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Introduction to Singular Points

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Introduction to Singular Points

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In these notes we shall give the definition of a singular point on an algebraic set and introduce the concept of desingularization after considering in some detail the case of complex plane curves.

1 Algebraic singular points

1.1 Algebraic sets

Let $\mathbb{K}$ be a field (commutative). We shall call it the base field (or also the ground field).

An **$\mathbb{K}$-algebraic set defined by the set of polynomials $(P_i)_{i\in I}$ of $\mathbb{K}[X_1,\ldots,X_n]$,** is the subset of the affine space $\mathbb{K}^n$ of the points $(x_1,\ldots,x_n)$, such that $P_i(x_1,\ldots,x_n)=0$, for $i\in I$.

When the base field is clear, we speak of algebraic set instead of $\mathbb{K}$-algebraic set. The polynomials P_i, with $i\in I$, are also called the **equations** of the algebraic set. We also write these equations $P_i=0$, $i\in I$.

Now notice that the set of polynomials of $\mathbb{K}[X_1,\ldots,X_n]$ which vanish on an algebraic subset E of $\mathbb{K}^n$ is an ideal $I(E)$ of the ring of polynomials $\mathbb{K}[X_1,\ldots,X_n]$. Hilbert finiteness theorem tells us that

Theorem 1 *The ideals of $\mathbb{K}[X_1,\ldots,X_n]$ are finitely generated.*

We express this property by saying that the ring of polynomials $\mathbb{K}[X_1,\ldots,X_n]$ is noetherian.

In particular an algebraic subset E of $\mathbb{K}^n$ is defined by a finite number of equations, since it is defined by a set of generators of the ideal $I(E)$.

The first obvious fact is that algebraic subsets of $\mathbb{K}^n$ are very particular subsets of $\mathbb{K}^n$. For instance, when $n=1$, the algebraic subsets of $\mathbb{K}$ are the finite subsets.

However, the study of algebraic sets may be in general complicated.

In this section we shall observe that the points of algebraic sets are of two types:

- **regular points** or **non-singular points**;
- **singular points**.

1.2 An exemple: Complex Hypersurfaces

Let assume that the base field is the field of complex numbers $\mathbb{C}$. We shall consider the case of algebraic sets defined by a non-constant polynomial.

Definition 1 *A $\mathbb{C}$-algebraic set defined by one non-constant polynomial is called a* **complex hypersurface**.

Let $x \in X$ be a point of a complex hypersurface $X \subset \mathbb{C}^n$. Let P be an equation of X. The differential of P at x is a linear form from $\mathbb{C}^n$ into $\mathbb{C}$. Suppose that this differential $dP(x)$ of P at x is a non-zero linear form of $\mathbb{C}^n$. Then, implicit function theorem implies that there is an open neighbourhood U of x in $\mathbb{C}^n$, such that $X \cap U$ is a complex analytic submanifold of U. This implies that at x, the hypersurface X has a tangent space defined by

$$dP(x) = 0,$$

i.e.

$$\sum_1^n \partial P/\partial X_j(x) X_j = 0.$$

In fact the linear form $dP(x)$ is $\neq 0$ if and only if there is j, $1 \leq j \leq n$, such that $\partial P/\partial X_j(x) \neq 0$. Therefore, the points of X where the differential of P vanishes is an algebraic subset of X.

Example: Consider a linear form ℓ of $\mathbb{C}^n$. It is the equation of a complex hyperplane H of $\mathbb{C}^n$. It is easy to show that at every point x of H, $d\ell(x) \neq 0$.
Now the same set H is defined by the equation $\varphi = \ell^2 = 0$, in which case at every point $x \in H$, we have $d\varphi(x) = 0$.
So the fact that the differential of the equation vanishes at a point x of X depends on the equation.

Let us recall Hilbert Nullstellensatz:

Theorem 2 *Let $\mathbb{K}$ be an algebraically closed field. Let E be a $\mathbb{K}$-algebraic set defined by the set of polynomials $(P_i)_{i \in I}$ of $\mathbb{K}[X_1, \ldots, X_n]$, the ideal $I(E)$ of polynomials in $\mathbb{K}[X_1, \ldots, X_n]$ which vanish on E is the radical ideal of the ideal generated by $(P_i)_{i \in I}$.*

Recall that the radical ideal of an ideal $\mathfrak{A}$ of a ring A is the ideal $\mathcal{R}(\mathfrak{A})$ of elements which have a power in $\mathfrak{A}$.

Since $\mathbb{C}$ is an algebraically closed field, we can apply Hilbert Nullstellensatz. It shows that if a hypersurface X is defined by an equation P, it is also defined by the reduced polynomial P_0 defined by P, because, if $P = Q_1^{a_1} \dots Q_r^{a_r}$ is the decomposition of P into irreducible components, the radical of the principal ideal (P) is generated by the associated reduced polynomial $P_0 = Q_1 \dots Q_r$.

Let X be a complex hypersurface defined by a reduced polynomial P_0.

Definition 2 *A point x of the hypersurface X is called a* **non-singular point** *of X (or a* **regular point***) of X, if $dP_0(x) \neq 0$. A point of X is called a* **singular point** *of X if $dP_0(x) = 0$.*

As noticed above, singular points of X define an algebraic subset of X defined by the equations

$$P_0 = \partial P_0/\partial X_1 = \dots = \partial P_0/\partial X_n = 0.$$

1.3 Definition of complex singular points

More generally, let E be a complex algebraic subset of $\mathbb{C}^n$. Let $I(E)$ be the ideal of all polynomials in $\mathbb{C}[X_1, \dots, X_n]$ which vanish on E.

Hilbert finiteness theorem tells that the ideal $I(E)$ is finitely generated

$$I(E) = (P_1, \dots, P_k)$$

Let us fix a system of generators of $I(E)$ and consider the Jacobian matrix $J(x)$:

$$\begin{pmatrix} \partial P_1/\partial X_1(x), & \dots & , \partial P_1/\partial X_n(x) \\ & \dots & \\ \partial P_k/\partial X_1(x), & \dots & , \partial P_k/\partial X_1(x) \end{pmatrix}$$

Denote $\rho(x)$ the rank of this matrix at $x \in E$.

Denote $\rho_E := \max_{x \in E} \rho(x)$.

In [W], H. Whitney proved:

Theorem 3 *The subset E^0 of points x of E where $\rho(x) = \rho_E$ is a complex analytic manifold of dimension $n - \rho_E$. The subset of $E_1 := E \setminus E_0$ of E is a proper algebraic subset of E.*

Example: Consider the complex algebraic subset V of $\mathbb{C}^3$ defined by

$$XY = XZ = 0.$$

One can check that V is the union of the plane $X = 0$ and of the line $Y = Z = 0$. The Jacobian matrix $J(x)$ is

$$\begin{pmatrix} Y(x) & X(x) & 0 \\ Z(x) & 0 & X(x) \end{pmatrix}$$

So, $\rho_V = 2$. In this case V_1 is the plane $X = 0$.

This example leads us to recall that an algebraic set E is the finite union of irreducible subsets $E(i)$, $1 \leq i \leq r$, such that for $i \neq j$, $E(i) \not\subset E(j)$. These subsets are called the (irreducible) **components** of E.

Definition 3 *An algebraic set E is irreducible if whenever E is the union of two algebraic subsets E_1 and E_2, then one of these is E itself.*

One can easily prove the following lemma:

Lemma 4 *Let $\mathbb{K}$ be a field. A $\mathbb{K}$-algebraic set is irreducible if and only if the ideal $I(E)$ is prime.*

Proof: Suppose that the algebraic subset E of $\mathbb{K}^n$ is the union $E_1 \cup E_1$ of two algebraic subsets of $\mathbb{K}^n$ both different from E. Since $E_1 \neq E_2$, we have $I(E_1) \neq I(E_2)$. Moreover $E_1 \not\subset E_2$, so there is $f_2 \in I(E_2)$ such that $f_2 \notin I(E_1)$. Similarly as $E_2 \not\subset E_1$, there is $f_1 \in I(E_1)$ and $f_1 \notin I(E_2)$. The product $f_1 f_2$ vanishes on $E_1 \cup E_2$, so $f_1 f_2 \in I(E)$, but $f_1 \notin I(E_2)$ and $I(E) \subset I(E_2)$, so $f_1 \notin I(E)$. Similarly $f_2 \notin I(E)$, so $I(E)$ is not prime.
Conversely, suppose that $I(E)$ is not prime. There is f and g in $\mathbb{K}[X_1, \ldots, X_n]$, such that $fg \in I(E)$, but $f \notin I(E)$ and $g \notin I(E)$. Let

$$E_1 := E \cap \{f(x) = 0\}$$

and

$$E_2 := E \cap \{g(x) = 0\}.$$

We have $E \neq E_1$ and $E \neq E_2$, however $E \subset \{f = 0\} \cup \{g = 0\} = \{fg = 0\}$, so

$$E_1 \cup E_2 = (E \cap \{f(x) = 0\}) \cup (E \cap \{g(x) = 0\}) = E \cap (\{f(x) = 0\} \cup \{g(x) = 0\}) = E.$$

Therefore, E is not irreducible.

A way to obtain the irreducible components of an algebraic subset of $\mathbb{K}^n$ is to look for the prime ideals of $\mathbb{K}[X_1, \ldots, X_n]$ minimal among those which contain $I(E)$.

A singular point of an irreducible complex algebraic set E is a point $x \in E$ where $\rho(x) \neq \rho_E$. So the set of singular points, defined by the vanishing of the minors of a matrix with algebraic entries, is an algebraic subset of E. One can prove that non-singular points of an irreducible complex algebraic set E is connected.

A singular point of a complex algebraic set

$$E = \cup_1^r E(i)$$

is a point $x \in E$ where either there is i, such that $x \in E(i)$ and $\rho(x) \neq \rho_{E(i)}$, or x belongs to two distinct irreducible components $E(i)$ and $E(j)$ $(i \neq j)$ of E.

A non-singular point (or a regular point) of E is a point which is not singular. One can observe that a point $x \in E \subset \mathbb{C}^n$ is non-singular if it has an open neighbourhood U in E such that U is a complex submanifold of $\mathbb{C}^n$. In particular at a non-singular point x, one can define the tangent space $T_{E,x}$ to E.

Consider the $\mathbb{C}$-algebra

$$A[E] := \mathbb{C}[X_1, \ldots, X_n]/I(E)$$

quotient of the $\mathbb{C}$-algebra of complex polynomials in n variables by the ideal $I(E)$ generated by all the polynomials of $\mathbb{C}[X_1, \ldots, X_n]$ which vanish on E.

The local ring $\mathcal{O}_{E,x}$ of E at x is the **localization** of $A[E]$ at the maximal ideal generated by $X_1 - a_1, \ldots, X_n - a_n$.

Recall that for a ring A and a multiplicative set $S \subset A$, i.e. a subset of $\mathcal{O}$ such that $1 \in S$ and, for $f, g \in S$, $fg \in S$, one can define the fraction ring $A[S^{-1}]$ with a map $j_S : A \to A[S^{-1}]$, such that, for any ring homomorphism $h : A \to B$, such that for any $s \in S$, $h(s)$ is invertible in B, the is a unique homorphism $\tilde{h} : A[S^{-1}] \to B$ such that $h = \tilde{h} \circ j_S$. We call $A[S^{-1}]$ the fraction ring of A with denominators in S.

Since the complement of a prime ideal P of a ring A is mutliplicative set, one can define $A[(A \setminus P)^{-1}]$. This fraction ring is in fact a local ring, i.e. it only has a maximal ideal, and its unique maximal ideal is generated by the image of P in $A[A \setminus P]$. In this case the fraction rings is also denoted $A_P := A[A \setminus P]$ and is also called the localization of A at the prime ideal P. The local ring $\mathcal{O}_{E,x}$ considered above is the localization of $A[X]$ at the maximal ideal of the point x in E, i.e. the one generated by $X_1 - a_1, \ldots, X_n - a_n$ in $A[E]$. The maximal ideal $\mathfrak{M}_{E,x}$ is the ideal generated by the image of the maximal ideal of x in E.

Whenever the multiplicative set S contains a zero divisor the corresponding ring of fractions is the trivial ring $\{0\}$. In the case of a domain of integrity, the ring of fractions with denominators in the multiplicative set S is the subring of the field of fractions of the integral domain of fractions having effectively their denominators in S.

Since the ring $A[E]$ can be identified with the ring of functions on E which are restrictions to E of a polynomial function (in this case of complex algebraic sets, this is a consequence of the fact that the field $\mathbb{C}$ has an infinite number of elements), when E is irreducible, the local ring $\mathcal{O}_{E,x}$ can be identified can be identified with the restrictions to E of rational functions the denominators of which do not vanish at x.

Now, notice that at a non-singular point $x \in E$ there is a natural complex linear map $d : \mathcal{O}_{E,x} \to T^*_{E,x}$ into the cotangent space of E at x which is defined by the differential of a function. This map induces an isomorphism of $\mathfrak{M}_{E,x}/\mathfrak{M}^2_{E,x}$ onto $T^*_{E,x}$. This observation will lead to an algebraic definition of non-singular points.

In the preceding example the only singular point of V is the origin of $\mathbb{C}^3$.

2 Algebraic characterizations of a singular point

2.1 The Multiplicity

In the case of a complex hypersurface, by definition one finds the singular points knowing the reduced equation: Let P_0 be a reduced equation of the hypersurface X, the point $x \in X$ is singular if and only if $dP_0(x) = 0$.

A way to check it is to consider the Taylor expansion of P_0 at the point $x := (a_1, \dots, a_n)$:

$$\begin{aligned} P_0(X_1, \dots, X_n) \quad &= P_0(x) + \textstyle\sum_{i=1}^n \partial P_0/\partial X_i(a_1, \dots, a_n)(X_i - a_i) \\ &+ \text{ terms in } X_1 - a_1, .., X_n - a_n \text{ of degree } \geq 2. \end{aligned}$$

The lowest degree of the non-zero non-constant terms in this Taylor expansion is called the **multiplicity** $m_{X,x}$ of X at x. The point $x := (a_1, \dots, a_n)$ is a singular point of the complex hypersurface X if and only if the multiplicity $m_{X,x} \geq 2$.

There is an algebraic way to compute the multiplicity.

Consider the following function on $\mathbb{N}$:

$$\forall \nu \in \mathbb{N}, \quad F(\nu) := \dim_{\mathbb{C}} \mathcal{O}_{X,x}/\mathfrak{M}_{X,x}^{\nu+1},$$

where $\mathfrak{M}_{X,x}$ is the maximal ideal of $\mathcal{O}_{X,x}$. This function F is called the **Samuel function** of $\mathcal{O}_{X,x}$.

Exercise: For $\nu \gg 0$, the function F is a polynomial in ν of degree $n-1$ and the term of highest degree is

$$\frac{m_{X,x}}{(n-1)!}\nu^{n-1}.$$

The corresponding polynomial is also called the **Hilbert-Samuel polynomial** of $\mathcal{O}_{X,x}$.

There is also a geometric way to understand the multiplicity of a complex hypersurface:

Let ℓ a "general" line through x. The line ℓ intersects X at x and the point x is isolated in the intersection $\ell \cap X$. Consider an open neighbourhood U such that

$$\ell \cap X \cap U = \{x\}$$

For "general" lines ℓ_t parallel to ℓ, ℓ_t intersects X transversally and the number of points of $\ell_t \cap X \cap U$ is the multiplicity $m_{X,x}$.

We can specify what we mean by "general" line. Let us consider again the Taylor expansion of the reduced equation P_0 of X at x:

$$P_0 = P_0(x) + (P_0)_m + (P_0)_{m+1} + \dots$$

where m is the multiplicity of X at x and $(P_0)_k$ is the homogeneous polynomial of degree k in the Taylor expansion of P_0. Then we choose a line not contained in the cone define by $(P_0)_m = 0$.

Consider the parallel line ℓ through x. It intersects X only at x locally at x, i.e. there is an open neighbourhood U of x such that $\ell \cap X \cap U = \{x\}$. For almost all lines ℓ_t, parallel to ℓ, not through x but near enough to ℓ, we have that ℓ_t intersects X transversally X in U and the number of points $U \cap X \cap \ell_t$ is $m_{X,x}$. The choice of ℓ_t is determined by a local discriminant. Namely, we choose local coodinates $(Z_1, \ldots, Z_n)$ at x in such a way ℓ is defined by $Z_1 = \ldots = Z_{n-1} = 0$, then, the Preparation theorem of Weierstrass tells that in a neighbourhood V of x in $\mathbb{C}^n$, $X \cap V$ is defined by

$$Z^r + \sum_1^r a_{r-i}(Z_1, \ldots, Z_{n-1}) Z^i = 0$$

where $a_{r-i}(Z_1, \ldots, Z_{n-1}) \in \mathbb{C}\{Z_1, \ldots, Z_{n-1}\}$, $1 \leq i \leq n$. The discriminant of this degree r polynomial is $\Delta(a_1, \ldots, a_n)$. Choosing $z_1, \ldots, z_{n-1}$ such that

$$\Delta(a_1(z_1, \ldots, z_{n-1}), \ldots, a_n(z_1, \ldots, z_{n-1})) \neq 0$$

and sufficiently near to 0, one can choose ℓ_t as $Z_1 - z_1 = \ldots = Z_{n-1} - z_{n-1} = 0$. In particular it shows that $r = m_{X,x}$.

The cone $(P_0)_m = 0$ is called the **tangent cone** of X and x and the homogeneous polynomial $(P_0)_m$ is called the **initial form** of P_0 at x.

Let E be a complex algebraic subset of $\mathbb{C}^n$. Let $E(1), \ldots, E(r)$ be the irreducible components of E. We define

$$\dim_x E := \max_{\{i, x \in E(i)\}} \{n - \rho_{E(i)}\}.$$

At a point $x \in E$ one consider $\mathcal{O}_{E,x}$ the localization at x of the complex algebra

$$A[E] := \mathbb{C}[X_1, \ldots, X_n]/I(E).$$

We have the following results

Theorem 5 *For $\nu \gg 0$, the function defined by*

$$\forall \nu \in \mathbb{N}, \quad F(\nu) := \dim_{\mathbb{C}} \mathcal{O}_{E,x}/\mathfrak{M}_{E,x}^{\nu+1},$$

is a polynomial of ν of degree $\dim_x E$.

For $\nu \gg 0$, the coefficient of the term of degree $\dim_x E$ is $e/(\dim_x E)!$

Definition 4 *The multiplicity $m_{E,x}$ of E at x is the number e.*

We have

Theorem 6 *The point $x \in E$ is singular if and only if $m_{E,x} \geq 2$.*

As above we have the following interpretation of the multiplicity for complex irreducible algebraic sets (see R. Draper, Math. Ann. 180 (1969)):

Let ℓ a "general" affine subspace of $\mathbb{C}^n$ of dimension ρ_E through x. The affine space ℓ intersects E at the isolated point x. Consider an open neighbourhood U such that

$$\ell \cap E \cap U = \{x\}$$

For "general" affine spaces ℓ_t parallel to ℓ, ℓ_t intersects E transversally and the number of points of $\ell_t \cap E \cap U$ is the multiplicity $m_{E,x}$.

In this case, the characterisation of a general affine subspace, although similar to the hypersurface case, is a bit more complex. Let $d := \dim E = n - \rho_E$ be the dimension of E and $x = (a_1, \ldots, a_n)$. The tangent cone of E at x is the subspace of $\mathbb{C}^n$ defined by all the initial forms of the elements $P \in I(E)$. The affine subspace ℓ through x has to intersect the translate at x of tangent cone of E at x only at the point x.

Now choose such a subspace ℓ. The linear projection of E into $\mathbb{C}^d$ parallel to ℓ is locally finite at the point x. To prove this, we need to use the viewpoint of analytic geometry. Let us call p_ℓ the projection parallel to ℓ. Since ℓ intersects the translate of the tangent cone of X at x only at x, one can prove that x is isolated in the intersection $\ell \cap X$. The analytic Hilbert Nullstellensatz (due to Rückert) tells us that x is isolated in the intersection $\ell \cap X$ if and only if the analytic local ring $\mathcal{O}^{an}_{E,x}/\ell^*(\mathfrak{M}_{\mathbb{C}^d,\ell(x)})$, quotient of

$$\mathcal{O}^{an}_{E,x} := \mathbb{C}\{X_1 - a_1, \ldots, X_n - a_n\}/I(E)\mathbb{C}\{X_1 - a_1, \ldots, X_n - a_n\}$$

by the ideal generated by the analytic functions $\varphi \circ p_\ell$, with φ in the maximal ideal $\mathfrak{M}_{\mathbb{C}^d,\ell(x)}$ of the local ring of $\mathbb{C}^d$ at $\ell(x)$, is an artinian ring, i.e. it is a $\mathbb{C}$-vector space of finite dimension. Now, the geometric version of Weierstrass Preparation theorem (see [Ho]) implies that locally at x the restriction of p_ℓ to X induces a map p of an open neighbourhood U of x in E onto an open neighbourhood V of $p_\ell(x)$ in $\mathbb{C}^d$ which is proper with finite fibers. For such a map one can define a discriminant, namely the image Δ by p of subspace Γ of $E \cap U$ of points which are either singular in E or where p has not rank d. For $y \notin \Delta$ sufficiently near to $p_\ell(x)$, the fiber $p_\ell(y)$ can be chosen as general space ℓ_t.

2.2 Regular Local Rings

There are other algebraic characterizations of non-singular points:

Theorem 7 *A point $x \in E$ is non-singular if and only if the $\mathfrak{M}_{E,x}$-completion of the local ring $\mathcal{O}_{E,x}$ is isomorphic to the $\mathbb{C}$-algebra of formal series $\mathbb{C}[[X_1, \ldots, X_d]]$, where $d = \dim_x E$.*

In fact there is also a version in complex analytic geometry of this theorem. A point $x \in E$ is non-singular if and only if the analytic local ring $\mathcal{O}^{an}_{E,x}$, defined in the preceding paragraph is isomorphic to the ring $\mathbb{C}\{X_1, \ldots, X_d\}$ of convergent series in d variables.

An important notion is the following.

Let $\mathcal{O}$ be a local noetherian ring. Call $\mathfrak{M}$ its maximal ideal. We can define the dimension $\dim \mathcal{O}$ of $\mathcal{O}$ as the degree of the polynomial defined by

$$F(\nu) := \mathrm{length}_{\mathcal{O}} \left(\mathcal{O}/\mathfrak{M}^{\nu+1}\right)$$

for $\nu \gg 0$. Recall that the length over $\mathcal{O}$ of an $\mathcal{O}$-module is defined by the length of a Jordan-Hölder sequence (also called composition series) of $\mathcal{O}$-submodules of the $\mathcal{O}$-module (see [SZ] Tome 1, Chapter III, §11).

The local ring $\mathcal{O}$ is **regular** if there are $\dim \mathcal{O}$ elements of $\mathcal{O}$ which generate the maximal ideal $\mathfrak{M}$ of $\mathcal{O}$.

Then:

Theorem 8 *A point $x \in E$ is non-singular if and only if the local ring $\mathcal{O}_{E,x}$ is regular.*

This type of theorem will allow us to define regular or non-singular points for algebraic sets over any field $\mathbb{K}$ by considering the local ring of the algebraic set at the point.

2.3 Singular points over an arbitrary field

Let E be a $\mathbb{K}$-algebraic subset of $\mathbb{K}^n$. We have defined the affine algebra of E as the quotient $\mathbb{K}$-algebra

$$A[E] := \mathbb{K}[X_1, \ldots, X_n]/I(E).$$

Since the ring $\mathbb{K}[X_1, \ldots, X_n]$ is noetherian, i.e. its ideals are finitely generated, $A[E]$ is also noetherian.

As noticed before $A[E]$ can be identified as the $\mathbb{K}$-algebra of polynomial functions on E, i.e. restriction to E of polynomial functions on $\mathbb{K}^n$, when the field $\mathbb{K}$ has an infinite number of elements. Most of the time we shall assume the field $\mathbb{K}$ to be algebraically closed, because we shall need Hilbert Nullstellensatz. In this case, we know that $\mathbb{K}$ has an infinite number of elements.

There is a maximal ideal $M_{E,x}$ of $A[E]$ generated in $A[E]$ by the images of $X_1 - a_1, \ldots X_n - a_n$. The localization $\mathcal{O}_{E,x}$ of $A[E]$ at the maximal ideal $M_{E,x}$ of $A[E]$ is, as defined before, the local ring of fractions with denominators in $A[E] \setminus M_{E,x}$. The ring $\mathcal{O}_{E,x}$ is a local noetherian ring and its maximal ideal $\mathfrak{M}_{E,x}$ is generated by the image of the ideal $M_{E,x}$ in $\mathcal{O}_{E,x}$.

To a noetherian local ring $\mathcal{O}$ with maximal ideal $\mathfrak{M}$, we have associated the Samuel function

$$F(\nu) := l\left(\mathcal{O}/\mathfrak{M}^{\nu+1}\right)$$

for $\nu \in \mathbb{N}$ and where l is the length as $\mathcal{O}$-module. We have observed that one can prove that for $\nu \gg 0$, this function coincides with a polynomial P whose degree is the dimension (Krull dimension)

d of $\mathcal{O}$ and its term of highest degree is

$$\frac{e}{d!}\nu^d$$

where, by definition, the positive integer e is the multiplicity of $\mathcal{O}_{E,x}$.

Therefore we can define the dimension of E at x as the Krull dimension of $\mathcal{O}_{E,x}$ and the multiplicity of E at x as the multiplicity of the local ring $\mathcal{O}$.

Theorem **??** gives a definition of non-singular points:

Definition 5 *A point x of a $\mathbb{K}$-algebraic set E is non-singular (we also say regular) if its local ring $\mathcal{O}_{E,x}$ is regular.*

A singular point is a point which is not non-singular.

A basic theorem in commutative algebra gives the following characterisation of a regular ring (see [S], Théorème 9, Chapitre IV):

Theorem 9 *Let $\mathcal{O}$ be a noetherian local ring and $\mathfrak{M}$ be its maximal ideal. The following assertions are equivalent:*

- *The local ring $\mathcal{O}$ is regular;*
- *The maximal ideal $\mathfrak{M}$ is generated by $d = \dim \mathcal{O}$ elements;*
- *The dimension over the residue field $k := \mathcal{O}/\mathfrak{M}$ of the k vector space $\mathfrak{M}/\mathfrak{M}^2$ equals d;*
- *The graded ring $G_{\mathfrak{M}}(\mathcal{O})$, of the ring $\mathcal{O}$ with the filtration $(\mathfrak{M}^n)_{n\in\mathbb{N}}$, is isomorphic to the polynomial k-algebra $k[X_1, \ldots, X_d]$.*

In particular, this theorem shows that a point $x \in E$ is non-singular if and only if the k-vector space $\mathfrak{M}_{E,x}/\mathfrak{M}^2_{E,x}$ has k-dimension equal to $\dim_x E$. We saw above that for complex algebraic sets the quotient $\mathfrak{M}_{E,x}/\mathfrak{M}^2_{E,x}$ is isomorphic to the cotangent space of E at x.

2.4 Complete local rings

Let $\mathcal{O}$ be a local ring and $\mathfrak{M}$ be its maximal ideal. The filtration $(\mathfrak{M}^n)_{n\in\mathbb{N}}$ defines a topology on the local ring $\mathcal{O}$. Take the filtration $(\mathfrak{M}^n)_{n\in\mathbb{N}}$ as fundamental system of neighbourhoods of 0 in $\mathcal{O}$ and, for any $a \in \mathcal{O}$, the family $(a + \mathfrak{M}^n)_{n\in\mathbb{N}}$ as fundamental system of neighbourhoods of a in $\mathcal{O}$. The topology defined by the filtration $(\mathfrak{M}^n)_{n\in\mathbb{N}}$ is the topology generated by these systems of neighbourhoods. Endowed with this topology, $\mathcal{O}$ is a topological ring, i.e. the addition, the multiplication, the opposite are continuous operations. This topology is called the **$\mathfrak{M}$-adic topology** of $\mathcal{O}$.

It is easy to show that this topology is Hausdorff if and only if

$$\cap_{n\in\mathbb{N}}\mathfrak{M}^n = \{0\}.$$

A sequence $(x_n)_{n\in\mathbb{N}}$ is a **Cauchy sequence** if, for any $m \in \mathbb{N}$, there is N_m such that

$$n_1, n_2 \geq N_m \Rightarrow x_{n_1} - x_{n_2} \in \mathfrak{M}^m.$$

In general Cauchy sequences do not converge, but, as in the case of metric spaces, one may define the completion of $\mathcal{O}$ for the $\mathfrak{M}$-adic topology. It is a morphism

$$j : \mathcal{O} \to \hat{\mathcal{O}}$$

of $\mathcal{O}$ into a local ring $\hat{\mathcal{O}}$ complete for the $\hat{\mathfrak{M}}$-adic topology generated by the maximal ideal $\hat{\mathfrak{M}}$ of $\hat{\mathcal{O}}$, such that j is a continuous ring homomorphism and for any continuous local ring homomorphism $h : \mathcal{O} \to \mathcal{O}_1$ into a complete local ring, there is a unique continuous local ring homomorphism $\hat{h}$, such that $h = \hat{h} \circ j$.

Definition 6 *A local ring is a complete local ring if it is complete for its $\mathfrak{M}$-topology, i.e. all Cauchy sequences converge.*

The notion of complete local ring is important, because it can be considered as an algebraic analogue of analytic local rings. Notice that the ring of formal power series $k[[X_1, \ldots, X_n]]$ is complete.

Above we have considered the local rings

$$\begin{array}{rcc} \mathcal{O}_{E,x} & = & A[E]_{\mathfrak{M}_{E,x}} \\ \mathcal{O}^{an}_{E,x} & = & \mathbb{C}\{X_1 - a_1, \ldots, X_n - a_n\}/ < I(E) > \end{array}$$

for a point x of a complex algebraic set E. These rings have the same completion

$$\mathbb{C}[[X_1 - a_1, \ldots, X_n - a_n]]/ < I(E) > .$$

Now, we can state the theorem of Cohen:

Theorem 10 *Let $\mathcal{O}$ a complete noetherian local ring with maximal ideal $\mathfrak{M}$. If the characteristic of $\mathcal{O}$ and the characteristic of the residue field $k := \mathcal{O}/\mathfrak{M}$ are equal, the following assertions are equivalent:*

- *the local ring $\mathcal{O}$ is regular;*
- *the local ring $\mathcal{O}$ is isomorphic to a ring of formal power series $k[[X_1, \ldots, X_d]]$.*

3 Plane Curves Singularities

As an example of how one studies singularities, let us first consider the case of complex plane curves, i.e. complex algebraic hypersurfaces of $\mathbb{C}^2$.

Let $F(X,Y)=0$ be a reduced equation of a plane curve C. Notice that all the irreducible components of the curve C have the dimension one.

According to Whitney theorem, the singular subset of C is a proper subset. From the definition of singular points, it is easy to prove that the singular subset of C has dimension 0, so it has only a finite number of points. Therefore C has only a finite number of singularities.

If $x \in C$ is non-singular, the implicit function theorem implies that there is an open neighbourhood of x in $\mathbb{C}^2$, such that $C \cap U$ is a complex analytic submanifold of U. So, locally at x is like the complex line. This means that there is an analytic isomorphism of an open disc D of $\mathbb{C}$ onto a neighbourhood V of x in C:

$$\pi : D \to V,$$

i.e. we have a local **parametrization** of C at the point $x \in C$.

Suppose that $0 \in C$ is a singular point of C. Puiseux theorem tells that locally at 0, we also have a local parametrization.

Theorem 11 *Suppose that $F(0,Y) \neq 0$. There is an integer m and a formal series $\Phi(X^{1/m})$ in $\mathbb{C}[[X^{1/m}]]$ such that*

$$F(X, \Phi(X^{1/m})) = 0.$$

In fact, the series Φ is convergent at 0.

Puiseux theorem gives a local parametrization $\pi : D \to C$ of C at 0 defined by $\pi(t) = (t^m, \Phi(t))$.

However if 0 is a singular point of C, π does not give an isomorphism of a disc D onto an open neighbourhood of 0 in C.

Examples: Consider the case

$$F(X,Y) = X^2 - Y^2.$$

In some cases, for instance with

$$F(X,Y) = Y^2 - X^3,$$

we have a local parametrization which is a homeomorphism of a disc D with an open neighbourhood of 0 in the curve. Here it is given by $Y = t^3$ and $X = t^2$.

The existence of a local parametrization helps to study functions on a singular curve. For instance, the rational function Y/X restricted to the curve $Y^2 - X^3 = 0$ defines a continuous function!

One can prove that, locally at the singular point of a complex curve, there are parametrizations for each **local branch**.

The definition of branches involves the complex analytic structure of C, this will lead to algebraic difficulties when considering arbitrary base fields.

Since the ring of complex polynomials $\mathbb{C}[X,Y]$ in two variables is a subring of the ring of convergent complex series in two variables $\mathbb{C}\{X,Y\}$ a reduced equation P_0 of C is also an element of $\mathbb{C}\{X,Y\}$. This latter ring is factorial. So, P_0 has a decomposition into irreducible factors in $\mathbb{C}\{X,Y\}$:

$$P_0 = f_1 \dots f_s.$$

The analytic curves $f_i = 0$ are the branches of C at 0.

Consider an irreducible element f in $\mathbb{C}\{X,Y\}$. We have an analytic version of Puiseux theorem:

Theorem 12 *Suppose that $f(0,Y) \neq 0$ and has valuation n. There is a convergent series Φ in $\mathbb{C}\{X\}$ such that*

$$f(X,Y) = u(X,Y) \prod_{\xi, \xi^n=1} (Y - \Phi(\xi X^{1/n})),$$

where $u(X,Y)$ is a unit of $\mathbb{C}\{X,Y\}$.

We call **Puiseux expansion** (or Puiseux series) of the branch f relatively to the coordinates X and Y the series

$$Y = \Phi(X^{1/n})$$

If an algebraic curve C defined by the reduced equation $P_0 = 0$ has several branches $f_i = 0$, $1 \leq i \leq s$ at the singular point 0, and the coordinates X, Y are such that none of the series $f_i(0,Y)$, $1 \leq i \leq s$ vanish identically, we have simultaneous Puiseux expansions

$$\begin{aligned} Y &= \Phi_1(X^{1/n_1}) \\ &\dots \\ Y &= \Phi_s(X^{1/n_s}) \end{aligned}$$

These Puiseux expansions determine the analytic structure of C at 0, since we have

$$P_0 = u(X,Y) \prod_{i=1}^{i=s} \left(\prod_{\xi, \xi^{n_i}=1} (Y - \Phi(\xi X^{1/n_i})) \right)$$

where $u(X,Y)$ is an invertible element of $\mathbb{C}\{X,Y\}$, i.e. the constant coefficient $u(0,0)$ of $u(X,Y)$ is $\neq 0$.

They also determine the topological structure of C at 0.

When C has several branches at 0, it is rather complicated to show how the Puiseux expansions of the branches determine the local topology of the curve.

We shall restrict ourselves to the case when C has only one branch. Let $\Phi(X^{1/n})$ be the Puiseux expansion of this branch relatively to X, Y.

First, notice that $\Phi(X^{1/n})$ is an element of the ring extension $\mathbb{C}\{X\}[X^{1/n}]$ of $\mathbb{C}\{X\}$. Let $\mathbb{C}\{\{X\}\}$ be the field of fractions of $\mathbb{C}\{X\}$. Then, the field $\mathbb{C}\{\{X\}\}[X^{1/n}]$ is an algebraic extension of the field $\mathbb{C}\{\{X\}\}$. The Galois group of this field extension is the cyclic group μ_n of order n. Let σ an element of μ_n. There is a unique root of unity $\xi(\sigma)$ such that

$$\sigma(\Phi(X^{1/n})) = \Phi(\xi(\sigma)X^{1/n}).$$

In the power series ring $\mathbb{C}\{X\}[X^{1/n}]$ we have a valuation v such that $v(X^{1/n}) = 1$. Consider the subgroup G_j of μ_n defined by

$$G_j := \{\sigma \in \mu_n \, , \, v(\sigma\Phi(X^{1/n}) - \Phi(X^{1/n})) \geq j\}.$$

Obviously $G_1 = \mu_n$ and $G_{j+1} \subset G_j$, for $j \geq 1$. Since μ_n is a finite group, for $k \gg 0$, $G_k = \{\varepsilon\}$, where ε is the neutral element of μ_n.

Let $\beta_1 < \ldots < \beta_g$ the sequence of integers such that

$$\mu_n = G_1 = \ldots = G_{\beta_1} \supsetneq G_{\beta_1+1} \ldots \supsetneq G_{\beta_g+1} = \{\varepsilon\}.$$

The integers $\beta_1, \ldots, \beta_g$ are called the **Puiseux exponents** relatively to the coordinates X, Y.

There is a unique sequence of pairs of relatively prime integers $(m_1, n_1), \ldots, (m_g, n_g)$ such that

$$\frac{\beta_1}{n} = \frac{m_1}{n_1}, \ldots, \frac{\beta_g}{n} = \frac{m_g}{n_1 \ldots n_g}.$$

We call these pairs the **Puiseux characteristic pairs** of C relatively to the coordinates X, Y.

These Puiseux pairs give a description of the local topology of C at the singular point 0.

Namely, for $0 < \epsilon \ll 1$, the real 3-sphere $S_\epsilon(0)$ centered at 0 with radius ϵ intersects C transversally. Since C has only one branch at 0, the intersection $C \cap S_\epsilon(0)$ is connected and is diffeomorphic to the circle $\mathbb{S}^1$. Therefore the embedding of $C \cap S_\epsilon(0)$ into the sphere $S_\epsilon(0)$ is a knot.

It is an iterated torus knot given by the Puiseux pairs $(m_1, n_1), \ldots, (m_g, n_g)$.

These Puiseux pairs determine the local topology of C at 0 in the following sense:
Let C_1 and C_2 be two plane curves having one branch at 0. Suppose that there exists a homeomorphism Ψ of a neighbourhood U_1 of 0 in $\mathbb{C}^2$ onto a neighbourhood U_2 of 0 in $\mathbb{C}^2$ such that $\Psi(U_1 \cap C_1) = U_2 \cap C_2$, then the Puiseux pairs of C_1 at 0 relatively to "general" coordinates of 0 in $\mathbb{C}^2$ at 0 are equal to the Puiseux pairs of C_2 at 0 relatively to "general" coordinates of 0 in $\mathbb{C}^2$ at 0

4 Parametrizations and Desingularization of curves

4.1 Newton approximation method

To obtain the Puiseux series relatively to given coordinates, we have mentioned Newton approximation method.

To indicate how to get the local parametrizations of branches, we shall briefly sketch this method used by Puiseux.

Let P_0 be a reduced equation of the complex plane curve C. Assume that the origin 0 of $\mathbb{C}^2$ is on C, i.e. $P_0(0) = 0$.

We have

$$P_0(X,Y) = \sum_{(\alpha,\beta)\in\mathbb{N}^2} c_{\alpha,\beta} X^\alpha Y^\beta.$$

We are looking for a series $aX^r + \ldots$ where $a \neq 0$ and the rational r is the lowest degree of non-zero terms of this series, such that

$$P_0(X, aX^r + \ldots) = 0.$$

By variable substitution, we must have

$$P_0(X, aX^r + \ldots) = \sum_{(\alpha,\beta)\in\mathbb{N}^2} a^\beta c_{\alpha,\beta} X^{\alpha+r\beta} + \ldots = 0.$$

In particular the terms of lowest degree of this series must vanish. To find them, consider the subset $\mathcal{Q}$ of $\mathbb{N}^2$ defined by

$$\mathcal{Q} := \{(\alpha,\beta) \in \mathbb{N}^2,\ c_{\alpha,\beta} \neq 0\}.$$

In Puiseux theorem we assume that $P_0(0,Y) \neq 0$. Therefore if n is the valuation of $P_0(0,Y)$, i.e. the lowest degree of the non-zero terms of $P(0,Y)$, the point $(0,n)$ belongs to $\mathcal{Q}$. By definition of the multiplicity, we have

$$m_{C,0} = \inf\{\alpha + \beta,\ (\alpha,\beta) \in \mathcal{Q}\}$$

and there is a point (α_0, β_0) in $\mathcal{Q}$ such that $\alpha_0 + \beta_0 = m_{C,0}$.

Let $\tilde{\mathcal{Q}}$ be the convex hull of $\mathcal{Q}$. The compact part of the boundary of $\tilde{\mathcal{Q}}$ facing 0 is called the **Newton polygon** $\mathcal{N}(P_0)$ of P_0 relatively to the coordinates X and Y. This Newton polygon is a union of segments having pairwise at most an extremity in common.

For a fix positive rational r the terms of $P_0(X,Y)$ which give the terms of lowest degree in

$$P_0(X, aX^r + \ldots) = \sum_{(\alpha,\beta)\in\mathbb{N}^2} a^\beta c_{\alpha,\beta} X^{\alpha+r\beta} + \ldots$$

necessarily lie on the Newton Polygon $\mathcal{N}(P_0)$. In fact these terms correspond to the points of $\mathcal{N}(P_0)$ where the linear function $\alpha + r\beta$ takes its lowest value. If $-1/r$ is not a slope of one of segments in

$\mathcal{N}(P_0)$, the lowest value of $\alpha + r\beta$ on $\mathcal{N}(P_0)$ is reached only in one point of $\mathcal{N}(P_0)$, in which case the lowest degree term of $P_0(X, aX^r + \ldots)$ do not vanish. So necessarily $-1/r$ is the slope of one of the non-trivial segments of the Newton polygon. Let $(\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)$ be the points of $\mathcal{Q}$ on this segment. So, by definition, $c_{\alpha_j, \beta_j} \neq 0$ and $\alpha_1 + r\beta_1 = \alpha_j + r\beta_j$, for $1 \leq j \leq k$. In order to obtain $P_0(X, aX^r + \ldots) = 0$, we must also have

$$g(a) = a^{\beta_1} c_{\alpha_1, \beta_1} + \ldots + a^{\beta_k} c_{\alpha_k, \beta_k} = 0.$$

Since the base field $\mathbb{C}$ is algebraically closed, we can find $a \neq 0$, so that $g(a) = 0$.

To find the rest of the series, let $r = p/q$, where p and q are relatively prime positive integers. Put

$$X = X_1^q,\ Y = X_1^p(a + Y_1).$$

By change of variables we have

$$P_0(X, Y) = P_0(X_1^q, X_1^p(a + Y_1)) = P_1(X_1, Y_1).$$

Since $P_1(0, Y_1) \neq 0$, we can repeat the arguments above and by induction one find a formal series which satisfies Puiseux theorem. It remains to prove that series is also convergent.

Details are left as an exercise.

4.2 Normalization

Another way to obtain the local parametrization is to observe that if the curve C, defined by the reduced polynomial P is analytically irreducible at 0, the local analytic ring $\mathcal{O}_{C,0}^{an} := \mathbb{C}\{X, Y\}/(P)$ of C at 0 is a domain of integrity. It embeds in its field of fractions.

It can be proved that the normalization of the local ring $\mathcal{O}_{C,0}^{an}$ is a regular local ring $\mathbb{C}\{t\}$. Recall that, for a domain of integrity A, an element of its field of fractions K is integral over A if it is the root of a unitary polynomial of $A[T]$, i.e. there are $a_0, \ldots, a_m \in A$ such that

$$\alpha^m + \sum_1^m a_{m-i}\alpha^i = 0.$$

The elements of K which are integral over A are a subring $\overline{A}$ of K which contains A. We call $\overline{A}$ the integral closure of A or the normalization of A. We say that A is normal if it is equal to its integral closure. The preceding result is consequence of the fact that an analytic local ring which is normal and of dimension one is a regular local ring.

The class x and y of X and Y in $\mathcal{O}_{C,0}^{an}$ are elements of its normalization, so there are series ϕ and ψ, such that

$$x = \phi(t),\ y = \psi(t).$$

By changing the variables, one may find an integer n and a series Φ such that

$$x = u^n,\ y = \Phi(u).$$

In the case of complete local ring of dimension one, one may have a similar argument by using the following

Theorem 13 *The normalization of a complete local domain of integrity of dimension one is a complete regular ring.*

Therefore in dimension one there is the possibility to define a parametrization over any field.

4.3 Blowing-up points

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A last way to obtain a local parametrization is to use blowing-ups of points.

Let us define the blowing-up of a point. We first give a complex analytic definition of the blowing-up of a point.

Let U be an open neighbourhood of 0 in $\mathbb{C}^2$. We have a natural map

$$\lambda : U \setminus \{0\} \to \mathbb{P}^1_{\mathbb{C}}$$

defined by $\lambda(x) = \{$the complex line from 0 to $x\}$.

The graph of λ is a subset $G(\lambda)$ of $U \times \mathbb{P}^1_{\mathbb{C}}$. Remember that $\mathbb{P}^1_{\mathbb{C}}$ is a complex analytic manifold which is the union of two affine spaces U_0 and U_1 isomorphic to $\mathbb{C}$ with respective coordinates u_0 and u_1. Then, $U \times \mathbb{P}^1_{\mathbb{C}}$ is the union of $U \times U_0$ and $U \times U_0$. The intersection $G(\lambda) \cap (U \times U_0)$ is contained in the set defined by

$$\frac{X}{Y} = u_0$$

Similarly $G(\lambda) \cap (U \times U_1)$ is contained in the set defined by

$$\frac{Y}{X} = u_1.$$

Then, the closure E of $\overline{G(\lambda)}$ of $G(\lambda)$ in $U \times \mathbb{P}^1_{\mathbb{C}}$ is defined by $X = u_0 Y$ in $U \times U_0$ and by $Y = u_1 X$ in $U \times U_1$. It is easy to see that E is a complex analytic manifold of complex dimension 2. The projection onto U induces a map $e : E \to U$ which is called the **blowing-up of 0 in U**.
The inverse image $e^{-1}(0)$ is called the **exceptional divisor** of the blowing-up.

In $(U \times U_0) \cap E$ the equation of the exceptional divisor is $Y = 0$. In $(U \times U_1) \cap E$ it is $X = 0$.

It is convenient to consider on $(U \times U_0) \cap E$ the two coordinates Y and u_0 and on $(U \times U_1) \cap E$ the coordinates X and u_1. The restriction of the blowing-up e to $(U \times U_0) \cap E$ is given by

$$e(Y, u_0) = (Y u_0, Y).$$

Similarly the restriction of e to $(U \times U_1) \cap E$ is given by

$$e(X, u_1) = (X, X u_1).$$

Now consider a curve C which a branch $f = 0$ at 0. Assume that the irreducible element f of $\mathbb{C}\{X, Y\}$ defines an analytic function on the open neighbourhood U of 0 in $\mathbb{C}^2$.

The intersection of the inverse image of C by the blowing-up of 0 in U with the subset of $U \times U_0$ is the set defined by $f(Yu_0, Y)$.

Consider the expansion of f by homogeneous forms:

$$f = f_m + f_{m+1} + \ldots,$$

where m is the multiplicity of f at 0. Then,

$$\begin{aligned} f(Yu_0, Y) &= Y^m f_m(u_0, 1) + Y^{m+1} f_{m+1}(u_o, 1) + \ldots \\ &= Y^m(f_m(u_0, 1) + Y f_{m+1}(u_0, 1) + \ldots) = Y^m f_0(Y, u_o). \end{aligned}$$

Similarly the set $e^{-1}(C) \cap (U \times U_1)$ is defined by $f(X, Xu_1) = 0$ and

$$\begin{aligned} f(X, Xu_1) &= X^m f_m(1, u_1) + X^{m+1} f_{m+1}(1, u_1) + \ldots \\ &= X^m(f_m(1, u_1) + X f_{m+1}(1, u_1) + \ldots) = X^m f_1(X, u_1). \end{aligned}$$

Remember that $Y = 0$ is the equation of the exceptional divisor in $U \times U_0$ and that $X = 0$ is the equation of the exceptional divisor in $U \times U_1$.

Therefore $e^{-1}(C)$ is the union of the exceptional divisor and a set C_1 whose intersections with $U \times U_0$ and $U \times U_1$ are defined respectively by $f_0 = 0$ and $f_1 = 0$.

The set C_1 is also the topological closure of $e^{-1}(C \setminus \{0\})$ in E. It is called the **strict transform** of C by the blowing-up e. The restriction of the blowing-up e to C_1 induces a map $e_0 : C_1 \to C$ which is called the **blowing-up of the curve** C at 0.

Obviously locally C_1 is isomorphic to analytic plane curve, but C_1 itself is the patch of two plane curves.

In fact the definition of blowing-up is applicable to the case $U = \mathbb{C}^2$. In which case, we observe that the restrictions of the blowing-up of $\mathbb{C}^2$ to two open subsets isomorphic to the affine space $\mathbb{C}^2$ are algebraic maps (maps whose components are polynomials).

Extending the notion of algebraic sets to objects which are "locally" algebraic sets, we have the notion of an algebraic variety or of a finitely generated reduced scheme over the complex field (see the usual litterature). In this context, a blowing-up is an algebraic map and the strict transform of an irreducible algebraic plane curve is a variety of dimension one (i.e. an algebraic curve). Furthermore, locally this curve is isomorphic to a plane curve.

Coming back to the notion of parametrization, one can prove that by a succession of plane blowing-ups the strict transform becomes non-singular.

The way to prove it is first to recall that after a blowing-up locally the blown-up curve is isomorphic to a plane curve. Then, we observe that after a point blowing-up the multiplicities of the singularities do not increase. In fact, in the case of a branch f at a point 0, the first Puiseux exponent

β_1 (defined above) relatively to coordinates X, Y such that the valuation of $f(0, Y)$ equals the multiplicity m of $f = 0$ at 0, can be interpreted in the following way:

We saw that the multiplicity is the intersection number of a general line through 0 with the curve. So, it is also the intersection number of a general non-singular curve (whose tangent at 0 is a general line) with the curve at 0. Now, if the line or the tangent of the non-singular curve at 0, is not general, this intersection number is strictly higher than the multiplicity. It can be shown that the highest value of the intersection is precisely β_1 (in contrast with the case the curve is non-singular at 0 when this number can be as high as one wishes).

Then, by one blowing-up one can prove that for a curve with one branch at 0, this number β_1 decreases strictly, in fact the new value at the singular point of the blown-up branch is $\beta_1 - m$.

Using these observations one can prove that after a finite number of point blowing-ups, the final strict transform of the given curve is non-singular.

In the case of a branch the composition of the successive blowing-ups of the singular points of the successive strict transforms

$$C \leftarrow C_1 \leftarrow C_2 \leftarrow \ldots \leftarrow C_k$$

gives a map π from a non-singular curve C_k onto C.

It is easy to check that a blowing-up is an isomorphism outside the blown-up point, so π induces an isomorphism of $C_k \setminus \pi^{-1}(0)$ onto $C \setminus \{0\}$ and $\pi^{-1}(0) = \{x_k\}$. Since x_k is a non-singular point of C_k an open neighbourhood of x_k in C_k is isomorphic to a disc D and π induces a parametrization $p : D \to C$ of C at 0.

Therefore, in the case of a plane branch the process of eliminating a singular point by successive point blowing-ups gives the local parametrization.

5 Desingularization

Historically algebraic geometers were looking for a "transformation" which could replace the local parametrization in the case of a variety of dimension ≥ 2.

As we have mentioned above, the normalization also leads to parametrization of plane branches. Unfortunaley the normalization of a surface is in general singular. In fact surfaces with non-singular normalization are very special. However, it can be shown that the singularities of normal surfaces, i.e. algebraic varieties for which all the local rings are equal to their normalization, are isolated. Then, the natural idea was then to blow-up singular points of normal surfaces. In general the blowing-up of a singular point of a normal surface is not a normal surface any more, since the singularities of the blown-up surface might not be isolated. R. Walker stated that a surface could be desingularized after a finite sequence of normalizations and point blowing-ups. The proof of this theorem was given by O. Zariski using valuation theory (see [Z]).

One can observe that a map $\pi : W \to V$, which is the composition of normalizations and point blowing-ups at singularities, is

1. a proper map (the inverse image of a compact subset is compact);
2. an isomorphism of $\pi^{-1}(V \setminus SingV)$ onto $V \setminus SingV$ where $SingV$ is the subset of singular points of V.

We are led to the following definition:

A topological space W is an $\mathbb{K}$-algebraic variety if there is a finite covering by open subsets U_i, $1 \leq i \leq s$ such that, for each i, there is a bijection $\sigma_i : U_i \to E_i$ onto a $\mathbb{K}$-algebraic set E_i, such that for any $1 \leq i, j \leq s$ the map from $\sigma_{i,j} : \sigma_i(U_i \cap U_j) \to \sigma_j(U_i \cap U_j)$ defined by $\sigma_{i,j}(x) = (\sigma_j \circ \sigma_i^{-1})(x)$ is an algebraic isomorphism.

A map φ of an algebraic variety W into an algebraic variety V is algebraic if there are coverings by open subsets isomorphic to an algebraic set U_i, $1 \leq i \leq s$ and V_j, $1 \leq j \leq t$, such that for any i there is j such that φ induces an algebraic map from U_i into V_j, i.e. a map which induces an algebraic map between the corresponding algebraic sets.

In particular strict transform of an algebraic curve by a point blowing-up is an algebraic variety and projective varieties are algebraic varieties. The point blowing-up defined above is an algebraic map.

An algebraic map $\pi : W \to V$ is a **desingularization** of a variety V (we also say a **resolution of singularities** of V) if:

1. W is a non-singular variety;
2. it is a proper map;
3. it is an isomorphism of $\pi^{-1}(V \setminus SingV)$ on $V \setminus SingV$ where $SingV$ is the subset of singular points of V;
4. $\pi^{-1}(V \setminus SingV)$ is dense in W.

In the case of $\mathbb{C}$-algebraic varieties we have a topological definition of the properness of a map, since complex algebraic varieties are also endowed with the topology induces by the one of the field of complex numbers. Over an arbitrary field one has to generalized the notion of properness. This will not be done here, but it will be necessary if we wish to find resolutions of singularity over arbitrary fields. The generalization of properness will come from the observation that a proper map is also a closed map. The topology involved with algebraic varieties over an arbitrary field will be a topology which generalized the Zariski topology of algebraic sets where the closed sets are precisely the algebraic subsets. In fact, we shall deal with projective varieties, i.e. closed subvarieties of the projective space. In which case the algebraic maps from a projective variety are proper.

Over an algebraically closed field of characteristic zero the existence of a resolution of singularities was obtained by H. Hironaka in [H].

One of the aim of this school on the resolution of singularities is to find a desingularization in the case of varieties over a non-zero characteristic field.

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