2 \geq \cdots \geq q_r \geq 1$,
(2) *for each $i, 1 \leq i \leq r$, $\exists D_i \in Diff_{R/\mathbb{K}}^{(m-q_i)}$ and $\exists f_i \in J$ such that $ord_M(f_i) = m$ and $in_M(D_i f_i) = \bar{y}_i^{q_i}$,*
(3) *for every $f \in J$ with $ord_M(f) = m$, we have $in_M(f) \in \kappa[\bar{y}_1^{q_1}, \cdots, \bar{y}_r^{q_r}], \forall i$.*

Corollary 4.2. *Let $\wp(E) = \sum_{j \geq 0} J_{max}(j) T^j$ with $E = (J, m)$ where J and m are the same as above. Then for every $g \in J_{max}(d)$ with $d > 1$ and $ord_M(g) = d$, we have a regular system of parameters $y = (y_1, \cdots, y_n)$ of R, an integer $r, 1 \leq r \leq n$, and a sequence of powers $q_i = p^{e_i}, e_i \geq 0, 1 \leq i \leq r$, satisfying the following conditions:*

(1) *for every i we have $q_i \leq d$ and $\exists\ g_i \in J_{max}(q_i)$ such that*
$$g_i \;=\; y_i^{q_i} + \text{ higher order terms } \;\in\; y_i^{q_i} + M^{q_i+1}$$
(2) *and*
$$g \;-\; \sum_{\substack{\alpha \in \mathbb{Z}_0^r \\ m = \sum_{j=1}^r q_j \alpha_j}} c_\alpha \prod g_j^{\alpha_j} \quad \in M^{m+1}$$
with suitably chosen $c_\alpha \in R, \forall \alpha$.

Thanks to the lemma (4.1) and corollary (4.2) above, we obtain

Theorem 4.3. *Consider $E = (J, b)$ on Z and let $m = max\{ord_\eta(J) \mid \eta \in Z\}$. Let $E^* = (J, m)$ and write $\wp(E^*) = \sum_{j \geq 0} J_{max}(j) T^j$. Pick any closed point $\xi \in Sing(E^*)$ so that $ord_\xi(J) = m$. Let $R = \mathcal{O}_{Z,\xi}$ and $M = max(R)$. Then we can find*

(1) *a local coordinate system $y = (y_1, \cdots, y_n)$ of Z, centered at ξ,*

(2) *an integer r with $0 < r \le n$, a system of non-negative powers of the $char(\mathbb{K})$*

$$q_i = p^{e_i}, e_1 \ge \cdots \ge e_r \ge 0$$

and a system of elements $g_i \in J_{max}(q_i)_\xi$

$$g_i = y_i^{q_i} + (\text{higher order terms}) \in y_i^{q_i} + M^{q_i+1}$$

such that for every $a \ge 0$

$$J_{max}(a)_\xi \quad \subset \quad M^{a+1} \;+\; \sum_{\substack{\alpha \in \mathbb{Z}_0^r \\ a=\sum_{j=1}^r q_j \alpha_j}} \Big(\prod_{j=1}^r g_j^{\alpha_j}\Big) R$$

It follows that we obtain the following type of equivalence which holds within a sufficiently small neighborhood of $\xi \in Z$:

$$E^* \sim \Big(\bigcap_{i=1}^r E_i\Big) \bigcap F$$

where $E_i = (g_i \mathcal{O}_Z, q_i), 1 \le i \le r$, and $F = (I, a)$ with $ord_\xi(I) > a$, which make sense within a neighborhood of $\xi \in Z$.

Remark 4.1. In the above Th.(4.3), if there exists no α with $a = \sum_{j=1}^r q_j \alpha_j$ for a given a, we then must have $ord_\xi(J_{max}(a)) > a$. In particular, we have $ord_\xi(J_{max}(k)) > k$ for all $k < q_r$.

Remark 4.2. The existence of F of Th.(4.3) is due to the fact that $\wp(E^*)$ is finitely presented as $\mathcal{O}_Z$-algebra, although its uniqueness is not true in general.

Proposition 4.4. *By the properties listed above of the decomposition of Th.(4.3), we have the uniqueness of the following objects:*

(1) *the system of numbers defined by*

$$\mathfrak{q}_\xi(E^*) \;=\; (n, n-r, q_1, \cdots, q_r)$$

which is also denoted by $\mathfrak{q}_\xi(E)$.

(2) *For each $d > 0$ the κ-module $\bar{J}_{max}(d)_\xi \subset M^d/M^{d+1}$, which is defined to be*

$$J_{max}(d)_\xi + M^{d+1} \qquad modulo \qquad J_{max}(d)_\xi^* + M^{d+1}$$

where, with $\iota(d) = min\{\, i \mid q_i < d \,\}$,

$$J_{max}(d)_\xi^* \quad = \quad \sum_{\substack{\alpha=(\alpha_\iota,\cdots,\alpha_r) \,\in\, \mathbb{Z}_0^{r-\iota(d)+1} \\ d \,=\, \sum_{\iota(d)\le i\le r} \alpha_i q_i}} \Big(\prod_{j=\iota(d)}^r g_j^{\alpha_j}\Big) \mathcal{O}_{Z,\xi}.$$

(3) *The flag of $\kappa(\xi)$-subspaces*

$$M/M^2 \supset F_1 \supset, \cdots, \supset F_k \subset F_{k+1} = (0) \quad with \quad F_j = \sum_{i > r(j-1)} \bar{y}_i \kappa(\xi)$$

where $\bar{y}_i = in_M(y_i)$ denotes the class in M/M^2 and $r(0) = 0$ and $q_1 = q_{r(1)} > q_{r(1)+1} = q_{r(2)} > \cdots = q_{r(k-1)} > q_{r(k-1)+1} = q_r$. The y_i's are those selected by Th.(4.3).

Remark 4.3. This concerns with the globalization of the invariants $\mathfrak{q}_\xi(E^*)$. Let S be the smooth part of the set of points $\eta \in Z$ with $ord_\eta(E^*) = 1$ or $ord_\eta(J) = m$. It is an open dense subset of a closed subset in Z. Let I be the idea induced by the diagonal ideal into $Z \times_{\mathbb{K}} S$. The graded algebra by the powers of I corresponds to the family of tangent spaces to Z parametrized by the points of S. In it we have the family of homogeneous ideals $J\mathcal{O}_{Z\times S} \mod I^{m+1}$ parametrized by the points of S. In this way we can describe the upper semicontinuity of $\mathfrak{q}_\eta(E^*), \eta \in S$, and an algebraic $\mathfrak{q}$-stratification of S.

5. Strategy for Induction (3)

We will now proceed to formulate our inductive approach in terms of the sequence $\mathfrak{q}_\xi(E^*)$ of Prop.(4.4) for an idealistic exponent $E = (J, b)$ in Z, where Z is an n-dimensional smooth irreducible scheme of finite type over a perfect base field $\mathbb{K}$ of characteristic $p > 0$. Recall that E^* is (J, m) with $m = max\{ord_\eta(J) \mid \eta \in Z\}$.

Our definition of *ordering* among systems of numbers is *lexicographical*. Namely, we say that

$$(a_1, a_2, \cdots, a_l) \ < \ (b_1, b_2, \cdots, b_m)$$

if and only if there exists an integer $k \geq 1$ such that $a_i = b_i, \forall i < k$, and $a_k < b_k$, where $b_k = \infty$ when $k > m$. (Think of adjoining a tail of unlimited length of repeated ∞ to every sequence.)

Let us now begin to examine the effects of permissible sequence of blow-ups to $\mathfrak{q}_\xi(E^*)$. Let us recall the decomposition of Th.(4.3) and the notation used there. We let $R = \mathcal{O}_{Z,\xi}$ and $M = max(R)$.

Lemma 5.1. *Let us pick $q = p^e$ with $e \geq 1$ and a regular system of parameters $y = (y_1, \cdots, y_n)$ of R. let us pick any $g \in M$ of the following form.*

$$g \ = \ {y_1}^q + \textit{ higher order terms } \ \in \ {y_1}^q + M^{q+1}$$

If $I \subset R$ is a prime ideal such that R/I is regular and $g \in I^q$, then there exists an element $h \in M^2$ such that $y_1 - h \in I$. Moreover it follows that $h^q + (g - {y_1}^q) \in I^q M$.

Corollary 5.2. *If a proper blow-up $\pi : Z' \longrightarrow Z$ with center D passing through ξ is permissible for E_i of Th.(4.3), then we have $y_i \in I(D, Z)_\xi + M^2$ with the same i where $I(D, Z)_\xi$ denotes the ideal of $D \subset Z$ at ξ.*

Remark 5.1. Let $y_i - h_i = z_i$ with $z_i \in I$ and $h_i \in M^2$ where $I = I(D, Z)_\xi$. In terms of the notation of Th.(4.3), the sequence $\mathfrak{q}_\xi(E^*)$ remains unchanged when z_i takes the place of y_i in the expression of g_i. If π of Cor.(5.2) is permissible for E^* and hence so it is for every one of the E_i, then we can let z_i take the place of y_i for all i in such a way that $\mathfrak{q}_\xi(E^*)$ remains unchanged. Indeed the decomposition of Th.(4.3) itself is unchanged.

Lemma 5.3. *Keep the notation and assumptions of both Lem.(5.1) and Rem.(5.1). If a closed point $\xi' \in \pi^{-1}(\xi) \subset Z'$ belongs to the singular locus $Sing(E_i')$ of the transform E_i' of E_i by π, then we have $z_i = y_i - h_i$ with $h_i \in M^2$ such that*

$$(z_i)R' \ \subset \ I'M' \quad \textit{for the same } \ i,$$

where $R' = \mathcal{O}_{Z',\xi'}$, $M' = max(R')$ and $I' = I(D', Z')_{\xi'} = IR'$ with $D' = \pi^{-1}(D)$ which is the exceptional divisor of π.

Remark 5.2. The key point of the proof of this lemma is that if we write $g_i = {z_i}^{q_i} + f_i$ then

$$g_i \in I^{q_i} \Longrightarrow f_i \in I^{q_i} \cap M^{q_i+1} = I^{q_i}M,$$

so that

$$(f_i)R' \subset I'^{q_i}MR' \subset I'^{q_i}M'.$$

If $I' = (v)R'$ with $v \in I$ then

$$g'_i = v^{-q_i}g_i = (v^{-1}z_i)^{q_i} + v^{-q_i}f_i$$

and therefore we must have $v^{-1}z_i \in M'$.

Lemma 5.4. *Maintain all the notation and assumptions of Lem.(5.3) for all $i, 1 \le i \le r$. In particular we are assuming $\xi' \in Sing(E'_i), \forall i$. Pick a regular system of parameters $z = (z_1, \cdots, z_n)$ of R such that $z_i = y_i - h_i, 1 \le i \le r$, $I = (z_1, \cdots, z_c)R, r < c \le n$, and $(z_{r+1})R' = I'$. Let $y'_i = z_i/z_{r+1} \in M', 1 \le i \le r$. Then the system $(y'_1, \cdots, y'_r, v, w)$ extends to a regular system of parameters y' of R' where $v = z_{r+1} = y'_{r+1}$ and $w = (z_{c+1}, \cdots, z_n) = (y'_{c+1}, \cdots, y'_n)$. Moreover, with $g'_i = v^{-q_i}g_i$ for each $i, 1 \le i \le r$, either one of the following is true:*

1. $g'_i - (y'_i)^{q_i} \in M'^{q_i+1}$.
2. $ord_{M'}\left(g'_i - (y'_i)^{q_i}\right) = q_i$ *and* $in_{M'}\left(g'_i - (y'_i)^{q_i}\right)$ *effectively contains variables* $in_{M'}(y'_j)$ *with* $j, r+1 \le j \le n$. *In other words, it is not a polynomial in* $\kappa(\xi')[in_{M'}(y'_1), \cdots, in_{M'}(y'_r)]$.

Remark 5.3. A proof of the last assertion of the lemma can be reduced to the following fact. Let us take the M-adic completion $\hat{R}$ of R. Letting κ denote the algebraic closure of $\mathbb{K}$ in $\hat{R}$, we have $\hat{R} = \kappa[[z]]$ with $z = (z_1, \cdots, z_n)$ of Lem.(5.4). Let us write $g_i = {z_i}^{q_i} + f_{i1} + f_{i2}$ where

1. $f_{i1} \in z(r)^{q_i}\kappa[[z]]$, with $z(r) = (z_1, \cdots, z_r)$, and
2. $f_{i2} \in \sum_{\beta \in \mathbb{Z}_0^r,\ |\beta| < q_i} z(r)^{\beta}\kappa[[z_{r+1}, \cdots, z_n]]$

We then must have $f_{i1} \in (z)z(r)^{q_i}\kappa[[z]]$, i.e., a linear combination of the monomials of degree q_i with coefficients in $M\hat{R}$. It follows that $ord_{\hat{M}'}(z_{r+1}^{-q_i}f_{i1}) > q_i$, where $\hat{M}' = M'\hat{R}'$ with the completion $\hat{R}'$ of R'.

Remark 5.4. In the last case of Lem.(5.4), we apply the *initial decomposition lemma* Lem.(4.1) to the transform g'_i. We then find a component whose initial is a q-power of a variable independent of those $y'_i, 1 \le i \le r$, where q is a power of p which is at most q_i. This is seen by means of Rem.(5.3). It therefore follows that the second number of the $\mathfrak{q}$ must then decrease lexicographically.

Theorem 5.5. *Let us consider a proper blow-up $\pi : Z' \longrightarrow Z$ with center D passing through ξ and assume that π is permissible for E^* of Th.(4.3). We can then change $y_i, 1 \le i \le r$, of Th.(4.3), if necessary, in such a way that*

1. *we have* $y_i \in I(D, Z)_\xi, \forall i$,
2. $in_M(y_i)$ *is unchaned for all* i, *and*
3. *the properties of the idealistic decomposition of Th.(4.3) are preserved.*

Theorem 5.6. *Under the same assumptions as Th.(5.5), if a closed point $\xi' \in \pi^{-1}(\xi)$ belongs to $Sing(E^{*\prime})$ of the transform $E^{*\prime}$ of E^* by π, then we can choose those $y_i, 1 \le i \le r$, of Th.T:before-q-trans and $v \in I(D, Z)_\xi$ with $(v)R' = I(D', Z')_{\xi'}$*

in such a way that $(y'_1, \cdots, y'_r, v)$ *with* $y'_i = v^{-1}y_i, \forall i$, *can be extended to a regular system of parameters of* $\mathcal{O}_{Z',\xi'}$. *In other words, the exceptional divisor* $\pi^{-1}(D)$ *has normal crossings with the strict trandforms in* Z' *of the hypersurfaces defined by* $y_i = 0$ *in* Z *for* $1 \le i \le r$, .

We will follow the * symbol of Prop(3.2).

Theorem 5.7. *If a proper blow-up* $\pi : Z' \longrightarrow Z$ *with center* D *with* $\xi \in D$ *is permissible for* E^* *of Th.*(4.3) *and if a closed point* $\xi' \in \pi^{-1}(\xi)$ *belongs to the* $Sing(E^{*\prime})$ *of the transform* $E^{*\prime}$ *of* E^* *by* π, *then we have*

$$\mathfrak{q}_{\xi'}(E^{*\prime}) \le \mathfrak{q}_{\xi}(E^*) \text{ in the sense of lexicographical ordering.}$$

Moreover, if $\mathfrak{q}_{\xi'}(E^{*\prime}) = \mathfrak{q}_{\xi}(E^*)$ *then the component-by-component transformation of a* $\mathfrak{q}_{\xi}(E^*)$ *by means of* π *is such a decomposition of* $\mathfrak{q}_{\xi'}(E^{*\prime})$ *in the sense of Th.*(4.3).

In view of Lem.(3.1), Prop.(3.2), Rem.(3.1), Th.(4.3), Ths.(5.5)+(5.6)+(5.7) and Prop(4.4), we now propose to search for an affirmative answer to the following second stage of *Desingularization Problems*:

Problem(II.$\mathfrak{q}$):
Let Z and $E = (J, b)$ be as before, say $Sing(E) \ne \emptyset$. Let $E^* = (J, m)$ with $m = max\{\ ord_\eta(J) \mid \eta \in Z\ \}$. The new task is then to prove the existence of a finite proper sequence of blow-ups over Z, permissible for both E^* and Γ, such that we have a strict inequality $\mathfrak{q}(\tilde{E}^*) < \mathfrak{q}(E^*)$ in the lexicographical ordering provided $Sing(\tilde{E}^*) \ne \emptyset$, where $\tilde{E}^*$ denotes the final transform of E^* by the sequence of blow-ups.

Remark 5.5. In order to make our inductive proof to work out, we have essentially two type of easy cases to start with as follows:

1. (*The case of ambient reduction.*) If $q_r = 1$, then we make repeated use of the *ambient reduction* by means of Ths.(1.2+1.7):
   - (a) Firstly, if the local smooth hypersurface $W : g_r = 0$ has normal crossings with the given NC which we called Γ then we apply the Th.(1.7) to the given E from Z to W together with the NC in W which is induced by Γ. This makes our task reduced to the *Problem*(I. $n-1$).
   - (b) If otherwise, we apply Th.(1.2) from Z to each of the Γ_i and solve *Problem*(I. $n-1$) in Γ_i for each $i, 1 \le i \le s$. Thanks to Th.(5.6) which garrantees the normal crossings with all new exceptional divisors, our probelm is the previous case.
2. (*The case of zero dimension.*) When we have $r = n$ in $\mathfrak{q}_\xi$, not only that the $Sing(E^*)$ is isolated at ξ but also there exists a unique canonical way of resolving all the infinitely near singulaities E^* above ξ by a finite succession of blow-ups with zero-dimenional centers (starting with ξ as the first center).

6. Review on differentiations

In this section we consider a base field which may not be perfect for the sake of technical convenience needed later. Let us use a symbol $\mathbb{L}$ for the base field, instead of the earlier symbol $\mathbb{K}$ which was assumed to be perfect. The same symbol Z will be used for an ambient scheme which is irreducible and smooth of finite type over $\mathbb{L}$. We let $n = \ dim\ Z$.

Pick a closed point $\xi \in Z$ whose residue field $\kappa(\xi)$ is separable algebraic over $\mathbb{L}$. Let $R = \mathcal{O}_{Z,\xi}$, $M = max(R)$ and $\kappa = R/M$. Pick any regular system of parameters of R, say $x = (x_1, \cdots, x_n)$. Knowing that $\kappa(\xi)$ is separable algebraic over $\mathbb{L}$, we can find a system of differential operators of R into itself, $\{\partial^{(a)} = \partial_x^{(a)}, a \in \mathbb{Z}_0^n\}$, uniquely determined by x, such that

$$\partial^{(\alpha)} x^\beta = \begin{cases} \binom{\beta}{\alpha} x^{\beta-\alpha} & \text{if } \beta \in \alpha + \mathbb{Z}_0^n \\ 0 & \text{if otherwise} \end{cases} \tag{6.1}$$

We thus obtain a system of free generators

$$\{\partial^{(a)}, a \in \mathbb{Z}_0^n\} \ \text{ for the } R\text{-module } Diff_{Z,\xi}\Big(= Diff_{R/\mathbb{L}} = \bigcup_\mu Diff_{R/\mathbb{L}}^{(\mu)}\Big)$$

Remark 6.1. The reference to x is essential in the definition of $\partial_x^{(a)}$ and the abbreviated symbol $\partial^{(a)}$ is used only when x is set and clear by the context.

Let us define the following notation:

$$Diff_{Z,\xi}^{(\mu)*} = Diff_{R/\mathbb{K}}^{(\mu)*} \ = \ \{\ \partial \in Diff_{R/\mathbb{K}}^{(\mu)} \mid \partial(c) = 0, \forall c \in \mathbb{L}.\ \} \tag{6.2}$$

Remark 6.2. There exists a unique direct sum decomposition:

$$Diff_{R/\mathbb{L}}^{(\mu)} \ = \ R \oplus Diff_{R/\mathbb{L}}^{(\mu)*}$$

where the elements of the first summand are meant to be the multiplication by elements $f \in R$, which will be sometimes expressed as $f\mathbf{1}$.

We will let ∂_i denote the partial derivation by x_i with respect to x,

$$\partial_i \ = \ \partial^{(\delta_i)}, 1 \le i \le n, \tag{6.3}$$

where δ_i is the one whoose components are one in the $(i-th)$ and zero in the others. Let $Der_{R/\mathbb{L}}$ denote the R-module of derivations which is freely generated by those ∂_i. Note that

$$Der_{R/\mathbb{L}} \ = \ Diff_{R/\mathbb{L}}^{(1)*} \ \subset \ Diff_{R/\mathbb{L}}^{(\mu)*}, \forall \mu > 0. \tag{6.4}$$

Remark 6.3. The $\mathcal{O}_Z$-module Der_Z denotes the sheaf of all derivations from $\mathcal{O}_Z$ into itself, which is coherent and whose stalk $Der_{Z,\xi}$ at the point ξ is *freely generated* as $\mathcal{O}_{Z,\xi}$-module by the following system of derivations:

$$\{\partial_i, 1 \le i \le n,\} \ \text{ which is the same as } \ \{\partial^{(\alpha)}, |\alpha| = 1\},$$

because $\kappa(\xi)$ is separable algebraic over $\mathbb{L}$. (See *Eq.*(6.1) and *Eq.*(6.3).) Therefore the *free generation* by the same system extends to hold at every point within a sufficently small neighborhood of $\xi \in Z$.

In this paper, our primary interest is in the case of $p > 0$ in which much richer contents will be seen in notions and reasonings. However $p = 0$ is not excluded. When $p = 0$, the powers $p^e, e \ge 0$, should normally be understood to be 1, except for few exceptional cases which will be specifically mentioned. For instance, we have the *Frobenius map* ρ from any $\mathbb{L}$-algebra into itself, which send $f \mapsto f^p$ when

$p > 0$. For $p = 0$, however, ρ will better be understood as the zero map rather than the identity map.

We have special kind of differential operators which are derived from derivations by means of Frobenius map ρ. We let

(6.5) $$R(e) = \mathbb{L}\rho^e(R) \textit{ for every integer } e \geq 0.$$

Note that $R(e)$ is an algebraic local ring over the base field $\mathbb{L}$, for $\mathbb{L}$ is purely inseparable over $\rho^e(\mathbb{L})$, and that if $\mathbb{L}$ is perfect then $R(e) = \rho^e(R)$.

Definition 6.1. Assume that $p > 0$. For an integer $e > 0$, we define

$$\rho^e(\partial) \in Diff^{(\mu)}_{R(e)/\mathbb{L}} \textit{ for every } \partial \in Diff^{(\mu)}_{R/\mathbb{L}},$$

as follows:

(1) If $\partial \in Diff^{(\mu)*}_{R/\mathbb{L}}$ then $\rho^e(\partial)(g) = \rho^e\Big(\partial\big(\rho^{-e}(g)\big)\Big)$, $\forall g \in \rho^e(R)$, and zero in $\mathbb{L}$.

(2) If $\partial = \sum_\alpha c_\alpha \partial^{(\alpha)}$ with $c_\alpha \in R$, then $\rho^e(\partial) = \sum_\alpha c_\alpha^{p^e} \rho^e(\partial^{(\alpha)})$ in terms of $\{\partial^{(\alpha)}\}$ of Eq.(6.1) with reference to the chosen x. In particular if $\partial = f \oplus \partial^*$ in the sense of Rem.(6.2) then we have

$$\rho^e(\partial) \;=\; f^{p^e} \oplus \rho^e(\partial^*).$$

Remark 6.4. The definition of $\rho^e(\partial)$ is intrinsic and is independent of the choice of x.

Definition 6.2. We know that any $\partial \in Diff^{(\mu)}_{R/\mathbb{L}}$ extends uniquely to a differential operator of any localization of R, in particular to an element of $Diff^{(\mu)}_{\mathfrak{F}/\mathbb{L}}$ where $\mathfrak{F}$ denotes the field of fractions of R that is the so-called *function field* of Z. We will use the same symbol ∂ for any such extention. Now let y be any separating transcendental base of $\mathfrak{F}/\mathbb{L}$. We then define

$$\rho^e_y(\partial) \in Diff^{(p^e\mu)}_{\mathfrak{F}/\mathbb{L}} \quad \textit{for every } \partial \in Diff^{(\mu)}_{\mathfrak{F}/\mathbb{L}},$$

by letting

$$\rho^e_y(\partial)(cy^\alpha h) \;=\; cy^\alpha \rho^e(\partial)(h),\ \forall c \in \mathbb{L}, \forall \alpha \in [0, p^e - 1]^n, \forall h \in \rho^e(\mathfrak{F}).$$

For instance, if $\partial \in Diff^{(\mu)*}_{R/\mathbb{L}}$, $\rho^e_y(\partial)$ acts identically zero in $\sum_{\alpha \in [0,p^e-1]^n} \mathbb{L}\ \mathfrak{F}^{\ p^d} y^\alpha$ for every integer $d, p^d > p^e\mu$. If $\partial = f \oplus \partial^*$ in the sense of Rem.(6.2) then we have

$$\rho^e_y(\partial) \;=\; f^{p^e} \oplus \rho^e_y(\partial^*).$$

Remark 6.5. It is important to note that $\rho^e_x(\partial)$ depends upon the choice of x unlike $\rho^e(\partial)$.

Remark 6.6. According to the notation of *Eq.*(6.1) with respect to a chosen regular system x of parameters at $\xi \in Z$, we have

$$\rho^k_x(\partial^{(\alpha)}) \;=\; \partial^{(p^k\alpha)},\ \forall k > 0,\ \forall \alpha \in \mathbb{Z}^n_0.$$

Remark 6.7. Assume $char(\mathbb{L}) = p > 0$. Following the notation of Eq.(6.1) with respect to x at $\xi \in Z$, we have:

(1) (*Commutativity*) $\partial^{(\alpha)}\partial^{(\beta)} = \partial^{(\beta)}\partial^{(\alpha)} = \binom{\alpha+\beta}{\alpha}\partial^{(\alpha+\beta)}$ for every pair of α and β in $\mathbb{Z}^n_0$,

(2) (*Distribution*) $\partial^{(\gamma)}(fg) = \sum_{\alpha+\beta=\gamma} \partial^{(\alpha)}(f)\,\partial^{(\beta)}(g)$, and
(3) (*Generation*) For every $\alpha = \sum_e p^e\alpha(e) \in \mathbb{Z}_0^n$, where $\alpha(e) \in [0, p-1]^n, \forall e \geq 0$,

$$\partial^{(\alpha)} = \prod_e \rho_x^e\Big(\prod_{i=1}^n \frac{1}{\alpha(e)_i!}(\partial_i)^{\alpha(e)_i}\Big).$$

Remark 6.8. For any separating transcendental base y of the function field of Z over $\mathbb{L}$ and for any $\partial \in Der_{Z,\xi}$, we have

$$\rho_y^e(\partial) = \rho_x^e(\partial) - \sum_{\substack{\beta\in\mathbb{Z}_0^n \\ 0\leq\beta_i<p^e, 0<|\beta|\leq p^e}} C_{y,x,e}(\partial,\beta)\,\partial_y{}^{(\beta)}$$

where symbols $\partial_x{}^{(\beta)}$ are after Eq.(6.1) and $C_{y,x,e}(\partial,\alpha)$ are defined inductively as follows:

(1) $C_{y,x,e}(\partial,(0)) = 0$,
(2) for every α with $|\alpha| = 1$ we have $C_{y,x,e}(\partial,\alpha) = \rho_x^e(\partial)y^\alpha$,
(3) for every $\alpha \in \mathbb{Z}_0^n$ we have

$$C_{y,x,e}(\partial,\alpha) = \rho_x^e(\partial)y^\alpha - \sum_{\beta\neq\alpha\in\beta+\mathbb{Z}_0^n} \binom{\alpha}{\beta} C_{y,x,e}(\partial,\beta)y^{\alpha-\beta}.$$

Here we may say $C_{y,x,e}(\partial,\alpha) = 0$ for all α with $|\alpha| > p^e$ and for all $\alpha \in p^e\mathbb{Z}_0^n$. Note that the $\rho_y^e(\partial), \rho_x^e(\partial)$ and indivisually $C_{y,x,e}(\partial,\alpha)$ possess certain $\rho^{e+1}(R)$-linearity in the following sense: For every $(A_1, A_2) \in \rho^{e+1}(R)^2$ and every $(\partial_1, \partial_2) \in Der_{Z,\xi}{}^2$,

(1) $\rho_y^e(A_1\partial_1 + A_2\partial_2) = \rho^e(A_1)\rho_y^e(\partial_1) + \rho^e(A_2)\rho_y^e(\partial_2)$, likewise for ρ_x^e, and
(2) $C_{y,x,e}(A_1\partial_1 + A_2\partial_2, \alpha) = \rho^e(A_1)C_{y,x,e}(\partial_1,\alpha) + \rho^e(A_2)C_{y,x,e}(\partial_2,\alpha)$, $\forall\alpha$.

Remark 6.9. Following the notation of Rem.(6.8), if $x' = fx$ where $f = h^{p^{e+1}}$ with a unit $h \in R$ then we have

$$C_{y,x',e}(\partial,\alpha) = f^{|\alpha|}C_{y,x',e}(\partial,\alpha), \forall\partial \textit{ and } \forall\alpha.$$

We also have $\rho_{x'}^e(\partial) = \rho_x^e(\partial)$. Moreover if $y' = fy$ with the same f then we have

$$C_{y',x,e}(\partial,\alpha) = C_{y',x,e}(\partial,\alpha), \forall\partial \textit{ and } \forall\alpha.$$

Example 6.1. Assume $p > 0$ and pick an integer $e \geq 1$. Let $x_1 \in R$ be a member of a regular system of parameters $x = (x_1, \cdots, x_n)$ of R. We will make use of the symbols such as ρ_x^e defined in Def.(6.1). We will then consider various $y_{da} = x_1 + x_1{}^{p^d-a+1}$ under the following conditions:

(1) d is an integer such that $e \geq d \geq 1$, and
(2) a is an integer such that $1 \leq a < p^d$ and $a \not\equiv 0 \mod p$.

We will write $y(da)$ for the new regular system of parameters $(y_{da}, x_2, \cdots, x_n)$.

$$\frac{\partial}{\partial x_1} = U_{da}\frac{\partial}{\partial y_{da}}.$$

where the multiplier $U_{da} = 1 + (p^d - a + 1)x_1{}^{p^d-a} \equiv 1 \mod (x_1)$ which is obviously a unit in R. We look at various powers of y_{da} and the effect of $\rho_x^e(\frac{\partial}{\partial x_1})$ to them.

Let b be any integer such that $1 \leq b < p^d$.

$$
\begin{aligned}
{y_{da}}^{bp^{e-d}} &= \\
& x_1^{bp^{e-d}} + b^{p^{e-d}} {x_1}^{p^e + p^{e-d}(b-a)} \\
& + \binom{b}{2}^{p^{e-d}} {x_1}^{p^e + p^{e-d}(b-a) + p^{e-d}(p^d - a)} \\
& + \binom{b}{3}^{p^{e-d}} {x_1}^{p^e + p^{e-d}(b-a) + 2p^{e-d}(p^d - a)} \\
& + \binom{b}{4}^{p^{e-d}} {x_1}^{p^e + p^{e-d}(b-a) + 3p^{e-d}(p^d - a)} \\
& \cdots \\
& + {x_1}^{p^e + p^{e-d}(b-a) + (b-1)p^{e-d}(p^d - a)}
\end{aligned}
$$

where $p^d - a > 0$ and the degrees are strictly increasing. It hence follows that

(1) if $p^d > b > a$, then

$$\Big(\rho_x^e(\frac{\partial}{\partial x_1})\Big)({y_{da}}^{bp^{e-d}}) \equiv 0 \mod (x_1)R$$

(2) if $b \leq a$ and if $\exists c \in \mathbb{Z}_0$ such that $a - b = c(p^d - a)$, then

$$\Big(\rho_x^e(\frac{\partial}{\partial x_1})\Big)({y_{da}}^{bp^{e-d}}) \equiv \binom{b}{c+1}^{p^{e-d}} \mod (x_1)R$$

(3) as a special case of the above, if $b = a$, then

$$\Big(\rho_x^e(\frac{\partial}{\partial x_1})\Big)({y_{da}}^{ap^{e-d}}) \equiv a^{p^{(e-d)}} \mod (x_1)R$$

(4) if $b < a$ and if $\nexists c$ as above, then

$$\Big(\rho_x^e(\frac{\partial}{\partial x_1})\Big)({y_{da}}^{bp^{e-d}}) \equiv 0 \mod (x_1)R$$

On the other hand we always have

$$\Big(\rho_{y(da)}^e(\frac{\partial}{\partial x_1})\Big)({y_{da}}^{bp^{e-d}}) = \Big(U_{da}^{p^e}\rho_{y(da)}^e(\frac{\partial}{\partial y_{da}})\Big)({y_{da}}^{bp^{e-d}}) = 0.$$

Therefore, letting $\partial_{y(da)}^{(\alpha)}, b > 0$, denote the differential operators in R such that

$$\partial_{y(da)}^{(\alpha)}(\textit{ every monomial in the } x_k \textit{ with } 2 \leq k \leq n,) \; = \; 0, \text{ and}$$

$$\partial_{y(da)}^{(\alpha)}(y_{da}^{\beta}) = \begin{cases} \binom{\beta}{b} y_{da}^{\beta-\alpha} & \textit{if } \beta \geq \alpha \\ 0 & \textit{if otherwise} \end{cases}$$

we obtain the following congruence among differential operators:

$$\begin{aligned}\rho_x^e\Big(\frac{\partial}{\partial x_1}\Big) &\equiv \rho_{y(da)}^e\Big(\frac{\partial}{\partial y_{da}}\Big) + a^{p^{(e-d)}}\partial_{y(da)}^{(ap^{e-d})} \\ &+ \sum_{\substack{b\in\mathbb{Z}_0-p\mathbb{Z}_0:1\le b<a,\\ \exists c\in\mathbb{Z}_0, a-b=c(p^d-a)}} \binom{b}{c+1}^{p^{(e-d)}} \partial_{y(da)}^{(bp^{e-d})} \\ &\bmod (x_1)Diff_{R/\mathbb{L}}\end{aligned}$$

for every integer $d, e \geq d \geq 1$, and for every integer a such that $1 \leq a < p^d$ and $a \not\equiv 0 \mod p$. Note that a can be any and the last summation is for only those with $b < a$. For the case of dimension one, i.e., $n = 1$, it follows from the above congruence that

$$Diff_{R/\mathbb{L}}^{(p^e)*} = \sum_{all\ y:M=yR} R\ \rho_y^e\Big(\frac{\partial}{\partial y}\Big) \quad \textit{for every integer } e \geq 1.$$

7. Diffcompanions and Fittcompanions

Let us go back to the case in which the base field $\mathbb{K}$ of Z is algebraically closed and of characteristic $p > 0$. Let $R = \mathcal{O}_{Z,\xi}$ and $M = max(R)$ as before. The objective of this section is roughly speaking as follows. Assume that we are given an element $g \in R$ such that

$$g = {y_1}^q + h \quad with \quad ord_M(h) > q$$

where y_1 is a member of a regular system of parameters $y = (y_1, \cdots, y_n)$ of R and $q = p^e$ with and integer $e \geq 1$. We then want to examine the effect of permissible blow-ups upon h viewed only up to q-power differences which can be absorbed into the first term ${y_1}^q$ of g. We thus make use of certain differential operators ∂ which kills all the q-power terms of g. For instance, we may take $\partial_y^\alpha \cdot y^\beta$ with $\alpha \in \beta + \mathbb{Z}_0^n, \alpha \neq \beta \in \mathbb{Z}_o^n$. We then investigate the transforms of $\partial(h)$ under such blow-ups in terms of suitably chosen ∂'s for the given g.

Recall that we are given an NC of Def.(3.1) in the ambient scheme Z, denoted by $\Gamma = (\Gamma_1, \cdots, \Gamma_s)$. From now on we will consider only those regular systems of parameters of R which are Γ-*adapted* in the sense of Def.(3.2).

We write $|\Gamma|$ for the set-theoretical union of the $\Gamma_i, 1 \leq i \leq s$. Let $H = \prod_{i=1}^s H_i$ with the ideal H_i of Γ_i in $\mathcal{O}_Z$. It is the ideal of $|\Gamma|$ in Z.

We will define and make use of what will be called *diffcompanion* which will be a certain type of coherent $\mathcal{O}_Z$-submodule

$$\mathfrak{D} \subset Diff_Z[H^{-1}] = \Big(\mathcal{O}_Z[H^{-1}]\Big)Diff_Z\ ,$$

They will be sheaves of differential operators in Z possibly having poles that are of finite orders along $|\Gamma|$ and no poles anywhere in $Z - |\Gamma|$.

Definition 7.1. Let e be a non-negative integer. For each closed point $\eta \in Z$, we define the following $\mathcal{O}_{Z,\eta}$-submodule of $Diff_{Z,\eta}^{(p^e)}$:

$$\mathfrak{E}(e)_{Z,\eta} = \sum_x \Big(\sum_{\partial\in Der_{Z,\xi}} \mathcal{O}_{Z,\eta}\ \rho_x^e(\partial) \Big) \subset Diff_{Z,\eta}^{(p^e)}$$

where x ranges over all those regular systems of parameters of $\mathcal{O}_{Z,\eta}$ which are Γ-adapted in the sense of Def.(3.2). As for the definition of $\rho_x^e(\partial)$, refer to Def.(6.1) and Def. (6.2).

Theorem 7.1. *We have a coherent $\mathcal{O}_Z$-module $\mathfrak{E}(e)_Z$ in Z, or $\mathfrak{E}(e)$ for short, whose stalk at any closed point $\eta \in Z$ is the $\mathcal{O}_{Z,\eta}$-module $\mathfrak{E}(e)_{Z,\eta}$ which is defined by Def.(7.1).*

A proof of this theorem is done as shown in the following three remarks.

Remark 7.1. For each $x = (x_1, \cdots, x_n)$ of Def.(7.1), there exists an affine open neighborhood U of $\eta \in Z$ in which $x - \zeta$, meaning $(x_1 - x_1(\zeta), \cdots, x_n(\zeta))$, is a regular system of parameters of $\mathcal{O}_{Z,\zeta}$ which are naturally Γ-adapted. Moreover for any $\partial \in Der_Z(U)$, we have $\delta \in Diff_Z^{(p^e)}$ such that $\delta_\zeta = \rho_{x-\zeta}^{\ e}(partial)$ for every $\zeta \in U$, for we have

$$\sum_{\alpha\in[0,p^e-1]^n\subset\mathbb{Z}_0^n} \mathbb{K}x^\alpha = \sum_{\alpha\in[0,p^e-1]^n\subset\mathbb{Z}_0^n} \mathbb{K}(x-\zeta)^\alpha$$

so that $\rho_{x-\zeta}^{\ e}(partial)$ coincides with $\rho_x^{\ e}(partial)$ wherever both continue to as differential operators. (Refer to Defs.(6.1)+ (6.2).)

Remark 7.2. The submodule $\mathfrak{E}(e)_{Z,\eta}$ defined in Def.(7.1) is finitely generated as $\mathcal{O}_{Z,\eta}$-module. Hence there can be found a neighborhood U of $\eta \in Z$ and a finitely generated $\mathcal{O}_Z$-submodule $\mathfrak{F}$ of $Diff_Z^{(p^e)}$ in U such that $\mathfrak{E}(e)_{Z,\eta}$ equals the stalk $\mathfrak{F}_\eta$ of $\mathfrak{F}$ at η and that the quotient $Diff_Z^{(p^e)}/\mathfrak{F}$ has zero $\mathcal{O}_Z$-torsion in U.

Remark 7.3. Pick $x, U, \zeta, \mathfrak{F}$, of Rems.(7.1)+(7.2). We may assume that each ideal H_i of Γ_i is principal in U, say $H_i|U = h_i\mathcal{O}_U, \forall i$. We may assume that $h_i(\zeta)$ is either 0 or $1, \forall i$. Let $f = \prod_{i:h_i(\zeta)=1} h_i^{p^{e+1}}$, provided $f(\eta) = 0$. If otherwise, pick any $c \in \mathcal{O}_Z(U)$ such that $c(\eta) = 0$ and $c(\zeta) = 1$, and replace f by $c^{p^{e+1}}f$. Call it f again. We next pick $g = \prod_{i:h_i(\eta)=1} h_i^{p^{e+1}}$, provided $g(\zeta) = 0$. If not, replace g by $ga^{p^(e+1)}$ with any a in the affine ring of U such that $a(\eta) = 1$ and $a(\zeta) = 0$. Now let y be any regular system of parameters of $\mathcal{O}_{Z,\zeta}$ which is Γ-adapted in case of $\zeta \in |\Gamma|$. We then let $z(m) = fy + g^m x$ with an arbitrary positive integer m, which is clearly a regular system of parameters admitted as a member to make the Def.(7.1) both at η and at ζ. In terms of remarks (6.8)+(6.9) with reference to fy and $z(m)$ (in the places of y and x in there) considered at the point ζ, we obtain

$$\rho_{z(m)}^e(\partial) = \rho_{(fy)}^e(\partial) - \sum_{\substack{\beta\in\mathbb{Z}_0^n \\ 0\le\beta_i<p^e, 0<|\beta|\le p^e}} C_{z(m),fy,e}(\partial,\beta)\, \partial_{z(m)}^{(\beta)}$$

where $C_{z(m),fy,e}(\partial,\alpha)$ are defined inductively as follows:

(1) $C_{z(m),fy,e}(\partial,(0)) = 0$,

(2) for every β with $|\beta| = 1$, $C_{z(m),fy,e}(\partial,\beta) = \rho_{(fy)}^e(\partial)(fy + g^m x)^\beta$,

(3) for every $\alpha \in \mathbb{Z}_0^n$ we let

$$\begin{aligned} &C_{z(m),fy,e}(\partial,\alpha) \\ =\ &\rho_{(fy)}^e(\partial)(fy+g^m x)^\alpha - \sum_{\beta\neq\alpha\in\beta+\mathbb{Z}_0^n} \binom{\alpha}{\beta} C_{z(m),fy,e}(\partial,\beta) z(m)^{\alpha-\beta}. \end{aligned}$$

Having chosen $g \in M_\zeta = max(\mathcal{O}_{Z,\zeta})$, we see that $\rho^e_{(fy)}(\partial)(fy + g^m x)^\gamma \in M_\zeta{}^m$ for every γ with $p^e \geq |\gamma| > 0$ and $\gamma \notin p^e\mathbb{Z}_0^n$ because $\rho^e_{(fy)}(\partial)(fy)^\gamma = 0$ and $\rho^e_{(fy)}(\partial)$ is $\rho^{e+1}(R_\zeta)$-linear in elements of $R_\zeta = \mathcal{O}_{Z,\zeta}$. It follows that $C_{z(m),fy,e}(\partial,\alpha) \in M_\zeta{}^m$ and hence $\rho^e_{z(m)}(\partial) - \rho^e_{(fy)}(\partial) \in M_\zeta{}^m$. Since $\rho^e_{z(m)}(\partial) \in \mathfrak{F}_\eta$ implies $\rho^e_{z(m)}(\partial) \in \mathfrak{F}_\zeta$ by Rem.(7.2), we must have

$$\rho^e_{(fy)}(\partial) \in \mathfrak{F}_\zeta \cap M_\zeta{}^m$$

for all $m \geq 1$. It follows by noetherian argument that $\rho^e_y(\partial) = \rho^e_{(fy)}(\partial) \in \mathfrak{F}_\zeta$.

Remark 7.4. According to Def.(7.1) and Th.(7.1), $\mathfrak{E}(0)$ is nothing but the sheaf of derivations Der_Z.

Definition 7.2. For an integer $d \geq 0$, a Γ-*diffcompanion of level* d in Z means a coherent $\mathcal{O}_Z$-submodule $\mathfrak{D}(d,\Gamma)$, $\mathfrak{D}(d)$ for short, of $Diff_Z^{(p^d)}[-H^{-1}]$ such that $\mathfrak{D}(d)|_{Z-|\Gamma|} = \mathfrak{E}(d)|_{Z-|\Gamma|}$ where $\mathfrak{E}(d)$ is the one defined in Def.(7.1).

Remark 7.5. If $\mathfrak{D}(e,\Gamma)$ is a Γ-diffcompanion of level d, then it is a coherent $\mathcal{O}_Z$-submodule of $\mathfrak{E}(d)[-H^{-1}]$.

Remark 7.6. For any Γ-permissible blow-up $\pi : Z' \to Z$, the natural pull-back by π of any Γ-diffcompanion of level d in Z is a Γ'-diffcompanion of level d in Z', where Γ' is the transform of Γ by π which was defined by Def.(3.3).

Let us now go back to the Th.(4.3): Given an idealistic exponent $E = (J,b)$ on Z and a closed point $\xi \in Sing(E)$, we let $E^* = (J,m)$ with $m = ord_\xi(J)$ and find systems

$$q_i = p^{e_i}, e_1 \geq \cdots \geq e_r \geq 0, \ \textit{and}\ (y_1, \cdots, y_r)$$

which is extendable to a regular system of parameters of $\mathcal{O}_{Z,\xi}$ and

$$g_i = y_i^{q_i} + h_i \in J_{max}(q_i)_\xi \ \textit{with}\ h_i \in max(\mathcal{O}_{Z,\xi}^{q_i+1})^{\ q_i+1}$$

such that, within a neighborhood of $\xi \in Z$,

$$E^* \sim \Big(\bigcap_{i=1}^{r} E_i\Big) \bigcap F$$

where $E_i = (g_i\mathcal{O}_Z, q_i), 1 \leq i \leq r$, and $F = (I,a)$ with $ord_\xi(I) > a$. We also have the uniqueness of the following system of numbers

$$\mathfrak{q}_\xi(E^*) \ = \ (n, n-r, q_1, \cdots, q_r)$$

by Prop.(4.4).

With $e = e_1$ of Th.(4.3), we are going to apply Γ-*diffcompanions of various levels* $\leq e-1$ to the following system:

$$H = (h_1, h_2^{p^{e_1-e_2}}, \cdots, h_r^{p^{e_1-e_r}})$$

and study the effect of their applications. Rewrite $H = (H_1, \cdots, H_r)$ for simplicity.

To begin with, we choose our Γ-*diffcompanions* to be $\mathfrak{D}(d) = \mathfrak{E}(d)_{Z,\eta}, 0 \leq d \leq e$, themselves.

Take the idealistic exponent

$$F(0) \; = \; E(0) \; \cap \; F \quad \textit{where} \quad E(0) \; = \; (\sum_{i=1}^{r} \mathfrak{D}(d) H_i, \; p^e \;)$$

to which we apply the processes of the theorems of the Strategies (1)-(3) to this $F(0)$. Pay sepcial attention to the theorems (5.5) and (5.6) vis-avis the invariants $\mathfrak{q}$ along the way. In the non-trivial case, we will end up with the ideal of the transform of $E(0)$ is locally generated by "monomials" with respect to the transform Γ' of Γ.

Thus, after the previous process, we now assume to have Γ-*diffcompanions* $\mathfrak{D}(d)'$'s (which are the pullbacks fo the earlier ones) together with the new system $H' = (H'_1, \cdots, H'_r)$ (which is the transform of the H we started with), such that

$$\sum_{i=1}^{r} \mathfrak{D}(d)' H'_i$$

is locally generated by Γ'-monomials. (In the process, make use of the fact that differential operators of order $< p^e$ commute with multiplications by p^e-powers of functions.)

For the next stage, we introduce another kind of companions called *Fitting companions* or *Fittcompanions* of lebel $d, 0 \le d \le e-1$.

Given a system of elements $H' = (H'_1, \cdots, H'_r)$ with $H'_i \in \mathcal{O}'_Z(U')$, we pick any point $\xi' \in U'$ corresponding to ξ and let

$$H'(d)_{\xi'} = \sum_{i=1}^{r} \rho^{e+1}(R_{\xi'}) H'_i,$$

where $0 \le d \le e-1$ and $R_{\xi'} = \mathcal{O}_{Z',\xi'}$.

Then, for each $h \in H'(d)$ and for each regular system of parameters x of $R_{\xi'}$, Γ'-adapted, we denote by $I(h,x,d)_{\xi'}$ to be the $(n-1) \times (n-1)$ Fitting ideal for the submodule

$$\{\partial \in \rho_x^d(Der_{Z',\xi'}) \mid \partial(h) = 0\} \quad \subset \quad \rho_x^d(Der_{Z',\xi'})$$

and let $\mathfrak{I}(d)_{\xi'} = \sum_{h,x} I(h,x,d)_{\xi'}$.

Remark 7.7. There exists a coherent ideal sheaf $\mathfrak{I}(d)$ in $\mathcal{O}'_Z|_{U'}$ such that the above $\mathfrak{I}(d)_{\xi'}$ is indeed the stalk of $\mathfrak{I}(d)$ at the point ξ'.

The strict transforms of $\mathfrak{I}(d)$ by any sequence of blow-ups are called the *Fittcompanion* of level d.

Theorem 7.2. *Assume that we are given a positive integer e and a system* $H = (H_1, \cdots, H_r)$ *of elements in* M_ξ *where* $R = \mathcal{O}_{Z,\xi}$). *Assume also that we are given a* Γ*-diffcompanion of level d for each* $d, 0 \le d \le -1$*, say* $\mathfrak{D}(d)$*. Pick a closed point* $\xi \in Z$*. If* $\sum_{j=1}^{r} \mathfrak{D}(d) H_j$ *is generated by a* Γ*-monomial at* ξ *for every d and if Fittcompanions* $\mathfrak{I}(d)$ *are locally principal at* ξ *for every d, then H must have the following properties:*

There exists a regular system of parameters x of R, Γ*-adapted at* ξ*, and a monomial* x^α *such that*

$$H_i \; = \; v^{p^c} \epsilon_i x^\alpha + \delta_i \quad \textit{with} \quad \delta_i \in \rho^{e+1}(R)$$

where

(1) c *is the largest integer under the conditions that* $0 \leq c \leq e$ *and* $H_i \in \rho^c(R), \forall i$,
(2) $\epsilon_i \in \rho^c(R), \forall i$, *and at least one of the* ϵ_i *is a unit in* $\rho^c(R)$,
(3) $\alpha \in p^c\mathbb{Z}_0^n - p^{c+1}\mathbb{Z}_0^n$, *and*
(4) v *is either a unit of* R *or it defines a hypersurface smooth outside* $|\Gamma|$ *within a neighborhood of* $\xi \in Z$.

Still to continue !

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