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# Introduction to Singularity Theory

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# Introduction to Singularity Theory

by Lê Dũng Tráng

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What is an algebraic singularity?

Let $\mathbb{K}$ be a field (commutative). It is often called the base field.

An $\mathbb{K}$-**algebraic set defined by the polynomials** $P_1, \ldots, P_k \in \mathbb{K}[X_1, \ldots, X_n]$ is the subset of the affine space $\mathbb{K}^n$ of the points $(x_1, \ldots, x_n)$, such that $P_i(x_1, \ldots, x_n) = 0$. for $1 \leq i \leq k$.

When the base field is clear, we speak of algebraic set instead of $\mathbb{K}$-algebraic set. The polynomials $P_i$, with $1 \leq i \leq k$, are also called the **equations** of the algebraic set. We also write these equations $P_i = 0$, $1 \leq i \leq k$.

The first obvious fact is that algebraic subsets of $\mathbb{K}^n$ are very particular subsets of $\mathbb{K}^n$. For instance, when $n = 1$, the algebraic subsets of $\mathbb{K}$ are the finite subsets.

However, the study of algebraic sets is not easy in general.

One of the first observations is that points of algebraic sets are of two types:

- **regular points** or **non-singular points**;
- **singular points**.

To understand the difference let consider the case of algebraic sets defined by one equation.

Let assume that the base field is the field of complex numbers $\mathbb{C}$.

A $\mathbb{C}$-algebraic set defined by one equation is called a **complex hypersurface**.

Let $x \in X$ be a point of a complex hypersurface $X \subset \mathbb{C}^n$. Let $P$ be an equation of $X$. Suppose that the differential $dP(x)$ of $P$ at $x$ is a linear form $\neq 0$. It is known that at $x$, the hypersurface $X$ has a tangent space defined by

$$dP(x) = 0,$$

i.e.

$$\sum_{1}^{n} \partial P/\partial X_j(x) X_j = 0.$$

In fact the linear form $dP(x)$ is $\neq 0$ iff there is $j$, $1 \leq j \leq n$, such that $\partial P/\partial X_j(x) \neq 0$.

The implicit function theorem implies that there is an open neighbourhood $U$ of $x$ in $\mathbb{C}^n$, such that $X \cap U$ is a complex analytic submanifold of $U$.

**Example**: Consider a linear form $\ell$ of $\mathbb{C}^n$. It is the equation of a complex hyperplane $H$ of $\mathbb{C}^n$. It is easy to show that at every point $x$ of $H$, $d\ell(x) \neq 0$.

Now the same set $H$ is defined by the equation $\varphi = \ell^2 = 0$, in which case at every point $x \in H$, we have $d\varphi(x) = 0$.

So the fact that the differential of the equation vanishes at a point $x$ of $X$ depends on the equation.

Since $\mathbb{C}$ is an algebraically closed field, Hilbert Nullstellensatz shows that if a hypersurface $X$ is defined by an equation $P$, it is also defined by the reduced polynomial $P_0$ defined by $P$.

Let $X$ be a complex hypersurface defined by a reduced polynomial $P_0$.

A point $x$ of the hypersurface $X$ is called a **non-singular point** of $X$ or, also, a **regular point** of $X$, if $dP_0(x) \neq 0$.

A point of $X$ is called a **singular point** of $X$ if $dP_0(x) = 0$.

Singular points of $X$ make an algebraic subset of $X$ defined by the equations

$$P_0 = \partial P_0 / \partial X_1 = \ldots = \partial P_0 / \partial X_n = 0.$$

More generally, let $E$ be a complex algebraic subset of $\mathbb{C}^n$. Let $I(E)$ the ideal of all polynomials in $\mathbb{C}[X_1, \ldots, X_n]$ which vanish on $E$.

Hilbert finiteness theorem shows that the ideal $I(E)$ is finitely generated

$$I(E) = (P_1, \ldots, P_k)$$

Now consider the Jacobian matrix $J(x)$:

$$\begin{pmatrix} \partial P_1/\partial X_1(x), & \ldots & , \partial P_1/\partial X_n(x) \\ & \ldots & \\ \partial P_k/\partial X_1(x), & \ldots & , \partial P_k/\partial X_1(x) \end{pmatrix}$$

Denote $\rho(x)$ the rank of this matrix at $x \in E$.

Let $\rho_E := \max_{x \in E} \rho(x)$.

In Ann. Math., 66 (1957), H. Whitney proved:

**Theorem 1** *The subset $E^0$ of points $x$ of $E$ where $\rho(x) = \rho_E$ is a complex analytic manifold of dimension $n - \rho_E$. The subset of $E_1 := E \backslash E_0$ of $E$ is a proper algebraic subset of $E$.*

**Example**: Consider the complex algebraic subset $V$ of $\mathbb{C}^3$ defined by

$$XY = XZ = 0.$$

One can check that $V$ is the union of the plane $X = 0$ and of the line $Y = Z = 0$. The Jacobian matrix $J(x)$ is

$$\begin{pmatrix} Y(x) & X(x) & 0 \\ Z(x) & 0 & X(x) \end{pmatrix}$$

So, $\rho_V = 2$. In this case $V_1$ is the plane $X = 0$.

This example leads us to notice that an algebraic set $E$ is the finite union of irreducible subsets $E(i)$, $1 \leq i \leq r$, such that for $i \neq j$, $E(i) \not\subset E(j)$.

These subsets are called the (irreducible) **components** of $E$.

A singular point of an irreducible complex algebraic set $E$ is a point $x \in E$ where $\rho(x) \neq \rho_E$. One can prove that the set of non-singular points of an irreducible complex algebraic set $E$ is connected.

A singular point of a complex algebraic set

$$E = \cup_1^r E(i)$$

is a point $x \in E$ where either $\rho(x) \neq \rho_{E(i)}$, for all $i, 1 \leq i \leq r$, or $x$ belongs to two distinct irreducible components $E(i)$ and $E(j)$ $(i \neq j)$ of $E$.

A non-singular point or a regular point of $E$ is a point which is not singular.

In the preceding example the only singular point of $V$ is the origin of $\mathbb{C}^3$.

How do we recognize a singular point?

In the case of a complex hypersurface, it is rather easy from the reduced equation: Let $P_0$ be a reduced equation of the hypersurface $X$, the point $x \in X$ is singular iff $dP_0(x) = 0$.

Another way to check it is to consider the Taylor expansion of $P_0$ at $x$ and notice that $x := (a_1, \ldots, a_n)$ is singular if and only if

$$P_0(X_1, \ldots, X_n) = P_0(x)$$

$+$ terms in $X_1 - a_1, .., X_n - a_n$ of degree $\geq 2$.

The lowest degree of the non-zero non-constant terms in this Taylor expansion is called the **multiplicity** $m_{X,x}$ of $X$ at $x$.

Therefore $x \in X$ is singular iff the multiplicity $m_{X,x}$ is $\geq 2$.

There is an algebraic way to compute the multiplicity.

Consider the $\mathbb{C}$-algebra

$$A[X] := \mathbb{C}[X_1, \ldots, X_n]/(P_0)$$

quotient of the $\mathbb{C}$-algebra of complex polynomials in $n$ variables by the principal ideal generated by the reduced equation $P_0$.

The local ring $\mathcal{O}_{X,x}$ of $X$ at $x$ is the **localization** of $A[X]$ at the maximal ideal generated by $X_1 - a_1, \ldots, X_n - a_n$.

Consider the following function on $\mathbb{N}$:

$$\forall \nu \in \mathbb{N}, \quad F(\nu) := \dim_{\mathbb{C}} \mathcal{O}_{X,x}/\mathfrak{M}_{X,x}^{\nu+1},$$

where $\mathfrak{M}_{X,x}$ is the maximal ideal of $\mathcal{O}_{X,x}$.

**Exercise**: For $\nu \gg 0$, the function $F$ is a polynomial in $\nu$ of degree $n-1$ and the term of highest degree is

$$\frac{m_{X,x}}{(n-1)!}\nu^{n-1}.$$

There is also a geometric way to understand the multiplicity of a hypersurface:

Let $\ell$ a "general" line through $x$. The line $\ell$ intersects $X$ at the isolated point $x$. Consider an open neighbourhood $U$ such that

$$\ell \cap X \cap U = \{x\}$$

For "general" lines $\ell_t$ parallel to $\ell$, $\ell_t$ intersects $X$ transversally and the number of points of $\ell_t \cap X \cap U$ is the multiplicity $m_{X,x}$.

Let $E$ be a complex algebraic subset of $\mathbb{C}^n$. Let $E(1), \ldots, E(r)$ be the irreducible components of $E$. We define

$$\dim_x E := \max_{\{i, x \in E(i)\}} \{n - \rho_{E(i)}\}.$$

At a point $x \in E$ one consider $\mathcal{O}_{E,x}$ the localization at $x$ of the complex algebra

$$A[E] := \mathbb{C}[X_1, \ldots, X_n]/I(E).$$

We have the following results

**Theorem 2** *For $\nu \gg 0$, the function defined by*

$$\forall \nu \in \mathbb{N}, \quad F(\nu) := \dim_{\mathbb{C}} \mathcal{O}_{X,x}/\mathfrak{M}_{X,x}^{\nu+1},$$

*is a polynomial of $\nu$ of degree* $\dim_x E$.

For $\nu \gg 0$, the coefficient of the term of degree $\dim_x E$ is $e/(\dim_x E)!$

By definition the multiplicity $m_{E,x}$ of $E$ at $x$ equals $e$.

We have

**Theorem 3** *The point* $x \in E$ *is singular iff* $m_{E,x} \geq 2$.

As above we have the following interpretation of the multiplicity for complex irreducible algebraic sets (see R. Draper, Math. Ann. 180 (1969)):

Let $\ell$ a "general" affine subspace of $\mathbb{C}^n$ of dimension $\rho_E$ through $x$. The affine space $\ell$ intersects $E$ at the isolated point $x$. Consider an open neighbourhood $U$ such that

$$\ell \cap E \cap U = \{x\}$$

For "general" affine spaces $\ell_t$ parallel to $\ell$, $\ell_t$ intersects $E$ transversally and the number of points of $\ell_t \cap E \cap U$ is the multiplicity $m_{E,x}$.

There are other algebraic characterizations of non-singular points:

**Theorem 4** *A point $x \in E$ is non-singular iff the $\mathfrak{M}_{E,x}$-completion of the local ring $\mathcal{O}_{E,x}$ is isomorphic to the $\mathbb{C}$-algebra of formal series $\mathbb{C}[[X_1, \ldots, X_d]]$, where $d = \dim_x E$.*

An important notion is the following.

Let $\mathcal{O}$ be a local noetherian ring. Call $\mathfrak{M}$ its maximal ideal. We can define the dimension $\dim \mathcal{O}$ of $\mathcal{O}$ as the degree of the polynomial defined by

$$F(\nu) := \text{length} \mathcal{O}/\mathfrak{M}^{\nu+1}$$

for $\nu \gg 0$.

The local ring $\mathcal{O}$ is **regular** if there $\dim \mathcal{O}$ elements of $\mathcal{O}$ which generate the maximal ideal $\mathfrak{M}$ of $\mathcal{O}$ .

Then:

**Theorem 5** *A point $x \in E$ is non-singular iff the local ring $\mathcal{O}_{E,x}$ is regular.*

This type of theorem will allow us to define regular or non-singular points for any algebraic sets over any field $\mathbb{K}$.