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Valuation theoretic aspects of local uniformization

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Lecture Notes for the
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These lecture notes contain parts of the papers [KU] and [KU]. They are meant to present the material in more detail than the lectures. The lectures will not cover all of the material presented here.

1 What does local uniformization mean?

Here is the bitter truth of mankind: In most cases we average human beings are too stupid to solve our problems globally. So we try to solve them locally. And if we are clever enough (and truly interested), we then may think of patching the local solutions together to obtain a global solution.

What is the problem we are considering here? It is the fact that an algebraic variety has singularities, and we want to get rid of them. That is, we are looking for a second variety having the same function field, and having no singularities. This would be the global solution of our problem. As we are too stupid for it, we are first looking for a local solution. Naively speaking, "local" means something like "at a point of the variety". So local solution would mean that we get rid of one singular point. We are looking for a new variety where our point becomes non-singular. But wait, this was nonsense. Because what is our old, singular point on the new variety? We cannot talk of the same points of two different varieties, unless we deal with subvarieties. But passing from varieties to subvarieties or vice versa will in general not provide the solution we are looking for. So do we have to forget about local solutions of our problem?

The answer is: no. Let us have a closer look at our notion of "point". Assume our variety V is given by polynomials $f_1,\dots,f_n \in K[X_1,\dots,X_\ell]$. Naively, by a point of V we then mean an ℓ-tupel $(a_1,\dots,a_\ell)$ of elements in an arbitrary extension field L of K such that $f_i(a_1,\dots,a_\ell) = 0$ for $1 \le i \le n$. This means that the kernel of the "evaluation homomorphism" $K[X_1,\dots,X_\ell] \to L$ defined by $X_i \mapsto a_i$ contains the ideal $(f_1,\dots,f_n)$. So it induces a homomorphism η from the coordinate ring $K[V] =$

$K[X_1,\dots,X_\ell]/(f_1,\dots,f_n)$ into L over K. (The latter means that it leaves the elements of K fixed.) However, if $a'_1,\dots,a'_\ell \in L'$ are such that $a_i \mapsto a'_i$ induces an isomorphism from $K(a_1,\dots,a_\ell)$ onto $K(a'_1,\dots,a'_\ell)$, then we would like to consider $(a_1,\dots,a_\ell)$ and $(a'_1,\dots,a'_\ell)$ as the same point of V. That is, we are only interested in η up to composition $\sigma\circ\eta$ with isomorphisms σ. This we can get by considering the kernel of η instead of η. This leads us to the modern approach: to view a point as a prime ideal of the coordinate ring.

But I wouldn't have told you all this if I intended to follow this modern approach. Instead, I want to build on the picture of homomorphisms. So I ask you to accept temporarily the convention that a point of V is a homomorphism of $K[V]$ over K (i.e., leaving K elementwise invariant), modulo composition with isomorphisms. Recall that $K[V] = K[x_1,\dots,x_\ell]$, where x_i is the image of X_i under the canonical epimorphism $K[X_1,\dots,X_\ell] \to K[X_1,\dots,X_\ell]/(f_1,\dots,f_n) = K[V]$. The function field $K(V)$ of V is the quotient field $K(x_1,\dots,x_\ell)$ of $K[V]$. It is generated by $x_1,\dots,x_\ell$ over K, hence it is finitely generated. Every finite extension of a field K of transcendence degree at least 1 is called an **algebraic function field** (over K), and it is in fact the function field of a suitable variety defined over K. When we talk of function fields in this paper, we will always mean algebraic function fields.

Now recall what it means to look for another variety V' having the same function field $F := K(V)$ as V (i.e., being birationally equivalent to V). It just means to look for another set of generators $y_1,\dots,y_k$ of F over K. Now the points of V' are the homomorphisms of $K[y_1,\dots,y_k]$ over K, modulo composition with isomorphisms. But in general, $y_1,\dots,y_k$ will not lie in $K[x_1,\dots,x_\ell]$, hence we do not see how a given homomorphism of $K[x_1,\dots,x_\ell]$ could determine a homomorphism of $K[y_1,\dots,y_k]$. But if we could extend the homomorphism of $K[x_1,\dots,x_\ell]$ to all of $K(x_1,\dots,x_\ell)$, then this extension would assign values to every element of $K[y_1,\dots,y_k]$. Let us give a very simple example.

Example 1 Consider the coordinate ring $K[x]$ of $V = \mathbb{A}^1_K$. That is, x is transcendental over K, and the function field $K(V)$ is just the rational function field $K(x)$ over K. A homomorphism of the polynomial ring $K[V] = K[x]$ is just given by "evaluating" every polynomial $g(x)$ at $x = a$. I have seen many people who suffered in school from the fact that one can also try to evaluate rational functions $g(x)/h(x)$. The obstruction is that a could be a zero of h, and what do we get then by evaluating $1/h(x)$ at a? (In fact, if our homomorphism is not an embedding, i.e., if a is not transcendental over K, then there will always be a polynomial h over K having a as a root.) So we have to accept that the evaluation will not only render elements in $K(a)$, but also the element ∞, in which case we say that the evaluated rational function has a pole at a. So we can extend our homomorphism to a map P on all of $K(x)$, taking into the bargain that it may not always render finite values. But on the subring $\mathcal{O}_P = \{g(x)/h(x) \mid h(a) \neq 0\}$ of $K(x)$ on which P is finite, it is still a homomorphism. $\diamondsuit$

What we have in front of our eyes in this example is one of the two basic classical examples for the concept of a **place**. (The other one, the p-adic place, comes from number theory.) Traditionally, the application of a place P is written in the form $g \mapsto gP$, where instead of gP also $g(P)$ was used in the beginning, reminding of the fact that P originated from an evaluation homomorphism. If you translate the German "g an der Stelle a auswerten" literally, you get "evaluate g at the place a", which explains the origin of the word "place".

Associated to a place P is its **valuation ring** $\mathcal{O}_P$, the maximal subring on which P is finite, and a valuation v_P. In our case, the value $v_P(g/h)$ is determined by computing the zero or pole order of g/h (pole orders taken to be negative integers). In this way, we obtain values in $\mathbb{Z}$, which is the value group of v_P. In general, given a field L with place P and associated valuation v_P, the valuation ring $\mathcal{O}_P = \{b \in L \mid bP \neq \infty\} = \{b \in L \mid v_P b \geq 0\}$ has a unique maximal ideal $\mathcal{M}_P = \{b \in L \mid bP = 0\} = \{b \in L \mid v_P b > 0\}$. The **residue field** is $LP := \mathcal{O}_P/\mathcal{M}_P$ so that P restricted to $\mathcal{O}_P$ is just the canonical epimorphism $\mathcal{O}_P \to LP$. The characteristic of LP is called the **residue characteristic** of (L, P). If P is the identity on $K \subseteq L$, then $K \subseteq LP$ canonically. The valuation v_P can be defined to be the homomorphism $L^\times \to L^\times/\mathcal{O}_P^\times$. The latter is an ordered abelian group, the **value group** of (L, v_P). We denote it by $v_P L$ and write it additively. Note that $bP \neq \infty \Leftrightarrow b \in \mathcal{O}_P \Leftrightarrow v_P b \geq 0$, and $bP = 0 \Leftrightarrow b \in \mathcal{M}_P \Leftrightarrow v_P b > 0$.

Instead of (L, P), we will often write (L, v) if we talk of valued fields in general. Then we will write av and Lv instead of aP and LP. If we talk of an **extension of valued fields** and write $(L|K, v)$ then we mean that v is a valuation on L and K is endowed with its restriction. If we only have to consider a single extension of v from K to L, then we will use the symbol v for both the valuation on K and that on L. Similarly, we use "$(L|K, P)$".

Observe that in Example 1, P is uniquely determined by the homomorphism on $K[x]$. Indeed, we can always write g/h in a form such that a is not a zero of both g and h. If then a is not a zero of h, we have that $(g/h)P = g(a)/h(a) \in K(a)$. If a is a zero of h, we have that $(g/h)P = \infty$. Thus, the residue field of P is $K(a)$, and the value group is $\mathbb{Z}$. On the other hand, we have the same non-uniqueness for places as we had for homomorphisms: also places can be composed with isomorphisms. If P, Q are places of an arbitrary field L and there is an isomorphism $\sigma : LP \to LQ$ such that $\sigma(bP) = bQ$ for all $b \in \mathcal{O}_P$, then we call P and Q **equivalent places**. In fact, P and Q are equivalent if and only if $\mathcal{O}_P = \mathcal{O}_Q$. Nevertheless, it is often more convenient to work with places than with valuation rings, and we will just identify equivalent places wherever this causes no problems.

Two valuations v and w are called **equivalent valuations** if they only differ by an isomorphism of the value groups; this holds if and only if v and w have the same valuation ring. As for places, we will identify equivalent valuations wherever this causes no problems, and we will also identify the isomorphic value groups.

At this point, we shall introduce a useful notion. Given a function field $F|K$, we will call P a **place of** $F|K$ if it is a place of F whose restriction to K is the identity. We say that P is **trivial** on K if it induces an isomorphism on K. But then, composing P with the inverse of this isomorphism, we find that P is equivalent to a place of F whose restriction to K is the identity. Note that a place P of F is trivial on K if and only if v_P is **trivial** on K, i.e., $v_P K = \{0\}$. This is also equivalent to $K \subset \mathcal{O}_P$. A place P of $F|K$ is said to be a **rational place** if $FP = K$. The **dimension** of P, denoted by $\dim P$, is the transcendence degree of $FP|K$. Hence, P is **zero-dimensional** if and only if $FP|K$ is algebraic.

Let's get back to our problem. The first thing we learn from our example is the following. Clearly, we would like to extend our homomorphism of $K[V]$ to a place of $K(V)$ because then, it will induce a map on $K[V']$. Then we have the chance to say that the point we have to look at on the new variety (e.g., in order to see whether this one is simple) is the point given by this map on $K[V']$. But this only makes sense if this map is a homomorphism of $K[V']$. So we have to require that

$$y_1, \ldots, y_k \in \mathcal{O}_P$$

(since then, $K[y_1, \ldots, y_k] \subseteq \mathcal{O}_P$, which implies that P is a homomorphism on $K[y_1, \ldots, y_k]$).

This being granted, the next question coming to our mind is whether to every point there corresponds exactly one place (up to equivalence), as it was the case in Example 1. To destroy this hope, I give again a very simple example. It will also serve to introduce several types of places and their invariants.

Example 2 Consider the coordinate ring $K[x_1, x_2]$ of $V = \mathbb{A}^2_K$. That is, x_1 and x_2 are algebraically independent over K, and the function field $K(V) = K(x_1, x_2)$ is just the rational function field in two variables over K. A homomorphism of the polynomial ring $K[V] = K[x_1, x_2]$ is given by "evaluating" every polynomial $g(x_1, x_2)$ at $x_1 = a_1$, $x_2 = a_2$. For example, let us take $a_1 = a_2 = 0$ and try to extend the corresponding homomorphism of $K[x_1, x_2]$ to $K(V) = K(x_1, x_2)$. It is clear that $1/x_1$ and $1/x_2$ go to ∞. But what about x_1/x_2 or even x_1^m/x_2^n? Do they go to 0, ∞ or some non-zero element in K? The answer is: all that is possible, and there are infinitely many ways to extend our homomorphism to a place of $K(x_1, x_2)$.

There is one way, however, which seems to be the most well-behaved. It is to construct a **place of maximal rank**; we will explain this notion later in full generality. The idea is to learn from Example 1 where we replace K by $K(x_2)$ and x by x_1, and extend the homomorphism defined on $K(x_2)[x_1]$ by $x_1 \mapsto 0$ to a unique place Q of $K(x_1, x_2)$. Its residue field is $K(x_2)$ since $x_1 Q = 0 \in K(x_2)$, and its value group is $\mathbb{Z}$. Now we do the same for $K(x_2)$, extending the homomorphism given on $K[x_2]$ by $x_2 \mapsto 0$ to a unique place $\overline{Q}$ of $K(x_2)$ with residue field K and value group $\mathbb{Z}$. We compose the two places, in the following way. Take $b \in K(x_1, x_2)$. If $bQ = \infty$, then we set $bQ\overline{Q} = \infty$. If $bQ \neq \infty$, then $bQ \in K(x_2)$, and we know what $bQ\overline{Q} = (bQ)\overline{Q}$ is. In this way, we obtain a place $P = Q\overline{Q}$ on $K(x_1, x_2)$ with residue field K. We observe that for every $g \in K[x_1, x_2]$, we have that $g(x_1,x_2)Q\overline{Q} = g(0,x_2)\overline{Q} = g(0,0)$, so our place P indeed extends the given homomorphism of $K[x_1,x_2]$. Now what happens to our critical fractions? Clearly, $(1/x_1)P = (1/x_1)Q\overline{Q} = (\infty)\overline{Q} = \infty$, and $(1/x_2)P = (1/x_2)Q\overline{Q} = (1/x_2)\overline{Q} = \infty$. But what interests us most is that for all $m > 0$ and $n \geq 0$, $(x_1^m/x_2^n)P = (x_1^m/x_2^n)Q\overline{Q} = 0\overline{Q} = 0$. We see that "$x_1$ goes more strongly to 0 than every x_2^n". We have achieved this by sending first x_1 to 0, and only afterwards x_2 to 0. We have arranged our action "lexicographically".

What is the associated value group? General valuation theory (cf. [V], §3 and §4, or [ZS]) tells us that for every composition $P = Q\overline{Q}$, the value group $v_{\overline{Q}}(FQ)$ of the place $\overline{Q}$ on FQ is a convex subgroup of the value group v_PF, and that the value group v_QF of P is isomorphic to $v_PF/v_{\overline{Q}}(FQ)$. If the subgroup $v_{\overline{Q}}(FQ)$ is a direct summand of v_PF (as it is the case in our example), then v_PF is the lexicographically ordered direct product $v_QF \times v_{\overline{Q}}(FQ)$. Hence in our case, $v_PK(x_1,x_2) = \mathbb{Z}\times\mathbb{Z}$, ordered lexicographically. The **rank of an abelian ordered group** G is the number of proper convex subgroups of G (or rather the order type of the chain of convex subgroups, ordered by inclusion, if this is not finite). The **rank of** (F,P) is defined to be the rank of v_PF. See under the name "hauteur" in [V]. In our case, the rank is 2. We will see in Section 6 that if P is a place of $F|K$, then the rank cannot exceed the transcendence degree of $F|K$. So our place $P = Q\overline{Q}$ has maximal possible rank.

There are other places of maximal rank which extend our given homomorphism, but there is also an abundance of places of smaller rank. In our case, "smaller rank" can only mean rank 1, i.e., there is only one proper convex subgroup of the value group, namely $\{0\}$. For an ordered abelian group G, having rank 1 is equivalent to being archimedean ordered and to being embeddable in the ordered additive group of $\mathbb{R}$. Which subgroups of $\mathbb{R}$ can we get as value groups? To determine them, we look at the **rational rank** of an ordered abelian group G. It is $\mathrm{rr}\, G := \dim_{\mathbb{Q}} \mathbb{Q}\otimes_{\mathbb{Z}} G$ (note that $\mathbb{Q}\otimes_{\mathbb{Z}} G$ is the **divisible hull** of G). This is the maximal number of rationally independent elements in G. We will see in Section 6 that for every place P of $F|K$ we have that

$$\mathrm{rr}\, v_PF \;\leq\; \mathrm{trdeg}\, F|K\ . \tag{1}$$

Hence in our case, also the rational rank of P can be at most 2. The subgroups of $\mathbb{R}$ of rank 2 are well known: they are the groups of the form $r\mathbb{Z}+s\mathbb{Z}$ where $r > 0$ and $s > 0$ are rationally independent real numbers. Moreover, through multiplication by $1/r$, the group is order isomorphic to $\mathbb{Z} + \frac{s}{r}\mathbb{Z}$. As we identify equivalent valuations, we can assume all rational rank 2 value groups (of a rank 1 place) to be of the form $\mathbb{Z}+r\mathbb{Z}$ with $0 < r \in \mathbb{R}\setminus\mathbb{Q}$. To construct a place P with this value group on $K(x_1,x_2)$, we proceed as follows. We want that $v_Px_1 = 1$ and $v_Px_2 = r$; then it will follow that $v_PK(x_1,x_2) = \mathbb{Z}+r\mathbb{Z}$ (cf. Theorem 37 below). We observe that for such P, $v_P(x_1^m/x_2^n) = m - nr$, which is > 0 if $m/n > r$, and < 0 if $m/n < r$. Hence, $(x_1^m/x_2^n)P = 0$ if $m/n > r$, and $(x_1^m/x_2^n)P = \infty$ if $m/n < r$. I leave it to you as an exercise to verify that this defines a unique place P of $K(x_1,x_2)|K$ with the desired value group and extending our given homomorphism.

Observe that so far every value group was finitely generated, namely by two elements. Now we come to the groups of rational rank 1. If such a group is finitely generated, then it is simply isomorphic to $\mathbb{Z}$. How do we get places P on $K(x_1, x_2)$ with value group $\mathbb{Z}$? A place with value group $\mathbb{Z}$ is called a **discrete place**. The idea is to first construct a place on the subfield $K(x_1)$. We know from Example 1 that every place of $K(x_1)|K$ (if it is not trivial on $K(x_1)$) will have value group $\mathbb{Z}$ (cf. Theorem 37). Then we can try to extend this place from $K(x_1)$ to $K(x_1, x_2)$ in such a way that the value group doesn't change.

There are many different ways how this can be done. One possibility is to send the fraction x_1/x_2 to an element z which is transcendental over K. You may verify that there is a unique place which does this and extends the given homomorphism; it has value group $\mathbb{Z}$ and residue field $K(z)$. If, as in this case, a place P of $F|K$ has the property that $\operatorname{trdeg} FP|K = \operatorname{trdeg} F|K - 1$, then P is called a **prime divisor** and v_P is called a **divisorial valuation**. The places $Q, \overline{Q}$ were prime divisors, one of F, the other one of FQ.

But maybe we don't want a residue field which is transcendental over K? Maybe we even insist on having K as a residue field? Well, then we can employ another approach. Having already constructed our place P on $K(x_1)$ with residue field K, we can consider the completion of $(K(x_1), P)$. The **completion** of an arbitrary valued field (L, v) is the completion of L with respect to the topology induced by v. Both v and the associated place P extend canonically to this completion, whereby value group and residue field remain unchanged. Let us give a more concrete representation of this completion.

Let t be any transcendental element over K. We consider the unique place P of $F|K$ with $tP = 0$. The associated valuation is called the **t-adic valuation**, denoted by v_t. It is the unique valuation v on $K(t)$ (up to equivalence) which is trivial on K and satisfies that $vt > 0$. We want to write down the completion of $(K(t), v_t)$. We define the **field of formal Laurent series** (I prefer **power series field**) over K. It is denoted by $K((t))$ and consists of all formal sums of the form

$$\sum_{i=n}^{\infty} c_i t^i \quad \text{with } n \in \mathbb{Z} \text{ and } c_i \in K\,. \tag{2}$$

I suppose I don't have to tell you in which way the set $K((t))$ can be made into a field. But I tell you how v_t extends from $K(t)$ to $K((t))$: we set

$$v_t \sum_{i=n}^{\infty} c_i t^i \;=\; n \quad \text{if } c_n \neq 0\,. \tag{3}$$

One sees immediately that $v_t K((t)) = v_t K(t) = \mathbb{Z}$. For $b = \sum_{i=n}^{\infty} c_i t^i$ with $c_n \neq 0$, we have that $bv_t = \infty$ if $m < 0$, $bv_t = 0$ if $m > 0$, and $bv_t = c_0 \in K$ if $m = 0$. So we see that $K((t))v_t = K(t)v_t = K$. General valuation theory shows that $(K((t)), v_t)$ is indeed the completion of $(K(t), v_t)$.

It is also known that the transcendence degree of $K((t))|K(t)$ is uncountable. If K is countable, this follows directly from the fact that $K((t))$ then has the cardinality of the continuum. But it is quite easy to show that the transcendence degree is at least one, and already this suffices for our purposes here. The idea is to take any $y \in K((t))$, transcendental over $K(t)$; then $x_1 \mapsto t$, $x_2 \mapsto y$ induces an isomorphism $K(x_1, x_2) \to K(t, y)$. We take the restriction of v_t to $K(t, y)$ and pull it back to $K(x_1, x_2)$ through the isomorphism. What we obtain on $K(x_1, x_2)$ is a valuation v which extends our valuation v_P of $K(x_1)$. As is true for v_t, also this extension still has value group $\mathbb{Z} = v_P K(x_1)$ and residue field $K = K(x_1)P$. The desired place of $K(x_1, x_2)$ is simply the place associated with this valuation v.

We have now constructed essentially all places on $K(x_1, x_2)$ which extend the given homomorphism of $K[x_1, x_2]$ and have a finitely generated value group (up to certain variants, like exchanging the role of x_1 and x_2). The somewhat shocking experience to every "newcomer" is that on this rather simple rational function field, there are also places extending the given homomorphism and having a value group which is not finitely generated. For instance, the value group can be $\mathbb{Q}$. (In fact, it can be any subgroup of $\mathbb{Q}$.) We postpone the construction of such a place to Section 13. $\diamondsuit$

After we have become acquainted with places and how one obtains them from homomorphisms of coordinate rings, it is time to formulate our problem of local desingularization. Instead of looking for a desingularization "at a given point" of our variety V, we will look for a desingularization at a given place P of the function field $F|K$ (we forget about the variety from which F originates). Suppose we have any V such that $K(V) = F$, that is, we have generators $x_1, \ldots, x_\ell$ of $F|K$ and the coordinate ring $K[x_1, \ldots, x_\ell]$ of V. If we talk about the **center of** P **on** V, we always tacitly assume that $x_1, \ldots, x_\ell \in \mathcal{O}_P$, so that the restriction of P is a homomorphism on $K[x_1, \ldots, x_\ell]$. With this provision, the center of P on V is the point $(x_1P, \ldots, x_\ell P)$ (or, if we so want, the induced homomorphism). We also say that P is **centered on** V **at** $(x_1P, \ldots, x_\ell P)$. If V is a variety defined over K with function field F, then we call V a **model of** $F|K$. Our problem now reads:

(LU) *Take any function field $F|K$ and a place P of $F|K$. Does there exist a model of $F|K$ on which P is centered at a simple point?*

This was answered in the positive by Oscar Zariski in [Z] for the case of K having characteristic 0. Instead of "local desingularization", he called this principle **local uniformization**.

2 Local uniformization and the Implicit Function Theorem

Let's think about what we mean by "simple point". I don't really have to tell you, so let me pick the most valuation theoretic definition, which will show us our way on our excursion.

It is the Jacobi criterion: Given our variety V defined by $f_1, \ldots, f_n \in K[X_1, \ldots, X_\ell]$ and having function field F, then a point $a = (a_1, \ldots, a_\ell)$ of V is called **simple** (or **smooth**) if $\mathrm{trdeg}\, F|K = \ell - r$, where r is the rank of the Jacobi matrix

$$\left(\frac{\partial f_i}{\partial X_j}(a) \right)_{\substack{1\leq i\leq n \\ 1\leq j\leq \ell}}$$

But wait — I have seen the Jacobi matrix long before I learned anything about algebraic geometry. Now I remember: I saw it in my first year calculus course in connection with the **Implicit Function Theorem**. Let's have a closer look. First, let us assume that we don't have too many f_i's. Indeed, when looking for a local uniformization we will construct varieties V defined by $\ell - \mathrm{trdeg}\, F|K$ many polynomial relations, whence $n = \ell - \mathrm{trdeg}\, F|K$. In this situation, if a is a simple point, then n is equal to r and after a suitable renumbering we can assume that for $k := \ell - n = \mathrm{trdeg}\, F|K$, the submatrix

$$\left(\frac{\partial f_i}{\partial X_j}(a) \right)_{\substack{1\leq i\leq n \\ k+1\leq j\leq \ell}}$$

is invertible. Then, assuming that we are working over the reals, the Implicit Function Theorem tells us that for every $(a'_1, \ldots, a'_k)$ in a suitably small neighborhood of $(a_1, \ldots, a_k)$ there is a unique $(a'_{k+1}, \ldots, a'_\ell)$ such that $(a'_1, \ldots, a'_\ell)$ is a point of V. Working in the reals, the existence is certainly interesting, but for us here, the main assertion is the uniqueness. Let's look at a very simple example.

Example 3 I leave it to you to draw the graph of the function $y^2 = x^3$ in $\mathbb{R}^2$. It only exists for $x \geq 0$. Starting from the origin, it has two branches, one positive, one negative. Now assume that we are sitting on one of these branches at a point (x, y), away from the origin. If somebody starts to manipulate x then we know exactly which way we have to run (depending on whether x increases or decreases). But if we are sitting at the origin and somebody increases x, then we have the freedom of choice into which of the two branches we want to run. So we see that everywhere but at the origin, y is an implicit function of x in a sufficiently small neighborhood. Indeed, with $f(x, y) = x^3 - y^2$, we have that $\frac{\partial f}{\partial x}(x, y) = 3x^2$. If $x \neq 0$, then this is non-zero, whence $r = 1$ while $\mathrm{trdeg}\, F|K = 1$ and $\ell = 2$, so for $x \neq 0$, (x, y) is a simple point. On the other hand, $\frac{\partial f}{\partial x}(0, 0) = 0$ and $\frac{\partial f}{\partial y}(0, 0) = 0$, so $(0, 0)$ is singular. ♢

We have now seen the connection between simple points and the Implicit Function Theorem. "Wait!" you will interrupt me. "You have used the topology of $\mathbb{R}$. What if we don't have such a topology at hand? What then do you mean by 'neighborhood'?" Good question. So let's look for a topology. My luck, that the Implicit Function Theorem is also known in valuation theory. Indeed, we have already remarked in connection with the notion "completion of a valued field" that every valuation induces a topology. And since we have our place on F, we have the topology right at hand. That is why I said that the Jacobi criterion renders the most valuation theoretical definition of “simple”.

But now this makes me think: haven't I seen the Jacobi matrix in connection with an even more famous valuation theoretical theorem, one of central importance in valuation theory? Indeed: it appears in the so-called “multidimensional version” of **Hensel's Lemma**. This brings us to our next sightseeing attraction on our excursion.

3 Hensel's Lemma

Hensel's Lemma is originally a lemma proved by Kurt Wilhelm Sebastian Hensel for the field of p-adic numbers $\mathbb{Q}_p$. It was then extended to all complete discrete valued fields and later to all maximal fields (see Corollary 29 below). A valued field (L, v) is called **maximal** (or **maximally complete**) if it has no proper extensions for which value group and residue field don't change. A complete field is not necessarily maximal, and if it is not of rank 1 (i.e., its value group is not archimedean), then it also does not necessarily satisfy Hensel's Lemma. However, complete discrete valued fields are maximal. In particular, $(K((t)), v_t)$ is maximal.

In modern valuation theory (and its model theory), Hensel's Lemma is rather understood to be a property of a valued field. The nice thing is that, in contrast to “complete” or “maximal”, it is an elementary property in the sense of model theory. We call a valued field **henselian** if it satisfies Hensel's Lemma. Here is one version of Hensel's Lemma for a valued field with valuation ring $\mathcal{O}_v$:

(Hensel's Lemma) *For every polynomial $f \in \mathcal{O}_v[X]$ the following holds: if $b \in \mathcal{O}_v$ satisfies*

$$vf(b) > 0 \quad \text{and} \quad vf'(b) = 0 , \tag{4}$$

then f admits a root $a \in \mathcal{O}_v$ such that $v(a-b) > 0$.

Here, f' denotes the derivative of f. Note that a more classical version of Hensel's Lemma talks only about monic polynomials.

For the multidimensional version, we introduce some notation. For polynomials $f_1, \ldots, f_n$ in variables $X_1, \ldots, X_n$, we write $f = (f_1, \ldots, f_n)$ and denote by J_f the Jacobian matrix $\left(\frac{\partial f_i}{\partial X_j}\right)_{i,j}$. For $a \in L^n$, $J_f(a) = \left(\frac{\partial f_i}{\partial X_j}(a)\right)_{i,j}$.

(Multidimensional Hensel's Lemma) *Let $f = (f_1, \ldots, f_n)$ be a system of polynomials in the variables $X = (X_1, \ldots, X_n)$ and with coefficients in $\mathcal{O}_v$. Assume that there exists $b = (b_1, \ldots, b_n) \in \mathcal{O}_v^n$ such that*

$$vf_i(b) > 0 \text{ for } 1 \leq i \leq n \quad \text{and} \quad v \det J_f(b) = 0 . \tag{5}$$

Then there exists a unique $a = (a_1, \ldots, a_n) \in \mathcal{O}_v^n$ such that $f_i(a) = 0$ and $v(a_i - b_i) > 0$ for all i.

And here is the valuation theoretical Implicit Function Theorem:

(Implicit Function Theorem) *Take $f_1,\dots,f_n \in L[X_1,\dots,X_\ell]$ with $n<\ell$. Set*

$$\tilde{J} := \begin{pmatrix} \frac{\partial f_1}{\partial X_{\ell-n+1}} & \cdots & \frac{\partial f_1}{\partial X_\ell} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial X_{\ell-n+1}} & \cdots & \frac{\partial f_n}{\partial X_\ell} \end{pmatrix} . \tag{6}$$

Assume that $f_1,\dots,f_n$ admit a common zero $a=(a_1,\dots,a_\ell)\in L^\ell$ and that $\det\tilde{J}(a)\neq 0$. Then there is some $\alpha\in vL$ such that for all $(a'_1,\dots,a'_{\ell-n})\in L^{\ell-n}$ with $v(a_i-a'_i)>2\alpha$, $1\leq i\leq \ell-n$, there exists a unique $(a'_{\ell-n+1},\dots,a'_\ell)\in L^n$ such that $(a'_1,\dots,a'_\ell)$ is a common zero of $f_1,\dots,f_n$, and $v(a_i-a'_i)>\alpha$ for $\ell-n<i\leq\ell$.

It is (not all too well) known that Hensel's Lemma holds in (L,v) if and only if the Multidimensional Hensel's Lemma holds in (L,v), and this in turn is true if and only if the Implicit Function Theorem holds in (L,v). For a proof, see [KU2] or [PZ]. The latter paper is particularly interesting since it shows the connection between the Implicit Function Theorem in henselian fields and the "real" Implicit Function Theorem in $\mathbb{R}$.

There are many more versions of Hensel's Lemma which all are equivalent to the above (the classical Hensel's Lemma for monic polynomials, Krasner's Lemma, Newton's Lemma, Hensel–Rychlik,...). See [R2] or [KU2] for a listing of them. It is indeed often very useful to have the different versions at hand. One particularly important is given in the following lemma:

Lemma 4 *A valued field (L,v) is henselian if and only if the extension of v to the algebraic closure $\tilde{L}$ of L is unique.*

Since any valuation of any field can always be extended to any extension field (cf. [V], §5), the following is an easy consequence of this lemma: (L,v) *is henselian if and only if v admits a unique extension to every algebraic extension field.* Also, we immediately obtain:

Corollary 5 *Every algebraic extension of a henselian field is again henselian.*

This is hard to prove if you use Hensel's Lemma instead of the unique extension property in the proof. On the other hand, the next lemma is hard to prove using the unique extension property, while it is immediate if you use Hensel's Lemma:

Lemma 6 *Take a henselian field (L,v) and a relatively algebraically closed subfield L' of L. Then also (L',v) is henselian.*

Let us take a short break to see how Hensel's Lemma can be applied. The following two examples will later have important applications.

Example 7 Assume that $\operatorname{char} L = p > 0$. A polynomial $f(X) = X^p - X - c$ with $c \in L$ is called an **Artin-Schreier polynomial** (over L). If ϑ is a root of f in some extension of L, then $\vartheta, \vartheta + 1, \ldots, \vartheta + p - 1$ are the distinct roots of f. Hence if f is irreducible over L, then $L(\vartheta)|L$ is a Galois extension of degree p. It is called an **Artin-Schreier extension**. Conversely, *every* Galois extension of degree p in characteristic p is generated by a root of a suitable Artin-Schreier polynomial, i.e., is an Artin-Schreier extension (see [KU2] for a proof).

Let us prove our assertion about the roots of f. We note that in characteristic $p > 0$, the map $x \mapsto x^p$ is a ring homomorphism (the **Frobenius**). Therefore, the polynomial $\wp(X) := X^p - X$ is an **additive polynomial**. A polynomial g is called additive if $g(a + b) = g(a) + g(b)$ for all a, b (for details, cf. [L], VIII, §11). Thus, if $i \in \mathbb{F}_p$, then $f(\vartheta + i) = \wp(\vartheta + i) - c = \wp(\vartheta) - c + \wp(i) = 0 + i^p - i = i - i = 0$ since $i^p = i$ for every $i \in \mathbb{F}_p$.

Now assume that (L, v) is henselian. Suppose first that $vc > 0$. Take $b = 0 \in \mathcal{O}_v$. Then $vf(b) = vc > 0$. On the other hand, $f'(X) = pX^{p-1} - 1 = -1$ since $p = 0$ in characteristic p. Hence, $vf'(b) = v(-1) = 0$. Therefore, Hensel's Lemma shows that f admits a root in L, which by our above observation about the roots of f means that f splits completely over L.

Suppose next that $vc = 0$. Then for $b \in \mathcal{O}_v$ we have that $v(b^p - b - c) > 0$ if and only if $0 = (b^p - b - c)v = (bv)^p - bv - cv$. Hence, $v(b^p - b - c) > 0$ if and only if bv is a root of the Artin-Schreier polynomial $X^p - X - cv \in Lv[X]$. If $cv = 0$, which is our previous case where $vc > 0$, then 0 is a root of $X^p - X - cv = X^p - X$ and we can choose $b = 0$. But in our present case, $cv \neq 0$, and everything depends on whether $X^p - X - cv$ has a root in Lv or not. If it has a root η in Lv, then we choose $b \in \mathcal{O}_v$ such that $bv = \eta$. We obtain that $(b^p - b - c)v = (bv)^p - bv - cv = \eta^p - \eta - cv = 0$, hence $vf(b) > 0$. Then by Hensel's Lemma, $X^p - X - c$ has a root $a \in \mathcal{O}_v$ with $v(a - b) > 0$, hence $av = bv = \eta$. Conversely, if $X^p - X - c$ has a root a in L, then one easily shows that $a \in \mathcal{O}_v$, and $0 = 0v = (a^p - a - c)v = (av)^p - av - cv$ yields that $X^p - X - cv$ has a root in Lv.

The only remaining case is that of $vc < 0$. In this case, $X^p - X - c \notin \mathcal{O}_v[X]$, so Hensel's Lemma doesn't give us any immediate information about whether f has a root in L or not. $\diamondsuit$

Example 8 Take a field K of characteristic $p > 0$. In the field $(K((t)), v_t)$ (which is henselian, cf. Corollary 29 below), the Artin-Schreier polynomial

$$X^p - X - t \tag{7}$$

has the root

$$a = \sum_{i=0}^{\infty} (-t)^{p^i} \tag{8}$$

since

$$a^p - a = \sum_{i=0}^{\infty} (-t)^{p^{i+1}} - \sum_{i=0}^{\infty} (-t)^{p^i} = \sum_{i=1}^{\infty} (-t)^{p^i} - \sum_{i=0}^{\infty} (-t)^{p^i} = t\,.$$

$\diamondsuit$

Take any polynomial $f \in \mathcal{O}_v[X]$. By fv we mean the **reduction of the polynomial** f **modulo** v, that is, the polynomial we obtain from f by replacing every coefficient c_i of f by its residue $c_i v$. As the residue map is a homomorphism on $\mathcal{O}_v$, we have that $f'v = (fv)'$. Suppose there is some $b \in L$ such that $vf(b) > 0$ and $vf'(b) = 0$. This is equivalent to $f(b)v = 0$ and $f'(b)v \neq 0$. But $f(b)v = fv(bv)$ and $f'(b)v = (fv)'(bv)$, so the latter is equivalent to bv being a simple root of fv. Conversely, if fv has a simple root ζ, find some b such that $bv = \zeta$, and you will have that $vf(b) > 0$ and $vf'(b) = 0$. Hence, Hensel's Lemma is also equivalent to the following version:

(Hensel's Lemma, Simple Root Version) *For every polynomial $f \in \mathcal{O}_v[X]$ the following holds: if fv has a simple root ζ in Lv, then f admits a root $a \in \mathcal{O}_v$ such that $av = \zeta$.*

Example 9 Take a henselian valued field (L, v) and a relatively algebraically closed subfield L' of L. Assume there is an element ζ of the residue field Lv which is algebraic over $L'v$, and denote its minimal polynomial over $L'v$ by $h \in L'v[X]$. Find a monic polynomial $f \in (\mathcal{O}_v \cap L')[X]$ such that $fv = h$.

If ζ is separable over $L'v$, then ζ is a simple root of h. As $\zeta \in Lv$ and (L, v) is henselian by assumption, the Simple Root Version of Hensel's Lemma tells us then that there is some $a \in L$ such that $h(a) = 0$ and $av = \zeta$. But as a is algebraic over L' we have that $a \in L'$, so that $\zeta = av \in L'v$. If on the other hand ζ is not separable over $L'v$, then it is quite possible that $\eta \notin L'v$. But at least we have proved:

Lemma 10 *If (L, v) is henselian and L' is relatively algebraically closed in L, then $L'v$ is* **relatively separable-algebraically closed** *in Lv, i.e., every element of Lv already belongs to $L'v$ if it is separable-algebraic over $L'v$.*

Something similar can be shown for the value groups, provided that $Lv = L'v$. Pick an element $\delta \in vL$ such that for some $n > 0$, $n\delta \in vL'$. Choose some $d \in L$ such that $vd = \delta$. Hence, $vd^n = nvd \in vL'$ and we can choose some $d' \in L'$ such that $vd'd^n = 0$. Assuming that $Lv = L'v$, we can also pick some $d'' \in L'$ such that $(d'd''d^n)v = 1$.

An element u with $uv = 1$ is called a **1-unit**. We consider the polynomial $X^n - u$. Its reduction modulo v is simply the polynomial $X^n - 1$. Obviously, 1 is a root of that polynomial, but is it a simple root? The answer is: 1 is a simple root of $X^n - 1$ if and only if the characteristic of Lv does not divide n. Hence in that case, Hensel's Lemma shows that there is a root $a \in L$ of the polynomial $X^n - u$ such that $av = 1$. This proves:

Lemma 11 *Take a 1-unit u in the henselian field (L, v) and $n \in \mathbb{N}$ such that the characteristic of Lv does not divide n. Then there is a unique 1-unit $a \in L$ such that $a^n = u$.*

In our present case, this provides an element $a \in L$ such that $a^n = d'd''d^n$. We find that $(a/d)^n = d'd'' \in L'$. Since $a/d \in L$ and L' is relatively algebraically closed in L, this implies that $a/d \in L'$. On the other hand, $v(a/d)^n = vd'd'' = vd' = n\alpha$ so that $v(a/d) = \alpha$. This proves that $\alpha \in vL'$. We have proved:

Lemma 12 *If (L,v) is henselian and L' is relatively algebraically closed in L and $Lv = L'v$, then the torsion subgroup of vL/vL' is trivial if* $\mathrm{char}\, Lv = 0$, *and it is a p-group if* $\mathrm{char}\, Lv = p > 0$.

It can be shown that the assertion is in general not true without the assumption that $Lv = L'v$. ♢

Let's return to our variety V which is defined by $f_1, \ldots, f_n \in K[X_1, \ldots, X_\ell]$ and has coordinate ring $K[x_1, \ldots, x_\ell]$. We have seen that a point $a = (a_1, \ldots, a_\ell)$ of V is simple if and only if after a suitable renumbering, the submatrix

$$\tilde{J} \;=\; \left(\frac{\partial f_i}{\partial X_j}(a)\right)_{\substack{1\leq i\leq n\\ k+1\leq j\leq \ell}} \tag{9}$$

of $J_f(a)$ is invertible, where $k := \ell - n = \mathrm{trdeg}\, F|K$. That means that $f_1, \ldots, f_n$ and a satisfy the assumptions of the Implicit Function Theorem.

Since we are interested in the question whether the center of P on V is simple, we have to look at $a = (x_1P, \ldots, x_\ell P)$. As P is a homomorphism on $\mathcal{O}_P$ and leaves the coefficients of the f_i invariant, we see that

$$\left(\frac{\partial f_i}{\partial X_j}(x_1P, \ldots, x_\ell P)\right) \;=\; \left(\frac{\partial f_i}{\partial X_j}(x_1, \ldots, x_\ell)\right) P\;. \tag{10}$$

We have omitted the indices since this holds for *every* submatrix of J_f. Again because P is a homomorphism, it commutes with taking determinants (since this operation remains inside the ring $\mathcal{O}_P$). Hence,

$$\det \tilde{J}(x_1P, \ldots, x_\ell P) \;=\; (\det \tilde{J}(x_1, \ldots, x_\ell))P\;. \tag{11}$$

Therefore, $\det \tilde{J}(x_1P, \ldots, x_\ell P) \neq 0$ is equivalent to $v \det \tilde{J}(x_1, \ldots, x_\ell) = 0$. This condition also appears in the Multidimensional Hensel's Lemma, but with J_f in the place of $\tilde{J}$. So we are led to the question: what is the connection? It is obvious that we have some variables too many for the case of the Multidimensional Hensel's Lemma. But they are exactly $\mathrm{trdeg}\, F|K$ too many, and on the other hand, at least the basic Hensel's Lemma obviously talks about algebraic elements (we will see that this is also true for the Multidimensional Hensel's Lemma). So why don't we just take $x_1, \ldots, x_k$ as a transcendence basis of $F|K$ and view $f_1, \ldots, f_n$ as polynomial relations defining the remaining $x_{k+1}, \ldots, x_\ell$, which are algebraic over $K(x_1, \ldots, x_k)$? But then, we should write every f_i as a polynomial $\tilde{f}_i$ in the variables $X_{k+1}, \ldots, X_\ell$ with coefficients in $K(x_1, \ldots, x_k)$, or actually, in $K[x_1, \ldots, x_k]$. Then we have that

$$f_i(x_1, \ldots, x_\ell)P \;=\; f_i(x_1P, \ldots, x_\ell P) \;=\; \tilde{f}_iP(x_{k+1}P, \ldots, x_\ell P) \;=\; \tilde{f}_i(x_{k+1}, \ldots, x_\ell)P\;.$$

With $\tilde{f} := (\tilde{f}_1, \ldots, \tilde{f}_n)$ and $\tilde{f}P := (\tilde{f}_1 P, \ldots, \tilde{f}_n P)$, it follows that

$$\det \tilde{J}(x_1P, \ldots, x_\ell P) \;=\; \det J_{\tilde{f}P}(x_{k+1}P, \ldots, x_\ell P) \;=\; (\det J_{\tilde{f}}(x_{k+1}, \ldots, x_\ell))P \;. \qquad (12)$$

Hence, $\det \tilde{J}(x_1P, \ldots, x_\ell P) \neq 0$ is equivalent to $v \det J_{\tilde{f}}(x_{k+1}, \ldots, x_\ell) = 0$, which means that the polynomials $\tilde{f}_1, \ldots, \tilde{f}_n$ and the elements $x_{k+1}, \ldots, x_\ell$ satisfy the assumption (5) of the Multidimensional Hensel's Lemma. Indeed, as $\tilde{f}_i(x_{k+1}, \ldots, x_\ell) = 0$, we have that $v\tilde{f}_i(x_{k+1}, \ldots, x_\ell) = \infty > 0$. So we see:

To find a model of $F|K$ on which P is centered at a simple point means to find generators $x_1, \ldots, x_\ell \in \mathcal{O}_P$ such that $x_1, \ldots, x_k$ form a transcendence basis of $F|K$ and $x_{k+1}, \ldots, x_\ell$ together with the polynomials which define them over $K[x_1, \ldots, x_k]$ satisfy the assumption of the Multidimensional Hensel's Lemma.

What we have derived now is still quite vague, and before we can make more out of it, I'm sorry, you have to go to a course again.

4 A crash course in ramification theory

Throughout, we assume that $L|K$ is an algebraic extension, not necessarily finite, and that v is a *non-trivial* valuation on K. If w is a valuation on L which extends v, then there is a natural embedding of the value group vK of v in the value group wL of w. Similarly, there is a natural embedding of the residue field Kv of v in the residue field Lw of w. If both embeddings are onto (which we just express by writing $vK = wL$ and $Kv = Lw$), then the extension $(L, w)|(K, v)$ is called **immediate**. WARNING: It may happen that $vK \cong wL$ or $Kv \cong Lw$ although the corresponding embedding is not onto and therefore, the extension is not immediate. For example, every finite extension of the p-adics $(\mathbb{Q}_p, v_p)$ will again have a value group isomorphic to $\mathbb{Z}$, but $v_p p$ may not be anymore the smallest positive element in this value group.

We choose an arbitrary extension of v to the algebraic closure $\tilde{K}$ of K. Then for every $\sigma \in \mathrm{Aut}\,(\tilde{K}|K)$, the map

$$\tilde{v}\sigma = \tilde{v} \circ \sigma : \; L \ni a \;\mapsto\; \tilde{v}(\sigma a) \in \tilde{v}\tilde{K} \qquad (13)$$

is a valuation of L which extends v.

Theorem 13 *The set of all extensions of v from K to L is*

$$\{\tilde{v}\sigma \mid \sigma \text{ an embedding of } L \text{ in } \tilde{K} \text{ over } K\} \;.$$

(We say that "all extensions of v from K to L are **conjugate**".)

Corollary 14 *If $L|K$ is finite, then the number* g *of distinct extensions of v from K to L is smaller or equal to the extension degree $[L:K]$. More precisely,* g *is smaller or equal to the degree of the maximal separable subextension of $L|K$. In particular, if $L|K$ is purely inseparable, then v has a unique extension from K to L.*

Theorem 15 *Assume that $n := [L:K]$ is finite, and denote the extensions of v from K to L by $v_1, \ldots, v_{\mathrm{g}}$. Then for every $i \in \{1, \ldots, \mathrm{g}\}$, the* **ramification index** $\mathrm{e}_i = (v_iL : vK)$ *and the* **inertia degree** $\mathrm{f}_i = [Lv_i : Kv]$ *are finite, and we have the* **fundamental inequality**

$$n \;\geq\; \sum_{i=1}^{\mathrm{g}} \mathrm{e}_i \mathrm{f}_i \;\;. \tag{14}$$

From now on, let us assume that $L|K$ is normal. Hence, the set of all extensions of v from K to L is given by $\{\tilde{v}\sigma \mid \sigma \in \mathrm{Aut}\,(L|K)\}$. For simplicity, we denote the restriction of $\tilde{v}$ to L again by v. The valuation ring of v on L will be denoted by $\mathcal{O}_L$. We define distinguished subgroups of $G := \mathrm{Aut}\,(L|K)$. The subgroup

$$G^d := G^d(L|K,v) := \{\sigma \in G \mid \forall x \in \mathcal{O}_L :\; v\sigma x \geq 0\} \tag{15}$$

is called the **decomposition group of** $(L|K,v)$. It is easy to show that σ sends $\mathcal{O}_L$ into itself if and only if the valuations v and $v\sigma$ agree on L. Thus,

$$G^d = \{\sigma \in G \mid v\sigma = v \text{ on } L\}\;. \tag{16}$$

Further, the **inertia group** is defined to be

$$G^i := G^i(L|K,v) := \{\sigma \in \mathrm{Aut}\,(L|K) \mid \forall x \in \mathcal{O}_L : v(\sigma x - x) > 0\}\;, \tag{17}$$

and the **ramification group** is

$$G^r := G^r(L|K,v) := \{\sigma \in \mathrm{Aut}\,(L|K) \mid \forall x \in \mathcal{O}_L : v(\sigma x - x) > vx\}\;. \tag{18}$$

Let S denote the maximal separable extension of K in L (we call it the **separable closure of** K **in** L). The fixed fields of G^d, G^i and G^r in S are called **decomposition field**, **inertia field** and **ramification field of** $(L|K,v)$. For simplicity, let us abbreviate them by Z, T and V. (These letters refer to the german words "Zerlegungskörper", "Trägheitskörper" and "Verzweigungskörper".)

Remark 16 WARNING: In contrast to the classical definition used by other authors, we take decomposition field, inertia field and ramification field to be the fixed fields of the respective groups *in the maximal separable subextension.* The reason for this will become clear in Section 7.

By our definition, V, T and Z are separable-algebraic extensions of K, and S|V, S|T, S|Z are (not necessarily finite) Galois extensions. Further,

$$1 \subset G^r \subset G^i \subset G^d \subset G \;\text{ and thus, }\; \mathrm{S} \supset \mathrm{V} \supset \mathrm{T} \supset \mathrm{Z} \supset K\;. \tag{19}$$

(For the inclusion $G^i \subset G^d$ note that $vx \geq 0$ and $v(\sigma x - x) > 0$ implies that $v\sigma x \geq 0$.)

Theorem 17 *G^i and G^r are normal subgroups of G^d, and G^r is a normal subgroup of G^i. Therefore,* $\mathrm{T}|\mathrm{Z}$, $\mathrm{V}|\mathrm{Z}$ *and* $\mathrm{V}|\mathrm{T}$ *are (not necessarily finite) Galois extensions.*

First, we consider the decomposition field Z. In some sense, it represents all extensions of v from K to L.

Theorem 18 *a) $v\sigma = v\tau$ on L if and only if $\sigma\tau^{-1}$ is trivial on* Z.
b) $v\sigma = v$ on Z *if and only if σ is trivial on* Z.
c) The extension of v from Z *to L is unique.*
d) The extension $(\mathrm{Z}|K, v)$ *is immediate.*

WARNING: It is in general not true that $v\sigma \neq v\tau$ holds already on Z if it holds on L.
a) and b) are easy consequences of the definition of G^d. c) follows from b) by Theorem 13. For d), there is a simple proof using a trick which is mentioned in the paper [AX] by James Ax.

Now we turn to the inertia field T. Let $\mathcal{M}_L$ denote the valuation ideal of v on L (the unique maximal ideal of $\mathcal{O}_L$). For every $\sigma \in G^d(L|K, v)$ we have that $\sigma\mathcal{O}_L = \mathcal{O}_L$, and it follows that $\sigma\mathcal{M}_L = \mathcal{M}_L$. Hence, every such σ induces an automorphism $\overline{\sigma}$ of $\mathcal{O}_L/\mathcal{M}_L = Lv$ which satisfies $\overline{\sigma}\,\overline{a} = \overline{\sigma a}$. Since σ fixes K, it follows that $\overline{\sigma}$ fixes Kv.

Lemma 19 *Since $L|K$ is normal, the same is true for $Lv|Kv$. Moreover, the map*

$$G^d(L|K, v) \ni \sigma \;\mapsto\; \overline{\sigma} \in \mathrm{Aut}\,(Lv|Kv) \tag{20}$$

is a group homomorphism.

Theorem 20 *a) The homomorphism (20) is onto and induces an isomorphism*

$$\mathrm{Aut}\,(\mathrm{T}|\mathrm{Z}) \;=\; G^d/G^i \;\cong\; \mathrm{Aut}\,(\mathrm{T}v|\mathrm{Z}v)\;. \tag{21}$$

b) For every finite subextension $F|\mathrm{Z}$ of $\mathrm{T}|\mathrm{Z}$,

$$[F : \mathrm{Z}] \;=\; [Fv : \mathrm{Z}v]\;. \tag{22}$$

c) We have that $v\mathrm{T} = v\mathrm{Z} = vK$. Further, T$v$ *is the separable closure of Kv in Lv, and therefore,*

$$\mathrm{Aut}\,(\mathrm{T}v|\mathrm{Z}v) \;=\; \mathrm{Aut}\,(Lv|Kv)\;. \tag{23}$$

If $F|\mathrm{Z}$ is normal, then b) is an easy consequence of a). From this, the general assertion of b) follows by passing from F to the normal hull of the extension $F|\mathrm{Z}$ and then using the multiplicativity of the extension degree. c) follows from b) by use of the fundamental inequality.

We set $p := \mathrm{char}\, Kv$ if this is positive, and $p := 1$ if $\mathrm{char}\, Kv = 0$. Given any extension $\Delta \subset \Delta'$ of abelian groups, the **p'-divisible closure of Δ in Δ'** is defined to be the subgroup $\{\alpha \in \Delta' \mid \exists n \in \mathbb{N} :\ (p, n) = 1 \wedge n\alpha \in \Delta\}$ of all elements in Δ' whose order modulo Δ is prime to p.

Theorem 21 *a) There is an isomorphism*

$$\mathrm{Aut}\,(\mathrm{V}|\mathrm{T}) \;=\; G^i/G^r \;\cong\; \mathrm{Hom}\left(v\mathrm{V}/v\mathrm{T}\,,\,(\mathrm{T}v)^\times\right)\;, \tag{24}$$

where the character group on the right hand side is the full character group of the abelian group $v\mathrm{V}/v\mathrm{T}$. Since this group is abelian, $\mathrm{V}|\mathrm{T}$ *is an abelian Galois extension.*
b) For every finite subextension $F|\mathrm{T}$ of $\mathrm{V}|\mathrm{T}$*,*

$$[F:\mathrm{T}] \;=\; (vF:v\mathrm{T})\;. \tag{25}$$

c) $\mathrm{V}v=\mathrm{T}v$*, and $v\mathrm{V}$ is the p'-divisible closure of vK in vL.*

b) follows from a) since for a finite extension $F|\mathrm{T}$, the group $vF/v\mathrm{T}$ is finite and therefore, there exists an isomorphism of $vF/v\mathrm{T}$ onto its full character group. The equality $\mathrm{V}v=\mathrm{T}v$ follows from b) by the fundamental inequality. The second assertion of c) follows from the next theorem and the fact that the order of all elements in $(\mathrm{T}v)^\times$ and thus also of all elements in $\mathrm{Hom}\,(v\mathrm{V}/v\mathrm{T}\,,\,(\mathrm{T}v)^\times)$ is prime to p.

Theorem 22 *The ramification group G^r is a p-group and therefore,* $\mathrm{S}|\mathrm{V}$ *is a p-extension. Further, $vL/v\mathrm{V}$ is a p-group, and the residue field extension $Lv|\mathrm{V}v$ is purely inseparable. If* $\mathrm{char}\,Kv=0$*, then* $\mathrm{V}=\mathrm{S}=L$*.*

Remark 23 Every p-extension is a tower of Galois extensions of degree p. In characteristic p, all of them are Artin–Schreier–extensions, as we have mentioned in Example 7.

From the last theorem it follows that there is a canonical isomorphism

$$\mathrm{Hom}\left(v\mathrm{V}/v\mathrm{T}\,,\,(\mathrm{T}v)^\times\right) \;\cong\; \mathrm{Hom}\left(vL/vK\,,\,(Lv)^\times\right)\;. \tag{26}$$

We summarize our main results in the following table:

Galois group	**field**		**value group**	**residue field**
	L		vL	Lv
	purely inseparable		division by p	purely inseparable
1	S	maximal separable subextension		
	Galois p-extension			
$G^r(L\mid K,v)$	V	ramification field	$(vL\mid vK)^{p'}$	$(Lv\mid Kv)^{\rm sep}$
Char	abelian Galois p'-extension		division prime to p	‖
$G^i(L\mid K,v)$	T	inertia field	vK	$(Lv\mid Kv)^{\rm sep}$
Aut $(Lv\mid Kv)$	Galois		‖	Galois
$G^d(L\mid K,v)$	Z	decomposition field	vK	Kv
	immediate		‖	‖
Aut $(L\mid K)$	K		vK	Kv

where $(vL|vK)^{p'}$ denotes the p'-divisible closure of vK in vL, $(Lv|Kv)^{\rm sep}$ denotes the separable closure of Kv in Lv, and Char denotes the character group (26).

We state two more useful theorems from ramification theory. If we have two subfields K, L of a field M (in our case, we will have the situation that $L \subset \tilde{K}$) then $K.L$ will denote the smallest subfield of M which contains both K and L; it is called the **field compositum of** K **and** L.

Theorem 24 *If $K \subseteq K' \subseteq L$, then the decomposition field of the normal extension $(L|K',v)$ is* $\mathrm{Z}.K'$*, its inertia field is* $\mathrm{T}.K'$*, and its ramification field is* $\mathrm{V}.K'$ *.*

Theorem 25 *If $E|K$ is a normal subextension of $L|K$, then the decomposition field of $(E|K,v)$ is* $\mathrm{Z}\cap E$*, its inertia field is* $\mathrm{T}\cap E$*, and its ramification field is* $\mathrm{V}\cap E$*.*

If we take for $L|K$ the normal extension $\tilde{K}|K$, then we speak of **absolute ramification theory**. The fixed fields K^d, K^i and K^r of $G^d(\tilde{K}|K,v)$, $G^i(\tilde{K}|K,v)$ and $G^r(\tilde{K}|K,v)$ in the separable-algebraic closure $K^{\rm sep}$ of K are called **absolute decomposition field**, **absolute inertia field** and **absolute ramification field of** (K,v) (with respect to the given extension of v from K to its algebraic closure $\tilde{K}$). If $\operatorname{char} Kv = 0$, then by Theorem 22, $K^r = K^{\rm sep} = \tilde{K}$.

Lemma 26 *Fix an extension of v from K to $\tilde{K}$. Then the absolute inertia field of (K,v) is the unique maximal extension of (K,v) within the absolute ramification field having the same value group as K.*

Proof: Let $(L|K,v)$ be any extension within the absolute ramification field s.t. $vL = vK$. Then $vL^i = vL = vK = vK^i$. By Theorem 24, $L^i = L.K^i$. Further, $L \subseteq K^r$ yields that $L.K^i \subseteq K^r$. If the subextension $L^i|K^i$ of $K^r|K^i$ were proper, it contained a proper finite subextension $L_1|K^i$, and by part b) of Theorem 21 we had that $vK^i \subsetneqq vL_1 \subseteq vL^i$. As this contradicts the fact that $vL^i = vK^i$, we find that $L^i = K^i$, that is, $L \subseteq K^i$. $\square$

From part c) of Theorem 18 we infer that the extension of v from K^d to $\tilde{K}$ is unique. On the other hand, if L is any extension field of K within K^d, then by Theorem 24, $K^d = L^d$. Thus, if $L \neq K^d$, then it follows from part b) of Theorem 18 that there are at least two distinct extensions of v from L to K^d and thus also to $\tilde{K} = \tilde{L}$. This proves that the absolute decomposition field K^d is a minimal algebraic extension of K admitting a unique extension of v to its algebraic closure. So it is the minimal algebraic extension of K which is henselian (cf. Lemma 4). We call it the **henselization of** (K,v) **in** $(\tilde{K},v)$. Instead of K^d, we also write K^h. A valued field is henselian if and only if it is equal to its henselization. Henselizations have the following universal property:

Theorem 27 *Let (K,v) be an arbitrary valued field and (L,v) any henselian extension field of (K,v). Then there is a unique embedding of (K^h,v) in (L,v) over K.*

From the definition of the henselization as a decomposition field, together with part d) of Theorem 18, we obtain another very important property of the henselization:

Theorem 28 *The henselization (K^h,v) is an immediate extension of (K,v).*

Corollary 29 *Every maximal valued field is henselian. In particular, $(K((t)),v_t)$ is henselian.*

Finally, we employ Theorem 24 to obtain:

Theorem 30 *If $K'|K$ is an algebraic extension, then the henselization of K' is $K'.K^h$.*

5 A valuation theoretical interpretation of local uniformization

We return to where we stopped before entering the crash course in ramification theory. The first question is: what does it mean that $x_{k+1},\dots,x_\ell$ together with the polynomials which define them over $K[x_1,\dots,x_k]$ satisfy the assumption of the Multidimensional Hensel's Lemma? First of all, general valuation theory tells us that a rational function field $K(x_1,\dots,x_k)$ is much too small to be henselian (unless the valuation is trivial). But we could pass to the henselization of $(K(x_1,\dots,x_k),P)$. So does it mean that $x_{k+1},\dots,x_\ell$ lie in this henselization? If we look closely, there is something fishy in the way we have satisfied the assumption of the Multidimensional Hensel's Lemma. Instead of talking about a so-called "approximative root" $b=(b_1,\dots,b_n)$ which lies in the henselian field we wish to work in, we have talked already about the actual root, and we do not know where it lies. Let us modify our Example 3 a bit to see that it does not always lie in the henselization of $(K(x_1,\dots,x_k),v)$.

Example 31 Let us consider the function field $\mathbb{Q}(x,y)$ where $y^2=x^3$. Take the place given by $xP=2$, $yP=2\sqrt{2}$. The minimal polynomial of y over $\mathbb{Q}(x)$ is $f(Y)=Y^2-x^3$. As $f(y)=0$, we have that $v_Pf(y)=\infty>0$. As $f'(Y)=2Y$, we have that $v_Pf'(y)=v_P2y=0$ (since $2yP=4\sqrt{2}\neq 0$). Hence, f and y satisfy the assumption (4) of Hensel's Lemma. But y does not lie in the henselization of $(\mathbb{Q}(x),P)$. Indeed, P on $\mathbb{Q}(x)$ is just the place coming from the evaluation homomorphism given by $x\mapsto 2$; hence, $\mathbb{Q}(x)^hP=\mathbb{Q}(x)P=\mathbb{Q}$. But $\mathbb{Q}(x,y)P\neq\mathbb{Q}$ since $yP=2\sqrt{2}\notin\mathbb{Q}$. $\diamondsuit$

So we see that extensions of the residue field can play a role. We could try to suppress them by requiring that K be algebraically closed. This works for those P for which $FP|K$ is algebraic, but if this is not the case, then we have no chance to avoid them. At least, we can show that they are the only reason why $x_{k+1},\dots,x_\ell$ may not lie in the henselization of $(K(x_1,\dots,x_k),P)$.

Theorem 32 *If $x_{k+1},\dots,x_\ell$ together with the polynomials f_i which define them over $K[x_1,\dots,x_k]$ satisfy the assumption (5) of the Multidimensional Hensel's Lemma, then the elements $x_{k+1},\dots,x_\ell$ lie in the absolute inertia field of $(K(x_1,\dots,x_k),P)$, and the extension $FP|K(x_1P,\dots,x_kP)$ is separable-algebraic. If in addition P is a rational place, then $x_{k+1},\dots,x_\ell$ lie in the henselization of $(K(x_1,\dots,x_k),P)$.*

Proof: Denote by (L,P) the absolute inertia field of $(K(x_1,\dots,x_k),P)$. First,

$$\det J_{\tilde{f}P}(x_{k+1}P,\dots,x_\ell P)\;=\;\det J_{\tilde{f}}(x_{k+1},\dots,x_\ell)P\;\neq\;0 \tag{27}$$

and the fact that the f_iP are polynomials over $K(x_1P,\dots,x_kP)$ imply that $x_{k+1}P,\dots,x_\ell P$ are separable algebraic over $K(x_1P,\dots,x_kP)$ (cf. [L], Chapter X, §7, Proposition 8). On the other hand, LP is the separable-algebraic closure of $K(x_1,\dots,x_k)P$. Therefore, there are elements $b_1, \dots, b_n$ in L such that $b_i P = x_{k+i} P$. Since (L, P) is henselian, the Multidimensional Hensel's Lemma now shows the existence of a common root $(b_1', \dots, b_n') \in L^n$ of the f_i such that $b_i' P = b_i P = x_{k+i} P$. But the uniqueness assertion of the Multidimensional Hensel's Lemma also holds in the algebraic closure $\tilde{L}$ of L (which is also henselian). So we find that $(b_1', \dots, b_n') = (x_{k+1}, \dots, x_\ell)$. Hence, $x_{k+1}, \dots, x_\ell$ are elements of L.

If we have in addition that P is a rational place, then $x_{k+1}P, \dots, x_\ell P \in K$. In this case, we can choose $b_1, \dots, b_n$ and $b_1', \dots, b_n'$ already in the henselization of $(K(x_1, \dots, x_k), P)$, which implies that also $x_{k+1}, \dots, x_\ell$ lie in this henselization. □

Since the absolute inertia field is a separable-algebraic extension and every rational function field is separable, we obtain:

Corollary 33 *If the place P of $F|K$ admits local uniformization, then $F|K$ is separable.*

We see that we are slowly entering the **structure theory of valued function fields**, that is, the algebraic theory of function fields $F|K$ equipped with a valuation (which may or may not be trivial on K). Later, we will see some main results from this theory (Theorems 63 and 65).

Given a place P of F, not necessarily trivial on K, we will say that $(F|K, P)$ is **inertially generated** if there is a transcendence basis T of $F|K$ such that (F, P) lies in the absolute inertia field of $(K(T), P)$. Similarly, $(F|K, P)$ is **henselian generated** if there is a transcendence basis T of $F|K$ such that (F, P) lies in henselization of $(K(T), P)$. Now we see a valuation theoretical interpretation of local uniformization:

Theorem 34 *If the place P of $F|K$ admits local uniformization, then $(F|K, P)$ is inertially generated. If in addition $FP = K$, then $(F|K, P)$ is henselian generated.*

So if local uniformization holds in arbitrary characteristic for every $F|K$ with perfect K, then for every place P of $F|K$, the valued function field $(F|K, P)$ is inertially generated. In the context of valuation theory, at least to me, this is a quite surprising assertion. Here is our first open problem:

Open Problem 1: Is the converse also true, i.e., if $(F|K, P)$ is inertially generated, does it then admit local uniformization?

A partial answer to this question is given in the papers [K5] and [K6]. What we see is that in order to get local uniformization, one has to avoid ramification. Indeed, ramification is the valuation theoretical symptom of branching, the violation of the Implicit Function Theorem at a point of the variety. Let us look again at our simple Example 3:

Example 35 Consider the function field $\mathbb{R}(x, y)$ where $y^2 = x^3$. Take the place given by $xP = 0 = yP$. As P on $K(x)$ originates from the evaluation homomorphism given by $x \mapsto 0$, we have that $v_P K(x) = \mathbb{Z}$, with $v_P x = 1$ the smallest positive element in the value group. Now compute $v_P y$. We have that $y^2 = x^3$, whence $2vy = vy^2 = vx^3 = 3$.

It follows that $vy = 3/2 \notin \mathbb{Z}$, that is, the extension $(K(x,y)|K(x),P)$ is **ramified**, or in other words, $(K(x,y),P)$ does not lie in the absolute inertia field of $(K(x),P)$. We see that we have ramification at the singular point $(0,0)$. As an exercise, you may check that $(K(x,y),Q)$ lies in the absolute inertia field of $(K(x),Q)$ whenever $xQ \neq 0$. $\diamondsuit$

6 The Abhyankar inequality for valued field extensions

We may now ask ourselves: How could we show that for a given place P of $F|K$, the valued function field $(F|K,P)$ is inertially generated? Throughout this section, we will write v for the valuation v_P associated with the place P.

Example 36 Let us start with the most simple case, where $\operatorname{trdeg} F|K = 1$. Assuming that P is not trivial on F (if it is trivial, then local uniformization is trivial if $F|K$ is separable), we pick some $z \in F$ such that $zP = 0$. As we have seen in Example 1, $vK(z) = \mathbb{Z}$. Since $z \notin K$ and $\operatorname{trdeg} F|K = 1$, we know that $F|K(z)$ is algebraic; since $F|K$ is finitely generated, it follows that $F|K$ is finite. From Theorem 15 we infer that the ramification index $(vF : vK(z))$ is finite. Therefore, vF is again isomorphic to $\mathbb{Z}$ and we can pick some $x \in F$ such that $x \in \mathcal{O}_P$ and $vF = \mathbb{Z}vx$.

We have achieved that $vF = vK(x)$. If the residue characteristic $\operatorname{char} FP = \operatorname{char} K$ is 0, then we know from Lemma 26 that the absolute inertia field $K(x)^i$ is the unique maximal extension still having the same value group as $K(x)$. In this case, we find that F must lie in this absolute inertia field, and we have proved that $(F|K,P)$ is inertially generated. But we are lost, it seems, if the characteristic is $p > 0$, since in this case, the absolute inertia field is not necessarily the maximal algebraic extension of $K(x)$ having the same value group. To solve this case, we yet have to learn some additional tools. $\diamondsuit$

In this example, the fact that vF was finitely generated played a crucial role. As we have shown, this is always the case if $\operatorname{trdeg} F|K = 1$. But in general, we can't expect this to hold. We will give counterexamples in Section 13. But prior to the negative, we want to start with the positive, i.e., criteria for the value group to be finitely generated.

The following theorem has turned out in the last years to be amazingly universal in many different applications of valuation theory. It plays an important role in algebraic geometry as well as in the model theory of valued fields, in real algebraic geometry, or in the structure theory of exponential Hardy fields (= nonarchimedean ordered fields which encode the asymptotic behaviour of real-valued functions including exp and log, cf. [KK]). (See also [V], Theorem 5.5, or [B], Chapter VI, §10.3, Theorem 1.)

Theorem 37 *Let $(L|K,P)$ be an extension of valued fields. Take elements $x_i, y_j \in L$, $i \in I$, $j \in J$, such that the values vx_i, $i \in I$, are rationally independent over vK, and the residues y_jP, $i \in J$, are algebraically independent over KP. Then the elements x_i, y_j,*

$i \in I$, $j \in J$, are algebraically independent over K. Then the elements x_i, y_j, $i \in I$, $j \in J$, are algebraically independent over K. Moreover, if we write

$$f \;=\; \sum_k c_k \prod_{i\in I} x_i^{\mu_{k,i}} \prod_{j\in J} y_j^{\nu_{k,j}} \in K[x_i, y_j \mid i \in I, j \in J]$$

in such a way that for every $k \neq \ell$ there is some i s.t. $\mu_{k,i} \neq \mu_{\ell,i}$ or some j s.t. $\nu_{k,j} \neq \nu_{\ell,j}$, then

$$vf \;=\; \min_k \left(v\,(c_k \prod_{i\in I} x_i^{\mu_{k,i}} \prod_{j\in J} y_j^{\nu_{k,j}}) \right) \;=\; \min_k \left(v\,c_k \;+\; \sum_{i\in I} \mu_{k,i} v x_i \right) \;.$$

That is, the value of each polynomial in $K[x_i, y_j \mid i \in I, j \in J]$ is equal to the least of the values of its monomial summands. In particular, this implies:

$$vK(x_i, y_j \mid i \in I, j \in J) \;=\; vK \oplus \bigoplus_{i\in I} \mathbb{Z} v x_i \tag{28}$$

$$K(x_i, y_j \mid i \in I, j \in J)P \;=\; KP\,(y_j P \mid j \in J)\;. \tag{29}$$

The valuation v on $K(x_i, y_j \mid i \in I, j \in J)$ is uniquely determined by its restriction to K and the values vx_i. The place P is uniquely determined by its restriction to K, the values vx_i and the residues y_jP.

Proof: Suppose that vf is not equal to the least of the values of the monomial summands. Then this will remain true if we omit all monomials that are not of minimal value; hence without loss of generality, we may assume that all monomials in f have equal value. But then it follows from the linear independence of the values vx_i, $i \in I$, that every element x_i has to appear to the same power $\mu_{k,i}$ in each of these monomials. Dividing f by $x_i^{\mu_{k,i}}$ for all $i \in I$, we obtain a polynomial $g \in K[y_j \mid j \in J]$ with summands of equal value, whose value is still bigger than the value of all of its monomials. Since the elements y_j all have value zero, we find that all nonzero coefficients must have equal value. After dividing by one of them, we can assume that all monomials have value zero, but that the value of g is bigger than 0. Passing to the residue field through the residue map P, we obtain a nontrivial polynomial $gP \in KP[Y_j \mid j \in J]$ such that $gP(y_jP \mid j \in J) = 0$. But this contradicts our assumption that the residues y_jP be algebraically independent over KP. So vf must be equal to the least of the values of the monomials. Hence for some k,

$$vf \;= v\,(c_k \prod_{i\in I} x_i^{\mu_{k,i}} \prod_{j\in J} y_j^{\nu_{k,j}}) \;=\; vc_k + \sum_i \mu_{k,i}\, v x_i \;\in\; vK \;\oplus\; \bigoplus_{i\in I} \mathbb{Z} v x_i$$

(the latter sum is direct by our assumption on the values vx_i). Since the value of a quotient f/g is $vf - vg$, we see that also these values lie in $vK \oplus \bigoplus_{i\in I} \mathbb{Z} v x_i$. By the choice of quotients of suitable polynomials, one shows that every value in $vK \oplus \bigoplus_{i\in I} \mathbb{Z} v x_i$ appears as a value of some element in $K(x_i, y_j \mid i \in I, j \in J)$. Observe that the prescription of the values vx_i determines which monomials in a given polynomial are the ones of least value, and that their value is computed by the above formula only by use of the restriction of v to K and the values vx_i. Hence, v is uniquely determined on $K(x_i, y_j \mid i \in I, j \in J)$ by these data.

Now assume that the value of some quotient f/g is zero; we want to determine its residue. Since $vf = vg$, the monomials of minimal value in f and g all contain x_i to the same power $\mu_{k,i}$, for every $i \in I$. After dividing f and g by $c\prod_i x_i^{\mu_{k,i}}$ with a suitable constant $c \in K$, we may assume that f and g have value zero and that the monomials of minimal value in f and g lie in $\mathcal{O}_P[y_j \mid j \in J]$. The residues of f and g are just the sum of the residues of these monomials of minimal value 0. For $c_k \prod_j y_j^{\nu_{k,j}} \in \mathcal{O}_P[y_j \mid j \in J]$ we have

$$(c_k \prod_j y_j^{\nu_{k,j}})P \;\in\; KP(y_jP \mid j \in J)\;.$$

It follows that $(f/g)P = fP/gP \in KP(y_jP \mid j \in J)$. By the choice of quotients of suitable polynomials, one shows that every element in $KP(y_jP \mid j \in J)$ appears as the residue of some element in $K(x_i, y_j \mid i \in I, j \in J)$. To determine whether an element of $K(x_i, y_j \mid i \in I, j \in J)$ lies in $\mathcal{O}_P$, we need to know its value, which is is uniquely determined by the restriction of v to K and the values vx_i. For sn element of $\mathcal{O}_P$, its residue is uniquely determined by the restriction of P to K and the residues y_jP. Altogether, we find that the place P is uniquely determined by its restriction to K, the values vx_i and the residues y_jP. □

Corollary 38 *Let $(L|K,P)$ be an extension of valued fields of finite transcendence degree. Then*

$$\mathrm{trdeg}\, L|K \;\geq\; \mathrm{trdeg}\, LP|KP \,+\, \mathrm{rr}\,(vL/vK)\;. \tag{30}$$

If in addition $L|K$ is a function field, and if equality holds in (30), then the extensions $vL|vK$ and $LP|KP$ are finitely generated. In particular, if P is trivial on K, then vL is a product of finitely many (namely, $\mathrm{rr}\, vL$) copies of $\mathbb{Z}$, and LP is again a function field over K.

Proof: Choose elements $x_1, \ldots, x_\rho, y_1, \ldots, y_\tau \in L$ such that the values $vx_1, \ldots, vx_\rho$ are rationally independent over vK and the residues $y_1P, \ldots, y_\tau P$ are algebraically independent over KP. Then by the foregoing theorem, $\rho + \tau \leq \mathrm{trdeg}\, L|K$. This proves that $\mathrm{trdeg}\, LP|KP$ and the rational rank of vL/vK are finite. Therefore, we may choose the elements x_i, y_j such that $\tau = \mathrm{trdeg}\, LP|KP$ and $\rho = \mathrm{rr}\,(vL/vK)$ to obtain inequality (30).

Assume that this is an equality. This means that for $L_0 := K(x_1, \ldots, x_\rho, y_1, \ldots, y_\tau)$, the extension $L|L_0$ is algebraic. Since $L|K$ is finitely generated, it follows that $L|L_0$ is finite. By the fundamental inequality, this yields that $vL|vL_0$ and $LP|L_0P$ are finite extensions. Since already $vL_0|vK$ and $L_0P|KP$ are finitely generated by the foregoing lemma, it follows that also $vL|vK$ and $LP|KP$ are finitely generated. □

If P is a place of $F|K$, then (30) reads as follows:

$$\operatorname{trdeg} F|K \;\geq\; \operatorname{trdeg} FP|K \;+\; \operatorname{rr} vF \;. \tag{31}$$

The famous **Abhyankar inequality** is a generalization of this inequality to the case of noetherian local rings (see [V]). We call P an **Abhyankar place** if equality holds in (31).

The rank of an ordered abelian group is always smaller or equal to its rational rank. This is seen as follows. If G_1 is a subgroup of G, then its divisible hull $\mathbb{Q} \otimes G_1$ lies in the convex hull of G_1 in $\mathbb{Q} \otimes G$. Hence if G_1 is a proper convex subgroup of G, then $\mathbb{Q} \otimes G_1$ is a proper convex subgroup of $\mathbb{Q} \otimes G$ and thus, $\dim_{\mathbb{Q}} \mathbb{Q} \otimes G_1 < \dim_{\mathbb{Q}} \mathbb{Q} \otimes G$. It follows that if $\{0\} = G_0 \subsetneqq G_1 \subsetneqq \ldots \subsetneqq G_n = G$ is a chain of convex subgroups of G, then $\operatorname{rr} G = \dim_{\mathbb{Q}} \mathbb{Q} \otimes G \geq n$. In view of (31), this proves that the rank of a place P of a function field $F|K$ cannot exceed $\operatorname{trdeg} F|K$ and thus is finite. We say that P is of **maximal rank** if the rank is equal to $\operatorname{trdeg} F|K$.

If $\operatorname{trdeg} F|K = 1$, then every place P of $F|K$ is an Abhyankar place. It is of maximal rank if and only if it is non-trivial. Indeed, if vF is not trivial, then $\operatorname{rr} vF \geq 1$, and it follows from (31) that $\operatorname{trdeg} F|K = 1 = \operatorname{rr} vF$. Then also the rank is 1 since a group of rational rank 1 is a non-trivial subgroup of $\mathbb{Q}$. If on the other hand vF is trivial, then P is an isomorphism on F so that $\operatorname{trdeg} F|K = 1 = \operatorname{trdeg} FP|K$.

Using Corollary 38, we can now generalize our construction given in Example 36. Let P be an arbitrary place of $F|K$. We set $\rho = \operatorname{rr} vF$ and $\tau = \operatorname{trdeg} FP|K$. We take elements $x_1, \ldots, x_\rho \in F$ such that $vx_1, \ldots, vx_\rho$ are rationally independent elements in vF. Further, we take elements $y_1, \ldots, y_\tau \in F$ such that $y_1P, \ldots, y_\tau P$ are algebraically independent over K. Then by Theorem 37, $x_1, \ldots, x_\rho, y_1, \ldots, y_\tau$ are algebraically independent over K. The restriction of P to $K(x_1, \ldots, x_\rho, y_1, \ldots, y_\tau)$ is an Abhyankar place. We fix this situation for later use. We call

$$\left.\begin{array}{l} F_0 \;:=\; K(x_1, \ldots, x_\rho, y_1, \ldots, y_\tau) \text{ with} \\ \rho = \operatorname{rr} vF \;\text{ and }\; \tau = \operatorname{trdeg} FP|K, \\ vx_1, \ldots, vx_\rho \text{ rationally independent in } vF\text{, and} \\ y_1P, \ldots, y_\tau P \text{ algebraically independent over } K \end{array}\right\} \tag{32}$$

an **Abhyankar subfunction field** of $(F|K, P)$.

Now let us assume in addition that P is an Abhyankar place of $F|K$. That is, $\rho + \tau = \operatorname{trdeg} F|K$. It follows that $x_1, \ldots, x_\rho, y_1, \ldots, y_\tau$ is a transcendence basis of $F|K$. We refine our choice of these elements as follows. From Corollary 38 we know that vF is product of ρ copies of $\mathbb{Z}$. So we can choose $x_1, \ldots, x_\rho \in \mathcal{O}_P$ in such a way that $vF = \mathbb{Z}vx_1 \oplus \ldots \oplus \mathbb{Z}vx_\rho$, which implies that $vF = vK(x_1, \ldots, x_\rho)$. From Corollary 38 we also know that $FP|K$ is finitely generated. We shall also assume that $FP|K$ is separable. Then it follows that there is a separating transcendence basis for $FP|K$. We choose $y_1, \ldots, y_\tau \in \mathcal{O}_P$ in such a way that $y_1P, \ldots, y_\tau P$ is such a separating transcendence basis. Now we can choose some $a \in FP$ such that $FP = K(y_1P, \ldots, y_\tau P, a)$. We take a monic polynomial f with coefficients in the valuation ring of (F_0, P) such that its reduction fv is the minimal polynomial of a over $F_0P = K(y_1P, \ldots, y_\tau P)$. Since $a \in FP$ is separable-algebraic over $K(y_1P, \ldots, y_\tau P)$, by Hensel's Lemma (Simple Root Version) there exists a root η of f in the henselization of (F, P) such that $\eta P = a$. Take $\sigma \in \mathrm{Aut}\,(\tilde{F}_0|F_0)$ such that $v(\sigma x - x) > 0$ for all x in the valuation ring of P on $\tilde{F}$. Then in particular, $v(\sigma\eta - \eta) > 0$. But if $\sigma\eta \neq \eta$, then it follows from $\deg(f) = \deg(fv)$ that $(\sigma\eta)P \neq \eta P$, i.e., $v(\sigma\eta - \eta) = 0$. Hence, $\sigma\eta = \eta$, which shows that η lies in the absolute inertia field of F_0.

Now the field $F_0(\eta)$ has the same value group and residue field as F, and it is contained in the henselization of F. Hence by Theorem 28,

$$(F^h|F_0(\eta)^h, P) \tag{33}$$

is an immediate algebraic extension. As η lies in the absolute inertia field of F_0 and this field is henselian, we have that $F_0(\eta)^h$ is a subfield of this absolute inertia field. If we could show that $F^h = F_0(\eta)^h$, then F itself would lie in this absolute inertia field, which would prove that $(F|K, P)$ is inertially generated. If the residue characteristic $\mathrm{char}\, FP = \mathrm{char}\, K$ is 0, then again Lemma 26 tells us that the absolute inertia field of (F_0, P) is the unique maximal extension having the same value group as F_0; so F^h must be a subfield of it. Hence in characteristic 0 we have now shown that $(F|K, P)$ is inertially generated. But what happens in positive characteristic? Can the extension (33) be non-trivial? To answer this question, we have to take a closer look at the main problem of valuation theory in positive characteristic.

7 The defect

Assume that (K, v) is henselian and $(L|K, v)$ is a finite extension of degree n. Then we have to deal only with a single ramification index e and a single inertia degree f. Hence, the fundamental inequality now reads as

$$n \geq \mathrm{e}\,\mathrm{f}\,. \tag{34}$$

If L is contained in K^i then by Theorem 20, $n = \mathrm{f}$. If $K = K^i$ and L is contained in K^r, then by Theorem 21, $n = \mathrm{e}$. Putting these observations together (using Theorems 24 and 25 and the fact that extension degree, ramification index and inertia degree are multiplicative), one finds:

Lemma 39 *If (K, v) is henselian and $L|K$ is a finite subextension of $K^r|K$, then it satisfies the* **fundamental equality**

$$n = \mathrm{e}\,\mathrm{f}\,. \tag{35}$$

Hence, an inequality can only result from some part of the extension which lies beyond the absolute ramification field. So Theorem 22 shows that the missing factor can only be a power of p. In this way, one proves the important **Lemma of Ostrowski**:

Theorem 40 *Set* $p := \mathrm{char}\, Kv$ *if this is positive, and* $p := 1$ *if* $\mathrm{char}\, Kv = 0$. *If* (K, v) *is henselian and* $L|K$ *is of degree* n, *then*

$$n = \mathrm{d}\,\mathrm{e}\,\mathrm{f}\,, \tag{36}$$

where d *is a power of* p. *In particular, if* $\mathrm{char}\, Kv = 0$, *then we always have the fundamental equality* $n = \mathrm{e}\,\mathrm{f}$.

The integer $\mathrm{d} \geq 1$ is called the **defect** of the extension $(L|K, v)$. This can also be taken as a definition for the defect if (K, v) is not henselian, but the extension of v from K to L is unique. We note:

Corollary 41 *If* (K, v) *is henselian and* $(L|K, v)$ *is a finite immediate extension, then the defect of* $(L|K, v)$ *is equal to* $[L : K]$.

A henselian field (K, v) is called **defectless** if every finite extension has trivial defect $\mathrm{d} = 1$. In rigid analysis, this is also called **stable**. A not necessarily henselian field is called defectless if for every finite extension of it, equality holds in the fundamental inequality (14) (if the field is henselian, this coincides with our first definition). A proof of the next theorem can be found in [KU2] and, partially, also in [E].

Theorem 42 *A valued field is a defectless field if and only if its henselization is.*

We also note the following fact, which is easy to prove:

Lemma 43 *Every finite extension field of a defectless field is again a defectless field.*

The following are examples of defectless fields:

(DF1) All valued fields with residue characteristic 0. This is a direct consequence of the Lemma of Ostrowski.

(DF2) Every discretely valued field of characteristic 0. An easy argument shows that every finite extension with non-trivial defect of a discretely valued field must be inseparable. In particular, the field $(\mathbb{Q}_p, v_p)$ of p-adic numbers with its p-adic valuation, and all of its subfields, are defectless fields.

(DF3) All maximal fields (and hence also all power series fields, see Section 8) are defectless fields. For the proof, see [R1] or [KU2].

We have seen that the extensions beyond the absolute ramification field are responsible for non-trivial defects. To get this picture, we have chosen a modified approach to ramification theory (see Remark 16). We have shifted the purely inseparable extensions to the top (cf. our table). In fact, that is where the purely inseparable extensions belong, because from the ramification theoretical point of view, they can be nasty, and in this respect, they have much in common with the extension S|V.

The defect, appearing only for positive residue characteristic, is essentially the cause of the problems that we have in algebraic geometry as well as in the model theory of valued fields, in positive characteristic. Therefore, it is very important that you get a feeling for what the defect is. Let us look at three main examples. The first one is the most basic and was probably already known to most of the early valuation theorists. But it seems reasonable to attribute it to F. K. Schmidt.

Example 44 We consider the power series field $K((t))$ with its t-adic valuation $v = v_t$. We have already remarked in Example 2 that $K((t))|K(t)$ is transcendental. So we can choose an element $s \in K((t))$ which is transcendental over $K(t)$. Since $vK((t)) = \mathbb{Z} = vK(t)$ and $K((t))v = K = K(t)v$, the extension $(K((t))|K(t), v)$ is immediate. The same must be true for the subextension $(K(t, s)|K(t), v)$ and thus also for $(K(t, s)|K(t, s^p), v)$. The latter extension is purely inseparable of degree p (since s, t are algebraically independent over K, the extension $K(s)|K(s^p)$ is linearly disjoint from $K(t, s^p)|K(s^p)$). Hence, Corollary 14 shows that there is only one extension of the valuation v from $K(t, s^p)$ to $K(t, s)$. Consequently, its defect is p. ♢

To give an example of a henselian field which is not defectless, we build on the foregoing example.

Example 45 By Theorem 27, there is a henselization $(K(t, s), v)^h$ of $(K(t, s), v)$ in the henselian field $K((t))$ and a henselization $(K(t, s^p), v)^h$ of $(K(t, s^p), v)$ in $(K(t, s), v)^h$. We find that the extension $(K(t, s), v)^h|(K(t, s^p), v)^h$ is again purely inseparable of degree p. Indeed, $K(t, s)|K(t, s^p)$ is linearly disjoint from the separable extension $K(t, s^p)^h|K(t, s^p)$, and by virtue of Corollary 30, $K(t, s)^h = K(t, s).K(t, s^p)^h$. Also for this extension, we have that $\mathrm{e} = \mathrm{f} = 1$ and again, the defect is p. Note that by what we have said earlier, an extension of degree p with non-trivial defect over a discretely valued field like $(K(t, s^p), v)^h$ can only be purely inseparable. ♢

Now we will give an example of a finite *separable* extension with non-trivial defect. It seems to be the generic example for our purposes since its importance is also known in algebraic geometry.

Example 46 Take an arbitrary field K of characteristic $p > 0$, and t to be transcendental over K. On $K(t)$ we take the t-adic valuation $v = v_t$. We set $L := K(t^{1/p^i} \mid i \in \mathbb{N})$. This is a purely inseparable extension of $K(t)$; if K is perfect, then it is the perfect hull of $K(t)$. By Corollary 14, v has a unique extension to L. We set $L_k := K(t^{1/p^k})$ for every $k \in \mathbb{N}$; so $L = \bigcup_{k\in\mathbb{N}} L_k$. We observe that $1/p^k = vt^{1/p^k} \in vL_k$, so $(vL_k : vK(t)) \geq p^k$. Now the fundamental inequality shows that $(vL_k : vK(t)) = p^k$ and that $L_k v = K(t)v = K$.

The former shows that $vL_k = \frac{1}{p^k}\mathbb{Z}$. We obtain that $vL = \bigcup_{k\in\mathbb{N}} vL_k = \frac{1}{p^\infty}\mathbb{Z}$ and that $Lv = \bigcup_{k\in\mathbb{N}} L_k v = K$.

We consider the extension $L(\vartheta)|L$ generated by a root ϑ of the Artin–Schreier–polynomial $X^p - X - \frac{1}{t}$. We set

$$\vartheta_k := \sum_{i=1}^{k} t^{-1/p^i} \tag{37}$$

and compute

$$\vartheta_k^p - \vartheta_k - \frac{1}{t} = \sum_{i=1}^{k} t^{-1/p^{i-1}} - \sum_{i=1}^{k} t^{-1/p^i} - t^{-1} = \sum_{i=0}^{k-1} t^{-1/p^i} - \sum_{i=1}^{k} t^{-1/p^i} - t^{-1} = -t^{-1/p^k} . \tag{38}$$

Therefore,

$$(\vartheta - \vartheta_k)^p - (\vartheta - \vartheta_k) = \vartheta^p - \vartheta - \frac{1}{t} - \left(\vartheta_k^p - \vartheta_k - \frac{1}{t}\right) = 0 + t^{-1/p^k} = t^{-1/p^k} . \tag{39}$$

If we have the equation $b^p - b = c$ and know that $vc < 0$, then we can conclude that $vb < 0$ since otherwise, $v(b^p - b) \geq 0 > vc$, a contradiction. But since $vb < 0$, we have that $vb^p = pvb < vb$, which implies that $vc = v(b^p - b) = pvb$. Consequently, $vb = \frac{vc}{p}$. In our case, we obtain that

$$v(\vartheta - \vartheta_k) = \frac{vt^{-1/p^k}}{p} = -\frac{1}{p^{k+1}} . \tag{40}$$

We see that $1/p^{k+1} \in vL_k(\vartheta)$, so that $(vL_k(\vartheta) : vL_k) \geq p$. Since $[L_k(\vartheta) : L_k] \leq p$, the fundamental inequality shows that $[L_k(\vartheta) : L_k] = p$, $vL_k(\vartheta) = 1/p^{k+1}\mathbb{Z}$, $L_k(\vartheta)v = L_k v = K$ and that the extension of the valuation v from L_k to $L_k(\vartheta)$ is unique. As $L(\vartheta) = \bigcup_{k\in\mathbb{N}} L_k(\vartheta)$, we obtain that $vL(\vartheta) = \bigcup_{k\in\mathbb{N}} 1/p^{k+1}\mathbb{Z} = \frac{1}{p^\infty}\mathbb{Z} = vL$ and that $L(\vartheta)v = \bigcup_{k\in\mathbb{N}} L_k v = K = Lv$. We have thus proved that the extension $(L(\vartheta)|L, v)$ is immediate. Since ϑ has degree p over every L_k, it must also have degree p over their union L. Further, as the extension of v from L_k to $L_k(\vartheta)$ is unique for every k, also the extension of v from L to $L(\vartheta)$ is unique. So we have $n = p$, $\mathrm{e} = \mathrm{f} = \mathrm{g} = 1$, and we find that the defect of $(L(\vartheta)|L, v)$ is p.

To obtain a defect extension of a henselian field, we show that L can be replaced by its henselization. By Theorem 30, $L^h(\vartheta) = L^h.L(\vartheta) = L(\vartheta)^h$. By Theorem 28, $vL(\vartheta)^h = vL(\vartheta) = vL = vL^h$ and $L(\vartheta)^h v = L(\vartheta)v = Lv = L^h v$. Hence, also the extension $(L^h(\vartheta)|L^h, v)$ is immediate. We only have to show that it is of degree p. This follows from the general valuation theoretical fact that if an extension $L'|L$ admits a unique extension of the valuation v from L to L', then $L'|L$ is linearly disjoint from $L^h|L$. But we can also give a direct proof. Again by Theorem 30, $L_k^h(\vartheta) = L_k(\vartheta)^h$, and by Theorem 28, $vL_k(\vartheta)^h = vL_k(\vartheta)$ and $vL_k^h = vL_k$. Therefore, $(vL_k^h(\vartheta) : vL_k^h) = (vL_k(\vartheta) : vL_k) = p$, showing that $[L_k^h(\vartheta) : L_k^h] = p$ for every k. Again by Theorem 30,

$L^h = L.L_1^h = (\bigcup_{k\in\mathbb{N}} L_k).L_1^h = \bigcup_{k\in\mathbb{N}} L_k.L_1^h = \bigcup_{k\in\mathbb{N}} L_k^h$. By the same argument as before, it follows that $[L^h(\vartheta) : L^h] = p$.

Hence, we have found an immediate Artin–Schreier extension of degree p and defect p of a henselian field which is only of transcendence degree 1 over K. $\diamondsuit$

A valued field (K, v) is called **algebraically maximal** if it admits no proper immediate algebraic extension, and it is called **separable-algebraically maximal** if it admits no proper immediate separable-algebraic extension. Since the henselization is an immediate separable-algebraic extension by Theorem 28, every separable-algebraically maximal field is henselian. The converse is not true, since the field (L^h, v) of our foregoing example is henselian but not separable-algebraically maximal. Corollary 41 shows that every henselian defectless field is algebraically maximal. The converse is not true, as was shown by Françoise Delon [D] (cf. also [KU2]).

Example 47 In the foregoing example, we may replace $K(t)$ by $K((t))$, taking L to be the field $K((t))(t^{1/p^k} \mid k \in \mathbb{N})$. It is not hard to show (by splitting up the power series in a suitable way) that $K((t^{1/p^k})) = K((t))[t^{1/p^k}]$, which is algebraic over $K((t))$. Hence, $L = \bigcup_{k\in\mathbb{N}} K((t^{1/p^k}))$, a union of an ascending chain of power series fields. All of them are henselian, and so also (L, v) is henselian. Thus, $(L(\vartheta)|L, v)$ gives an instant example of an immediate extension of a henselian field. But this L is "very large": it is of infinite transcendence degree over K. On the other hand, this version of our example shows that an infinite algebraic extension of a maximal field (or a union over an ascending chain of maximal fields) is in general not even defectless (and hence also not maximal). The example can also be transformed into the p-adic situation, showing that there are infinite extensions of $(\mathbb{Q}_p, v_p)$ which are not defectless fields (cf. [KU2]). $\diamondsuit$

8 Maximal immediate extensions

Based on our examples, we can observe another obstruction in positive characteristic. In many applications of valuation theory, one is interested in the embedding of a given valued field in a power series field which, if possible, should have the same value group and residue field. (We give an example relevant for algebraic geometry in the next section.) Then this power series field would be an immediate extension of our field, and since every power series field is maximal, it would be a **maximal immediate extension** of our field. So we see that we are led to the problem of determining maximal immediate extensions, in particular, whether maximal immediate extensions of a given valued field are unique up to valuation preserving isomorphism. It was shown by Wolfgang Krull [KR] that maximal immediate extensions exist for every valued field. The proof uses Zorn's Lemma in combination with an upper bound for the cardinality of valued fields with prescribed value group and residue field. Krull's deduction of this upper bound is hard to read; later, Kenneth A. H. Gravett [GRA] gave a nice and simple proof.

The uniqueness problem for maximal immediate extensions was considered by Irving Kaplansky in his important paper [KA]. He showed that if the so-called **hypothesis A** holds, then the field has a unique maximal immediate extension (up to valuation preserving isomorphism). For a Galois theoretic interpretation of hypothesis A and more information about the uniqueness problem, see [KUPR]. Let us mention a problem which was only partially solved in [KUPR] and in [WA1]:

Open Problem 2: If a valued field does not satisfy Kaplansky's hypothesis A, does it then admit two non-isomorphic maximal immediate extensions?

We can give a quick example of a valued field with two non-isomorphic maximal immediate extensions.

Example 48 In the setting of Example 47, assume that K is not **Artin–Schreier closed**, that is, there is an element $c \in K$ such that $X^p - X - c$ is irreducible over K. Take ϑ_c to be a root of $X^p - X - (\frac{1}{t} + c)$; note that $v(\frac{1}{t} + c) = v\frac{1}{t} < 0$ since $vc = 0 > v\frac{1}{t}$. Then in exactly the same way as for ϑ, one shows that the extension $(L(\vartheta_c)|L, v)$ is immediate of degree p and defect p. So we have two distinct immediate extensions of L. We take (M_1, v) to be a maximal immediate extension of $L(\vartheta)$, and (M_2, v) to be a maximal immediate extension of $L(\vartheta_c)$. Then (M_1, v) and (M_2, v) are also maximal immediate extensions of (L, v). If they were isomorphic over L, then M_1 would also contain a root of $X^p - X - (\frac{1}{t} + c)$; w.l.o.g., we can assume that it is the one called ϑ_c. Now we compute:

$$(\vartheta_c - \vartheta)^p - (\vartheta_c - \vartheta) \;=\; \vartheta_c^p - \vartheta_c - (\vartheta^p - \vartheta) \;=\; \frac{1}{t} + c - \frac{1}{t} \;=\; c \,.$$

Since $vc = 0$, we also have that $v(\vartheta_c - \vartheta) = 0$ (you may prove this along the lines of an argument given earlier). Applying the residue map to $\vartheta_c - \vartheta$, we thus obtain a root of $X^p - X - c$. But by our assumption, this root is not contained in $K = Lv$. Consequently, $M_1 v \neq Lv$, contradicting the fact that (M_1, v) was an immediate extension of (L, v). This proves that (M_1, v) and (M_2, v) cannot be isomorphic over (L, v).

We have used that K is not Artin–Schreier closed. And in fact, one of the consequences of hypothesis A for a valued field (L, v) is that its residue field be Artin–Schreier closed (see Section 10). $\diamondsuit$

We will need a generalization of the field of formal Laurent series, called **(generalized) power series field**. Take any field K and any ordered abelian group G. We take $K((G))$ to be the set of all maps μ from G to K with well-ordered **support** $\{g \in G \mid \mu(g) \neq 0\}$. You can visualize the elements of $K((G))$ as formal power series $\sum_{g\in G} c_g t^g$ for which the support $\{g \in G \mid c_g \neq 0\}$ is well-ordered. Using this condition one shows that $K((G))$ is a field in a similar way as it is done for $K((t))$. Also, one uses it to introduce the valuation:

$$v \sum_{g\in G} c_g t^g \;=\; \min\{g \in G \mid c_g \neq 0\} \tag{41}$$

(the minimum exists because the support is well-ordered). This valuation is often called the **canonical valuation of** $K((G))$, and sometimes called the **minimum support valuation**. With this valuation, $K((G))$ is a maximal field.

The fields L constructed in Examples 46 and 47 are subfields of $K((\mathbb{Q}))$ in a canonical way. It is interesting to note that the element ϑ is an element of $K((\mathbb{Q}))$:

$$\vartheta \;=\; \sum_{i\in\mathbb{N}} t^{-1/p^i} \;=\; t^{-1/p} + t^{-1/p^2} + \ldots + t^{-1/p^i} + \ldots \;. \tag{42}$$

Indeed,

$$\vartheta^p - \vartheta - \frac{1}{t} \;=\; \sum_{i\in\mathbb{N}} t^{-1/p^{i-1}} \;-\; \sum_{i\in\mathbb{N}} t^{-1/p^i} \;-\; t^{-1} \;=\; \sum_{i=0}^{\infty} t^{-1/p^i} \;-\; \sum_{i=1}^{\infty} t^{-1/p^i} \;-\; t^{-1} \;=\; 0\;.$$

Note that the values $vt^{-1/p^n} = -1/p^n$ converge from below to 0. Therefore, ϑ does not even lie in the completion of L. In fact, there cannot be a root of $X^p - X - 1/t$ in the completion; if a would be such a root, then there would be some $b \in L$ such that $v(a-b) > 0$. We would have that

$$(a-b)^p - (a-b) \;=\; a^p - a - (b^p - b) \;=\; \frac{1}{t} - (b^p - b)\;. \tag{43}$$

Because of $v(a-b) > 0$, the left hand side and consequently also the right hand side has value > 0. But as we have seen in Example 7, the polynomial $X^p - X - c$ splits over every henselian field containing c if $vc > 0$. Hence, in the cases where L is henselian, there exists a root $a' \in L$ of $X^p - X - 1/t + b^p - b$. It follows that $(a'+b)^p - (a'+b) - 1/t = 1/t - (b^p - b) + b^p - b - 1/t = 0$. As $a' + b \in L$, this would imply that $X^p - X - 1/t$ splits over L, a contradiction.

Let us illustrate the influence of the defect by considering an object which is well-known in algebraic geometry.

9 A quick look at Puiseux series fields

Recall that $K((\mathbb{Q}))$ is the field of all formal sums $\sum_{q\in\mathbb{Q}} c_q t^q$ with $c_q \in K$ and well-ordered support. The subset

$$\mathrm{P}(K) \;:=\; \left\{ \sum_{i=n}^{\infty} c_i t^{i/k} \mid c_i \in K,\, n \in \mathbb{Z},\, k \in \mathbb{N} \right\} \;=\; \bigcup_{k\in\mathbb{N}} K((t^{1/k})) \;\subset\; K((\mathbb{Q})) \tag{44}$$

is itself a field, called the **Puiseux series field over** K. Here, the valuation v on $K((t^{1/k}))$ is again the minimum support valuation, in particular, we have that $vt^{1/k} = 1/k$. In this way, the valuation v on every $K((t^{1/k}))$ is an extension of the t-adic valuation v_t of $K((t))$ and of the valuation of every subfield $K((t^{1/m}))$ where m divides k.

$\mathrm{P}(K)$ can also be written as a union of an ascending chain of power series fields in the following way. We take p_i to be the i-th prime number and set $m_k := \prod_{i=1}^{k} p_i^k$. Then $m_k|m_{k+1}$ and thus $K((t^{1/m_k})) \subset K((t^{1/m_{k+1}}))$ for every $k \in \mathbb{N}$, and every natural number will divide m_k for large enough k. Therefore,

$$\mathrm{P}(K) \;=\; \bigcup_{k\in\mathbb{N}} K((t^{1/m_k}))\;. \tag{45}$$

If one does not want to work in the power series field $K((\mathbb{Q}))$, then one simply has to choose a compatible system of k-th roots $t^{1/k}$ of t (that is, for $k = \ell m$ we must have $(t^{1/k})^\ell = t^{1/m}$; this is automatic for the elements $t^{1/k} \in K((\mathbb{Q}))$ by definition of the multiplication in this field). Then (44) can serve as a definition for the Puiseux series field over K.

Lemma 49 *The Puiseux series field $P(K)$ is an algebraic extension of $K((t))$, and it is henselian with respect to its canonical valuation v. Its residue field is K and its value group is $\mathbb{Q}$.*

Proof: For every $k \in \mathbb{N}$, the element $t^{1/k}$ is algebraic over $K((t))$. Similarly as in Example 47, we have that $K((t^{1/k})) = K((t))[t^{1/k}]$, which is algebraic over $K((t))$. Consequently, also the union $\mathrm{P}(K)$ of the $K((t^{1/k}))$ is algebraic over $K((t))$. By Corollary 29, $K((t))$ is henselian w.r.t. its canonical valuation v_t. As the canonical valuation v of $\mathrm{P}(K)$ is an extension of v_t, Corollary 5 yields that $(\mathrm{P}(K), v)$ is henselian.

The value group of every $(K((t^{1/k})), v)$ is $\mathbb{Z}vt^{1/k} = \mathbb{Z}\frac{vt}{k} = \frac{1}{k}\mathbb{Z}$, so the union over all $K((t^{1/k}))$ has value group $\bigcup_{k\in\mathbb{N}} \frac{1}{k}\mathbb{Z} = \mathbb{Q}$. The residue field of every $(K((t^{1/k})), v)$ is K, hence also the residue field of their union is K. □

Theorem 50 *Let K be a field of characteristic* 0. *Then* $(\mathrm{P}(K), v)$ *is a defectless field. Further, $P(K)$ is the algebraic closure of $K((t))$ if and only if K is algebraically closed.*

Proof: The residue field of $(\mathrm{P}(K), v)$ is K, hence if $\mathrm{char}\, K = 0$, then $(\mathrm{P}(K), v)$ is a defectless field by **(DF1)** in Section 7.

For the second assertion, we use the following valuation theoretical fact (try to prove it, it is not hard):

Let (L, v) be a valued field and choose any extension of v to the algebraic closure $\tilde{L}$. Then $v\tilde{L}$ is the divisible hull of vL, and $\tilde{L}v$ is the algebraic closure of Lv.

Hence, $v\widetilde{K((t))} = \mathbb{Q} = v\mathrm{P}(K)$ and $\widetilde{K((t))}P = \tilde{K}$. Thus if $\widetilde{K((t))} = \mathrm{P}(K)$, then $\tilde{K} = \mathrm{P}(K) = K$, which shows that K must be algebraically closed. For the converse, note that by the foregoing lemma, $\mathrm{P}(K) \subseteq \widetilde{K((t))}$. Assume that $\tilde{K} = K$. Then the extension

$(\widetilde{K((t))}|\mathrm{P}(K), v)$ is immediate. But since $(\mathrm{P}(K), v)$ is henselian (by the foregoing lemma) and defectless, every finite subextension must be trivial by Theorem 40. This proves that $\widetilde{K((t))} = \mathrm{P}(K)$, i.e., $\mathrm{P}(K)$ is algebraically closed. □

The assertion of this theorem does not hold if K has positive characteristic:

Example 51 In Example 46, we can replace L_k by $K((t^{1/k}))$ for every $k \in \mathbb{N}$ (as opposed to $K((t^{1/p^k}))$, which we used in Example 47). Still, everything works the same, producing the henselian Puiseux series field $L = \mathrm{P}(K)$ with an immediate Artin–Schreier extension $(L(\vartheta)|L, v)$ of degree p and defect p.

By construction, $\mathrm{P}(K)$ is a subfield of $K((\mathbb{Q}))$. Hence, the arguments at the end of the last section show that there is no root of $X^p - X - 1/t$ in the completion of $\mathrm{P}(K)$. The arguments of Example 48 show that $\mathrm{P}(K)$ has non-isomorphic maximal immediate extensions if K is not Artin–Schreier closed. ♢

Our example proves:

Theorem 52 *Let K be a field of characteristic $p > 0$. Then $(\mathrm{P}(K), v)$ is not defectless. In particular, $\mathrm{P}(K)$ is not algebraically closed, even if K is algebraically closed. Not even the completion of $\mathrm{P}(K)$ is algebraically closed.*

There is always a henselian defectless field extending $K((t))$ and having residue field K and divisible value group, even if K has positive characteristic. We just have to take the power series field $K((\mathbb{Q}))$. But in contrast to the Puiseux series field, this field is "very large": it has uncountable transcendence degree over $K((t))$. Nevertheless, having serious problems with the Puiseux series field in positive characteristic, we tend to replace it by $K((\mathbb{Q}))$. But this seems problematic since it might not be the unique maximal immediate extension of the Puiseux series field. However, if K is perfect and does not admit afinite extension whose degree is divisible by p (and in particular if K is algebraically closed), then Kaplansky's uniqueness result shows that the maximal immediate extension is unique. On the other hand, our example shows that the assumption "K is perfect" alone is not sufficient, since there are perfect fields which are not Artin–Schreier closed.

10 The tame and the wild valuation theory

Before we carry on, let us describe some advanced ramification theory based on the material of Sections 4 and 7. Throughout, we let (K, v) be a henselian non-trivially valued field. We set $p = \mathrm{char}\, Kv$ if this is positive, and $p = 1$ otherwise. If $(L|K, v)$ is an algebraic extension, then we call $(L|K, v)$ a **tame extension** if for every finite subextension $L'|K$ of $L|K$,

1) $(vL' : vK)$ is not divisible by the residue characteristic $\mathrm{char}\, Kv$,

2) $L'v|Kv$ is separable,
3) $[L' : K] = (vL' : vK)[L'v : Kv]$, i.e., $(L'|K, v)$ has trivial defect.

From the ramification theoretical facts presented in Section 4, one derives:

Theorem 53 *If (K, v) is henselian, then its absolute ramification field (K^r, v) is the unique maximal tame extension of (K, v), and its absolute inertia field (K^i, v) is the unique maximal tame extension of (K, v) having the same value group as K.*

An extension $(L|K, v)$ is called **purely wild** if $L|K$ is linearly disjoint from $K^r|K$. An ordered group G is called p-divisible if for every $\alpha \in G$ and $n \in \mathbb{N}$ there is $\beta \in G$ such that $p^n\beta = \alpha$. The **p-divisible hull** of G, denoted by $\frac{1}{p^\infty}G$, is the smallest subgroup of the divisible hull $\mathbb{Q} \otimes G$ which contains G and is p-divisible; it can be written as $\{\alpha/p^n \mid \alpha \in G\,,\ n \in \mathbb{N}\}$. The following was proved by Matthias Pank (cf. [KUPR]):

Theorem 54 *If (K, v) is henselian, then there exists a field complement W to K^r over K, that is, $W.K^r = \tilde{K}$ and $W \cap K^r = K$. The degree of every finite subextension of $W|K$ is a power of p. Further, vW is the p-divisible hull $\frac{1}{p^\infty}vK$ of vK, and Wv is the perfect hull of Kv.*

So (W, v) is a maximal purely wild extension of (K, v). It was shown by Pank and is shown in [KUPR] via Galois theory that W is unique up to isomorphism over K if Kv does not admit finite separable extensions whose degree are divisible by p. On the other hand, if vK is p-divisible and Kv is perfect, then $(W|K, v)$ is an immediate extension, and since every subextension of $K^r|K$ has trivial defect, it follows that the field complements W of K^r over K are precisely the maximal immediate algebraic extensions of (K, v).

It was shown by George Whaples [WH] and by Francoise Delon [D] that Kaplansky's original hypothesis A consists of the following three conditions:

1) Kv does not admit finite separable extensions whose degree are divisible by p,
2) vK is p-divisible,
3) Kv is perfect.

So if (K, v) satisfies Kaplansky's hypothesis A, then it follows from what we said above that the maximal immediate algebraic extensions of (K, v) are unique up to isomorphism over K. But this is the kernel of the uniqueness problem for the maximal immediate extensions: using Theorem 2 of [KA], one can easily show that the maximal immediate extensions are unique as soon as the maximal immediate algebraic extensions are.

Since all finite tame extensions have trivial defect, the defect is located in the purely wild extensions $(W|K, v)$. So we are interested in their structure. Here is one amazing result, due to Florian Pop (for the proof, see [KU2], and for the notion of "additive polynomial", see Example 7):

Theorem 55 *Let $(L|K, v)$ be a minimal purely wild extension, i.e., there is no subextension $L'|K$ of $L|K$ such that $L \neq L' \neq K$. Then there is an additive polynomial $\mathcal{A} \in K[X]$ and some $c \in K$ such that $L|K$ is generated by a root of $\mathcal{A}(X) + c$.*

The degree of $\mathcal{A}$ is a power of p (since it is additive), and in general it may be larger than p.

Now we shall quickly develop the theory of tame fields. The henselian field (K,v) is said to be a **tame field** if all of its algebraic extensions are tame extensions. By Theorem 54, this holds if and only if K^r is algebraically closed. Similarly, (K,v) is said to be a **separably tame field** if all of its separable-algebraic extensions are tame extensions. This holds if and only if K^r is separable-algebraically closed.

By Theorem 24, $\tilde{K} = K^r.W$ is the absolute ramification field of W. If $W'|K$ is a proper subextension, then $\tilde{K} \neq K^r.W'$. This proves:

Lemma 56 *Every maximal purely wild extension W is a tame field. No proper subextension of $W|K$ is a tame field. The maximal separable subextension is a separably tame field.*

By Theorem 22, $K^{\mathrm{sep}}|K^r$ is a p-extension. Hence if $\mathrm{char}\, Kv = 0$, then this extension is trivial. Since then also $\mathrm{char}\, K = 0$, it follows that $K^r = K^{\mathrm{sep}} = \tilde{K}$. Therefore,

Lemma 57 *Every henselian field of residue characteristic 0 is a tame field.*

Suppose that $K_1|K$ is an algebraic extension. Then by Theorem 24, $K^r \subseteq K_1^r$. Hence if K^r is algebraically closed, then so is K_1^r, and if K^r is separable-algebraically closed, then so is K_1^r. This proves:

Lemma 58 *Every algebraic extension of a tame field is again a tame field. Every algebraic extension of a separably tame field is again a separably tame field.*

If $K^r = \tilde{K}$, then every finite extension of (K,v) is a tame extension and thus has trivial defect, which shows that (K,v) is a defectless field. If $K^r = K^{\mathrm{sep}}$, then every finite separable extension has trivial defect. So we note:

Lemma 59 *Every tame field is henselian defectless and perfect. Every separably tame field is henselian and all of its finite separable extensions have trivial defect.*

We give a characterization of tame and separably tame fields (for the proof, see [KU2]):

Lemma 60 *A valued field (K,v) is tame if and only if it is algebraically maximal, vK is p-divisible and Kv is perfect. If $\mathrm{char}\, K = \mathrm{char}\, Kv$ then (K,v) is tame if and only if it is algebraically maximal and perfect.*

A non-trivially valued field (K,v) is separably tame if and only if it is separable-algebraically maximal, vK is p-divisible and Kv is perfect.

This lemma together with Lemma 59 shows that for perfect valued fields (K,v) with $\mathrm{char}\, K = \mathrm{char}\, Kv$, the properties "algebraically maximal" and "henselian and defectless" are equivalent.

Corollary 61 *Assume that* $\operatorname{char} K = \operatorname{char} Kv$*. Then every maximal immediate algebraic extension of the perfect hull of* (K,v) *is a tame field (and no proper subextension of it is a tame field). If* $\operatorname{char} Kv = 0$ *then already the henselization* (K^h,v) *is a tame field.*

The following is a crucial lemma in the theory of tame fields. For its proof, see [KU1] or [KU2].

Lemma 62 *Let* (L,v) *be a tame field and* $K \subset L$ *a relatively algebraically closed subfield. If in addition* $Lv|Kv$ *is an algebraic extension, then* (K,v) *is also a tame field and moreover,* vK *is pure in* vL *and* $Kv = Lv$*. The same holds for "separably tame" in the place of "tame".*

11 The Generalized Grauert–Remmert Stability Theorem

Let us return to our problem of inertial generation as considered at the end of Section 6. Our problem was to show that the finite immediate extension (33) of henselian fields is trivial. If it is not, then by Corollary 41 it has non-trivial defect (which then is equal to its degree). So we would like to show that the field $F_0(\eta)^h$ is a defectless field. The reason for this would have to lie in the special way we have constructed this field.

At this point, let us invoke a deep and important theorem from the theory of valued function fields ([KU1], [KU2]). For historical reasons, I call it the **Generalized Grauert–Remmert Stability Theorem** although I do not like the notion "stable". It is one of those words in mathematics which is very often used in different contexts, but in most cases does not reflect its meaning. I replace it by "defectless".

If $(F|K,v)$ is an extension of valued fields of finite transcendence degree, then by inequality (30) of Corollary 38, $\operatorname{trdeg} F|K - \operatorname{trdeg} Fv|Kv - \operatorname{rr}(vF/vK)$ is a non-negative integer. It will be called the **transcendence defect** of $(F|K,v)$. We say that $(F|K,v)$ is **without transcendence defect** if the transcendence defect is 0.

Theorem 63 *Let* $(F|K,v)$ *be a valued function field without transcendence defect. If* (K,v) *is a defectless field, then also* (F,v) *is a defectless field.*

This theorem has a long and interesting history. Hans Grauert and Reinhold Remmert [GR] first proved it in a very restricted case, where (K,v) is a complete discrete valued field and (F,v) is discrete too. There are generalizations by Laurent Gruson [GRU], Michel Matignon, and Jack Ohm [O4]. All of these generalizations are restricted to the case $\operatorname{trdeg} F|K = \operatorname{trdeg} Fv|Kv$, the case of **constant reduction**. The classical origin of it is the study of curves over number fields and the idea to reduce them modulo a p-adic valuation. Certainly, the reduction should again render a curve, this time over a finite field. This is guaranteed by the condition $\operatorname{trdeg} F|K = \operatorname{trdeg} Fv|Kv$, where F is the function field of the curve and Fv will be the function field of its reduction. Naturally, one seeks to relate the genus of $F|K$ to that of $Fv|Kv$. Several authors proved **genus inequalities**. To illustrate the use of the defect, we cite an inequality proved by Barry Green, Michel Matignon and Florian Pop in [GMP1]. Let $F|K$ be a function field of transcendence degree 1 and assume that K coincides with the constant field of $F|K$ (the relative algebraic closure of K in F). Let $v_1, \ldots, v_s$ be distinct constant reductions of $F|K$ which have a common restriction to K. Then:

$$1 - g_F \leq 1 - s + \sum_{i=1}^{s} \delta_i e_i r_i (1 - g_i) \tag{46}$$

where g_F is the genus of $F|K$ and g_i the genus of $Fv_i|Kv_i$, r_i is the degree of the constant field of $Fv_i|Kv_i$ over Kv_i, δ_i is the defect of $(F^{h(v_i)}|K^{h(v_i)}, v_i)$ where "$.^{h(v_i)}$" denotes the henselization with respect to v_i, and $e_i = (v_iF : v_iK)$ (which is always finite in the constant reduction case by virtue of Corollary 38). It follows that constant reductions v_1, v_2 with common restriction to K and $g_1 = g_2 = g_F \geq 1$ must be equal. In other words, for a fixed valuation on K there is at most one extension v to F which is a **good reduction**, that is, (i) $g_F = g_{Fv}$, (ii) there exists $f \in F$ such that $vf = 0$ and $[F : K(f)] = [Fv : Kv(fv)]$, (iii) Kv is the constant field of $Fv|Kv$. An element f as in (ii) is called a **regular function**.

More generally, f is said to have the **uniqueness property** if fv is transcendental over Kv and the restriction of v to $K(f)$ has a unique extension to F. In this case, $[F : K(f)] = \delta\, e\, [Fv : Kv(fv)]$ where δ is the defect of $(F^h|K^h, v)$ and $e = (vF : vK(f)) = (vF : vK)$. If K is algebraically closed, then $e = 1$, and it follows from the Stability Theorem that $\delta = 1$; hence in this case, every element with the uniqueness property is regular.

It was proved in [GMP2] that F has an element with the uniqueness property already if the restriction of v to K is henselian. Elements with the uniqueness property also exist if vF is a subgroup of $\mathbb{Q}$ and Kv is algebraic over a finite field. This follows from work in [GMP4] where the uniqueness property is related to the **local Skolem property** which gives a criterion for the existence of algebraic v-adic integral solutions on geometrically integral varieties.

As an application to rigid analytic spaces, the Stability Theorem is used to prove that the quotient field of the free Tate algebra $T_n(K)$ is a defectless field, provided that K is. This in turn is used to deduce the **Grauert–Remmert Finiteness Theorem**, in a generalized version due to Gruson; see [BGR].

Surprisingly, it was not before the model theory of valued fields developed in positive characteristic that an interest in a generalized version of the Stability Theorem arose. There, one is forced to deal with arbitrary extensions of arbitrarily large valued fields. For instance, it is virtually impossible to restrict oneself to rank 1 in order to prove model completeness or completeness of a class of valued fields. And the extensions $(L|K, v)$ in question won't obey a restriction like "vL/vK is a torsion group". This is the reason why I proved the above Generalized Stability Theorem. At that time, I had not heard of the Grauert–Remmert Theorem, so I gave a purely valuation theoretic proof ([KU1], [KU2]), not based on the original proofs of Grauert–Remmert or Gruson like the other cited generalizations.

Later, I was amazed to see that the Generalized Stability Theorem is also the suitable version for an application to the problem of local uniformization. (If your valuation v is trivial on the base field K and you ask that $\operatorname{trdeg} F|K = \operatorname{trdeg} Fv|Kv$, then $vL/\{0\}$ is torsion, so $vL = \{0\}$ and v is also trivial on F; this is not quite the case we are interested in.) So let's now describe this application. By our assumption at the end of Section 6, P is an Abhyankar place on F and hence also on $F_0(\eta)$. That is, $(F_0(\eta)|K, P)$ is a function field without transcendence defect. As P is trivial on K, also $v = v_P$ is trivial on K. But a trivially valued field (K, v) is always a defectless field since for every finite extension $L|K$ we have that $[L : K] = [Lv : Kv]$. Hence by the Generalized Stability Theorem, $(F_0(\eta), v)$ is a defectless field. By Theorem 42, also $(F_0(\eta)^h, v)$ is a defectless field. Therefore, since $(F^h|F_0(\eta)^h, v)$ is an immediate extension, Corollary 41 shows that it must be trivial. We have proved that $F^h = F_0(\eta)^h$. By construction, $F_0(\eta)^h$ was a subfield of the absolute inertia field of (F_0, P). Hence also F is a subfield of that absolute inertia field, showing that (F, P) is inertially generated. We have thus proved the first part of the following theorem (I leave the rest of the proof to you as an exercise; cf. [KNKU1]):

Theorem 64 *Assume that P is an Abhyankar place of $F|K$ and that $FP|K$ is a separable extension. Then $(F|K, P)$ is inertially generated. If in addition $FP = K$ or $FP|K$ is a rational function field, then $(F|K, P)$ is henselian generated. In all cases, if $vF = \mathbb{Z}vx_1 \oplus \ldots \oplus \mathbb{Z}vx_\rho$ and $y_1P, \ldots, y_\tau P$ is a separating transcendence basis of $FP|K$, then $\{x_1, \ldots, x_\rho, y_1, \ldots, y_\tau\}$ is a generating transcendence basis.*

To conclude this section, we give a short sketch of a main part of the proof of the Generalized Stability Theorem. This is certainly interesting because very similar methods have been used by Shreeram Abhyankar for the proof of his results in positive characteristic (see, e.g., [A1], [A2], [A3]). We have to prove that a certain henselian field (L, v) is a defectless field. We take an arbitrary finite extension $(L'|L, v)$ and have to show that it has trivial defect. We may assume that this extension is separable since the case of purely inseparable extensions can be considered separately and is much easier. Looking at $(L'|L, v)$, we are completely lost since we have not the slightest chance to develop a good structure theory. But we only have to deal with the defect, and we remember that a defect only appears if extensions beyond the absolute ramification field L^r are involved. So instead of $(L'|L, v)$ we consider the extension $(L'.L^r|L^r, v)$ which has the same defect as $(L'|L, v)$, although it will in general not have the same degree (the use of this fact reminds of Abhyankar's "Going Up" and "Coming Down"; cf. [A1]). Now we use the fact that by Theorem 22 the separable-algebraic closure of L^r is a p-extension. It follows that its subextension $L'.L^r|L^r$ is a tower of Artin–Schreier extensions (cf. Remark 23). Since the defect is multiplicative, to prove that $(L'.L^r|L^r, v)$ has trivial defect it suffices to show that each of these Artin–Schreier extensions has trivial defect. So we take such an extension, generated by a root ϑ of an irreducible polynomial $X^p - X - c$ over some field L'' in the tower. By what we learned in Example 7, $vc \leq 0$. If $vc = 0$, the extension (if it is not trivial) would correspond to a proper separable extension of the residue field; but as we are working beyond the absolute ramification field, our residue field is already separable-algebraically closed. So we see that $vc < 0$. If $b \in L''$, then also the element $\vartheta - b$ generates the same extension. By the additivity of the polynomial $X^p - X$, $\vartheta - b$ is a root of the Artin–Schreier polynomial $X^p - X - (c - b^p + b)$. The idea now is to use this principle to deduce a "normal form" for c from which we can read off that the extension has trivial defect. Still, we are quite lost if we do not make some reductions beforehand. First, it is clear that one can proceed by induction on the transcendence degree; so we can reduce to the case of $\operatorname{trdeg} L|K = 1$. Second, as v may not be trivial on K, it may have a very large rank. By general valuation theory, one has to reduce first to finite rank and then to rank 1. This being done, one can show that c can be taken to be a polynomial $g \in K[x]$, where $x \in L''$ is transcendental over K. Now the idea is the following: if $k = p \cdot \ell$ and g contains a non-zero summand $c_k x^k$, then we replace it by $c_k^{1/p} x^\ell$. This is done by setting $b = c_k^{1/p} x^\ell$ in the above computation. In this way one eliminates all p-th powers in g, and the thus obtained normal form for c will show that the extension has trivial defect. This method (which I call "Artin–Schreier surgery") seems to have several applications; I used it again to prove a quite different result (Theorem 65 below). It can also be found in the paper [EPP].

Let us note that the Artin–Schreier polynomials appear in Abhyankar's work in a somewhat disguised form. This is because the coefficients have to lie in the local ring he is working in. For example, if $vc < 0$, we would rather prefer to have a polynomial having coefficients in the valuation ring, defining the same extension as $X^p - X - c$. Setting $X = cY$, we find that if ϑ is a root of $X^p - X - c$, then ϑ / c is a root of $Y^p - c^{1-p} Y - c^{1-p}$ with $vc^{1-p} = (1-p)vc > 0$. Therefore, Abhyankar considers polynomials of the form $Z^p - c_1 Z - c_2$ (cf. e.g., [A1], page 515, [A2], Theorem (2.2), or [A3], page 34). In an extension obtained from L by adjoining a $(p-1)$th root of c_1 (if (L, v) is henselian, then such an extension is tame), this polynomial can be transformed back into an Artin–Schreier polynomial.

12 Non-Abhyankar places and the Henselian Rationality of immediate function fields

What can we do if the place P of $F|K$ is *not* an Abhyankar place? Still, the place may be nice. Assume for instance that vF is finitely generated and $FP = K$. Then we can choose $x_1, \ldots, x_\rho$ such that $vF = \mathbb{Z}vx_1 \oplus \ldots \oplus \mathbb{Z}vx_\rho$, and set $F_0 := K(x_1, \ldots, x_\rho)$. Consequently, $(F|F_0, v)$ is an immediate extension. If P is not an Abhyankar place, then this extension is not algebraic. So the question arises: how can we avoid the defect in the case of immediate transcendental extensions? The answer is a theorem that I proved in [KU1] (cf. [KU2]). As for the Generalized Stability Theorem, the proof uses ramification theory and the deduction of normal forms for Artin-Schreier extensions. It also uses significantly a theory of immediate extensions which builds on Kaplansky's paper [KA].

Theorem 65 (Henselian Rationality of Immediate Function Fields) *Let (K,P) be a tame field and $(F|K,P)$ an immediate function field of transcendence degree 1. Then*

$$\text{there is } x \in F \text{ such that } (F^h,P) = (K(x)^h,P)\;, \tag{47}$$

that is, $(F|K,P)$ is henselian generated. The same holds over a separably tame field (K,P) if in addition $F|K$ is separable.

Since the assertion says that F^h is equal to the henselization of a rational function field, we also call F **henselian rational**. For valued fields of residue characteristic 0, the assertion is a direct consequence of the fact that every such field is defectless. Indeed, take any $x \in F \setminus K$. Then $K(x)|K$ cannot be algebraic since otherwise, $(K(x)|K,P)$ would be a proper immediate algebraic extension of the tame field (K,P), a contradiction to Lemma 60. Hence, $F|K(x)$ is algebraic and immediate. Therefore, $(F^h|K(x)^h,P)$ is algebraic and immediate too. But since it cannot have a non-trivial defect, it must be trivial. This proves that $(F,P) \subset (K(x)^h,P)$. In contrast to this, in the case of positive residue characteristic only a very carefully chosen $x \in F \setminus K$ will do the job.

A little bit of horror makes a story even more interesting. So let's watch out for bad places.

13 Bad places

In this section we will show that there are places of function fields $F|K$ whose value group or residue field are not finitely generated. By combining the methods you can construct examples where both is the case. The following two examples can already be found in [ZS], Chapter VI, §15. But our approach (using Hensel's Lemma) is somewhat easier and more conceptual.

Example 66 We construct a place on the rational function field $K(x_1,x_2)|K$ whose value group $G \subset \mathbb{Q}$ is not finitely generated, assuming that the order of every element in $G/\mathbb{Z}$ is prime to $\operatorname{char} K$. To this end, we just find a suitable embedding of $K(x_1,x_2)$ in $K((G))$. We do this by setting $S := \{n \in \mathbb{N} \mid 1/n \in G\}$ and

$$x_1 := t \quad \text{and} \quad x_2 := \sum_{n\in S} t^{-1/n}\;. \tag{48}$$

Further, take the valuation v on $K(x_1,x_2)$ to be the restriction of the canonical valuation v of $K((G))$. We wish to show that $1/S \subseteq vK(x_1,x_2)$, so that $G \subset vK(x_1,x_2)$. Since

$(K(x_1,x_2),v)\subset(K((G)),v)$, it follows that $G=vK(x_1,x_2)$. If x_2 were algebraic over $K(x_1)$, we would know by Corollary 38 that $vK(x_1,x_2)$ is finitely generated. Hence if it is not, then x_2 must be transcendental over $K(x_1)$, so that $K(x_1,x_2)$ is indeed the rational function field over K in two variables.

Suppose that $\mathrm{char}\,K=0$; then we can get $G=\mathbb{Q}$. Also in positive characteristic one can define the valuation in such a way that the value group becomes $\mathbb{Q}$; since then we have to deal with inseparability, our construction has to be refined slightly, which we will not do here.

Now let us prove our assertion. We take (L,v) to be the henselization of $(K(x_1,x_2),v)$. We are going to show that $t^{1/n}\in L$ for all $n\in S$. Suppose we have shown this for all $n<k$, where $k\in S$ (we can assume that $k>1$). Then also $s_k:=\sum_{n\in S,n<k}t^{-1/n}\in L$. We write

$$x_2-s_k\;=\;\sum_{n\in S,n\geq k}t^{-1/n}\;=\;t^{-1/k}(1+c) \tag{49}$$

where $c\in K((G))$ with $vc>0$. Hence, $1+c$ is a 1-unit. We have that $(1+c)^k=t(x_2-s_k)^k\in L$. On the other hand, $(1+c)^kv=((1+c)v)^k=1^k=1$, which shows that $(1+c)^k$ is again a 1-unit. Since $k\in S$ we know that $\mathrm{char}\,LP=\mathrm{char}\,K$ does not divide k. Hence by Lemma 11, $1+c\in L$. This proves that $t^{1/k}=(1+c)(x_2-s_k)^{-1}\in L$.

We have now proved that $t^{1/k}\in L$ for all $k\in S$. Hence, $1/k=vt^{1/k}\in vL$ for all $k\in S$. But since the henselization is an immediate extension, we know that $vL=vK(x_1,x_2)$, so we have proved that $1/S\subset vK(x_1,x_2)$. ♢

Example 67 We take a field K for which the separable-algebraic closure K^{sep} is an infinite extension (i.e., K is neither separable-algebraically closed nor real closed). We construct a place of the rational function field $K(x_1,x_2)|K$ whose residue field is not finitely generated. We choose a sequence a_n, $n\in\mathbb{N}$ of elements which are separable-algebraic over K of degree at least n. We define an embedding of $K(x_1,x_2)$ in $K^{\mathrm{sep}}((t))$ by setting

$$x_1\;:=\;t\quad\text{and}\quad x_2:=\sum_{n\in\mathbb{N}}a_nt^n\,. \tag{50}$$

Further, we take the valuation v on $K(x_1,x_2)$ to be the restriction of the valuation of $K^{\mathrm{sep}}((t))$. We wish to show that $a_n\in K(x_1,x_2)v$ for all $n\in\mathbb{N}$, so that $K(x_1,x_2)v|K$ cannot be finitely generated. If x_2 were algebraic over $K(x_1)$, we would know by Corollary 38 that $K(x_1,x_2)v|K$ is finitely generated. So if it is not, then x_2 must be transcendental over $K(x_1)$, so that $K(x_1,x_2)$ is indeed the rational function field over K in two variables. By a modification of the construction, one can also generate infinite inseparable extensions of K. If K is countable, one can generate every algebraic extension of K as a residue field of $K(x_1,x_2)$.

We take again (L,v) to be the henselization of $(K(x_1,x_2),v)$. We are going to show that $a_n\in L$ for all $n\in\mathbb{N}$. Suppose we have shown this for all $n<k$, where $k\in\mathbb{N}$. Then

also $s_k := \sum_{n=1}^{k-1} a_n t^n \in L$. We write

$$\frac{x_2 - s_k}{t^k} \;=\; \frac{1}{t^k}\sum_{n=k}^{\infty} a_k t^k \;=\; a_k(1+c) \tag{51}$$

where $c \in K^{\mathrm{sep}}((t))$ with $vc > 0$. Take $f \in K[X]$ to be the minimal polynomial of a_k over K and note that $f = fv$. Since $a_k \in K^{\mathrm{sep}}$, we know that a_k is a simple root of f. On the other hand, $a_k = a_k(1+c)v \in Lv$. Hence by Hensel's Lemma (Simple Root Version) there is a root a of f in L such that $av = a_k$. As we may assume that the place associated with v is the identity on K, this will give us that $a = a_k$; so $a_k \in L$.

We have now proved that $a_n \in L$ for all $n \in \mathbb{N}$. Hence, $a_n \in Lv = K(x_1, x_2)v$ for all $n \in \mathbb{N}$. $\diamondsuit$

14 Pseudo Cauchy sequences

Take any valued field (K, v). A sequence of elements $a_\nu \in K$, $\nu < \lambda$ (λ some limit ordinal), is called a **pseudo Cauchy sequence** in (K, v) if $v(a_\rho - a_\sigma) < v(a_\sigma - a_\tau)$ for all ρ, σ, τ with $\rho < \sigma < \tau < \lambda$. It follows from the ultrametric triangle law that $v(a_\nu - a_\tau) = v(a_\nu - a_{\nu+1})$ whenever $\nu < \tau < \lambda$. The element a is called a (pseudo) limit of this pseudo Cauchy sequence if $v(a_\nu - a) = v(a_\nu - a_{\nu+1})$ for all $\nu < \lambda$. In general, there may be several distinct limits:

Lemma 68 *Let a be a limit of $(a_\nu)_{\nu<\lambda}$. Then b is also a limit of $(a_\nu)_{\nu<\lambda}$ if and only if $v(a - b) > v(a_\nu - a_{\nu+1})$ for all $\nu < \lambda$.*

The following lemma describes the connection between pseudo Cauchy sequences and immediate extensions:

Lemma 69 *Let $(L|K, v)$ be an extension of valued fields. Then $(L|K, v)$ is immediate if and only if for every $a \in L^\times$ there is some $c \in K$ such that $v(a - c) > va$.*

Proof: Suppose that $(L|K, v)$ is immediate, and let $0 \neq a \in L$. Then $va \in vL = vK$ and thus, there is some $c' \in K$ such that $va = vc'$. Hence, $v(a/c') = 0$. Then $\overline{a/c'} \in \overline{L} = \overline{K}$ and thus, there is some $c'' \in K$ such that $\overline{a/c'} = \overline{c''}$. That is, $v(a/c' - c'') > 0$, which yields that $v(a - c'c'') > vc' = va$. Hence $c = c'c''$ is the element that we have looked for.

For the converse, let $\alpha \in vL$ and $a \in L$ such that $va = \alpha$. If there is $c \in K$ such that $v(a - c) > va$, then $\alpha = va = vc \in vK$. Now let $\zeta \in \overline{L}$ and $a \in L$ such that $\overline{a} = \zeta$. If there is $c \in K$ such that $v(a - c) > va = 0$, then $\zeta = \overline{a} = \overline{c} \in \overline{K}$. $\square$

By a repeated application of this lemma, one proves:

Lemma 70 *Assume that $(K, v) \subset (L, v)$ is immediate and that $a \in L \setminus K$. Then there is a pseudo Cauchy sequence in (K, v) with limit a, but not having a limit in K.*

A pseudo Cauchy sequence $\mathbf{A} = (a_\nu)_{\nu<\lambda}$ in (K,v) (where λ is some limit ordinal) is **of transcendental type** if for every $g(x) \in K(x)$, the value $vg(a_\nu)$ is eventually constant, that is, there is some $\nu_0 < \lambda$ such that

$$vg(a_\nu) \;=\; vg(a_{\nu_0}) \quad \text{for all } \; \nu \geq \nu_0 \,,\; \nu < \lambda \,. \tag{52}$$

Otherwise, **A** if **of algebraic type**.

Take a pseudo Cauchy sequence **A** in (K,v) of transcendental type. We define an extension $v_{\mathbf{A}}$ of v from K to the rational function field $K(x)$ as follows. For each $g(x) \in K[x]$, we choose $\nu_0 < \lambda$ such that (52) holds. Then we set

$$v_{\mathbf{A}}\, g(x) \;:=\; vg(a_{\nu_0}) \;.$$

We extend $v_{\mathbf{A}}$ to $K(x)$ by setting $v_{\mathbf{A}}(g/h) := v_{\mathbf{A}}g - v_{\mathbf{A}}h$. The following is Theorem 2 of [KA]:

Theorem 71 *Let* **A** *be a pseudo Cauchy sequence in (K,v) of transcendental type. Then $v_{\mathbf{A}}$ is a valuation on the rational function field $K(x)$. The extension $(K(x)|K,v_{\mathbf{A}})$ is immediate, and x is a pseudo limit of* **A** *in $(K(x),v_{\mathbf{A}})$. If $(K(y),w)$ is any other valued extension of (K,v) such that y is a pseudo limit of* **A** *in $(K(y),w)$, then $x \mapsto y$ induces a valuation preserving K-isomorphism from $(K(x),v_{\mathbf{A}})$ onto $(K(y),w)$.*

From this theorem we deduce:

Lemma 72 *Suppose that in some valued field extension of (K,v), x is the pseudo limit of a pseudo Cauchy sequence in (K,v) of transcendental type. Then $(K(x)|K,v)$ is immediate and x is transcendental over K.*

Proof: Assume that $(a_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence in (K,v) of transcendental type. Then by Theorem 71 there is an immediate extension w of v to the rational function field $K(y)$ such that y becomes a pseudo limit of $(a_\nu)_{\nu<\lambda}$; moreover, if also x is a pseudo limit of $(a_\nu)_{\nu<\lambda}$ in $(K(x),v)$, then $x \mapsto y$ induces a valuation preserving isomorphism from $K(x)$ onto $K(y)$ over K. Hence, $(K(x)|K,v)$ is immediate and x is transcendental over K. □

Lemma 73 *A pseudo Cauchy sequence of transcendental type in a valued field remains a pseudo Cauchy sequence of transcendental type in every algebraic valued field extension of that field.*

Proof: Assume that $(a_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence in (K,v) of transcendental type and that $(L|K,v)$ is an algebraic extension. If $(a_\nu)_{\nu<\lambda}$ were of algebraic type over (L,v), then by Theorem 3 of [KA] there would be an algebraic extension $L(y)|L$ and an immediate extension of v to $L(y)$ such that y is a pseudo limit of $(a_\nu)_{\nu<\lambda}$ in $(L(y),v)$. But then, y is also a pseudo limit of $(a_\nu)_{\nu<\lambda}$ in $(K(y),v)$. Hence by the foregoing lemma, y must be transcendental over K. This is a contradiction to the fact that $L(y)|L$ and $L|K$ are algebraic. $\square$

The following is Theorem 3 of [KA]:

Theorem 74 *Let* $\mathbf{A}$ *be a pseudo Cauchy sequence in* (K,v) *of algebraic type. Take a polynomial* $f(X)\in K[X]$ *of minimal degree whose value is not fixed by* $\mathbf{A}$*, and choose a root* y *of* f *in the algebraic closure of* K*. Then there exists an extension of* v *from* K *to* $K(y)$ *such that* $(K(y)|K,v)$ *is an immediate extension and* y *is a limit of* $\mathbf{A}$ *in* $(K(y),v)$*. If* $(K(z),v)$ *is another valued extension field of* (K,v) *such that* $\mathrm{appr}(z,K)=\mathbf{A}$*, then any field isomorphism between* $K(y)$ *and* $K(z)$ *over* K *sending* y *to* z *will preserve the valuation. (Note that there exists such an isomorphism if and only if* z *is also a root of* f*.)*

If (K,v) admits no immediate extensions, then by Theorems 71 and 74, every pseudo Cauchy sequence in (K,v) must have a limit in K. On the other hand, if every pseudo Cauchy sequence in (K,v) has a limit in K, then (K,v) admits no proper immediate extensions (cf. Lemma 70). This proves the following theroem, which is Theorem 4 of [KA]:

Corollary 75 *A valued field* (K,v) *is maximal if and only if every pseudo Cauchy sequence in* (K,v) *has a limit in* K*.*

15 Valuations on $K(x)$

15.1 A basic classification

In this section, we wish to classify all extensions of the valuation v of K to a valuation of the rational function field $K(x)$. As

$$1 \;=\; \mathrm{trdeg}\, K(x)|K \;\geq\; \mathrm{rr}\, vK(x)/vK \;+\; \mathrm{trdeg}\, K(x)v|Kv \tag{53}$$

holds by Lemma 37, there are the following mutually exclusive cases:

- $(K(x)|K,v)$ is **valuation-algebraic**:
  $vK(x)/vK$ is a torsion group and $K(x)v|Kv$ is algebraic,
- $(K(x)|K,v)$ is **value-transcendental**:
  $vK(x)/vK$ has rational rank 1, but $K(x)v|Kv$ is algebraic,
- $(K(x)|K,v)$ is **residue-transcendental**:
  $K(x)v|Kv$ has transcendence degree 1, but $vK(x)/vK$ is a torsion group.

We will combine the value-transcendental case and the residue-transcendental case by saying that

- $(K(x)|K,v)$ is **valuation-transcendental**:
  $vK(x)/vK$ has rational rank 1, or $K(x)v|Kv$ has transcendence degree 1.

A special case of the valuation-algebraic case is the following:

- $(K(x)|K,v)$ is **immediate**:
  $vK(x) = vK$ and $K(x)v = Kv$.

Remark 76 It was observed by several authors that a valuation-algebraic extension of v from K to $K(x)$ can be represented as a limit of an infinite sequence of residue-transcendental extensions. See, e.g., [APZ3]. A "higher form" of this approach is found in [S]. The approach is particularly important because residue-transcendental extensions behave better than valuation-algebraic extensions: the corresponding extensions of value group and residue field are finitely generated (Corollary 38), and they do not generate a defect: see the Generalized Stability Theorem (Theorem 3.1) and its application in [KNKU1].

If K is algebraically closed, then the residue field Kv is algebraically closed, and the value group vK is divisible. So we see that for an extension $(K(x)|K,v)$ with algebraically closed K, there are only the following mutually exclusive cases:
$(K(x)|K,v)$ is **immediate**: $vK(x) = vK$ and $K(x)v = Kv$,
$(K(x)|K,v)$ is **value-transcendental**: $\mathrm{rr}\, vK(x)/vK = 1$, but $K(x)v = Kv$,
$(K(x)|K,v)$ is **residue-transcendental**: $\mathrm{trdeg}\, K(x)v|Kv = 1$, but $vK(x) = vK$.

Let us fix an arbitrary extension of v to $\tilde{K}$. Every valuation w on $K(x)$ can be extended to a valuation on $\tilde{K}(x)$. If v and w agree on K, then this extension can be chosen in such a way that its restriction to $\tilde{K}$ coincides with v. Indeed, if w' is any extension of w to $\tilde{K}(x)$ and v' is its restriction to $\tilde{K}$, then there is an automorphism τ of $\tilde{K}|K$ such that $v'\tau = v$ on $\tilde{K}$. We choose σ to be the (unique) automorphism of $\tilde{K}(x)|K(x)$ whose restriction to $\tilde{K}$ is τ and which satisfies $\sigma x = x$. Then $w'\sigma$ is an extension of w from $K(x)$ to $\tilde{K}(x)$ whose restriction to $\tilde{K}$ is v. We conclude:

Lemma 77 *Take any extension of v from K to its algebraic closure $\tilde{K}$. Then every extension of v from K to $K(x)$ is the restriction of some extension of v from $\tilde{K}$ to $\tilde{K}(x)$.*

Now extend v to $\widetilde{K(x)}$. We know that $v\widetilde{K(x)}/vK(x)$ and $v\tilde{K}/vK$ are torsion groups, and also $v\tilde{K}(x)/vK(x) \subset v\widetilde{K(x)}/vK(x)$ is a torsion group. Hence,

$$\mathrm{rr}\, v\tilde{K}(x)/v\tilde{K} \;=\; \mathrm{rr}\, vK(x)/vK\;.$$

Since $v\tilde{K}$ is divisible, $vK(x)/vK$ is a torsion group if and only if $v\tilde{K}(x) = v\tilde{K}$.

Further, the extensions $\widetilde{K(x)}v|K(x)v$ and $\tilde{K}v|Kv$ are algebraic, and also the subextension $\tilde{K}(x)v|K(x)v$ of $\widetilde{K(x)}v|K(x)v$ is algebraic. Hence,

$$\mathrm{trdeg}\, \tilde{K}(x)v|\tilde{K}v \;=\; \mathrm{trdeg}\, K(x)v|Kv\;.$$

Since $\tilde{K}v$ is algebraically closed, $K(x)v|Kv$ is algebraic if and only if $\tilde{K}(x)v = \tilde{K}v$. We have proved:

Lemma 78 *$(K(x)|K,v)$ is valuation-algebraic if and only if $(\tilde{K}(x)|\tilde{K},v)$ is immediate. $(K(x)|K,v)$ is valuation-transcendental if and only if $(\tilde{K}(x)|\tilde{K},v)$ is not immediate. $(K(x)|K,v)$ is value-transcendental if and only if $(\tilde{K}(x)|\tilde{K},v)$ is value-transcendental. $(K(x)|K,v)$ is residue-transcendental if and only if $(\tilde{K}(x)|\tilde{K},v)$ is residue-transcendental.*

The proof can easily be generalized to show:

Lemma 79 *Let $(F|K,v)$ be any valued field extension. Then $vF|vK$ and $Fv|Kv$ are algebraic if and only if $(F.\tilde{K}|\tilde{K},v)$ is immediate, for some (or any) extension of v from F to $F.\tilde{K}$.*

15.2 Countability of value group and residue field extensions

The algebraic analogue of the transcendental case discussed in Theorem 37 is the following lemma (see [R1] or [E]):

Lemma 80 *Let $(L|K,v)$ be an extension of valued fields. Take $\eta_i \in L$ such that $v\eta_i$, $i \in I$, belong to distinct cosets modulo vK. Further, take $\vartheta_j \in \mathcal{O}_L$, $j \in J$, such that $\vartheta_j v$ are Kv-linearly independent. Then the elements $\eta_i\vartheta_j$, $i \in I$, $j \in J$, are K-linearly independent, and for every choice of elements $c_{ij} \in K$, only finitely many of them nonzero, we have that*

$$v\sum_{i,j} c_{ij}\eta_i\vartheta_j \;=\; \min_{i,j}\, vc_{ij}\eta_i\vartheta_j \;=\; \min_{i,j}\,(vc_{ij} \,+\, v\eta_i)\;.$$

If the elements $\eta_i\vartheta_j$ form a K-basis of L, then $v\eta_i$, $i \in I$, is a system of representatives of the cosets of vL modulo vK, and $\vartheta_j v$, $j \in J$, is a basis of $Lv|Kv$.

The following is an application which is important for our description of all possible value groups and residue fields of valuations on $K(x)$. The result has been proved with a different method in [APZ3] (Corollary 5.2); cf. Remark 76 in Section 15.1.

Theorem 81 *Let K be any field and v any valuation of the rational function field $K(x)$. Then $vK(x)/vK$ is countable, and $K(x)v|Kv$ is countably generated.*

Proof: Since $K(x)$ is the quotient field of $K[x]$, we have that $vK(x) = vK[x] - vK[x]$. Hence, to show that $vK(x)/vK$ is countable, it suffices to show that the set $\{\alpha + vK \mid \alpha \in vK[x]\}$ is countable. If this were not true, then by Lemma 80 (applied with $J = \{1\}$ and $\vartheta_1 = 1$), we would have that $K[x]$ contains uncountably many K-linearly independent elements. But this is not true, as $K[x]$ admits the countable K-basis $\{x^i \mid i \geq 0\}$.

Now assume that $K(x)v|Kv$ is not countably generated. Then by Corollary 38, $K(x)v|Kv$ must be algebraic. It also follows that $K(x)v$ has uncountable Kv-dimension.

Pick an uncountable set κ and elements $f_i(x)/g_i(x)$, $i \in \kappa$, with $f_i(x), g_i(x) \in K[x]$ and $vf_i(x) = vg_i(x)$ for all i, such that their residues are Kv-linearly independent. As $vK(x)/vK$ is countable, there must be some uncountable subset $\lambda \subset \kappa$ such that for all $i \in \lambda$, the values $vf_i(x) = vg_i(x)$ lie in the same coset modulo vK, say $vh(x) + vK$ with $h(x) \in K[x]$. The residues $(f_i(x)/g_i(x))v$, $i \in \lambda$, generate an algebraic extension of uncountable dimension. Choosing suitable elements $c_i \in K$ such that

$$vc_i f_i(x) \;=\; vh(x) \;=\; vc_i g_i(x) \;,$$

we can write

$$\frac{f_i(x)}{g_i(x)} \;=\; \frac{c_i f_i(x)}{h(x)} \cdot \frac{h(x)}{c_i g_i(x)} \;=\; \frac{c_i f_i(x)}{h(x)} \cdot \left(\frac{c_i g_i(x)}{h(x)}\right)^{-1}$$

for all $i \in \lambda$. Therefore,

$$\frac{f_i(x)}{g_i(x)}\, v \;=\; \left(\frac{c_i f_i(x)}{h(x)}\, v\right) \cdot \left(\frac{c_i g_i(x)}{h(x)}\, v\right)^{-1}$$

for all $i \in \lambda$. In order that these elements generate an algebraic extension of Kv of uncountable dimension, the same must already be true for the elements $(c_i f_i(x)/h(x))v$, $i \in \lambda$, or for the elements $(c_i g_i(x)/h(x))v$, $i \in \lambda$. It follows that at least one of these two sets contains uncountably many Kv-linearly independent elements. But then by Lemma 80 (applied with $I = \{1\}$ and $\eta_1 = 1$), there are uncountably many K-linearly independent elements in the set

$$\frac{1}{h(x)}\, K[x]$$

and hence also in $K[x]$, a contradiction. $\square$

Let me also mention the following lemma which combines the algebraic and the transcendental case. It generalizes a special case of Theorem 37. I leave its easy proof to the reader.

Lemma 82 *Let $(L|K,v)$ be an extension of valued fields. Take $x \in L$. Suppose that for some $e \in \mathbb{N}$ there exists an element $d \in K$ such that $vdx^e = 0$ and $dx^e v$ is transcendental over Kv. Let e be minimal with this property. Then for every $f = c_n x^n + \ldots + c_0 \in K[x]$,*

$$vf \;=\; \min_{1\leq i\leq n} vc_i x^i \;.$$

Moreover, $K(x)v = Kv(dx^e v)$ is a rational function field over Kv, and we have

$$vK(x) \;=\; vK + \mathbb{Z}vx \quad \textit{with} \quad (vK(x) : vK) \;=\; e \;.$$

15.3 Pure and weakly pure extensions

Take $t \in K(x)$. If vt is not a torsion element modulo vK, then t will be called a **value-transcendental element**. If $vt = 0$ and tv is transcendental over Kv, then t will be called a **residue-transcendental element**. An element will be called a **valuation-transcendental element** if it is value-transcendental or residue-transcendental. We will call the extension $(K(x)|K,v)$ **pure (in** x**)** if one of the following cases holds:

- for some $c,d \in K$, $d\cdot(x-c)$ is valuation-transcendental,
- x is the pseudo limit of some pseudo Cauchy sequence in (K,v) of transcendental type.

We leave it as an exercise to the reader to prove that $(K(x)|K,v)$ is pure in x if and only if it is pure in any other generator of $K(x)$ over K; we will not need this fact in the present paper.

If $(K(x)|K,v)$ is pure in x then it follows from Lemma 37 and Lemma 72 that x is transcendental over K. If $d\cdot(x-c)$ is value-transcendental, then $vK(x) = vK \oplus \mathbb{Z}v(x-c)$ and $K(x)v = Kv$ by Lemma 37 (in this case, we may in fact choose $d = 1$). If $d\cdot(x-c)$ is residue-transcendental, then again by Lemma 37, we have $vK(x) = vK$ and that $K(x)v = Kv(d(x-c)v)$ is a rational function field over Kv. If x is the pseudo limit of some pseudo Cauchy sequence in (K,v) of transcendental type, then $(K(x)|K,v)$ is immediate by Lemma 72. This proves:

Lemma 83 *If $(K(x)|K,v)$ is pure, then vK is pure in $vK(x)$ (i.e., $vK(x)/vK$ is torsion free), and Kv is relatively algebraically closed in $K(x)v$.*

Here is the "prototype" of pure extensions:

Lemma 84 *If K is algebraically closed and $x \notin K$, then $(K(x)|K,v)$ is pure.*

Proof: Suppose that the set

$$v(x-K) \;:=\; \{v(x-b) \mid b \in K\} \tag{54}$$

has no maximum. Then there is a pseudo Cauchy sequence in (K,v) with pseudo limit x, but without a pseudo limit in K. Since K is algebraically closed, Theorem 3 of [KA] shows that this pseudo Cauchy sequence must be of transcendental type. The extension therefore satisfies the third condition for being pure.

Now assume that the set $v(x-K)$ has a maximum, say, $v(x-c)$ with $c \in K$. If $v(x-c)$ is a torsion element over vK, then $v(x-c) \in vK$ because vK is divisible as K is algebraically closed. It then follows that there is some $d \in K$ such that $vd(x-c) = 0$. If $d(x-c)v$ were algebraic over Kv, then it were in Kv since K and thus also Kv is algebraically closed. But then, there were some $b_0 \in K$ such that $v(d(x-c)-b_0) > 0$. Putting $b := c + d^{-1}b_0$, we would then obtain that $v(x-b) = v((x-c)-d^{-1}b_0) > -vd = v(x-c)$, a contradiction to the maximality of $v(x-c)$. So we see that either $v(x-c)$ is non-torsion over vK, or there is some $d \in K$ such $vd(x-c) = 0$ and $d(x-c)v$ is transcendental over Kv. In both cases, this shows that $(K(x)|K,v)$ is pure. □

We will call the extension $(K(x)|K,v)$ **weakly pure (in** x**)** if it is pure in x or if there are $c,d \in K$ and $e \in \mathbb{N}$ such that $vd(x-c)^e = 0$ and $d(x-c)^e v$ is transcendental over Kv.

Lemma 85 *Assume that the extension $(K(x)|K,v)$ is weakly pure. If we take any extension of v to $\widetilde{K(x)}$ and take K^h to be the henselization of K in $(\widetilde{K(x)},v)$, then K^h is relatively algebraically closed in $K(x)^h$.*

Proof: Denote by K' the relative algebraic closure of K in K^h. By Lemma 6, K^h is contained in $K(x)^h$, hence it is contained in K'. Since $K(x)^h$ is the fixed field of the decomposition group $G^d_x := G^d(K(x)^{\mathrm{sep}}|K(x),v)$ in the separable-algebraic closure $K(x)^{\mathrm{sep}}$ of $K(x)$, we know that K' is the fixed field in K^{sep} of the subgroup

$$G_{\mathrm{res}} \;:=\; \{\sigma|_{K^{\mathrm{sep}}} \mid \sigma \in G^d_x\}$$

of $\mathrm{Gal}\,K$. In order to show our assertion, it suffices to show that $K' \subseteq K^h$, that is, that the decomposition group $G^d := G^d(K^{\mathrm{sep}}|K,v)$ is contained in G_{res}. So we have to show: if τ is an automorphism of $K^{\mathrm{sep}}|K$ such that $v\tau = v$ on K^{sep}, then τ can be lifted to an automorphism σ of $K(x)^{\mathrm{sep}}|K(x)$ such that $v\sigma = v$ on $K(x)^{\mathrm{sep}}$. In fact, it suffices to show that τ can be lifted to an automorphism σ of $K^{\mathrm{sep}}(x)|K(x)$ such that $v\sigma = v$ on $K^{\mathrm{sep}}(x)$. Indeed, then we take any extension σ' of σ from $K^{\mathrm{sep}}(x)$ to $K(x)^{\mathrm{sep}}$. Since the extensions $v\sigma'$ and v of v from $K^{\mathrm{sep}}(x)$ to $K(x)^{\mathrm{sep}}$ are conjugate, there is some $\rho \in \mathrm{Gal}\,(K(x)^{\mathrm{sep}}|K^{\mathrm{sep}}(x))$ such that $v\sigma'\rho = v$ on $K(x)^{\mathrm{sep}}$. Thus, $\sigma := \sigma'\rho \in G^d$ is the desired lifting of τ to $K(x)^{\mathrm{sep}}$.

We take σ on $K^{\mathrm{sep}}(x)$ to be the unique automorphism which satisfies $\sigma x = x$ and $\sigma|_{K^{\mathrm{sep}}} = \tau$. Using that $(K(x)|K,v)$ is weakly pure, we have to show that $v\sigma = v$ on $K^{\mathrm{sep}}(x)$. Assume first that for some $c,d \in K$ and $e \in \mathbb{N}$, $d(x-c)^e$ is valuation-transcendental. Since $K(x-c) = K(x)$, we may assume w.l.o.g. that $c = 0$. Every element of $K^{\mathrm{sep}}(x)$ can be written as a quotient of polynomials in x with coefficients from K^{sep}. For every polynomial $f(x) = a_n x^n + \ldots + a_1 x + a_0 \in K^{\mathrm{sep}}[x]$,

$$\begin{aligned} v\sigma f(x) &= v\left(\sigma(a_n)x^n + \ldots + \sigma(a_1)x + \sigma(a_0)\right) \\ &= \min_i (v\sigma(a_i) + ivx) \;=\; \min_i (v\tau(a_i) + ivx) \\ &= \min_i (va_i + ivx) \;=\; vf(x)\;, \end{aligned}$$

where the second equality holds by Lemma 37 and Lemma 82. This shows that $v\sigma = v$ on $K^{\mathrm{sep}}(x)$.

Now assume that x is the pseudo limit of a pseudo Cauchy sequence in (K,v) of transcendental type. By Lemma 73, this pseudo Cauchy sequence is also of transcendental type over $(K^{\rm sep}, v)$. Observe that x is still a pseudo limit of this pseudo Cauchy sequence in $(K^{\rm sep}(x), v\sigma)$, because $v\sigma(x-a) = v(\sigma x - \sigma a) = v(x-a)$ for all $a \in K$. But $v\sigma = v\tau = v$ on $K^{\rm sep}$, and the extension of v from $K^{\rm sep}$ to $K^{\rm sep}(x)$ is uniquely determined by the pseudo Cauchy sequence (cf. Theorem 71). Consequently, $v\sigma = v$ on $K^{\rm sep}(x)$. □

15.4 Construction of nasty examples

I will now prove the following nasty fact:

Theorem 86 *Let K be any algebraically closed field of positive characteristic. Then there exists a valuation v on the rational function field $K(x,y)|K$ whose restriction to K is trivial, such that $(K(x,y),v)$ admits an infinite chain of immediate Galois extensions of degree p and defect p.*

Proof: Let K be any algebraically closed field of characteristic $p > 0$. On $K(x)$, we take v to be the x-adic valuation. We work in the power series field $K((\frac{1}{p^\infty}\mathbb{Z}))$ of all power series in x with exponents in $\frac{1}{p^\infty}\mathbb{Z}$, the p-divisible hull of $\mathbb{Z}$. We choose y to be a power series

$$y = \sum_{i=1}^{\infty} x^{-p^{-e_i}} \tag{55}$$

where e_i is any increasing sequence of natural numbers such that $e_{i+1} \geq e_i + i$ for all i. We then restrict the canonical (x-adic) valuation of $K((\frac{1}{p^\infty}\mathbb{Z}))$ to $K(x,y)$ and call it again v. We show first that $vK(x,y) = \frac{1}{p^\infty}\mathbb{Z}$. Indeed, taking p^{e_j}-th powers and using that the characteristic of K is p, we find

$$y^{p^{e_j}} - \sum_{i=1}^{j} x^{-p^{e_j - e_i}} = \sum_{i=j+1}^{\infty} x^{-p^{e_j - e_i}} .$$

Since $e_j - e_i \geq 0$ for $i \leq j$, the left hand side is an element in $K(x,y)$. The right hand side has value

$$-p^{e_j - e_{j+1}} vx \;;$$

since $e_j - e_{j+1} \leq -j$, we see that $\frac{1}{p^j} vx$ lies in $vK(x,y)$. Hence, $\frac{1}{p^\infty}\mathbb{Z} \subseteq vK(x,y)$. On the other hand, $vK(x,y) \subseteq vK((\frac{1}{p^\infty}\mathbb{Z})) = \frac{1}{p^\infty}\mathbb{Z}$ and therefore, $vK(x,y) = \frac{1}{p^\infty}\mathbb{Z}$.

By definition, y is a pseudo limit of the pseudo Cauchy sequence

$$\left(\sum_{i=1}^{\ell} x^{-p^{-e_i}} \right)_{\ell \in \mathbb{N}}$$

in the field $L = K(x^{1/p^i} \mid i \in \mathbb{N}) \subset K((\frac{1}{p^\infty}\mathbb{Z}))$. Suppose it were of algebraic type. Then by [KA], Theorem 3, there would exist some algebraic extension $(L(b)|L, v)$ with b a pseudo limit of the sequence. But then b would also be algebraic over $K(x)$ and hence the extension $K(x,b)|K(x)$ would be finite. On the other hand, since b is a pseudo limit of the above pseudo Cauchy sequence, it can be shown as before that $vK(x,b) = \frac{1}{p^\infty}\mathbb{Z}$ and thus, $(vK(x,b) : vK(x)) = \infty$. This contradiction to the fundamental inequality shows that the sequence must be of transcendental type. Hence by Lemma 85, L^h is relatively algebraically closed in $L(y)^h$. Since $L^h = L.K(x)^h$ is a purely inseparable extension of $K(x)^h$ and $K(x,y)^h|K(x)^h$ is separable, this shows that $K(x)^h$ is relatively algebraically closed in $K(x,y)^h$.

Now we set $\eta_0 := \frac{1}{x}$, and by induction on i we choose $\eta_i \in \widetilde{K(x)}$ such that $\eta_i^p - \eta_i = \eta_{i-1}$. Then we have

$$v\eta_i \;=\; -\frac{1}{p^i}vx$$

for every i. Since $vK(x)^h = vK(x) = \mathbb{Z}vx$, this shows that $K(x)^h(\eta_i)|K(x)^h$ has ramification index at least p^i. On the other hand, it has degree at most p^i and therefore, it must have degree and ramification index equal to p^i. Note that for all $i \geq 0$, $K(x,\eta_i) = K(x,\eta_1,\ldots,\eta_i)$ and every extension $K(x,\eta_{i+1})|K(x,\eta_i)$ is a Galois extension of degree and ramification index p. By what we have shown, the chain of these extensions is linearly disjoint from $K(x)^h|K(x)$. Since $K(x)^h$ is relatively algebraically closed in $K(x,y)^h$ and the extensions are separable, it follows that the chain is also linearly disjoint from $K(x,y)^h|K(x)$.

We will now show that all extensions $K(x,y,\eta_i)|K(x,y)$ are immediate. First, we note that $K(x,y)v = K$ since $K \subset K(x,y) \subset K((\frac{1}{p^\infty}\mathbb{Z}))$ and $K((\frac{1}{p^\infty}\mathbb{Z}))v = K$. Since K is algebraically closed, the inertia degree of the extensions must be 1. Further, as the ramification index of a Galois extension is always a divisor of the extension degree, the ramification index of these extensions must be a power of p. But the value group of $K(x,y)$ is p-divisible, which yields that the ramification index of the extensions is 1. By what we have proved above, they are linearly disjoint from $K(x,y)^h|K(x,y)$, that is, the extension of the valuation is unique. This shows that the defect of each extension $K(x,y,\eta_i)|K(x,y)$ is equal to its degree p^i. □

Remark 87 Instead of defining y as in (55), we could also use any power series

$$y \;=\; \sum_{i=1}^{\infty} x^{n_i p^{-e_i}} \tag{56}$$

where $n_i \in \mathbb{Z}$ are prime to p and the sequence $n_i p^{-e_i}$ is strictly increasing. The example in [CP] is of this form. But in this example, the field $K(x,y)$ is an extension of degree p^2 of a field $K(u,v)$ such that the extension of the valuation from $K(u,v)$ to $K(x,y)$ is unique. Since the value group of $K(x,y)$ is $\frac{1}{p^\infty}\mathbb{Z}$, it must be equal to that of $K(u,v)$. Since K is algebraically closed, both have the same residue field. Therefore, the extension has defect p^2. This shows that we can also use subfields instead of field extensions to produce defect extensions, in quite the same way.

A special case of (56) is the power series

$$y = \sum_{i=1}^{\infty} x^{i-p^{-e_i}} = \sum_{i=1}^{\infty} x^i x^{-p^{-e_i}} \tag{57}$$

which now has a support that is cofinal in $\frac{1}{p^\infty}\mathbb{Z}$.

To conclude this section, I will use Lemma 85 to construct an example about relatively closed subfields in henselian fields. The following fact is well known and can be proced using Hensel's Lemma (I leave this as an exercize to the reader):
Suppose that K is relatively closed in a henselian valued field (L, v) of residue characteristic 0 and that $Lv|Kv$ is algebraic. Then vL/vK is torsion free.
I show that the assumption "$Lv|Kv$ is algebraic" is necessary.

Example 88 On the rational function field $\mathbb{Q}(x)$, let's take v to be the x-adic valuation. Extend v to the rational function field $\mathbb{Q}(x, y)$ in such a way that $vy = 0$ and yv is transcendental over $\mathbb{Q}(x)v = \mathbb{Q}$. So by Lemma 37 we have $v\mathbb{Q}(x, y) = v\mathbb{Q}(x) = \mathbb{Z}vx$ and $\mathbb{Q}(x, y)v = \mathbb{Q}(yv)$. We pick any integer $n > 1$. Then $v\mathbb{Q}(x^n) = \mathbb{Z}nvx$ and $\mathbb{Q}(x^n)v = \mathbb{Q}$. Further, $v\mathbb{Q}(x^n, xy) = \mathbb{Z}vx$ since $vx = vxy \in v\mathbb{Q}(x^n, xy) \subseteq \mathbb{Z}vx$. Also, $\mathbb{Q}(x^n, xy)v = \mathbb{Q}(y^n v)$ by Lemma 82. From Lemma 85 we infer that $\mathbb{Q}(x^n)^h$ is relatively algebraically closed in $\mathbb{Q}(x^n, xy)^h$. But

$$v\mathbb{Q}(x^n, xy)^h / v\mathbb{Q}(x^n)^h = v\mathbb{Q}(x^n, xy)/v\mathbb{Q}(x^n) = \mathbb{Z}vx/\mathbb{Z}nvx \cong \mathbb{Z}/n\mathbb{Z}$$

is a non-trivial torsion group. ♢

15.5 All valuations on $K(x)$

In this section, we will explicitly define extensions of a given valuation on K to a valuation on $K(x)$. First, we define valuation-transcendental extensions, using the idea of valuation independence. Let (K, v) be an arbitrary valued field, and x transcendental over K. Take $a \in K$ and an element γ in some ordered abelian group extension of vK. We define a map $v_{a,\gamma} : K(x)^\times \to vK + \mathbb{Z}\gamma$ as follows. Given any $g(x) \in K[x]$ of degree n, we can write

$$g(x) = \sum_{i=0}^{n} c_i (x-a)^i . \tag{58}$$

Then we set

$$v_{a,\gamma}\, g(x) := \min_{0 \le i \le n} vc_i + i\gamma . \tag{59}$$

We extend $v_{a,\gamma}$ to $K(x)$ by setting $v_{a,\gamma}(g/h) := v_{a,\gamma}g - v_{a,\gamma}h$.

For example, the valuation $v_{0,0}$ is called **Gauß valuation** or **functional valuation** and is given by

$$v_{0,0}\,(c_n x^n + \ldots + c_1 x + c_0) = \min_{0 \le i \le n} vc_i .$$

Lemma 89 *$v_{a,\gamma}$ is a valuation which extends v from K to $K(x)$. It satisfies:*

1) If γ is non-torsion over vK, then $v_{a,\gamma}K(x) = vK \oplus \mathbb{Z}\gamma$ and $K(x)v_{a,\gamma} = Kv$.

2) If γ is torsion over vK, e is the smallest positive integer such that $e\gamma \in vK$ and $d \in K$ is some element such that $vd = -e\gamma$, then $d(x-a)^e v_{a,\gamma}$ is transcendental over Kv, $K(x)v_{a,\gamma} = Kv(d(x-a)^e v_{a,\gamma})$ and $v_{a,\gamma}K(x) = vK + \mathbb{Z}\gamma$. In particular, if $\gamma = 0$ then $(x-a)v_{a,\gamma}$ is transcendental over Kv, $K(x)v_{a,\gamma} = Kv\left((x-a)v_{a,\gamma}\right)$ and $v_{a,\gamma}K(x) = vK$.

Proof: It is a straightforward exercise to prove that $v_{a,\gamma}$ is a valuation and that 1) and 2) hold. However, one can also deduce this from Lemma 37. It says that if we assign a non-torsion value γ to $x-a$ then we obtain a unique valuation which satisfies (59). Since this defines a unique map $v_{a,\gamma}$ on $K[x]$, we see that $v_{a,\gamma}$ must coincide with the valuation given by Lemma 37, which in turn satisfies assertion 1). Similarly, if $\gamma \in vK$, $d \in K$ with $vd = -\gamma$ and we assign a transcendental residue to $d(x-a)$, then Lemma 37 gives us a valuation on $K(x)$ which satisfies (59) and hence must coincide with $v_{a,\gamma}$. This shows that $v_{a,\gamma}$ is a valuation and satisfies 2).

If $e > 1$, then we can first use Lemma 37 to see that $v_{a,\gamma}$ is a valuation on the subfield $K(d(x-a)^e)$ of $K(x)$ and that $v_{a,\gamma}K(d(x-a)^e) = vK$ and $K(d(x-a)^e)v_{a,\gamma} = Kv(d(x-a)^e v_{a,\gamma})$ with $d(x-a)^e v_{a,\gamma}$ transcendental over Kv. We know that there is an extension w of $v_{a,\gamma}$ to $K(x)$. It must satisfy $w(x-a) = -vd/e = \gamma$. So $0, w(x-a)$, $w(x-a)^2, \ldots, w(x-a)^{e-1}$ lie in distinct cosets modulo vK. From Lemma 80 it follows that w satisfies (59) on $K(x)$, hence it must coincide with the valuation $v_{a,\gamma}$ on $K(x)$. Assertion 2) for this case follows from Lemma 80. □

Now we are able to prove:

Theorem 90 *Take any valued field (K, v). Then all extensions of v to the rational function field $\tilde{K}(x)$ are of the form*

- *$\tilde{v}_{a,\gamma}$ where $a \in \tilde{K}$ and γ is an element of some ordered group extension of vK, or*
- *$\tilde{v}_{\mathbf{A}}$ where $\mathbf{A}$ is a pseudo Cauchy sequence in $(\tilde{K}, \tilde{v})$ of transcendental type,*

where $\tilde{v}$ runs through all extensions of v to $\tilde{K}$. The extension is of the form $\tilde{v}_{a,\gamma}$ with $\gamma \notin \tilde{v}\tilde{K}$ if and only if it is value-transcendental, and with $\gamma \in \tilde{v}\tilde{K}$ if and only if it is residue-transcendental. The extension is of the form $\tilde{v}_{\mathbf{A}}$ if and only if it is valuation-algebraic.

All extensions of v to $K(x)$ are obtained by restricting the above extensions, already from just one fixed extension $\tilde{v}$ of v to $\tilde{K}$.

Proof: By Lemma 89 and Theorem 71, $\tilde{v}_{a,\gamma}$ and $\tilde{v}_{\mathbf{A}}$ are extensions of $\tilde{v}$ to $\tilde{K}(x)$. For the converse, let w be any extension of v to $\tilde{K}(x)$ and set $\tilde{v} = w|_{\tilde{K}}$. From Lemma 84 we know that $(\tilde{K}(x)|\tilde{K}, w)$ is always pure. Hence, either $d(x-c)$ is valuation-transcendental for some $c, d \in \tilde{K}$, or x is the pseudo limit of some pseudo Cauchy sequence $\mathbf{A}$ in $(\tilde{K}, \tilde{v})$ of transcendental type. In the first case, Lemma 37 shows that

$$w \sum_{i=0}^{n} d_i (d(x-c))^i \;=\; \min_{0 \le i \le n} vd_i + iwd(x-c) \;=\; \min_{0 \le i \le n} vd_i d^i + iw(x-c)$$

for all $d_i \in \tilde{K}$. This shows that $w = \tilde{v}_{c,\gamma}$ for $\gamma = w(x-c)$. If this value is not in $\tilde{v}\tilde{K}$, then it is non-torsion over $\tilde{v}\tilde{K}$ and thus, the extension of $\tilde{v}$ to $\tilde{K}(x)$, and hence also the extension of v to $K(x)$, is value-transcendental. If it is in $\tilde{v}\tilde{K}$, then the residue of $d(x-c)$ is not in $\tilde{K}\tilde{v}$, and the extension of $\tilde{v}$ to $\tilde{K}(x)$, and hence also the extension of v to $K(x)$, is residue-transcendental.

In the second case, we know from Theorem 71 that $\mathbf{A}$ induces an extension $\tilde{v}_{\mathbf{A}}$ of $\tilde{v}$ to $\tilde{K}(x)$ such that x is a pseudo limit of $\mathbf{A}$ in $(\tilde{K}(x), \tilde{v}_{\mathbf{A}})$. Since x is also a pseudo limit of $\mathbf{A}$ in $(\tilde{K}(x), w)$, we can infer from Lemma 72 that $w = \tilde{v}_{\mathbf{A}}$. It also follows from Theorem 71 that $(\tilde{K}(x)|\tilde{K}, \tilde{v}_{\mathbf{A}})$ is immediate and consequently, $(\tilde{K}(x)|K, \tilde{v}_{\mathbf{A}})$ is valuation-algebraic.

For the last assertion, we invoke Lemma 77. Now it just remains to show that it suffices to take the restrictions of the valuations $\tilde{v}_{a,\gamma}$ and $\tilde{v}_{\mathbf{A}}$ for one fixed $\tilde{v}$. Suppose that $\tilde{w}$ is another extension of v to $\tilde{K}$. Since all such extensions are conjugate, there is $\sigma \in \operatorname{Gal} K$ such that $\tilde{w} = \tilde{v} \circ \sigma$. Let $g(x) \in K[x]$ be given as in (58). Extend σ to an automorphism of $\tilde{K}(x)$ which satisfies $\sigma x = x$. Since g has coefficients in K, we then have

$$g(x) \;=\; \sigma g(x) \;=\; \sum_i \sigma c_i (x - \sigma a)^i$$

and therefore,

$$\tilde{w}_{a,\gamma}\, g(x) \;=\; \min_i (\tilde{w} c_i + i\gamma) \;=\; \min_i (\tilde{v} \sigma c_i + i\gamma) \;=\; \tilde{v}_{\sigma a,\gamma}\, g(x)\;.$$

This shows that $\tilde{w}_{a,\gamma} = \tilde{v}_{\sigma a,\gamma}$ on $K(x)$.

Given a pseudo Cauchy sequence $\mathbf{A}$ in $(\tilde{K}, \tilde{w})$, we set $\mathbf{A}_\sigma = (\sigma a_\nu)_{\nu<\lambda}$. This is a pseudo Cauchy sequence in $(\tilde{K}, \tilde{v})$ since $\tilde{v}(\sigma a_\mu - \sigma a_\nu) = \tilde{v}\sigma(a_\mu - a_\nu) = \tilde{w}(a_\mu - a_\nu)$. For every polynomial $f(x) \in \tilde{K}[x]$, we have $\tilde{v} f(\sigma a_\nu) = \tilde{w}\sigma^{-1}(f(\sigma a_\nu)) = \tilde{w}(\sigma^{-1}(f))(a_\nu)$, where $\sigma^{-1}(f)$ denotes the polynomial obtained from $f(x)$ by applying σ^{-1} to the coefficients. So we see that $\mathbf{A}_\sigma$ is of transcendental type if and only if $\mathbf{A}$ is. If $g(x) \in K[x]$, then $\sigma^{-1}(g) = g$ and the above computation shows that $\tilde{v} g(\sigma a_\nu) = \tilde{w} g(a_\nu)$. This implies that $\tilde{w}_{\mathbf{A}} = \tilde{v}_{\mathbf{A}_\sigma}$ on $K(x)$. □

Remark 91 If v is trivial on K, hence $Kv = K$ (modulo an isomorphism), and if we choose $\gamma > 0$, then the restriction w of $\tilde{v}_{a,\gamma}$ to $K(x)$ will satisfy $xw = aw = a$. It follows that $K(x)w = K(a)$. Further, $wK(x) \subseteq \mathbb{Z}\gamma$ and thus, $wK(x) \cong \mathbb{Z}$.

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