The Abdus Salam
International Centre for Theoretical Physics

United Nations
Educational, Scientific
and Cultural Organization

International Atomic
Energy Agency

SMR1757/7

Summer School on Resolution of Singularities

(12 - 30 June 2006)

Basic Algebraic Geometry

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BASIC ALGEBRAIC GEOMETRY

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1. Affine Varieties

These notes are a crash course in algebraic geometry. They are intended to quickly introduce the language of algebraic varieties and schemes.

Our basic reference for the first part of these notes is the book "Algebraic Geometry" by Robin Hartshorne (Springer-Verlag). Another book which is helpful for intuition is "Basic Algebraic Geometry" by Igor Shafarevich (Springer-Verlag). A reference for later parts of these notes is my book "Resolution of singularities" (American Mathematical Society).

Carefully working through the exercises in books such as these is highly recommended as a means of obtaining a command of the basics of algebraic geometry.

We will assume throughout these lectures that k is an algebraically closed field.

Define the affine n-space $\mathbf{A}_k^n$ over k to be the set of all n-tuples $(\alpha_1, \dots, \alpha_n)$ such that $\alpha_1, \dots, \alpha_n \in \mathbf{k}$.

Let $R = k[x_1, \dots, x_n]$, the ring of polynomials in the indeterminates $x_1, \dots, x_n$ with coefficients in k. The maximal ideals of R are precisely the ideals

$$(x_1 - \alpha_1, x_2 - \alpha_2, \dots, x_n - \alpha_n)$$

with $\alpha_1, \dots, \alpha_n \in k$. We can thus identify $\mathbf{A}_k^n$ with the set of maximal ideals of R.

Define the prime spectrum $\mathrm{Spec}(T)$ of any commutative ring T to be the set of prime ideals of T. There is a topology on $\mathrm{Spec}(T)$ (the Zariski topology) defined by taking the closed sets to be

$$V(\Lambda) = \{P \in \mathrm{Spec}(T) \mid \Lambda \subset P\}.$$

for $\Lambda \subset T$. If I is the ideal in T generated by Λ, then $V(I) = V(\Lambda)$.

A subset $X \subset \mathrm{Spec}(T)$ is irreducible if whenever $X = Y \cup Z$ where Y and Z are closed irreducible subsets of $\mathrm{Spec}(T)$, we have $X = Y$ or $X = Z$. It follows that X is irreducible if and only if $X = V(P)$ for a prime ideal $P \subset T$.

We saw that we can "identify" $\mathbf{A}^1$ with $\mathrm{Spec}(R)$ (where R is the polynomial ring in n variables). We see that the irreducible subsets of $\mathbf{A}^1$ are $\mathbf{A}^1$ and the maximal ideals $(x_1 - \alpha_1)$ with $\alpha_1 \in \mathbf{k}$ (which can be thought of as the point α_1).

The irreducible closed subsets of $\mathbf{A}^2$ are a little more complicated. They are $\mathbf{A}^2$, the maximal ideals $(x_1 - \alpha_1, x_2 - \alpha_2)$ with $\alpha_1, \alpha_2 \in \mathbf{k}$ (which can be thought of as the point (α_1, α_2), and the sets $V(f(x_1, x_2))$ where $f(x_1, x_2)$ is an irreducible polynomial in R. The maximal ideals of R contained in $V(f(x_1, x_2))$ are precisely the ideals

$$\{(x_1 - \alpha_1, x_2 - \alpha_2) \mid f(\alpha_1, \alpha_2) = 0\},$$

which can be thought of as the set of points (α_1, α_2) which lie on the curve $f(x, y) = 0$.

The irreducible subsets of $\mathbf{A}^3$ are $\mathbf{A}^3$, a point in $\mathbf{A}^3$, a surface in $\mathbf{A}^3$ (which can be described as the set of points which satisfy an irreducible equation $f(x_1, x_2, x_3) = 0$), and the curves in $\mathbf{A}^3$. There is no simple way to describe most curves in $\mathbf{A}^3$. Some of them are defined by exactly two equations, but there are examples where the prime ideal of the curve requires arbitrarily many generators.

Let us return to $\mathbf{A}^n = \mathrm{Spec}(R)$. An irreducible closed subset $X \subset \mathbf{A}^n$ is called an affine variety. We know that a variety X has the expression $X = V(P)$ for some prime ideal $P \subset R$. Notice that we have an identification

$$\begin{array}{ll} X & = V(P) \\ & = \text{ prime ideals of } R \text{ containing } P \\ & = \text{ prime ideals of } R/P \\ = & \mathrm{Spec}(R/P). \end{array}$$

A polynomial $f(x_1, \ldots, x_n) \in R$ defines a continuous mapping (in the Zariski topology) $f : \mathbf{A}^n \to \mathbf{A}^1$ by

$$(\alpha_1, \ldots, \alpha_n) \mapsto f(\alpha_1, \ldots, \alpha_n).$$

We define the the regular functions on $\mathbf{A}^n$ to be R.

Now given an affine variety $X = V(P) \subset \mathbf{A}^n$, and $f \in R$, we can restrict f to X, and get a continuous function on X. We define the regular functions on X to be the restrictions of the functions of R to X. This gives us a surjective ring homomorphism from R onto the regular functions of X. The kernel of this homomorphism is P. Certainly any function which is contained in P must vanish on X. It is Hilbert's nullstellensatz that if a function vanishes on X then it is contained in P.

Suppose that X and Y are two affine varieties. We say that a mapping $\Psi : X \to Y$ is a morphism if Ψ is continuous and for all regular functions $f : Y \to \mathbf{A}^1$, $f \circ \Psi : X \to \mathbf{A}^1$ is a regular function.

Example 1.1. *Let C be the curve $C = Spec(k[x,y]/(y^2 - x^3))$. Define a morphism $\Psi : \mathbf{A}^1 \to C$ by $\Psi(t) = (t^2, t^3)$.*

Let $X = V(P)$ be an affine variety, and let K be the quotient field of the regular functions $T = R/P$ of X. We define the sheaf $\mathcal{O}_X$ of regular functions on X as follows. To a point $q \in X$, we associate the local ring $\mathcal{O}_{X,q} = T_{m_q}$, where m_q is the maximal ideal of S associated to q. To an open subset $U \subset X$, we define

$$\mathcal{O}_X(U) = \cap_{q \in U} \mathcal{O}_{X,q} \subset K.$$

K is the set of rational functions on X, functions which are regular on some open subset of X. $\mathcal{O}_{X,q}$ are the functions which are regular in some neighborhood of q.

Suppose that $\Psi : X \to Y$ is a morphism. Then for any open subset $U \subset Y$, $\Psi^{-1}(U)$ is open in X, and thus composition with Ψ gives us a ring homorphism

$$\Psi^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(\Psi^{-1}(U)).$$

In Example 1.1, we have an inclusion

$$\Psi^* : T = \mathcal{O}_C(C) = k[x,y]/(y^2 - x^3) \to \mathcal{O}_{\mathbf{A}^1}(\mathbf{A}^1) = k[t].$$

$k[t]$ and T have the same quotient field, and t is integral over T so $k[t]$ is the normalization of T. $\Psi : \mathbf{A}^1 \to C$ is the resolution of singularities of C.

2. Projective Varieties

The projective space $\mathbf{P}^n_k$ is defined to be the set of equivalence classes $(\alpha_0 : \alpha_1 : \cdots : \alpha_n)$ of $(\alpha_0, \alpha_1, \ldots, \alpha_n) \in k^{n+1} - \{0\}$ by the relation $(\alpha_0, \alpha_1, \ldots, \alpha_n) \sim (\beta_0, \beta_1, \ldots, \beta_n)$ if there exists $0 \neq \lambda \in k$ such that $\lambda(\alpha_0, \alpha_1, \ldots, \alpha_n) = (\beta_0, \beta_1, \ldots, \beta_n)$.

Let S be the polynomial ring $k[x_0, \ldots, x_n]$. S is a graded ring, where we define the degree of a monomial $x_0^{i_0} \cdots x_n^{i_n}$ to be $i_0 + \cdots + i_n$. Let $S_+ = x_0 S + x_1 S + \cdots x_n S$. $F \in S$ is homogeneous of degree d if

$$F = \sum_{i_0 + \cdots + i_n = d} a_{i_0 \cdots i_n} x_0^{i_0} \cdots x_n^{i_n},$$

for some $a_{i_0\cdots i_n} \in k$.

We define

$$\mathrm{Proj}(S) = \{ \text{ homogeneous prime ideals P in } S \text{ such that } P \neq S_+\}.$$

The maximal primes P in $\mathrm{Proj}(S)$ are of the form

$$P = \{\alpha_j x_i - \alpha_i x_j \mid i \neq j\}$$

for some $\alpha_0, \alpha_1, \ldots, \alpha_n \in k$, which are not all zero.

P thus corresponds to the point $(\alpha_0 : \ldots : \alpha_n) \in \mathbf{P}^n$. We may thus "identify" $\mathbf{P}^n_k$ with $\mathrm{Proj}(S)$.

Suppose that F is homogeneous, and $(\alpha_0, \alpha_1, \ldots, \alpha_n) \sim (\beta_0, \beta_1, \ldots, \beta_n)$. Then $F(\alpha_0, \alpha_1, \ldots, \alpha_n) = 0$ if and only if $F(\beta_0, \beta_1, \ldots, \beta_n) = 0$. the notion of a homogeneous polynomial vanishing at a point of $\mathbf{P}^n$ is thus well defined.

Now suppose that $I \subset S$ is a homogeneous ideal (I is generated by homogeneous polynomials). We define the closed subsets of $\mathbf{P}^n$ to be the sets

$$V(I) = \{Q \in \mathrm{Proj}(S) \mid I \subset Q\}.$$

We will say that X is a projective variety if $X = V(P)$ for some homogeneous prime ideal P of S. We can "identify" X with the points $(\alpha_0 : \cdots : \alpha_n)$ such that $F(\alpha_0, \ldots, \alpha_n) = 0$ for all homogeneous $F \in P$.

S/P is a graded ring. We have

$$\begin{aligned} X &= V(P) \\ &= \text{ the homogeneous prime ideals in } S \text{ containing } P \\ &= \text{ the homogeneous prime ideals in } S/P \\ &= \mathrm{Proj}(S/P). \end{aligned}$$

We now define the sheaf of regular functions on a projective variety X.

Suppose that $X = \mathrm{Proj}(S/I)$. The rational functions on X are defined to be $K = (S/I)_{(0)}$, the elements of degree zero in the quotient field of S/I. For $q \in \mathrm{Proj}(S/I)$, define $\mathcal{O}_{X,q}$ to be the local ring

$$\mathcal{O}_{X,q} = (S/I)_{(m_q)} \subset K,$$

the elements of degree zero in the localization of S/I with respect to the homogeneous prime ideal m_q associated to q.

For an open subset U of X, we define

$$\mathcal{O}_X(U) = \cap_{q \in U} \mathcal{O}_{X,q} \subset K.$$

Example 2.1. *Suppose that $X = Proj(S/I) \subset \mathbf{P}^n$ is a projective variety. Let $U_i = X - V(x_i)$ for $0 \leq i \leq n$. $\{U_0, \ldots, U_n\}$ is an open cover of X. We have that*

$$\mathcal{O}_X(U_i) = (S/I)_{(x_i)} = k[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_0}]/I_i$$

where

$$I_i = \{F(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_0}) \mid F \in I \textit{ is homogeneous}\}.$$

We may identify

$$U_i = Spec(k[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_0}]/I_i) = V(I_i) \subset \mathbf{A}^n.$$

$\{U_0, \ldots, U_n\}$ is an example of an affine cover of X.

Suppose that X and Y are projective varieties. A mapping $\Psi : X \to Y$ is a morphism if Ψ is continuous, and for every open subset $U \subset Y$ and regular function $f : U \to \mathbf{A}^1$, $f \circ \Psi : f^{-1}(U) \to \mathbf{A}^1$ is regular.

Example 2.2. *(The Veronese mapping) Suppose that* $d \in \mathbf{N}$*. There are* $r = \binom{n+d}{n}$ *monomials* $M = x_0^{i_0} \cdots x_n^{i_n}$ *with* $i_0 + \cdots + i_n = d$*. For* $(\alpha_0 :, \ldots, : \alpha_n) \in \mathbf{P}^n$*, define* $M(\alpha_0, \ldots, \alpha_n) = \alpha_0^{i_0} \cdots \alpha_n^{i_n}$*. Let* $M_1, \ldots, M_r$ *be the distinct monomials of degree* d*. Define a morphism* $\mathbf{P}^n \rightarrow \mathbf{P}^{r-1}$ *by*

$$(\alpha_0 :, \ldots, : \alpha_n) \mapsto (M_1(\alpha_0, \ldots, \alpha_n) :, \ldots, : M_r(\alpha_0, \ldots, \alpha_n)).$$

3. Varieties

We will call X a variety if it is an affine or a projective variety.

Suppose that X and Y are varieties. We define $X \times Y$ to be the set of pairs $\{(p, q)\}$ such that $p \in X$ and $q \in Y$. It can be shown that $X \times Y$ has a structure of a variety. If X and Y are projective then $X \times Y$ is projective. If X and Y are affine, then $X \times Y$ is affine.

The projection mappings $\pi_1 : X \times Y \rightarrow X$ defined by $\pi_1(p, q) = p$ and $\pi_2 : X \times Y \rightarrow Y$ defined by $\pi_2(p, q) = q$ are morphisms.

We now mention a couple of extremely important examples of sheaves on a variety X.

Suppose that $Z \subset X$ is a subvariety. We then define the ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$ to be the ideal of functions which vanish on Z.

We say that a sheaf $\mathcal{L}$ of $\mathcal{O}_X$ modules is invertible if there exists an affine cover $U_1, \ldots, U_r$ of X such that $\mathcal{L} \mid U_i$ is isomorphic as an $\mathcal{O}_{U_i}$ module to $\mathcal{O}_{U_i}$ for all i.

$\mathbf{P}^n$ has the invertible sheaf $\mathcal{O}_{\mathbf{P}^n}(1)$ defined by

$$\mathcal{O}_{\mathbf{P}^n}(1) \mid U_i = x_i(\mathcal{O}_{\mathbf{P}^n} \mid U_i)$$

for $0 \leq i \leq n$, where U_i is the open set $\mathbf{P}^n - V(x_i)$.

Example 3.1. *Suppose that* X *is nonsingular (all of the local rings* $\mathcal{O}_{X,q}$ *of* X *are regular local rings) and* $Z \subset X$ *is a codimension one subvariety of* X *(the dimension is one less than the dimension of* X*). Then for all* $q \in X$*,* $\mathcal{I}_{Z,q}$ *is a principal ideal in* $\mathcal{O}_{X,q}$*, since regular local rings are factorial. Thus there exists an affine cover* $\{U_1, \ldots, U_r\}$ *of* X *and* $f_i \in \mathcal{O}_X(U_i)$ *such that* $\mathcal{I}_Z \mid U_i = f_i \mathcal{O}_X(U_i)$*. We see that* $\mathcal{I}_Z \mid U_i \cong \mathcal{O}_{U_i}$ *for all* i*, so that* $\mathcal{I}_Z$ *is invertible.*

If $\mathcal{I}_Z$ is invertible, we will write $\mathcal{I}_Z^r = \mathcal{O}_X(-rZ)$ for all $r \geq 1$.

We conclude this section by defining an operation on sheaves. Suppose that $f : X \rightarrow Y$ is a morphism and $\mathcal{F}$ is a sheaf on X. $f_*\mathcal{F}$ is the sheaf on Y defined by $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$ for open subsets $U \subset Y$.

We sometimes will write $\Gamma(X, \mathcal{F}) = \mathcal{F}(X)$.

4. Projective Morphisms

Suppose that X and Y are varieties. A morphism $f : X \rightarrow Y$ is projective if there is a closed embedding $i : X \rightarrow Y \times \mathbf{P}^n$ for some n such that $f = \pi_1 \circ i$. Here $\pi_1 : Y \times \mathbf{P}^n \rightarrow Y$ is the first projection.

Any morphism $f : X \rightarrow Y$ of projective varieties is projective. We construct the closed embedding i as the composition

$$X \rightarrow Y \times X \rightarrow Y \times \mathbf{P}^n.$$

where the first map is $p \mapsto (f(p), p)$, and the second map is obtained from a closed embedding (possible since X is projective) $X \subset \mathbf{P}^n$.

Assume now that $Y = \mathrm{Spec}(A)$ is an affine variety, and $X \subset Y \times \mathbf{P}^n$ is closed. $Y \times \mathbf{P}^n = \mathrm{Proj}(A[x_0, \ldots, x_n])$. Then there exists a homogeneous prime ideal $I \subset$

$A[x_0, \dots, x_n]$ such that

$$X = V(I) = \mathrm{Proj}(A[x_0, \dots, x_n]/I).$$

Set $S = \sum_{i \geq 0} S_i = A[x_0, \dots, x_n]/I$, $S_0 = A/I \cap A$.

We see that $X \to \mathrm{Spec}(A)$ is projective if and only if $X = \mathrm{Proj}(S)$ for some graded ring S where S_0 is a quotient of A, and S is finitely generated by S_1 as an A algebra.

We will look a little closer at this case.

Let $\mathcal{L} = \mathcal{O}_{Y \times \mathbf{P}^n}(1) \otimes \mathcal{O}_X$, a very ample invertible sheaf on X (this is the definition of being very ample). Define $T = \oplus_{m \geq 0} \Gamma(X, \mathcal{L})$. We have a homomorphism

$$\Psi : A[x_0, \dots, x_n] = \oplus_{m \geq 0} \Gamma(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m)) \to T$$

defined by $\sigma \mapsto \sigma \mid X$. The image of Ψ is S. T is finite over S, and there exists an r_0 such that for $m \geq r_0$, $T_m = S_m$. Thus $\oplus_{m \geq 0} T_{mr_0} = \oplus_{m \geq 0} S_{mr_0}$.

We have the Veronese embedding

$$X \to \mathbf{P}^n \to \mathbf{P}^{\binom{n+r_0}{n}-1}.$$

Thus the closed embedding $X \to Y \times \mathbf{P}^{\binom{n+r_0}{n}-1}$ realizes $X \cong \mathrm{Proj}(\oplus_{m \geq 0} T_{mr_0})$.

Now assume Y is an arbitrary variety and $f : X \to Y$ is a projective morphism. We thus have a closed embedding $X \subset Y \times \mathbf{P}^n$. Let $\mathcal{L} = \mathcal{O}_{Y \times \mathbf{P}^n}(1) \otimes \mathcal{O}_X$. $\mathcal{A} = \oplus_{m \geq 0} f_* \mathcal{L}^m$ is a sheaf of algebras on Y. Cover Y be open affine subsets $U_1, \dots, U_t$. For $1 \leq i \leq t$,

$$\Gamma(U_i, \mathcal{A}) = \oplus_{m \geq 0} \Gamma(\pi_1^{-1}(U_i), \mathcal{L}^m).$$

There exists an r_0 such that $\oplus_{m \geq 0} \Gamma(\pi_1^{-1}(U_i), \mathcal{L}^{r_0 m})$ is generated in degree 1 and $\pi_1^{-1}(U_i) \cong \mathrm{Proj}(\oplus_{m \geq 0} \Gamma(\pi_1^{-1}(U_i), \mathcal{L}^{r_0 m})$ for all i. Set $\mathcal{A}^{(r_0)} = \oplus_{d \geq 0} f_* \mathcal{L}^{dr_0}$. $\pi_1^{-1}(U_i) \cong \mathrm{Proj}(\Gamma(U_i, \mathcal{A}^{(r_0)})$ for all i.

We have shown that all projective morphisms $f : X \to Y$ can be written as $X = \mathrm{Proj}(\mathcal{A})$ with projection $\mathrm{Proj}(\mathcal{A}) \to Y$ where $\mathcal{A}$ is a quotient of $\mathcal{O}_Y$, and $\mathcal{A}$ is locally finitely generated in degree 1 by $\mathcal{A}$ as an $\mathcal{O}_Y$ algebra.

We mention without proof a strengthening of this construction.

Theorem 4.1. *Suppose that $X \to Y$ is projective and birational (an isomorphism on a dense open set). Then there exists an ideal sheaf $\mathcal{I} \subset \mathcal{O}_Y$ such that $X \cong$ Proj$(\oplus_{m \geq 0} \mathcal{I}^m)$. We say that $X = B(\mathcal{I})$ is the blow up of $\mathcal{I}$.*

We also give a useful application.

Theorem 4.2. *(Weak resolution of indeterminacy) Suppose that $\phi : X \to Y$ is a rational map of projective varieties (there exists a dense open subset of X on which ϕ is a morphism). Then there exists an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ and a commutative diagram*

$$\begin{array}{ccc} B(\mathcal{I}) & f & \\ \pi \downarrow & \searrow & \\ X & \xrightarrow{\phi} & Y \end{array}$$

where f is a morphism.

Proof. Let U be the largest open subset of X on which ϕ is a morphism. The graph of ϕ, $\Gamma \subset X \times Y$ is the Zariski closure of the image of the morphisms $U \to X \times Y$ which is defined by $p \to (p, f(p))$. $\Gamma \to X$ is a birational morphism of projective varieties. Thus $\Gamma = B(\mathcal{I})$ for some ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. $\square$

5. Monoidal transforms

Suppose that Y is a nonsingular variety, $Z \subset Y$ is a nonsingular subvariety. Let $\pi : X = B(\mathcal{I}_Z) = B(Z) \rightarrow Y$ be the blow up of Z (of $\mathcal{I}_Z$). We say that π is the monoidal transform of Y with center Z.

Let $p \in Z$ be a point. There exist regular parameters $x_1, \dots, x_n \in \mathcal{O}_{Y,p}$ and $r \le n$ such that $x_1 = \cdots = x_r = 0$ are local equations of Z in an affine neighborhood U of p in Y. Let

$$I = (x_1, \dots, x_r) = \Gamma(U, \mathcal{I}_Z) \subset A = \Gamma(U, \mathcal{O}_Y).$$

Since X is the blow up of $\mathcal{I}_Z$,

$$\begin{aligned} \pi^{-1}(U) &= \mathrm{Proj}(\oplus_{n \ge 0} I^n) \\ &= \mathrm{Proj}(A[It]) \text{ where } t \text{ is an indeterminate, the grading is by } \deg t = 1 \\ &= \cup_{i=1}^r \mathrm{Spec}(A[tI]_{(tx_i)}), \text{ take elements of degree 0 in localization by } tx_i \\ &= \cup_{i=1}^r \mathrm{Spec}(A[\tfrac{x_1}{x_i}, \dots, \tfrac{x_r}{x_i}]). \end{aligned}$$

Let us look more closely at a particular open set $\mathrm{Spec}(A[\frac{x_1}{x_i}, \dots, \frac{x_r}{x_i}])$ in this affine cover of $\pi^{-1}(U)$. We may as well assume that $i = 1$.

If $q \in \mathrm{Spec}(A[\frac{x_2}{x_1}, \dots, \frac{x_r}{x_1}])$, and $\pi(q) = p$, then $m_q \cap A = m_p = (x_1, \dots, x_r)$. Let $\overline{m} = A/m_p[\frac{x_2}{x_1}, \dots, \frac{x_r}{x_1}]$. We have

$$A[\frac{x_2}{x_1}, \dots, \frac{x_r}{x_1}]/m_q \cong (A/m_p[\frac{x_2}{x_1}, \dots, \frac{x_r}{x_1}])/\overline{m} \cong k[\frac{x_2}{x_1}, \dots, \frac{x_r}{x_1}]/\overline{m}.$$

$k[\frac{x_2}{x_1}, \dots, \frac{x_r}{x_1}]$ is a polynomial ring over k (since $\mathcal{O}_{Y,p}$ is a regular local ring, $x_1, \dots, x_r$ is a regular sequence in $\mathcal{O}_{Y,p}$). Thus there exist $\alpha_1, \dots, \alpha_r \in k$ such that $\overline{m} = (\frac{x_2}{x_1} - \alpha_2, \dots, \frac{x_r}{x_1} - \alpha_r)$, so that $m_q = (x_1, \dots, x_n, \frac{x_2}{x_1} - \alpha_2, \dots, \frac{x_r}{x_1} - \alpha_r)$. Since $x_i = \frac{x_i}{x_1} x_1$ for $1 \le i \le r$, we have that

$$m_q = (x_1, \frac{x_2}{x_1} - \alpha_2, \dots, \frac{x_r}{x_1} - \alpha_r, x_{r+1}, \dots, x_n).$$

We have

$$n = \dim\, \mathcal{O}_{X,q} = \dim_k m_q/m_q^2 \le n.$$

Thus $\dim_k m_q/m_q^2 = n$, and q is a nonsingular point on Y. We conclude that Y is nonsingular.

Set

$$y_1 = x_1, y_2 = \frac{x_2}{x_1} - \alpha_2, \dots, y_r = \frac{x_r}{x_1} - \alpha_r, y_{r+1} = x_{r+1}, \dots, y_n = x_n.$$

For $2 \le i \le r$, we have $y_1 y_i = x_i - \alpha_i y_1$. Thus $x_i = y_1(y_i + \alpha_i)$. The point q can thus be described by the equations

$$x_1 = y_1, x_2 = y_1(y_2 + \alpha_2), \dots, x_r = y_1(y_r + \alpha_r), x_{r+1} = y_{r+1}, \dots, x_n = y_n.$$

We have since $x_i = y_1(y_i + \alpha_i)$ for $1 \le i \le r$, that

$$\mathcal{I}_Z \mathcal{O}_{X,q} = (x_1, \dots, x_r)\mathcal{O}_{X,q} = y_1 \mathcal{O}_{X,q}.$$

Thus $y_1 = 0$ is a local equation of $\pi^{-1}(Z)$ at q. We thus have that

(1) $\mathcal{I}_Z \mathcal{O}_X$ is an invertible ideal sheaf.
(2) $\pi^{-1}(Z)$ is a nonsingular codimension 1 subvariety of X, the "exceptional divisor" of π.

Example 5.1. *We blow up the origin p in $\mathbf{A}^2$. $\mathbf{A}^2 = \mathit{Spec}(k[x,y])$. Let the blow up be $\pi : X = B(p) \rightarrow \mathbf{A}^2$. We write $X = U_1 \cup U_2$ where $U_1 = \mathit{Spec}(k[x, \frac{y}{x}]) \cong \mathbf{A}^2$, and $U_2 = \mathit{Spec}(k[y, \frac{y}{x}]) \cong \mathbf{A}^2$. $\pi \mid U_1$ is the morphism $\mathbf{A}^2 \rightarrow \mathbf{A}^2$ defined by $(\alpha, \beta) \mapsto (\alpha, \alpha\beta)$ and $\pi \mid U_2$ is the morphism $\mathbf{A}^2 \rightarrow \mathbf{A}^2$ defined by $(\gamma, \delta) \mapsto (\gamma\delta, \gamma)$.*

The exceptional divisor $E = \pi^{-1}(p)$ is defined by $E \cap U_1 = V(\alpha)$, $E \cap U_2 = V(\gamma)$, from which it follows that $E \cong \mathbf{P}^1$.

We now define the strict transform. Suppose that $\pi : X_1 = B(Z) \to X$ is the monoidal transform obtained by blowing up a nonsingular subvariety Z of X. Suppose that $Y \subset X$ is a subvariety. The strict transform Y_1 of Y on X_1 is the subvariety of X_1 which is the Zariski closure of $\pi^{-1}(Y - Z)$ in X_1.

The strict transform can be calculated by the following formula.

Lemma 5.2. *For $q \in X_1$,*

$$\begin{aligned} \mathcal{I}_{Y_1,q} &= \cup_{n\ge 0}(\mathcal{I}_Y\mathcal{O}_{X_1,q} : \mathcal{I}_Z^n\mathcal{O}_{X_1,q}) \\ &= \{f \in \mathcal{O}_{X_1,q} \mid f\mathcal{I}_Z^n \subset \mathcal{I}_Y\mathcal{O}_{X_1,q}\}. \end{aligned}$$

Example 5.3. *Suppose that $C = V(f(x,y)) \subset \mathbf{A}^2 = Spec(k[x,y])$ is an irreducible curve (a one dimensional variety). Let p be the origin in $\mathbf{A}^2$. Let $\pi : X = B(p) \to \mathbf{A}^2$ be the blow up of p, with exceptional divisor E. As shown in the previous example, we have a cover $X = U_1 \cup U_2$. We will describe the strict transform $\overline{C}$ of C on the open set U_1. There is a similar description on U_2.*

$U_1 \cong \mathbf{A}^2$, and U_1 has coordinates x_1, y_1 such that $U_1 \to \mathbf{A}^2$ is given by $x = x_1, y = x_1y_1$. $E \cap U_1 = V(x_1)$. Let

$$r = \mathrm{ord}(f) = \min\{n \mid f \in m_p^n\}.$$

We can write

$$f = \sum_{i+j\ge r} a_{ij}x^iy^j,$$

where $a_{ij} \in k$ for all i, j, and $a_{ij} \neq 0$ for some i, j with $i + j = r$. Substituting the equations $x = x_1, y = x_1y_1$, we obtain

$$f = \sum_{i+j\ge r} a_{ij}x_1^{i+j}y_1^j = x_1^rf_1,$$

where $f_1 = \sum a_{ij}x_1^{i+j-r}y_1^j$ is irreducible in $k[x_1, y_1]$.

$$\pi^{-1}(C) \cap U_1 = V(x_1^rf_1) = V(x_1) \cup V(f_1) = (E \cap U_1) \cup V(f_1).$$

We see that $\overline{C} \cap U_1 = V(f_1)$, so that $f_1 = 0$ is a local equation of the strict transform of C on U_1.

In fact, we have that

$$\mathcal{I}_C\mathcal{O}_X = \mathcal{I}_{\overline{C}}\mathcal{I}_E^r = \mathcal{I}_{\overline{C}}\mathcal{O}_X(-rE).$$

The computations of this example generalize to arbitary monoidal transforms. Suppose that $\pi : B(Z) \to X$ is the blow up of a nonsingular subvariety Z of X and $Y \subset X$ is a codimension one subvariety of X containing Z.

Suppose that $q \in B(Z)$, and $p = \pi(q) \in Z$. There exist regular parameters $y_1, \ldots, y_n$ at q, regular parameters $x_1, \ldots, x_n$ at p and $r \in \mathbf{N}$ such that $x_1 = \cdots = x_r = 0$ are local equations of Z at p, and

$$x_1 = y_1, x_2 = y_1y_2, \ldots, x_r = y_1y_r, x_{r+1} = y_{r+1}, \ldots, x_n = y_n.$$

Let $f = 0$ be a local equation of Y at p. We have a commutative diagram of inclusions

$$\begin{array}{lcl} \mathcal{O}_{X,p} & \to & \hat{\mathcal{O}}_{X,p} = k[[x_1, \ldots, x_n]] \\ \downarrow & & \downarrow \\ \mathcal{O}_{B(Z),q} & \to & \hat{\mathcal{O}}_{B(Z),q} = k[[y_1, \ldots, y_n]]. \end{array}$$

Suppose that Y has order s along Z ($f \in \mathcal{I}_{Z,p}^s$, $f \notin \mathcal{I}_{Z,p}^{s+1}$). We can then write, using the fact that $f(x_1, \dots, x_n) = f(y_1, y_1y_2, \dots, y_1y_r, y_{r+1}, \dots, y_n)$,

$$f = \sum_{i_1+\cdots+i_r \geq s} a_{i_1,\dots,i_r}(x_{r+1}, \dots, x_n)x_1^{i_1}x_2^{i_2}\cdots x_r^{i_r} = y_1^s f_1$$

where

$$f_1 = \sum_{i_1+\cdots+i_r \geq s} a_{i_1,\dots,i_r}(y_{i+1}, \dots, y_n)y_1^{i_1+\cdots+i_r-s}y_2^{i_2}\cdots y_r^{i_r}.$$

f_1 is a local equation of the strict transform $\overline{Y}$ of Y on $B(Z)$ at q, and $y_1 = 0$ is a local equation of the exceptional divisor E at q. We have

$$\mathcal{I}_Y\mathcal{O}_{B(Z)} = \mathcal{O}_{B(Z)}(-rE)\mathcal{I}_{\overline{Y}}.$$

We conclude this section with one more definition.

Suppose that $\pi : X_1 = B(Z) \rightarrow X$ is a monoidal transform obtained by blowing up a nonsingular subvariety Z, $Y \subset X$ is a subvariety, and suppose that Y has order r along Z ($\mathcal{I}_Y \subset \mathcal{I}_Z^r$ and $\mathcal{I}_Y \not\subset \mathcal{I}_Z^{r+1}$). Let $E = \pi^{-1}(Z)$ be the exceptional divisor. There exists an ideal sheaf $\mathcal{I} \subset \mathcal{O}_{X_1}$ such that $\mathcal{O}_{X_1}(-rE)\mathcal{I} = \mathcal{I}_Y\mathcal{O}_{X_1}$. The "scheme" $\tilde{Y}_1 = V(\mathcal{I} \subset X_1$ is the weak transform of Y.

The strict transform $\overline{Y}_1$ of Y is always contained in $\tilde{Y}_1$, and they are equal if and only if $\tilde{Y}_1$ is a variety.

In the important case where Y has codimension one in X, the weak transform and strict transform of Y are equal, as we computed above.

6. Monomialization and toroidalization

An important problem is to find a factorization of an arbitrary morphism by simple, well understood morphisms.

We have encountered one simple type of morphism, the monoidal transforms. We now define another type.

Definition 6.1. *Suppose that $\Phi : X \rightarrow Y$ is a dominant morphism of nonsingular integral finite type k schemes. Φ is* **monomial** *if for every $p \in X$ there exist regular parameters $(y_1, \dots, y_m)$ in $\mathcal{O}_{Y,\Phi(p)}$, and an étale cover U of an affine neighborhood of p, uniformizing parameters $(x_1, \dots, x_n)$ on U and a matrix a_{ij} such that*

$$\begin{array}{rl} y_1 = & x_1^{a_{11}}\cdots x_n^{a_{1n}} \\ & \vdots \\ y_m = & x_1^{a_{m1}}\cdots x_n^{a_{mn}} \end{array}$$

Since Φ is dominant (the image of Φ contains a dense open set), the matrix (a_{ij}) must have maximal rank m.

This concept generalizes to the notion of a toroidal morphism.

Definition 6.2. *Suppose that $\Phi : X \rightarrow Y$ is a dominant morphism of k-varieties. A morphism $\Psi : X_1 \rightarrow Y_1$ is a* **monomialization** *of Φ if there are sequences of monoidal transforms $\alpha : X_1 \rightarrow X$ and $\beta : Y_1 \rightarrow Y$, and a morphism $\Psi : X_1 \rightarrow Y_1$ such that the diagram*

$$\begin{array}{ccc} X_1 & \stackrel{\Psi}{\rightarrow} & Y_1 \\ \downarrow & & \downarrow \\ X & \stackrel{\Phi}{\rightarrow} & Y \end{array}$$

commutes, and Ψ is a monomial morphism.

This definition generalizes to the concept of a toroidalization.

Monomialization and toroidalization can be deduced for arbitary morphisms to a curve (the case when Y has dimension 1) from embedded resolution of hypersurface singularities. The case when X is a surface is also known, and is not so difficult to work out.

In our papers "Monomialization of morphisms from 3-folds to surfaces" (SLN 1786, 2002, Springer-Verlag), and "Toroidalization of dominant morphisms of 3-folds" (to appear in Memoirs of the AMS), we prove the following theorem.

Theorem 6.3. *Suppose that $\Phi : X \rightarrow Y$ is a dominant morphism from a 3 fold X to a nonsingular variety Y (over an algebraically closed field k of characteristic zero). Then there exist sequences of blow ups of nonsingular subvarieties $X_1 \rightarrow X$ and $Y_1 \rightarrow Y$ such that the induced map $\Phi_1 : X_1 \rightarrow S_1$ is a monomial morphism. That is, morphisms from a 3-fold can be monomialized.*

We also prove that it is possible to toroidalize such morphisms.

The general cases of monomialization and toroidalization are still open in higher dimension.