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Introduction to Singularity Theory

(continuation)

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How can we study singularities?

Let us consider the case of complex plane curves, i.e. complex algebraic hypersurfaces of $\mathbb{C}^2$.

Let $F(X,Y) = 0$ be a reduced equation of a plane curve C.

According to Whitney theorem, C has only a finite number of singularities.

If $x \in C$ is non-singular, the implicit function theorem implies that there is an open neighbourhood of x in $\mathbb{C}^2$, such that $C \cap U$ is a complex analytic submanifold of U. So, locally at x is like the complex line. This means that there is an analytic isomorphism of an open disc D of $\mathbb{C}$ onto a neighbourhood V of x in C:

$$\pi : D \to V,$$

i.e. we have a local **parametrization** of C at the point $x \in C$.

Suppose that $0 \in C$ is a singular point of C. Puiseux theorem tells that locally at 0, we also have a local parametrization.

Theorem 6 *Suppose that $F(0,Y) \neq 0$. There is an integer m and a formal series $\Phi(X^{1/m})$ in $\mathbb{C}[[X^{1/m}]]$ such that*

$$F(X, \Phi(X^{1/m})) = 0.$$

In fact, the series Φ is convergent at 0.

Puiseux theorem gives a local parametrization $\pi : D \to C$ of C at x defined by $\pi(t) = (t^m, \Phi(t))$.

However in general π does not gives an isomorphism of a disc D onto an open neighbourhood of x in C.

Example: Consider the case

$$F(X,Y) = X^2 - Y^2.$$

In some cases, for instance with

$$F(X,Y) = Y^2 - X^3,$$

we have a local parametrization which is a homeomorphism of a disc D with an open neighbourhood of 0 in the curve. Here it is $Y = t^3$ and $X = t^2$.

The existence of a local parametrization helps to study functions on a singular curve.

For instance, the rational function Y/X restricted to the curve $Y^2 - X^3 = 0$ defines a continuous function!

In fact locally at the singular point of a complex curve there are parametrizations for each local branches.

The definition of branches involves the complex analytic structure of C, this will lead to algebraic difficulties when one considers arbitrary base fields.

Since the ring of complex polynomials $\mathbb{C}[X,Y]$ in two variables is a subring of the ring of convergent complex series in two variables $\mathbb{C}\{X,Y\}$ a reduced equation P_0 of C is also an element of $\mathbb{C}\{X,Y\}$. This latter ring is factorial. So, P_0 has a decomposition into irreducible factors in $\mathbb{C}\{X,Y\}$:

$$P_0 = f_1 \dots f_s.$$

The analytic curves $f_i = 0$ are the branches of C at 0.

Consider an irreducible element f in $\mathbb{C}\{X,Y\}$. We have an analytic version of Puiseux theorem:

Theorem 7 *Suppose that $f(0,Y) \neq 0$ and has valuation n. There is a convergent series Φ in $\mathbb{C}\{X\}$ such that*

$$f(X,Y) = u(X,Y) \prod_{\xi,\xi^n=1} (Y - \Phi(\xi X^{1/n})),$$

where $u(X,Y)$ is a unit of $\mathbb{C}\{X,Y\}$.

We call **Puiseux expansion** (or Puiseux series) of the branch f relatively to the coordinates X and Y the series

$$Y = \Phi(X^{1/n})$$

If an algebraic curve C defined by the reduced equation $P_0 = 0$ has several branches $f_i = 0$, $1 \leq i \leq s$ at the singular point 0, and the coordinates X, Y are such that none of the series $f_i(0, Y)$, $1 \leq i \leq s$ vanish identically, we have simultaneous Puiseux expansions

$$\begin{aligned} Y &= \Phi_1(X^{1/n_1}) \\ &\dots \\ Y &= \Phi_s(X^{1/n_s}) \end{aligned}$$

These Puiseux expansions determine the analytic structure of C at 0, since we have

$$P_0 = u(X,Y) \prod_{i=1}^{i=s} \left(\prod_{\xi, \xi^{n_i}=1} (Y - \Phi(\xi X^{1/n_i})) \right)$$

where $u(X,Y)$ is an invertible element of $\mathbb{C}\{X,Y\}$.

They also determine the topological structure of C at 0.

When C has several branches at 0, it is rather complicated to show how the Puiseux expansions of the branches determine the local topology of the curve.

We shall restrict ourselves to the case when C has only one branch. Let $\Phi(X^{1/n})$ be the Puiseux expansion of this branch relatively to X, Y.

First, notice that $\Phi(X^{1/n})$ is an element of the ring extension $\mathbb{C}\{X\}[X^{1/n}]$ of $\mathbb{C}\{X\}$. Let $\mathbb{C}\{\{X\}\}$ be the field of fractions of $\mathbb{C}\{X\}$. Then, the field $\mathbb{C}\{\{X\}\}[X^{1/n}]$ is an algebraic extension of the field $\mathbb{C}\{\{X\}\}$. The Galois group of this field extension is the cyclic group μ_n of order n. Let σ an element of μ_n. There is a unique root of unity $\xi(\sigma)$ such that

$$\sigma(\Phi(X^{1/n})) = \Phi(\xi(\sigma)X^{1/n}).$$

In the power series ring $\mathbb{C}\{X\}[X^{1/n}]$ we have a valuation v such that $v(X^{1/n}) = 1$. Consider the subgroup G_j of μ_n defined by

$$G_j := \{\sigma \in \mu_n\,,\ v(\sigma\Phi(X^{1/n}) - \Phi(X^{1/n})) \geq j\}.$$

Obviously $G_1 = \mu_n$ and $G_{j+1} \subset G_j$, for $j \geq 1$. Since μ_n is a finite group for $k \gg 0$, $G_k = \{\varepsilon\}$, where ε is the neutral element of μ_n.

Let $\beta_1 < \ldots < \beta_g$ the sequence of integers such that

$$\mu_n = G_1 = \ldots = G_{\beta_1} \supsetneq G_{\beta_1+1} \cdots \supsetneq G_{\beta_g+1} = \{\varepsilon\}.$$

The integers $\beta_1, \ldots, \beta_g$ are called the **Puiseux exponents** relatively to the coordinates X, Y.

There are a unique sequence of pairs of relatively prime integers $(m_1, n_1), \ldots, (m_g, n_g)$ such that

$$\frac{\beta_1}{n} = \frac{m_1}{n_1}, \ldots, \frac{\beta_g}{n} = \frac{m_g}{n_1 \ldots n_g}.$$

We call these pairs the **Puiseux characteristic pairs** of C relatively to the coordinates X, Y.

These Puiseux pairs give a description of the local topology of C at the singular point 0.

Namely, for $0 < \epsilon \ll 1$, the real 3-sphere $S_\epsilon(0)$ centered at 0 with radius ϵ intersects C transversally. Since C has only one branch at 0, the intersection $C \cap S_\epsilon(0)$ is connected and is diffeomorphic to the circle $\mathbb{S}^1$. Therefore the embedding of $C \cap S_\epsilon(0)$ into the sphere $S_\epsilon(0)$ is a knot.

It is an iterated torus knot given by the Puiseux pairs $(m_1, n_1), \ldots, (m_g, n_g)$.

These Puiseux pairs determine the local topology of C at 0 in the following sense:

Let C_1 and C_2 be two plane curves having one branch at 0. Suppose that there exists a homeomorphism Ψ of a neighbourhood U_1 of 0 in $\mathbb{C}^2$ onto a neighbourhood U_2 of 0 in $\mathbb{C}^2$ such that $\Psi(U_1 \cap C_1) = U_2 \cap C_2$, then the Puiseux pairs of C_1 at 0 relatively to "general" coordinates of 0 in $\mathbb{C}^2$ at 0 are equal to the Puiseux pairs of C_2 at 0 relatively to "general" coordinates of 0 in $\mathbb{C}^2$ at 0

How do we obtain local parametrization of a branch?

1. The method of Puiseux consists to apply Newton approximation method.

2. Another way is to observe that if C, defined by the reduced polynomial P analytically irreducible at 0, the local integral ring $\mathbb{C}\{X,Y\}/(P)$ embed in its normalization which is a regular local ring $\mathbb{C}\{t\}$.

3. A third way is obtained by blowing-up singular points.

Let us define the blowing-up of a point. We first give an analytic definition of the blowing-up of a point.

Let U be an open neighbourhood of 0 in $\mathbb{C}^2$. We have a natural map

$$\lambda : U \setminus \{0\} \to \mathbb{P}^1_{\mathbb{C}}$$

defined by $\lambda(x) = \{\text{the complex line from 0 to } x\}$.

The graph of λ is a subset $G(\lambda)$ of $U \times \mathbb{P}^1_{\mathbb{C}}$. Remember that $\mathbb{P}^1_{\mathbb{C}}$ is a complex analytic manifold which is the union of two affine spaces U_0 and U_1 isomorphic to $\mathbb{C}$ with respective coordinates u_0 and u_1. Then, $U \times \mathbb{P}^1_{\mathbb{C}}$ is the union of $U \times U_0$ and $U \times U_0$. The intersection $G(\lambda) \cap (U \times U_0)$ is contained in the set defined by

$$\frac{X}{Y} = u_0$$

Similarly $G(\lambda) \cap (U \times U_1)$ is contained in the set defined by

$$\frac{Y}{X} = u_1.$$

Then, the closure E of $\overline{G(\lambda)}$ of $G(\lambda)$ in $U \times \mathbb{P}^1_{\mathbb{C}}$ is defined by $X = u_0 Y$ in $U \times U_0$ and by $Y = u_1 X$ in $U \times U_1$. It is easy to see that E is an complex analytic manifold of complex dimension 2. The projection onto U induces a map $e : E \to U$ which is called the **blowing-up of** 0 **in** U. The inverse image $e^{-1}(0)$ is called the **exceptional divisor** of the blowing-up.

In $(U \times U_0) \cap E$ the equation of the exceptional divisor is $Y = 0$. In $(U \times U_1) \cap E$ it is $X = 0$.

It is convenient to consider on $(U \times U_0) \cap E$ the two coordinates Y and u_0 and on $(U \times U_1) \cap E$ the coordinates X and u_1. The restriction of the blowing-up e to $(U \times U_0) \cap E$ is given by

$$e(Y, u_0) = (Y u_0, Y).$$

Similarly the restriction of e to $(U \times U_1) \cap E$ is given by

$$e(X, u_1) = (X, X u_1).$$

Now consider a curve C which a branch $f = 0$ at 0. Assume that the irreducible element f of $\mathbb{C}\{X, Y\}$ defines an analytic function on the open neighbourhood U of 0 in $\mathbb{C}^2$.

The intersection of the inverse image of C by the blowing-up of 0 in U with the subset of $U \times U_0$ is the set defined by $f(Yu_0, Y)$.

Consider the expansion of f by homogeneous forms:

$$f = f_m + f_{m+1} + \cdots,$$

where m is the multiplicity of f at 0. Then,

$$\begin{aligned} & f(Yu_0, Y) \\ & = Y^m f_m(u_0, 1) + Y^{m+1} f_{m+1}(u_o, 1) + \ldots \\ & = Y^m(f_m(u_0, 1) + Y f_{m+1}(u_0, 1) + \ldots) \\ & = Y^m f_0(Y, u_o). \end{aligned}$$

Similarly the set $e^{-1}(C) \cap (U \times U_1)$ is defined by $f(X, Xu_1) = 0$ and

$$\begin{aligned} & f(X, Xu_1) \\ & = X^m f_m(1, u_1) + X^{m+1} f_{m+1}(1, u_1) + \ldots \\ & = X^m(f_m(1, u_1) + X f_{m+1}(1, u_1) + \ldots) \\ & = X^m f_1(X, u_1). \end{aligned}$$

Remember that $Y = 0$ is the equation of the exceptional divisor in $U \times U_0$ and that $X = 0$ is the equation of the exceptional divisor in $U \times U_1$.

Therefore $e^{-1}(C)$ is the union of the exceptional divisor and a set C_1 whose intersections with $U \times U_0$ and $U \times U_1$ are defined respectively by $f_0 = 0$ and $f_1 = 0$.

The set C_1 is also the topological closure of $e^{-1}(C \setminus \{0\})$ in E. It is called the **strict transform** of C by the blowing-up e. The restriction of the blowing-up e to C_1 induces a map $e_0 : C_1 \to C$ which is called the **blowing-up of the curve** C at 0.

Obviously locally C_1 is isomorphic to analytic plane curve, but C_1 itself is the patch of two plane curves.

In fact the definition of blowing-up is applicable to the case $U = \mathbb{C}^2$. In which case, we observe that the restrictions of the blowing-up of $\mathbb{C}^2$ to two open subsets isomorphic to the affine space $\mathbb{C}^2$ are algebraic maps (maps whose components are polynomials).

Extending the notion of algebraic sets to objects which are "locally" algebraic sets, we have the notion of an algebraic variety or of a finitely generated reduced scheme over the complex field (see the usual litterature). In this context, a blowing-up is an algebraic map and the strict transform of an irreducible algebraic plane curve is a variety of dimension one (i.e. an algebraic curve). Furthermore, locally this curve is isomorphic to a plane curve.

Coming back to the notion of parametrization, one can prove that by a succession of plane blowing-ups the strict transform becomes non-singular.

The way to prove it is first to recall that after a blowing-up locally the blown-up curve is isomorphic to a plane curve. Then, we observe that after a point blowing-up the multiplicities of the singularities do not increase. In fact, in the case of a branch f at a point 0, the first Puiseux exponent β_1 (defined above) relatively to coordinates X, Y such that the valuation of $f(0, Y)$ equals the multiplicity m of $f = 0$ at 0, can be interpreted in the following way:

We saw that the multiplicity is the intersection number of a general line through 0 with the curve. So, it is also the intersection number of a general non-singular curve (whose tangent at 0 is a general line) with the curve at 0. Now, if the line or the tangent of the non-singular curve at 0, is not general, this intersection number is strictly higher than the multiplicity. It can be shown that the highest value of the intersection is precisely β_1 (in contrast with the case the curve is non-singular at 0 when this number can be as high as one wishes).

Then, by one blowing-up one can prove that for a curve with one branch at 0, this number β_1 decreases strictly, in fact the new value at the singular point of the blown-up branch is $\beta_1 - m$.

Using these observations one can prove that after a finite number of point blowing-ups, the final strict transform of the given curve is non-singular.

In the case of a branch the composition of the successive blowing-ups of the singular points of the successive strict transforms

$$C \leftarrow C_1 \leftarrow C_2 \leftarrow \ldots \leftarrow C_k$$

gives a map π from a non-singular curve C_k onto C.

It is easy to check that a blowing-up is an isomorphism outside the blown-up point, so π induces an isomorphism of $C_k \setminus \pi^{-1}(0)$ onto $C \setminus \{0\}$ and $\pi^{-1}(0) = \{x_k\}$. Since x_k is a non-singular point of C_k an open neighbourhood of x_k in C_k is isomorphic to a disc D and π induces a parametrization $p : D \to C$ of C at 0.

Therefore, in the case of a plane branch the process of eliminating a singular point by successive point blowing-ups gives the local parametrization.

What is a desingularization?

Historically algebraic geometers were looking for a "transformation" which could replace the local parametrization.

As we mentioned above the normalization also leads to parametrization of plane branches. Unfortunaley the normalization of a surface is in general singular. In fact surfaces with non-singular normalization are very special. However, it can be shown that the singularities of normal surfaces, i.e. algebraic varieties for which all the local rings are equal to their normalization, are isolated. Then, the natural idea was then to blow-up singular points of normal surfaces. In general the blowing-up of a singular point of a normal surface is not a normal surface any more, since the singularities of the blown-up surface might not be isolated. R. Walker stated that a surface could be desingularized after a finite sequence of normalizations and point blowing-ups. The proof of this theorem was given by O. Zariski using valuation theory (see Ann. Math. 40 (1939)).

One can observe that a map $\pi : W \to V$, which is the composition of normalizations and point blowing-ups at singularities, is

1. a proper map (the inverse image of a compact subset is compact);

2. an isomorphism of $\pi^{-1}(V \setminus SingV)$ onto $V \setminus SingV$ where $SingV$ is the subset of singular points of V.

We are led to the following definition:

A map $\pi : W \to V$ is a **desingularization** of a variety V (we also say a **resolution of singularities** of V) if:

1. W is a non-singular variety;

2. it is a proper map;

3. it is an isomorphism of $\pi^{-1}(V \setminus SingV)$ on $V \setminus SingV$ where $SingV$ is the subset of singular points of V;

4. $\pi^{-1}(V \setminus SingV)$ is dense in W.

As one can see, in order to find resolutions of singularity, in particular, one will have to generalize the notion of properness to maps between varieties defined over arbitrary fields.

Over an algebraically closed field of characteristic zero the existence of a resolution of singularities was obtained by H. Hironaka in Ann. Math. 79 (1964).

One of the aim of this school on the resolution of singularities is to find a desingularization in the case of varieties over a non-zero characteristic field.