

Flips for 3-folds and 4-folds

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CHAPTER 1

Introduction

ALESSIO CORTI

1.1. Minimal models of surfaces

In this book, we generalise the following theorem to 3-folds and 4-folds:

THEOREM 1.1.1 (Minimal model theorem for surfaces). *Let X be a nonsingular projective surface. There is a birational morphism $f: X \rightarrow X'$ to a nonsingular projective surface X' satisfying one of the following conditions:*

X' is a minimal model: $K_{X'}$ is nef, that is $K_{X'} \cdot C \geq 0$ for every curve $C \subset X'$; or

X' is a Mori fibre space: $X' \cong \mathbb{P}^2$ or X' is a $\mathbb{P}^1$ -bundle over a nonsingular curve T .

This result is well known; the classical proof runs more or less as follows. If X does not satisfy the conclusion, then X contains a -1 -curve, that is, a nonsingular rational curve $E \subset X$ such that $K_X \cdot E = E^2 = -1$. By the Castelnuovo contractibility theorem, a -1 -curve can always be contracted: there exists a morphism $f: E \subset X \rightarrow P \in X_1$ which maps $E \subset X$ to a nonsingular point $P \in X_1$ and restricts to an isomorphism $X \setminus E \rightarrow X_1 \setminus \{P\}$. Viewed from $P \in X_1$, this is just the blow up of $P \in X_1$. Either X_1 satisfies the conclusion, or we can continue contracting a -1 -curve in X_1 . Every time we contract a -1 -curve, we decrease the rank of the Néron-Severi group; hence the process must terminate (stop).

1.2. Higher dimensions and flips

The conceptual framework generalising this result to higher dimensions is the well known Mori program or Minimal Model Program. The higher dimensional analog $f: X \rightarrow X_1$ of the contraction of a -1 -curve is an extremal divisorial contraction. Even if we start with X nonsingular, X_1 can be singular. This is not a problem: we now know how to handle the relevant classes of singularities, for example the class of terminal singularities. The problem is that, in higher dimensions, we meet a new type of contraction:

DEFINITION 1.2.1. A *small* contraction is a birational morphism $f: X \rightarrow Z$ with connected fibres such the exceptional set $C \subset X$ is of codimension $\text{codim}_X C \geq 2$.

When we meet a small contraction $f: X \rightarrow Z$, the singularities on Z are so bad that the canonical class of Z does not even have a Chern class in $H^2(Z, \mathbb{Q})$: it does not make sense to form the intersection number $K_Z \cdot C$ with algebraic curves $C \subset Z$. We need a new type of operation, called a flip:

DEFINITION 1.2.2. A small contraction $f: X \rightarrow Z$ is a *flipping contraction* if K_X is anti-ample along the fibres of f . The *flip* of f is a new small contraction $f': X' \rightarrow Z$ such that $K_{X'}$ is ample along the fibres of f' .

It is conjectured that flips exist and that every sequence of flips terminates (that is, there exists no infinite sequence of flips).

1.3. The work of Shokurov

Mori [Mor88] first proved that flips exist if X is 3-dimensional. It was known (and, in any case, it is easy to prove) that 3-fold flips terminate, thus Mori's theorem implied the minimal model theorem in dimension 3.

Recently, Shokurov [Sho03] completed in dimension ≤ 4 a program to construct flips started in [Sho92, FA92]. In fact, Shokurov shows that a more general type of flip exists. We consider perturbations of the canonical class K_X of the form $K_X + \sum b_i B_i$, where $B_i \subset X$ are prime divisors and $0 \leq b_i < 1$. For the program to work, the pair $(X, \sum b_i B_i)$ needs to have klt singularities. The precise definition is given elsewhere in this book and it is not important for the present discussion; the condition is satisfied, for example, if X is nonsingular and the support of $B = \sum b_i B_i$ is a simple normal crossing divisor.

DEFINITION 1.3.1. A small contraction $f: X \rightarrow Z$ is a *klt flipping contraction* if the pair (X, B) has klt singularities and $K_X + B$ is anti-ample on the fibres of f . The *flip* of f is a new small contraction $f': X' \rightarrow Z$ such that $K_{X'} + B$ is ample on the fibres of f' .

Shokurov [Sho03] proves the following:

THEOREM 1.3.2. *The flips of klt flipping contractions $f: (X, B) \rightarrow Z$ exist if $\dim X \leq 4$.*

It is known from work of Kawamata [Kaw92, FA92] that 3-fold klt flips terminate. It is not known that klt flips terminate in dimension ≥ 4 ; the work of Shokurov leaves the following open.

PROBLEM 1.3.3. *Show that klt flips terminate in dimension ≥ 4 .*

1.4. Minimal models of 3-folds and 4-folds

It is known from [KMM87] that ordinary (that is, terminal) 4-fold flips terminate. It follows from Theorem 1.3.2 that ordinary 4-fold flips exist, thus we have the following consequence:

THEOREM 1.4.1 (Minimal model theorem for 3-folds and 4-folds). *Let X be a nonsingular projective variety of dimension ≤ 4 . There exists a birational map $X \dashrightarrow X'$ to a projective variety X' (with terminal singularities) satisfying one of the following conditions:*

X' is a minimal model: $K_{X'}$ is nef, that is $K_{X'} \cdot C \geq 0$ for every curve $C \subset X'$; or

X' is a Mori fibre space: There exists a morphism $\varphi: X' \rightarrow T$ to a variety T of smaller dimension, such that $K_{X'}$ is anti-ample on the fibres of φ . A morphism with these properties is called a Mori fibre space.

Because 3-fold klt flips terminate [Kaw92, FA92], we have the following:

THEOREM 1.4.2 (Minimal model theorem for klt 3-folds). *Let X be a nonsingular projective 3-fold and $B = \sum b_i B_i$ a $\mathbb{Q}$ -divisor on X where $0 < b_i < 1$ and the support of B is a simple normal crossing divisor. There exists a birational map $f: X \dashrightarrow X'$ to a projective 3-fold X' such that the pair $(X', B' = f_*B)$ has klt singularities and satisfies one of the following conditions:*

- (X', B') is a **klt minimal model**: $K_{X'} + B'$ is nef, that is $(K_{X'} + B') \cdot C \geq 0$ for every curve $C \subset X'$; or*
- (X', B') is a **klt Mori fibre space**: There exists a morphism $\varphi: X' \rightarrow T$ to a variety T of smaller dimension, such that $K_{X'} + B'$ is anti-ample on the fibres of φ . A morphism with these properties is called a klt Mori fibre space.*

If we knew that 4-fold klt flips terminated, then we could immediately generalise Theorem 1.4.2 to 4-folds.

1.5. The aim of this book

A large part of this book is a digest of the great work of Shokurov [Sho03]; in particular, we give a complete and essentially self-contained construction of 3-fold and 4-fold klt flips.

Shokurov has introduced many new ideas in the field and has made huge progress on the construction of higher dimensional flips. However, [Sho03] is very difficult to understand; in this book, we rewrite the entire subject from scratch.

Shokurov's construction of 3-fold flips is conceptual. Chapter 2 on 3-fold flips aims to give a concise, complete, and pedagogical proof of the existence of 3-fold flips. I have written up the material in great detail, with the goal to make it accessible to graduate students and algebraic geometers not working in higher dimensions. I assume little prior knowledge of Mori theory; the reader who is willing to take on trust a few general results can get away with almost no knowledge of higher dimensional methods.

The construction of 4-fold flips in [Sho03] is much harder than that of 3-fold flips. It uses everything from the 3-fold case and much more. We tried our best to understand this proof and, after years of work, there are still a few details that we couldn't figure out. Fortunately, Hacon and McKernan [HM] have found a much better approach which proves a much stronger theorem valid in all dimensions. Their ideas are a natural development of Shokurov's 3-dimensional proof. In Chapter 5, Hacon and McKernan give an account of their work, showing in particular the existence of 4-fold flips.

In the rest of the Introduction, I explain the main ideas of the construction of 3-fold and 4-fold flips, and briefly discuss the contents of the individual chapters.

1.6. Pl flips

The main result of [Sho92], reworked and generalised to higher dimension in [FA92, Chapter 18], is a reduction of klt flips to pl flips. We review the proof in Chapter 4. I briefly recall the basic definitions.

In what follows, I consider a normal variety X and a $\mathbb{Q}$ -divisor $S + B$ on X . In this notation, S is a prime Weil divisor and $B = \sum b_i B_i$ is a $\mathbb{Q}$ -divisor having no component in common with S . The pair $(X, S + B)$ needs to have dlt singularities. This notion is similar to klt singularities, but it is more general: the main difference

is that components with coefficient 1 are allowed in the boundary. The precise definition is discussed later in the book and it is not crucial for understanding the outline of the proof; for example, the condition holds if X is nonsingular and the support of $S + B$ is a simple normal crossing divisor.

DEFINITION 1.6.1. A *pl flipping contraction* is a flipping contraction $f: X \rightarrow Z$ for the divisor $K + S + B$, such that S is f -negative. The flip of a pl flipping contraction is called a *pl flip*.

THEOREM 1.6.2. (See [Sho92, FA92] and Chapter 4.) *If n -dimensional pl flips exist and terminate, then n -dimensional klt flips exist.*

Termination of n -dimensional pl flips is essentially a $n - 1$ -dimensional problem; this matter is treated in detail in Chapter 4.

The idea of pl flips is this. Because S is negative on the fibres of f , S contains all the positive dimensional fibres of f and hence the whole exceptional set. We may hope to reduce the existence of the flip to a problem which we can state in terms of S alone. Then we can hope to use the birational geometry of S to solve this problem. With luck we can hope eventually to construct flips by induction on $\dim X$. We are still far from realising all these hopes, but the finite generation conjecture of Shokurov [Sho03], which is stated purely in terms of S , implies the existence of pl flips. We state this conjecture below, after some preliminaries on b-divisors.

1.7. b-divisors

I give a short introduction to Shokurov's notion of b-divisors. The language of b-divisors is not required for the construction of 3-fold flips in Chapter 2, and it is not used in the work of Hacon and McKernan on higher dimensional flips. This material is included in the book for several reasons. First, Shokurov himself, and several others, use b-divisors extensively in their work; in particular, Shokurov's important finite generation conjecture is stated in terms of b-divisors. Second, I am convinced that b-divisors are useful and that they are here to stay. More detail can be found in Chapter 2. We always work with normal varieties. A *model* of a variety X is a proper birational morphism $f: Y \rightarrow X$ from a (normal) variety Y .

DEFINITION 1.7.1. A *b-divisor* on X is an element:

$$\mathbf{D} \in \mathbf{Div} X = \lim_{Y \rightarrow X} \mathbf{Div} Y$$

where the (projective) limit is taken over all models $f: Y \rightarrow X$ under the push forward homomorphism $f_*: \mathbf{Div} Y \rightarrow \mathbf{Div} X$. A b-divisor $\mathbf{D}$ on X has an obvious *trace* $\mathbf{D}_Y \in \mathbf{Div} Y$ on every model $Y \rightarrow X$.

Natural constructions of divisors in algebraic geometry often give rise to b-divisors. For example, the divisor $\mathbf{div}_X \varphi$ of a rational function, and the divisor $\mathbf{div}_X \omega$ of a rational differential, are b-divisors. Indeed, if $f: Y \rightarrow X$ is a model, and $E \subset Y$ is a prime divisor, both $\text{mult}_E \varphi$ and $\text{mult}_E \omega$ are defined.

A b-divisor on X gives rise to a sheaf $\mathcal{O}_X(\mathbf{D})$ of $\mathcal{O}_X$ -modules in a familiar way; if $U \subset X$ is a Zariski open subset, then

$$\mathcal{O}_X(\mathbf{D})(U) = \{\varphi \in k(X) \mid \mathbf{D}|_U + \mathbf{div}_U \varphi \geq 0\}$$

In general, this sheaf is not quasicoherent; however, it is a coherent sheaf in all cases of interest to us. We write $H^0(X, \mathbf{D})$ for the group of global sections of $\mathcal{O}_X(\mathbf{D})$ and denote by $|\mathbf{D}| = \mathbb{P}H^0(X, \mathbf{D})$ the associated “complete” linear system. It is crucial to understand that $H^0(X, \mathbf{D}) \subsetneq H^0(X, \mathbf{D}_X)$; the language of b-divisors is a convenient device to discuss linear systems with base conditions.

EXAMPLE 1.7.2. The $\mathbb{Q}$ -Cartier closure of a $\mathbb{Q}$ -Cartier ($\mathbb{Q}$ -)divisor D on X is the b-divisor $\overline{D}$ with trace

$$\overline{D}_Y = f^*(D)$$

on models $f: Y \rightarrow X$.

If $f: Y \rightarrow X$ is a model and D is a $\mathbb{Q}$ -Cartier ($\mathbb{Q}$ -)divisor on Y , we abuse notation slightly and think of $\overline{D}$ as a b-divisor on X . Indeed, f_* identifies b-divisors on Y with b-divisors on X .

DEFINITION 1.7.3. A b-divisor $\mathbf{D}$ on X is *b-($\mathbb{Q}$)-Cartier* if it is the Cartier closure of a ($\mathbb{Q}$)-Cartier divisor D on a model $Y \rightarrow X$.

1.8. Restriction and mobile b-divisors.

DEFINITION 1.8.1. Let $\mathbf{D}$ be a b- $\mathbb{Q}$ -Cartier b-divisor on X and $S \subset X$ an irreducible normal subvariety of codimension 1 not contained in the support of $\mathbf{D}_X$. I define the *restriction* $\mathbf{D}^0 = \text{res}_S \mathbf{D}$ of $\mathbf{D}$ to S as follows. Pick a model $f: Y \rightarrow X$ such that $\mathbf{D} = \overline{\mathbf{D}_Y}$; let $S' \subset Y$ be the proper transform. I define

$$\text{res}_S \mathbf{D} = \overline{\mathbf{D}_{Y|S'}}$$

where $\mathbf{D}_{Y|S'}$ is the ordinary restriction of divisors. (Strictly speaking, $\overline{\mathbf{D}_{Y|S'}}$ is a b-divisor on S' ; as already noted, b-divisors on S' are canonically identified with b-divisors on S via push forward.) It is easy to see that the restriction does not depend on the choice of the model $Y \rightarrow X$.

DEFINITION 1.8.2. An integral b-divisor $\mathbf{M}$ is *mobile* if there is a model $f: Y \rightarrow X$, such that

- (1) $\mathbf{M} = \overline{\mathbf{M}_Y}$ is the Cartier closure of $\mathbf{M}_Y$, and
- (2) the linear system (of ordinary divisors) $|\mathbf{M}_Y|$ is free on Y .

REMARK 1.8.3. The restriction of a mobile b-divisor is a mobile b-divisor.

1.9. Pbd-algebras

DEFINITION 1.9.1. A sequence $\mathbf{M}_\bullet = \{\mathbf{M}_i \mid i > 0 \text{ integer}\}$ of mobile b-divisors on X is *positive sub-additive* if $\mathbf{M}_1 > 0$ and

$$\mathbf{M}_{i+j} \geq \mathbf{M}_i + \mathbf{M}_j.$$

for all positive integers i, j . The associated *characteristic sequence* is the sequence $\mathbf{D}_i = (1/i)\mathbf{M}_i$ of b- $\mathbb{Q}$ -Cartier b-divisors. We say that the characteristic sequence is *bounded* if there is a (ordinary) $\mathbb{Q}$ -Cartier divisor D on X such that all $\mathbf{D}_i \leq \overline{D}$.

REMARK 1.9.2. The characteristic sequence of a positive sub-additive sequence is *positive convex*; that is, $\mathbf{D}_1 > 0$ and

$$\mathbf{D}_{i+j} \geq \frac{i}{i+j}\mathbf{D}_i + \frac{j}{i+j}\mathbf{D}_j \quad \text{for all positive integers } i, j.$$

DEFINITION 1.9.3. A *pbd-algebra* is a graded algebra

$$R = R(X, \mathbf{D}_\bullet) = \bigoplus_{i \geq 0} H^0(X, i\mathbf{D}_i)$$

where $\mathbf{D}_i = (1/i)\mathbf{M}_i$ is a bounded characteristic sequence of a positive sub-additive sequence $\mathbf{M}_\bullet$ of b-divisors.

1.10. Restricted systems and 3-fold pl flips

Consider a pl flipping contraction $f: X \rightarrow Z$ for the divisor $K + S + B$. Let $r > 0$ be a positive integer and $D \sim r(K + S + B)$ a Cartier divisor on X ; it is well known that the flip of f exists if and only if the algebra

$$R = R(X, D) = \bigoplus_{i \geq 0} H^0(X, iD)$$

is finitely generated (in fact, in that case, the flip is the Proj of this algebra). The first step is to interpret R as a suitable pbd-algebra.

DEFINITION 1.10.1. Let D be a Cartier divisor on X . The *mobile b-part* of D is the divisor $\mathbf{Mob} D$ with trace

$$(\mathbf{Mob} D)_Y = \mathbf{Mob} f^* D$$

on models $f: Y \rightarrow X$, where $\mathbf{Mob} f^* D$ is the mobile part of the divisor $f^* D$ (the part of D which moves in the linear system $|f^* D|$).

Choose, as above, a Cartier divisor $D \sim r(K + S + B)$. Denote by $\mathbf{M}_i = \mathbf{Mob} iD$ the mobile part and let $\mathbf{D}_i = (1/i)\mathbf{M}_i$; then, tautologically,

$$R = R(X, D) = R(X, \mathbf{D}_\bullet)$$

is a pbd-algebra. Now I come to the punchline. As I explained above, provided that S is not contained in the support of D (which is easily arranged), it makes sense to form the restriction $\mathbf{D}_i^0 = \text{res}_S \mathbf{D}_i$; we consider the associated pbd-algebra on S :

$$R(S, \mathbf{D}_\bullet^0).$$

It is easy to see, though not trivial, that $R(X, \mathbf{D}_\bullet)$ is finitely generated if and only if $R(S, \mathbf{D}_\bullet^0)$ is finitely generated.

1.11. Shokurov's finite generation conjecture

In this section, I state the finite generation conjecture of Shokurov. The statement is technical; the reader may wish to just skim over it.

The key issue is to state a condition on the system $\mathbf{D}_\bullet^0$ that, under suitable conditions, ensures that the pbd-algebra $R(S, \mathbf{D}_\bullet^0)$ is finitely generated. The condition is the following:

DEFINITION 1.11.1. Let (X, B) be a pair of a variety X and divisor $B \subset X$. A system $\mathbf{D}_\bullet$ of b-divisors on X is *canonically asymptotically saturated* (*canonically a-saturated*, for short) if for all i, j , there is a model $Y(i, j) \rightarrow X$ such that

$$\mathbf{Mob}[(j\mathbf{D}_i + \mathbf{A})_Y] \leq j\mathbf{D}_{jY}$$

on all models $Y \rightarrow Y(i, j)$.

In a sense, the saturation condition is a reformulation of the Kawamata technique; see the discussion in §2.3.5.

PROPOSITION 1.11.2 (see Lemma 2.3.43 and Lemma 2.4.3). *Let $(X, S+B) \rightarrow Z$ be a pl flip; the restricted system*

$$\mathbf{D}_\bullet^0 = \text{res}_S \mathbf{D}_\bullet$$

of Section 1.10 is canonically a -saturated.

FINITE GENERATION CONJECTURE 1.11.3. *Let (X, B) be a klt pair, $f: X \rightarrow Z$ a birational contraction to an affine variety Z . Assume that $K + S + B$ is anti-ample on the fibres of f . If $\mathbf{D}_\bullet$ is a canonically a -saturated positive convex bounded characteristic system of b -divisors on X , then the pbd-algebra $R(X, \mathbf{D}_\bullet)$ is finitely generated.*

A good way to get a feeling for this conjecture and the concept of canonical asymptotic saturation, is to work out the one-dimensional case; this is done in §2.3.10 and I encourage the reader to read that section now. To prove this conjecture in the case $\dim X = 2$ is a major theme in this book, see Theorem 2.4.10 and Corollary 7.5.2; in Chapter 9, we even prove a generalisation to non-klt surface pairs. In §2.4, the conjecture is proved assuming that $f: X \rightarrow Z$ is birational; this is sufficient for the construction of 3-fold pl flips.

It is important to realise that, in the light of the work of Hacon and McKernan on adjoint algebras and higher dimensional flips, see §1.14 below, the finite generation conjecture is no longer crucial for the construction of flips. Nevertheless, I feel that the conjecture says something deep about the structure of Fano varieties, and that it will be important in future work on finite generation.

1.12. What is log terminal?

Fujino's Chapter 3 is an essay on the definition of log terminal singularities of pairs. The category of pairs (X, B) of a variety X and a divisor $B \subset X$ was first introduced by Iitaka and his school. In the early days, $B = \sum B_i$ was a reduced integral divisor and one was really interested in the noncompact variety $U = X \setminus B$. It is an easy consequence of Hironaka's resolution theorem that, if X is nonsingular and B is a simple normal crossing divisor, then the *log plurigenera* $h^0(X, n(K+B))$ depend only on U and not on the choice of the compactification X and boundary divisor B . This suggests that it should be possible to generalise birational geometry, minimal models, etc. to noncompact varieties, or rather *pairs* (X, B) of a variety X and *boundary* divisor $B \subset X$. As in the absolute case, it is necessary to allow some singularities. There should be a notion of *log terminal singularities of pairs*, corresponding to terminal singularities of varieties. To define such a notion turned out to be a very subtle technical problem. Many slightly different inequivalent definitions were proposed; for example [FA92] alone contains a dozen variants. Over the years, we believe, one particular notion, called *divisorially log terminal* pairs in [KM98], has proved itself to be the most useful. In this book, we work exclusively with divisorially log terminal (abbreviated *dlt*) pairs. The book [KM98] contains a clear and technically precise exposition of divisorially log terminal singularities and we adopt it as our main reference; however, the professional in higher dimensional geometry must be able to read the literature; in particular, a good knowledge of at least the fundamental texts [KMM87, Sho92, FA92] is essential. There is no agreement on the basic definitions among these texts. This state of affairs creates very serious difficulties for the beginner and the expert alike. The chapter by Fujino is a guide to the different definitions existing in the literature; it discusses

their properties and respective merits as well as the state of the art on the various implications existing among them; it also provides illustrative examples.

1.13. Special termination and reduction to pl flips.

The goal of Chapter 4, written by Fujino, is to provide a self-contained proof of the statement, already alluded to in this Introduction, that, if the log MMP holds in dimension $n - 1$, and pl flips exist in dimension n , then klt flips exist in dimension n .

1.14. The work of Hacon and McKernan: adjoint algebras

Chapter 5 is an exposition of the brilliant work of Hacon and McKernan on higher dimensional flips. Let $f: (X, S+B) \rightarrow Z$ be a pl flipping contraction; Hacon and McKernan realised that the restricted algebra $R(S, \mathbf{D}_\bullet^0)$ has one additional crucial property, namely it is an *adjoint algebra*. This realisation is based on their lifting lemma, which they first discovered in their study of pluricanonical maps of varieties of general type, *see* [HM06, Tak06].

The starting point is the following elementary observation. If $f: (X, S+B) \rightarrow Z$ is a pl flipping contraction, fix an integer I such that $I(K + S + B)$ is a Cartier divisor, and consider the restriction

$$\begin{aligned} \rho: R(X, K_X + S + B)^{(I)} &= \bigoplus_{n=0}^{\infty} H^0(X, nI(K_X + S + B)) \rightarrow \\ &\bigoplus_{n=0}^{\infty} H^0(S, nI(K_S + B_S)) = R(S, K_S + B_S)^{(I)} \end{aligned}$$

(If $R = \bigoplus R_n$ is a graded algebra, $R^{(I)} = \bigoplus_n R_{In}$ denotes the I -th *truncation* of R .) It is easy to show that the kernel of ρ is a principal ideal, *see* the proof of Lemma 2.3.6. The observation is this: If the restriction homomorphisms

$$\rho_n: H^0(X, nI(K_X + S + B)) \rightarrow H^0(S, nI(K_S + B_S))$$

were surjective (perhaps for n sufficiently divisible), then finite generation of the log canonical algebra $R(X, K_X + S + B)$ would be a consequence of finite generation of the log canonical algebra $R(S, K_S + B_S)$; the latter algebra can be assumed to be finitely generated by induction on dimension. In general, when the pair $(X, S + B)$ has dlt singularities, the restriction maps ρ_n are not surjective (explicit examples, however, are not easy to find). Hacon and McKernan, building on previous work by Siu, Kawamata, Tsuji, and others, discovered that, if the pair $(X, S + B)$ has canonical singularities, and some additional technical conditions are satisfied, then the ρ_n are surjective; *see* Theorem 5.4.21 for a precise statement of their lifting lemma. Using the lifting lemma, Hacon and McKernan show that the restricted algebra is an adjoint algebra:

DEFINITION 1.14.1 (*see* Definition 5.3.10). Let T be a smooth variety, W an affine variety and $\pi: T \rightarrow W$ be a projective morphism. An *adjoint algebra* is an algebra of the form

$$R = \bigoplus_{m \in \mathbb{N}} H^0(Y, \mathcal{O}_Y(N_m))$$

where $N_\bullet$ is an additive sequence such that

- (1) there exists an integer $k > 0$ such that $N_m = mk(K_Y + B_m)$ where B_m is an effective, bounded and eventually convex sequence of $\mathbb{Q}$ -divisors on T with limit $B \in \text{Div}_{\mathbb{R}}(T)$ such that (T, B) is klt,

- (2) let $M_m = \text{Mob}(N_m)$ be the mobile sequence and $D_m = M_m/m$ the characteristic sequence. Then D_m is saturated, that is there exists a $\mathbb{Q}$ -divisor F on Y with $\lceil F \rceil \geq 0$ such that

$$\text{Mob}(\lceil jD_i + F \rceil) \leq jD_j$$

for all $i \geq j \gg 0$,

- (3) $D = \lim D_m$ is semiample.

These results lead to a proof that the minimal model program in dimension n implies existence of flips in dimension $n + 1$.

1.15. Mobile b-divisors on weak klt del Pezzo surfaces

Chapter 6 is a detailed study of mobile b-divisors on weak del Pezzo klt surfaces. On the one hand, the discussion illustrates a key example of some of the notions and ideas introduced in the construction of 3-fold and 4-fold pl flips; on the other hand, the main result is a prototype of the very interesting conjecture on the canonical confinement of singularities—the CCS conjecture—which constitutes the starting point for a possible attack on Shokurov’s finite generation conjecture. The main result is the following:

THEOREM 1.15.1. *Let $(X, B) \rightarrow Z$ be a relative weak klt del Pezzo surface pair. Denote by $(X', B') \rightarrow (X, B)$ the terminal model. There are*

- (1) *finitely many normal varieties T_i/Z and Z -morphisms $\varphi_i: X' \rightarrow T_i$,*
- (2) *when $Z = \{pt\}$, finitely many normal surfaces Y_j together with projective birational morphisms $h_j: Y_j \rightarrow X$ and elliptic fibrations $\chi_j: Y_j \rightarrow \mathbb{P}^1$, and*
- (3) *a bounded algebraic family $\mathcal{F}$ of mobile $\mathbf{A}(X, B)$ -saturated b-divisors on X ,*

such that the following holds: If $\mathbf{M}$ is any mobile $\mathbf{A}(X, B)$ -saturated b-divisor on X , then either

- (a) *$\mathbf{M}$ descends to the terminal model $(X', B') \rightarrow (X, B)$ and there is an index i such that $\mathbf{M}_{X'} = \varphi_i^*(\text{ample divisor on } T_i)$, or*
- (b) *for some j , $\mathbf{M}$ descends to Y_j and $\mathbf{M}_{Y_j} = \varphi_j^*(\text{ample divisor on } \mathbb{P}^1)$, or*
- (c) *$\mathbf{M}$ belongs to the bounded family $\mathcal{F}$.*

1.16. The CCS Conjecture

In Chapter 1.16, we state Shokurov’s CCS conjecture (“CCS” stands for “canonical confinement of singularities”) and show that it implies the finite generation conjecture. The CCS conjecture is a higher dimensional generalisation of the classification of canonically saturated mobile b-divisors on weak del Pezzo klt surfaces stated in §1.15 and treated in Chapter 6. The statement is technically very subtle and still tentative: Shokurov himself has several slightly different a priori inequivalent formulations. To make matters worse, as far as I know, not a single truly higher dimensional example has been worked out. McKernan’s Chapter 7 is an introduction to the simplest form of the CCS conjecture. Using the results of Chapter 6, he proves the conjecture in dimension 2 and obtains as a corollary the general form of the finite generation conjecture in dimension 2. It remains to be seen if the CCS conjecture is capable of playing a role in the proof of the finite generation conjecture in general, or in the study of the geometry of Fano varieties.

1.17. Kodaira's canonical bundle formula and subadjunction

If X is a normal variety and $S \subset X$ a reduced Cartier divisor, the adjunction formula states that $K_S = K_X + S|_S$. If S is reduced of codimension 1, but not necessarily a Cartier divisor, under mild assumptions, there is a canonically defined effective $\mathbb{Q}$ -divisor $\text{Diff} > 0$ on S such that the *subadjunction formula*

$$K_S + \text{Diff} = (K_X + S)|_S$$

holds. For example, consider the case of a quadric cone $X \subset \mathbb{P}^3$ and a line $S \subset X$; the line must pass through the vertex $P \in S$ of the cone, and $\text{Diff} = (1/2)P$. The central theme of Kollár's Chapter 8 is a vast generalisation of the subadjunction formula. Consider a pair (X, B) with log canonical singularities. The *non-klt locus* $\text{nklt}(X, B)$ is the subset of points of X where the pair does not have klt singularities. A Zariski closed subset $W \subset X$ is called a *log canonical centre*, or LC centre, of the pair (X, B) , if $W = \bar{c}_X E$ is the closure of the centre of a geometric valuation E with discrepancy $a(E, B) = -1$. The main results, Theorem 8.6.1, states that a form of the subadjunction formula holds on W .

1.18. Finite generation on non-klt surfaces

The main purpose of Chapter 9 is a generalisation of the finite generation conjecture to the case of a surface pair with worse than log canonical singularities. The main statement is as follows.

THEOREM 1.18.1. *Let (X, B) be a pair of a normal variety X of dimension ≤ 2 and an effective $\mathbb{Q}$ -divisor $B = \sum b_i B_i$ on X . We do not assume that the pair (X, B) is klt or even log canonical. Denote by $\text{nklt}(X, B) \subset X$ the Zariski closed subset where the pair (X, B) is not klt.*

Let $\mathbf{D}_i = (1/i)\mathbf{M}_i$ be a bounded positive convex $\mathbf{A}(X, B)$ -saturated sequence of b -divisors on X where all $\mathbf{M}_i$ are mobile b -divisors.

Assume given a morphism $f: X \rightarrow Z$ to an affine normal variety Z such that $-(K_X + B)$ is f -nef and big.

Assume there is a Zariski open subset $U \subset X$ such that $\text{nklt}(X, B) \subset U$ and all restrictions $\mathbf{D}_i|_U$ descend to U and $\mathbf{D}_\bullet|_U$ is constant.

Then $\mathbf{D}_\bullet$ is eventually constant on X .

1.19. The glossary

The last chapter in the book is a minimal glossary of technical terms. I tried to collect in one place common terminology for which a precise definition may exist somewhere in the book but which is elsewhere in the book used freely and taken for granted.

1.20. The book as a whole

The chapters of this book have been written as stand-alone papers by their authors. Nevertheless, the collection of chapters has a clear theme and a unified topic. The reader should expect differences in emphasis, repetitions and even occasional discrepancies in terminology between the chapters.

1.21. Prerequisites

Different chapters have different sets of prerequisites. For instance, prerequisites are minimal for Chapter 2. On the whole, I hope that at least the first half of the book can be read by someone with knowledge of basic algebraic geometry, e.g. [Har77], but little experience of higher dimensional methods. To have a fighting chance with some of the harder bits, the reader will need some knowledge of higher dimensional techniques as can be obtained from one of the books [CKM88, KM98, Deb01, Mat02]. Our main reference is [KM98]; we try consistently to follow the terminology in use there. The paper [KMM87] is still the best concise technical reference on the minimal model program, though some of its terminology is out of date and, in any case, it is different from terminology adopted in this book.

1.22. Acknowledgements

I am grateful to my co-authors for their outstanding contributions to this book. The first segment of 3-folds at the Newton Institute Feb-Jul 2002 was a working seminar on Shokurov's paper "Pre-limiting flips" (Mar 2002 draft, 247 pp.); I want to thank all of the participants.

We received comments and suggestions to the individual chapters from many friends and colleagues; rather than list them all here, I refer to the appropriate section in each chapter.

CHAPTER 2

3-fold flips after Shokurov

ALESSIO CORTI

2.1. Introduction

2.1.1. Statement and brief history of the problem. The flipping conjecture asserts that, under rather restrictive conditions, certain codimension 2 surgery operations, called flips, exist in the projective category. More precisely, let X be a normal variety with canonical divisor K and $B = \sum b_i B_i \subset X$ a $\mathbb{Q}$ -divisor such that the pair (X, B) has klt singularities. This is a technical assumption which I discuss below in detail. In particular, $K + B$ is $\mathbb{Q}$ -Cartier, and it therefore makes sense to intersect it with 1-dimensional cycles in X and ask, for example, whether it is ample. Let $f: X \rightarrow Z$ be a small contraction with $K + B$ anti-ample along f . Recall that small means that the exceptional set of f has codimension at least 2. A morphism with these properties is called a *flipping contraction*. The fact that flipping contractions exist is itself a nontrivial discovery and a central feature of Mori theory. By definition, the *flip* of f is a small birational contraction $f': X' \rightarrow Z$ such that $K' + B'$ is $\mathbb{Q}$ -Cartier and ample along f' . The precise form of the flipping conjecture asserts that the flip of f exists. Together with the conjecture on termination of flips (stating that there can be no infinite sequence of flips) and the cone and contraction theorems, which are standard results in higher dimensional algebraic geometry, the conjecture implies the existence of minimal models of projective algebraic varieties.

Kulikov was the first to use codimension 2 surgery systematically with the aim of constructing minimal models [Kul77a, Kul77b].

In the mid eighties, four people were working on flips: Kawamata, Mori, Shokurov and Tsunoda. They all independently showed existence of semistable flips [Kaw88, Mor02, Sho93a, Tsu87]. Kawamata thought his initial approach could construct 3-fold flips in general. In the end he did not succeed, and showed existence of semistable 3-fold flips by another method. He later revived his initial approach in his construction of semistable 3-fold flips in positive and mixed characteristic [Kaw94]. Mori constructed semistable 3-fold flips, but never published the details of his proof, some of which is now part of the more recent work [Mor02]; instead, he went on to construct 3-fold flips in general [Mor88]. Shokurov constructed semistable 3-fold flips around the same time; he published his proof much later [Sho93a]. Tsunoda's paper [Tsu87] also works out semistable 3-fold flips. Two general principles emerge from the work of these pioneers. First, semistable 3-fold flipping contractions can be understood in fairly explicit terms. With hindsight, it is not difficult to collect enough information to construct the flip. Second, 3-fold flipping contractions can also be classified to the extent that one has enough

information to construct the flip, but the general case is much harder than the case of semistable flips. The classification of 3-fold flips was taken a step further in the monumental paper [KM92].

In the early nineties, Shokurov discovered a new approach to 3-fold flips which was eventually worked out, revised, and corrected in several papers [Sho92, FA92, Sho93b, Takb]. This proof of Shokurov works in two stages. The first stage is a very conceptual reduction of flips to a special case called *pl flips*. The general features of this reduction work in all dimensions; in particular, they work unconditionally in dimension 4. In the second stage, 3-fold pl flips are constructed by a lengthy explicit analysis similar in spirit to the construction of semistable flips. In the end, we still lack a conceptual approach to the construction of pl flips.

In the intervening years, a few people started looking into 4-fold flips and proved existence in special cases [Kaw89, Kac98, Kac97, Taka].

Shokurov's new work [Sho03] constructs 4-fold flips in full generality by constructing 4-fold pl flips. Shokurov's proof in the 4-fold case is very convoluted and complicated; however, it rests on a rather appealing construction of 3-fold pl flips. The purpose of this chapter is to explain this construction.

2.1.2. Summary of the chapter. This chapter is divided into three large sections.

Section 2 is a brief introduction to log terminal singularities, the flipping problem, and the reduction of 3-fold klt flips to 3-fold pl flips. This material is included here primarily for pedagogical reasons. The section ends with a rather sketchy outline of the construction of 3-fold pl flips. I have written this section so it can be used as an introduction to the subject for beginners. The expert will find nothing new here.

Section 3 introduces the language of $\mathbb{Q}$ -divisors and the algebras naturally associated to them. Most of this material is very elementary and it could have been written by Zariski; see for example [Zar62] and [Hir73]. I have written up this subject in painful detail because it does not exist in this form anywhere else in the literature. I also discuss the key property of $\mathbb{Q}$ -saturation introduced by Shokurov and state the finite generation conjecture. It is easy to see that the finite generation conjecture in dimension $n - 1$ implies the existence of pl flips in dimension n . The section ends with a proof of the finite generation conjecture in dimension 1. While this has no relevance to flips, it shows in a completely elementary case that the conjecture is plausible. Section 3 can be used as a second, more full, introduction to the construction of 3-fold pl flips. The expert can probably skim through most of this section quickly and come back to it when needed.

The final Section 4 contains complete details of the construction of 3-fold pl flips. It opens with the proof that the finite generation conjecture implies the existence of pl flips, and ends with a proof of the finite generation conjecture in dimension 2, using the techniques developed in the previous section.

2.1.3. Other surveys. Shokurov's ideas on flips are published in [Sho03], *see also* the other papers in the same volume; in particular, the 3-fold case is surveyed in [Isk03].

2.2. Background

2.2.1. Summary. The first goal of this section is to give the definition of two flavours of log terminal singularities: klt and plt. To the beginner, the definition of log terminal singularities is one of the most confusing places in the theory. The psychological difficulty is that the characterisation of classes of singularities by means of discrepancies is indirect and nonintuitive. The classes of klt (Kawamata log terminal) and plt (purely log terminal) singularities are defined in terms of discrepancies. Their great advantage is that they are uncontroversial. Fortunately, we will not need in this chapter any of the more sophisticated flavours of log terminal singularities (*see* Chapter 3 for a discussion of these).

The second goal of this section is to state the flipping conjecture and briefly explain the reduction to pl flips and sketch the strategy of Shokurov's construction of 3-fold pl flips.

See [Kol91] for a very accessible introduction to codimension-2 surgery and flips. There are now several books on the minimal model program and higher dimensional complex geometry, *see* for instance [CKM88, KM98, Deb01, Mat02]; the paper [KMM87] is still the best concise technical reference on the minimal model program. My policy is to use [KM98] as my main reference; in particular, I try consistently to use their terminology.

CONVENTION 2.2.1. In this chapter, we always work over an algebraically closed field of characteristic zero.

2.2.2. Discrepancy. Let X be a normal variety with function field $k(X)$. I denote by K_X , or, when there is no danger of confusion, just K , the *canonical divisor* class of X . The best introduction to the canonical class is in [Rei80, Appendix to §1, pg. 281–285]. If X is a normal variety, the *dualising sheaf* ω_X^0 of [Har77, III.7 Definition on pg. 241] (sometimes called *predualising* sheaf in the literature) is a rank 1 reflexive sheaf or *divisorial* sheaf. This means that $\omega_X^0 \cong \mathcal{O}_X(K_X)$ for some divisor K_X which is well defined up to linear equivalence. By definition, the linear equivalence class of K_X is the canonical divisor class. In the literature and in what follows it is common to abuse language and say things like “let K be the canonical class of X ” or “let K be the canonical divisor of X ”.

DEFINITION 2.2.2. A rank 1 valuation $\nu: k(X) \rightarrow \mathbb{Z}$ is *geometric* if there exists a normal variety Y , a birational map $f: Y \dashrightarrow X$ and a prime Weil divisor $E \subset Y$ such that $\nu(-) = \nu_E(-) = \text{mult}_E(-)$ is the valuation given by the order of vanishing along E . It is customary to abuse language, and notation, identifying ν with E .

I say that ν has *centre on X* if f is regular at (the generic point of) E or, equivalently, by restricting the domain of f , if f can be taken to be a morphism. The *centre* $c_X(\nu)$ of ν on X is the scheme-theoretic point $c_X(\nu) = f(E) \in X$. I denote by $\overline{c_X}(\nu)$ the Zariski closure of the centre.

I say that the valuation is *exceptional over X* , or that it has *small centre* on X , if $c_X(\nu)$ is not a divisor or, equivalently, E is an exceptional divisor.

NOTATION AND CONVENTIONS 2.2.3. I work with Weil divisors $B = \sum b_i B_i \subset X$ with rational coefficients $b_i \in \mathbb{Q}$. I denote by

$$[B] = \sum [b_i] B_i, \quad \lfloor B \rfloor = \sum \lfloor b_i \rfloor B_i \quad \text{and} \quad \{B\} = B - \lfloor B \rfloor$$

the *round up*, the *round down* and the *fractional part* of B .

DEFINITION 2.2.4. I say that the divisor $B \subset X$ is a *boundary* if all $0 \leq b_i \leq 1$; a boundary is *strict* if all $b_i < 1$; B is a *subboundary* if all $b_i \leq 1$ (that is, the b_i can be negative).

In this chapter, I never use subboundaries. Subboundaries are used in Shokurov [Sho03] and occasionally in this book.

I now introduce the key notion of discrepancy of a geometric valuation. Various classes of singularities are defined in terms of discrepancies. In order for discrepancies even to be defined, it is necessary to assume that the divisor $K + B$ is $\mathbb{Q}$ -Cartier. This is a subtle condition on the singularities of the pair (X, B) ; in most cases of interest to us, it is a topological condition:

REMARK 2.2.5. Let X be a normal variety with rational singularities (this holds, for example, when the pair (X, B) has klt or plt singularities); denote by $U = X \setminus \text{Sing } X$ the nonsingular locus of X . Then, $K + B$ is $\mathbb{Q}$ -Cartier if and only if the first Chern class

$$c_1(\mathcal{O}(K_U + B|_U)) \in H^2(U, \mathbb{Z}) \subset H^2(U, \mathbb{Q})$$

is in the image of the (injective) restriction map $H^2(X, \mathbb{Q}) \rightarrow H^2(U, \mathbb{Q})$. For a proof, see [Kol91, Proposition 2.1.7]. Equivalently, if X has rational singularities and $\dim X = n$, $K_X + B$ is $\mathbb{Q}$ -Cartier if and only if the cycle class $\text{cl}(K_X + B) \in H_{2n-n}(X, \mathbb{Q})$ is in the image of the Poincaré map

$$P: H^2(X, \mathbb{Q}) \rightarrow H_{2n-2}(X, \mathbb{Q})$$

(An algebraic variety X is always a *pseudomanifold*, in other words it has a *fundamental homology class* $\text{cl } X \in H_{2n}(X, \mathbb{Q})$. The Poincaré map is given by cap product with $\text{cl } X$.)

DEFINITION 2.2.6. Let X be a normal variety, and $B \subset X$ a rational Weil divisor. Assume that $K + B$ is $\mathbb{Q}$ -Cartier.

Let ν be a geometric valuation with centre on X . Let $f: Z \rightarrow X$ be a birational morphism with divisor $E \subset Z$ such that $\nu = \nu_E$. By restricting Z , we may assume that E is the only f -exceptional divisor, and then we may write

$$K_Z = f^*(K + B) + aE$$

where a is a rational number. It is easy to see that a only depends on the valuation ν . I call $a = a(\nu, B)$ the *discrepancy* of the valuation ν . When identifying ν with the divisor E , I denote it $a(E, B)$.

Usually one only uses discrepancies in a context where it is known that B is a boundary. However, inductive formulae for several blow ups use the general case where one only assumes that $K + B$ is $\mathbb{Q}$ -Cartier.

2.2.3. Klt and plt.

DEFINITION 2.2.7. The pair (X, B) has *klt* (*Kawamata log terminal*) *singularities* if $a(\nu, B) > -1$ for all geometric valuations ν with centre on X .

The pair (X, B) has *plt* (*purely log terminal*) *singularities* if $a(\nu, B) > -1$ for all geometric valuations ν with small centre on X .

Sometimes, following established usage, I abuse language and say that the divisor $K + B$ is klt (plt) meaning that the pair (X, B) has klt (plt) singularities.

REMARK 2.2.8. If (X, B) has klt singularities, then all $b_i < 1$. It is known that, if (X, B) has plt singularities, then $[B]$ is normal, that is, it is the disjoint union of normal irreducible components. If X is nonsingular, $B = \sum b_i B_i \subset X$ is a strict boundary and $\text{Supp } B = \cup B_i \subset X$ is a simple normal crossing divisor, then (X, B) has klt singularities (this is, of course, the basic example of klt singularities).

DEFINITION 2.2.9. The pair (X, B) has *terminal singularities* if $a(\nu, B) > 0$ for all geometric valuations ν with small centre on X .

REMARK 2.2.10. In Shokurov's terminology, and elsewhere in the literature, terminal singularities are called "terminal singularities in codimension 2".

If X is a surface, then (X, B) has terminal singularities if and only if X is nonsingular and $\text{mult}_x B = \sum b_i \text{mult}_x B_i < 1$ for all points $x \in X$.

2.2.4. Inversion of adjunction.

NOTATION AND CONVENTIONS 2.2.11. In what follows, I consider a normal variety X and a $\mathbb{Q}$ -divisor $S + B$ on X . In this chapter, the notation always means that S is a prime Weil divisor and $B = \sum b_i B_i$ is a $\mathbb{Q}$ -divisor having no component in common with S . I always assume that $K + S + B$ is $\mathbb{Q}$ -Cartier. I also often assume that B is a strict boundary, that is, $0 < b_i < 1$, and that $K + S + B$ is plt.

As explained in [FA92, Section 16], under mild assumptions, one can define the *different* $\text{Diff} = \text{Diff}_S B \geq 0$ of B along S , and make sense of the formula

$$(K_X + S + B)|_S = K_S + \text{Diff}_S B.$$

When $K + S$ and B are each $\mathbb{Q}$ -Cartier, it is true that

$$\text{Diff}_S B = B|_S + \text{Diff}_S 0 \quad \text{where} \quad \text{Diff}_S 0 \geq 0,$$

but, unless $K + S$ is Cartier, $\text{Diff}_S 0$ is usually nonzero. (See [FA92, Section 16] for details.) The example to keep in mind is S a ruling inside a surface quadric cone X . In this case $\text{Diff}_S 0 = (1/2)P$ where P is the vertex of the cone.

THEOREM 2.2.12. [FA92, Theorem 17.6] *Let X be a normal variety, $S + B = S + \sum b_i B_i \subset X$ a $\mathbb{Q}$ -divisor on X where S is a prime divisor and B is a strict boundary. If $K_X + S + B$ is $\mathbb{Q}$ -Cartier, then $K_X + S + B$ is plt in a neighbourhood of S if and only if $K_S + \text{Diff}_S(B)$ is klt.*

REMARK 2.2.13. It is easy to show that $K + S + B$ plt implies $K_S + \text{Diff}_S B$ klt. The converse is a case of "inversion of adjunction". The book [FA92, Chapter 17] contains several statements of this type.

2.2.5. The flipping conjecture.

DEFINITION 2.2.14. Let (X, B) be a pair with klt (plt) singularities. A *flipping contraction* is a projective birational morphism $f: X \rightarrow Z$ such that the following properties hold.

- (1) $K + B$ is anti-ample relative to f .
- (2) The morphism f is *small*, that is, the exceptional set $\text{Exc } f$ has codimension at least 2 in X .
- (3) The relative Picard group $\text{Pic}(X/Z)$ has rank $\rho(X/Z) = 1$.

I sometimes say that $f: X \rightarrow Z$ is a flipping contraction for $K + B$ to mean all the above.

DEFINITION 2.2.15. Let $f: X \rightarrow Z$ be a flipping contraction. The *flip* of f is a small projective birational morphism $f': X' \rightarrow Z$ such that $K' + B'$ is $\mathbb{Q}$ -Cartier and f' -ample.

REMARK 2.2.16. The flip is unique if it exists. Ideed:

$$X' = \text{Proj } \bigoplus_{i \geq 0} f_* \mathcal{O}_X(i(K + B))$$

where, by definition, $f_* \mathcal{O}_X(i(K + B)) = f_* \mathcal{O}_X(iK + [iB])$, provided that the algebra is finitely generated. The formula makes it clear that the construction of the flip is local on Z in the Zariski topology. For this reason, in this chapter, I almost always assume that Z is affine.

THEOREM 2.2.17. [KM98, Corollary 3.42] *If the pair (X, B) has klt (plt) singularities, then so does the pair (X', B') .* $\square$

CONJECTURE 2.2.18 (Flip conjecture I). *The flip of a flipping contraction always exists.*

CONJECTURE 2.2.19 (Flip conjecture II). *There is no infinite sequence of flips.*

2.2.6. Reduction to pl flips.

DEFINITION 2.2.20. Let $(X, S + B)$ be a plt pair. The notation means that S is a prime Weil divisor and $B = \sum b_i B_i$ a $\mathbb{Q}$ -divisor having no component in common with S . I also assume that B is a strict boundary, that is $0 < b_i < 1$. A *pl (pre limiting) contraction* is a flipping contraction $f: X \rightarrow Z$ such that the following additional properties hold.

- (1) The variety X is $\mathbb{Q}$ -factorial.
- (2) The divisor S is irreducible and f -negative.

The flip of a pre limiting contraction is called a *pl (pre limiting) flip*.

REMARK 2.2.21. My definition of pl flip is slightly more restrictive than [FA92, Definition 18.6], which allows $K + S + B$ to be divisorially log terminal: the key difference is that there S is allowed to have several components, one of which is required to be f -negative. If $f: X \rightarrow Z$ is a pl flip in the sense of [FA92] and $S = \sum S_i$ with S_0 f -negative, then $K + S_0 + (1 - \varepsilon) \sum_{i>0} S_i + B$ is plt and f -negative if $0 < \varepsilon \ll 1$. Thus, the construction (but not the termination) of pl flips in the sense of [FA92] is reduced to the construction of pl flips in the slightly more restricted sense used here.

REMARK 2.2.22. If $f: X \rightarrow Z$ is a pl flipping contraction for $K + S + B$, then S contains all of the f -exceptional set. This is because if a curve $C \subset X$ is contracted by f , then $S \cdot C < 0$, hence $C \subset S$.

THEOREM 2.2.23. (See [FA92, 18.11] and Chapter 4.) *Assume that pl flips exist, and that any sequence of them terminates (to be more precise, we need to assume special termination of pl flips in the slightly more general sense of [FA92, Definition 18.6], see [FA92] and Chapter 4 for a detailed treatment). If (X, B) is a klt pair and $f: X \rightarrow Z$ is a flipping contraction, then the flip of f exists.* $\square$

This result in the 3-fold case is the key contribution of Shokurov's chapter [Sho92]. The proof is conceptual; the general framework is generalised to higher dimensions in [FA92] and discussed in Chapter 4 of this book.

REMARK 2.2.24. [FA92, Theorem 7.1] implies that there is no infinite sequence of 3-fold pl flips. The corresponding statement for 4-folds is proved in Fujino's Chapter 4.

2.2.7. Plan of the proof. This chapter is devoted to the construction of 3-fold pl flips.

THEOREM 2.2.25. *Flips of 3-fold pl contractions exist.*

Here I give an overview of the key steps of the proof. The starting point is a pl flipping contraction $f: X \rightarrow Z$ for $K + S + B$.

Restricted algebra. As we know, the flip exists if the algebra

$$R = R(X, K + S + B) = \bigoplus_{i \geq 0} H^0(X, \mathcal{O}_X(i(K + S + B)))$$

is finitely generated. The first step is to define a *restricted algebra* $R^0 = \text{res}_S R(X, D)$. Roughly speaking, $R^0 = \bigoplus_i R_i^0$ where

$$R_i^0 = \text{Im} \left(H^0(X, \mathcal{O}_X(i(K + S + B))) \rightarrow H^0(S, \mathcal{O}_S(i(K_S + \text{Diff}_S B))) \right).$$

It is easy to see, and it is shown in Lemma 2.3.6 below, that R is finitely generated if R^0 is.

Pbd-algebras. The restriction of $|i(K + S + B)|$ to S is not (a priori) a complete linear system; instead, it is a linear system with base conditions. To keep track of the base conditions, Shokurov develops the language of b-divisors. In the language of b-divisors, the restricted algebra R^0 (or rather, more precisely, an algebra integral over it) is a pbd-algebra (pseudo b-divisorial algebra).

Shokurov algebras. We still don't have a good reason to believe that R^0 is finitely generated. Shokurov introduces two key properties, boundedness and canonical a-saturation. Boundedness has a natural meaning and it is easy to verify. On the other hand, canonical a-saturation is a very subtle property whose meaning is poorly understood. It is not difficult to show that a suitable integral extension of the restricted algebra is bounded and canonically a-saturated. We call an algebra satisfying these properties a Shokurov algebra (Fano graded algebra, or FGA algebra in Shokurov's own terminology). Shokurov conjectures that a Shokurov algebra on a variety admitting a weak Fano contraction is finitely generated.

The surface case. To construct 3-fold pl flips, we show that a Shokurov algebra on a surface admitting a weak Fano contraction is finitely generated. The proof uses some features of birational geometry and linear systems which are specific to surfaces. The key point is Theorem 2.4.6 below, which states that a saturated mobile b-divisor $\mathbf{M}$ on a nonsingular surface admitting a weak Fano contraction, is base point free.

2.2.8. Log resolution. Recall that $D \subset Y$ or, more precisely, the pair (Y, D) is *simple normal crossing* if Y is nonsingular and the components of D are nonsingular and cross normally. In this chapter we work with the following definition of log resolution.

DEFINITION 2.2.26. A *log resolution* of the pair (X, B) is a proper birational morphism $f: Y \rightarrow X$ satisfying the following conditions.

- (1) The exceptional set of f is a divisor $E = \sum E_i \subset Y$, where we denote by E_i the irreducible components.

- (2) The space Y is nonsingular and the support of $f^{-1}B \cup E$ is a simple normal crossing divisor.

Sometimes it is also required that there exists an f -ample divisor $A = -\sum \varepsilon_i E_i$ supported on the exceptional divisor (necessarily all $\varepsilon_i > 0$). This can always be achieved by a further blow up and it is never at issue in this chapter. For a more general discussion, see Chapter 3.

2.3. The language of function algebras and b-divisors

2.3.1. Function algebras. In what follows, we consider a normal variety X and a birational contraction $f: X \rightarrow Z$ to an affine variety Z . We denote by $A = H^0(Z, \mathcal{O}_Z)$ the affine coordinate ring of Z .

DEFINITION 2.3.1. A *function algebra* on X is a graded A -subalgebra

$$V = \bigoplus_{i \geq 0} V_i$$

of the polynomial algebra $k(X)[T]$ where $V_0 = A$ and each V_i is a coherent A -module. In other words, each V_i is a coherent A -module with a given inclusion $V_i \subset k(X)$ and multiplication in V is induced by multiplication in $k(X)$; in particular, $V_i V_j \subset V_{i+j}$.

A function algebra is *bounded* if there is an (integral) Weil divisor $D \subset X$ such that $V_j \subset H^0(X, \mathcal{O}(jD))$ for all j . In this case, we also say that V is bounded by D . We are only interested in bounded algebras.

A *truncation* of a graded algebra $V = \bigoplus V_i$ is an algebra of the form $V^{(d)} = \bigoplus_j V_{jd}$

CONVENTION 2.3.2. Occasionally, we abuse language and say that a property holds for V when it actually holds, or even makes sense, only for a truncation of V .

LEMMA 2.3.3. A function algebra is *finitely generated* if and only if any of its truncations is *finitely generated*.

PROOF. $V^{(d)}$ is the subalgebra of invariants under an obvious action of μ_d on V . If V is finitely generated, then so is $V^{(d)}$ by E. Noether's Theorem on the finite generation of rings of invariants under finite group actions. In the opposite direction, every homogeneous element $f \in V$ satisfies a monic equation

$$X^d - f^d = 0$$

where $f^d \in V^{(d)}$; this implies that V is integral over $V^{(d)}$. If $V^{(d)}$ is finitely generated, then so is V by E. Noether's theorem on the finiteness of the integral closure. $\square$

DEFINITION 2.3.4. Let X be a normal variety and $S \subset X$ an irreducible normal subvariety of codimension 1. Denote by $\mathcal{O}_{X,S}$ the local ring of X at S , that is, the rank 1 valuation subring of $k(X)$ corresponding to S , and by

$$m_{X,S} = \{f \in k(X) \mid \text{mult}_S f > 0\} \subset \mathcal{O}_{X,S}$$

the maximal ideal. Note that $k(S) = \mathcal{O}_{X,S}/m_{X,S}$.

A function algebra $V = \bigoplus V_i$ is *regular along S* if it satisfies the following conditions.

- (1) All $V_i \subset \mathcal{O}_{X,S} \subset k(X)$, that is to say, the homogeneous elements of the algebra V are rational functions which are defined (regular) at the generic point of S .
- (2) $V_1 \not\subset m_{X,S}$.

If V is regular along S , the *restricted algebra* $V^0 = \text{res}_S V$ is the function algebra:

$$V^0 = \oplus V_i^0 \quad \text{where} \quad V_i^0 = \text{Im}(V_i \rightarrow k(S)).$$

By construction, this is a function algebra on S .

REMARK 2.3.5. If V is bounded by D and $S \not\subset \text{Supp } D$, then the restricted algebra $V^0 = \text{res}_S V$ is also bounded.

Let $(X, S + B)$ be a plt pair and $f: X \rightarrow Z$ a pl flipping contraction. As we know, the flip exists if and only if the canonical algebra

$$R = R(X, K + S + B) = \oplus_i H^0(X, \mathcal{O}_X(i(K + S + B)))$$

is finitely generated. More generally, the flip exists if $R(X, D)$ is finitely generated for some f -negative $\mathbb{Q}$ -divisor D . Indeed, since $\rho(X/Z) = 1$, $D \sim r(K + S + B)$ for a positive rational $r \in \mathbb{Q}_+$, hence $R(X, K + S + B)$ and $R(X, D)$ have a common truncation.

LEMMA 2.3.6. *Let $f: X \rightarrow Z$ be a pl flipping contraction for $K + S + B$. Let D be an effective f -negative $\mathbb{Q}$ -Cartier integral Weil divisor on X ; assume that $S \not\subset \text{Supp } D$. The flip exists if the restricted algebra $R^0 = \text{res}_S R(X, D)$ is finitely generated.*

PROOF. Denote $R = R(X, D)$; by what we said, we may assume that $D \sim S$ is linearly equivalent to S . In particular, there is a rational function $t \in k(X)$ with $\text{div } t + D = S$; by definition, $t \in R_1$. The statement clearly follows from the:

Claim. The kernel of the restriction map $R \rightarrow R^0$ is the principal ideal generated by t .

Indeed, let $\varphi \in R_n \subset k(X)$ restrict to $0 \in R^0$. This means that $\text{div } \varphi + nD \geq 0$ has a zero along S , that is:

$$\text{div } \varphi + nD - S \geq 0.$$

In view of this, we can write $\varphi = t\varphi'$ where $\text{div } \varphi' + (n-1)D = \text{div } \varphi + D - S + (n-1)D \geq 0$; in other words, $\varphi' \in R_{n-1}$ and $\varphi \in (t)$. $\square$

2.3.2. b-divisors.

TERMINOLOGY 2.3.7. Let X be a normal variety, not necessarily proper. We often work in the category of normal varieties Y , together with a proper birational morphism $f: Y \rightarrow X$. We say that $f: Y \rightarrow X$, or simply Y , is a *model proper over* X , or simply a *model* of X . A morphism in this category is a morphism $Y \rightarrow Y'$ defined over X , that is, a commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\quad} & Y' \\ & \searrow & \swarrow \\ & X & \end{array}$$

DEFINITION 2.3.8. Let X be a normal variety. An (integral) *b-divisor* on X is an element:

$$\mathbf{D} \in \mathbf{Div} X = \lim_{Y \rightarrow X} \text{Div } Y$$

where the (projective) limit is taken over all models $f: Y \rightarrow X$ proper over X , under the push forward homomorphism $f_*: \text{Div } Y \rightarrow \text{Div } X$.

Divisors with coefficients in $\mathbb{Q}$ are defined similarly.

If $f: Y \rightarrow X$ is a model proper over X , the natural push-forward homomorphism $f_*: \mathbf{Div} Y \rightarrow \mathbf{Div} X$ is an isomorphism. Because of this, we can safely abuse notation and identify b-divisors on Y with b-divisors on X .

REMARK 2.3.9. The “b” in b-divisor stands for “birational”.

NOTATION AND CONVENTIONS 2.3.10. If $\mathbf{D} = \sum d_\Gamma \Gamma$ is a b-divisor on X , and $Y \rightarrow X$ is a model of X , the *trace* of $\mathbf{D}$ on Y is the divisor

$$\mathbf{D}_Y = \text{tr}_Y \mathbf{D} = \sum_{\Gamma \text{ is a divisor on } Y} d_\Gamma \Gamma.$$

DEFINITION 2.3.11. We define the b-divisor of a rational function $f \in k(X)^\times$ by the formula

$$\mathbf{div}_X f = \sum \nu_E(f) E$$

where we sum over all geometric valuations E with centre on X .

Two b-divisors $\mathbf{D}_1, \mathbf{D}_2$ on X are *linearly equivalent* if their difference

$$\mathbf{D}_1 - \mathbf{D}_2 = \mathbf{div}_X f$$

is the b-divisor of a rational function $f \in k(X)^\times$.

EXAMPLE 2.3.12. Many familiar constructions lead naturally to b-divisors.

- (1) The *$\mathbb{Q}$ -Cartier closure* of a $\mathbb{Q}$ -Cartier ($\mathbb{Q}$ -)divisor D on X is the b-divisor $\overline{D}$ with trace

$$\overline{D}_Y = f^*(D)$$

on models $f: Y \rightarrow X$ of X . We call a b-divisor of this form a *$\mathbb{Q}$ -Cartier b-divisor*. If $\varphi \in k(X)^\times$ is a rational function, then

$$\mathbf{div}_X \varphi = \overline{\mathbf{div}_X \varphi}.$$

- (2) The *canonical divisor* of a normal variety X is a b-divisor. Indeed, the divisor $\mathbf{K} = \mathbf{div}_X \omega$ of a meromorphic differential $\omega \in \Omega_{k(X)}$ naturally makes sense as a b-divisor, for Zariski teaches how to take the order of vanishing of ω along a geometric valuation of X . I follow established usage and say that $\mathbf{K}$ is the canonical divisor of X when, more precisely, I can only make sense of the canonical divisor class; $\mathbf{K}$ is a divisor in the canonical class.
- (3) If X is a normal variety, and $B = \sum b_i B_i \subset X$ is a $\mathbb{Q}$ -divisor, the *discrepancy* of the pair (X, B) is the b-divisor $\mathbf{A} = \mathbf{A}(X, B)$ with trace $\mathbf{A}_Y$ defined by the formula:

$$K_Y = f^*(K_X + B) + \mathbf{A}_Y$$

on models $f: Y \rightarrow X$ of X . In order for $\mathbf{A}$ to be defined, we need to assume that $K_X + B$ is $\mathbb{Q}$ -Cartier.

- (4) For an ordinary divisor D on X , we denote by $\widehat{D}$ the *proper transform* b-divisor; its trace on models $f: Y \rightarrow X$ is

$$\widehat{D}_Y = f_*^{-1}D.$$

I usually abuse notation and simply write D instead of $\widehat{D}$.

REMARK 2.3.13. (1) If $\mathbf{D} = \sum m_E E$ is a b-divisor on a variety X (for instance affine) we can define a sheaf on X :

$$\mathcal{O}_X(\mathbf{D}) = \{f \in k(X) \mid v_E(f) + m_E \geq 0 \quad \forall E \text{ with centre on } X\}.$$

Note that this definition makes sense when $\mathbf{D}$ is a $\mathbb{Q}$ -b-divisor, and it means $\mathcal{O}_X(\mathbf{D}) = \mathcal{O}_X(\lfloor \mathbf{D} \rfloor)$. From the definition, $\mathcal{O}_X(\mathbf{D})$ is a subsheaf of the constant sheaf $k(X)$. Note that $\mathbb{P}H^0(X, \mathcal{O}_X(\mathbf{D})) \subset \mathbb{P}H^0(X, \mathcal{O}_X(\mathbf{D}_X))$ is a *linear system with base conditions on X* . The key point about b-divisors is that they are a convenient language to keep track of linear systems with base conditions. For instance, consider $X = \mathbb{P}^2$, and let $f: E \subset Y \rightarrow P \in X$ be the blow up of a point $P \in X$. Let $L \subset X$ be a line through P and $L' \subset Y$ the proper transform. Then the $\mathbb{Q}$ -Cartier closure $\mathbf{D} = \overline{L'}$ is a b-divisor on X and $\mathbb{P}H^0(X, \mathcal{O}_X(\mathbf{D}))$ is the linear system of lines passing through P .

- (2) In general, the sheaf $\mathcal{O}_X(\mathbf{D})$ is not coherent; in fact, it is often not even quasi-coherent. However, if $f: Y \rightarrow X$ is a model and $D \geq 0$ is an effective $\mathbb{Q}$ -Cartier divisor on Y , then $\mathcal{O}_X(\overline{D}) = f_* \mathcal{O}_Y(D)$ is a coherent sheaf ($\overline{D}$ denotes the $\mathbb{Q}$ -Cartier closure of D). More generally if $0 \leq \mathbf{D} \leq \overline{D}$, then $\mathcal{O}_X(\mathbf{D})$ is also coherent.

We only use $\mathcal{O}_X(\mathbf{D})$ when it is coherent. Lemma 2.3.15 below is a simple criterion that, under natural conditions, guarantees that $\mathcal{O}_X(\mathbf{D})$ is coherent.

- (3) The sheaf $\mathcal{O}_X(\mathbf{D})$ can be coherent even if $\mathbf{D}$ has infinitely many nonzero coefficients. When this happens, of course, it is possible to choose a finite $\mathbf{D}'$ for which $\mathcal{O}_X(\mathbf{D}') = \mathcal{O}_X(\mathbf{D})$. However, this may not be a natural thing to do, e. g. when $\mathbf{D} = \mathbf{K}$ is the canonical b-divisor.
- (4) At the heart of the construction of 3-fold pl flips we find ourselves in the following situation. We have a sequence of b-divisors $\mathbf{D}_i$ on X and we want to find a fixed model $f: Y \rightarrow X$ such that $\mathcal{O}_X(\mathbf{D}_i) = f_* \mathcal{O}_Y(\mathbf{D}_{iY})$ for all i .

LEMMA 2.3.14. *Let X be a normal variety, $D = \sum d_i D_i \subset X$ a $\mathbb{Q}$ -divisor, and $\mathbf{A} = \mathbf{A}(X, D)$ the discrepancy b-divisor. Let Y be a nonsingular variety and $f: Y \rightarrow X$ a proper birational morphism. Assume that $\text{Supp } \mathbf{A}_Y$ is a simple normal crossing divisor. If Y' is a normal variety and $g: Y' \rightarrow Y$ is a proper birational morphism, then*

$$[\mathbf{A}_{Y'}] = g^*[\mathbf{A}_Y] + \sum \delta_i E_i$$

where the E_i s are the g -exceptional divisors and all $\delta_i \geq 0$.

PROOF. By definition, $K_Y = f^*(K_X + D) + \mathbf{A}_Y$; therefore, we may write:

$$\begin{aligned} K_{Y'} &= g^* f^*(K_X + D) + \mathbf{A}_{Y'} = g^*(K_Y - \mathbf{A}_Y) + \mathbf{A}_{Y'} = \\ &= g^*(K_Y + \{-\mathbf{A}_Y\} + \lfloor -\mathbf{A}_Y \rfloor) + \mathbf{A}_{Y'} = g^*(K_Y + \{-\mathbf{A}_Y\}) - g^*[\mathbf{A}_Y] + \mathbf{A}_{Y'}. \end{aligned}$$

Now, Remark 2.2.8 states that $(X, \{-\mathbf{A}_X\})$ is a klt pair; we get that $-g^*[\mathbf{A}_Y] + \mathbf{A}_{Y'} > -\sum E_i$, and the statement follows. $\square$

LEMMA 2.3.15. *If X is a normal variety, $D = \sum d_i D_i \subset X$ a $\mathbb{Q}$ -divisor and $\mathbf{A} = \mathbf{A}(X, D)$ the discrepancy b -divisor, then $\mathcal{O}_X(\lceil \mathbf{A} \rceil)$ is a coherent sheaf.*

REMARK 2.3.16. It is useful to note that, in this lemma, the divisor D is an arbitrary $\mathbb{Q}$ -divisor (provided that $\mathbf{A}$ is defined, that is, $K + D$ is $\mathbb{Q}$ -Cartier). In other words, we are making no assumptions on the singularities of the pair (X, D) . This is just as well, since $\mathbf{A}(X, D) = \mathcal{O}_X$ if and only if the pair (X, D) has klt singularities.

REMARK 2.3.17. When $D \geq 0$, the sheaf $\mathcal{O}_X(\lceil \mathbf{A}(X, D) \rceil)$ is the same as the multiplier ideal sheaf $\mathcal{J}(D) = \mathcal{J}(X, D)$ in the sense of [Laz04, Definition 9.2.1], see also Chapter 5.

PROOF. We may assume that X is affine. Choose a log resolution $f: Y \rightarrow X$, and write

$$K_Y = f^*(K_X + D) + \sum a_i A_i$$

where the support $A = \cup A_i$ is a simple normal crossing divisor. Note that, by definition, $\mathbf{A}_Y = \sum a_i A_i$. By the previous Lemma 2.3.14, if $g: Y' \rightarrow Y$ is another model of X , then

$$\lceil \mathbf{A}_{Y'} \rceil = g^* \lceil \mathbf{A}_Y \rceil + (\text{effective \& exceptional}).$$

This shows that $f_* \mathcal{O}_Y(\lceil \mathbf{A}_Y \rceil) = f'_* \mathcal{O}_{Y'}(\lceil \mathbf{A}_{Y'} \rceil)$ (where $f' = fg$), from which it follows that $\mathcal{O}_X(\lceil \mathbf{A} \rceil) = f_* \mathcal{O}_Y(\sum \lceil a_i \rceil A_i)$ is a coherent sheaf. $\square$

2.3.3. Saturated divisors and b -divisors. Saturation is an important and natural property of b -divisors discovered by Shokurov.

DEFINITION 2.3.18. Let D be a $\mathbb{Q}$ -divisor on a variety X and

$$(0) \neq V \subset H^0(X, D) = \{f \mid \operatorname{div} f + D \geq 0\} \subset k(X)$$

a vector subspace; the *mobile part of D with respect to V* is the divisor

$$\operatorname{Mob}_V D = \sum m_E E \quad \text{where} \quad m_E = - \inf_{f \in V} \operatorname{mult}_E f$$

and we sum over all divisors of X . When $V = H^0(X, D)$, we simply speak of *the mobile part* $\operatorname{Mob} D$ of D .

REMARK 2.3.19. In the applications in this chapter, D is always integral and effective, but neither is assumed in the definition. If D is not integral, then the definition says that $\operatorname{Mob}_V D = \operatorname{Mob}_V \lfloor D \rfloor$. If D is integral, then $\operatorname{Mob} D = D - F$ where $F = \operatorname{Fix} |D|$ is the *fixed part* of the complete linear system $|D|$. In particular, when D is effective, $\operatorname{Mob} D$ is also effective. If $D \sim D'$, then, upon choosing a rational function φ such that $\operatorname{div} \varphi = D - D'$, we can identify $V \subset H^0(X, \mathcal{O}(D))$ with $V' = \varphi V \subset H^0(X, \mathcal{O}(D'))$, and then

$$\operatorname{Mob}_{V'} D' = -\operatorname{div} \varphi + \operatorname{Mob}_V D.$$

REMARK 2.3.20. If $V_1 \subset H^0(X, D_1)$ and $V_2 \subset H^0(X, D_2)$, then

$$\operatorname{Mob}_{V_1 \otimes V_2} (D_1 + D_2) = \operatorname{Mob}_{V_1} D_1 + \operatorname{Mob}_{V_2} D_2.$$

However, in general,

$$\text{Mob}(D_1 + D_2) \geq \text{Mob } D_1 + \text{Mob } D_2.$$

($H^0(X, D_1 + D_2)$ is not, in general, generated by $H^0(X, D_1) \otimes H^0(X, D_2)$.)

DEFINITION 2.3.21. Let D and C be $\mathbb{Q}$ -divisors on X . We say that D is *C-saturated* if $\text{Mob}[D + C] \leq D$.

REMARK 2.3.22. In this chapter, we only use saturation with D effective and $[D + C]$ effective.

TERMINOLOGY 2.3.23. Let X be a normal variety. We say that a property P holds *on high models* if P holds on a particular model $f: Y \rightarrow X$ and on every “higher” model, that is, every model $Y' \rightarrow Y \rightarrow X$.

DEFINITION 2.3.24. A b-divisor $\mathbf{D}$ on X is *C-saturated* if $\mathbf{D}_Y$ is $\mathbf{C}_Y$ -saturated on high models $Y \rightarrow X$ of X .

If $Y \rightarrow X$ is a model of X and $\mathbf{D}_Y$ is $\mathbf{C}_Y$ -saturated, we say that *saturation holds on Y* .

When $\mathbf{C} = \mathbf{A}(X, B)$ is the discrepancy b-divisor of a klt pair (X, B) , we say that $\mathbf{D}$ is *canonically saturated*.

REMARK 2.3.25. By definition, (X, B) is klt if and only if $\lceil \mathbf{A}(X, B)_Y \rceil \geq 0$ on all models $Y \rightarrow X$. Therefore, if $\mathbf{D}$ is effective, $\lceil \mathbf{D}_Y + \mathbf{A}(X, B)_Y \rceil$ is effective on every model.

When $(X, S + B)$ is plt, we work with $\mathbf{A}'(X, S + B) = \mathbf{A}(X, S + B) + S$. By definition, $(X, S + B)$ is plt if and only if $\lceil \mathbf{A}'(X, S + B)_Y \rceil \geq 0$ on all models $Y \rightarrow X$. Therefore, if $\mathbf{D}$ is effective, $\lceil \mathbf{D} + \mathbf{A}'(X, S + B)_Y \rceil$ is effective on every model.

The following definition names the most common example:

DEFINITION 2.3.26. A b-divisor $\mathbf{D}$ on X is *exceptionally saturated* over X if it is $\mathbf{E}$ -saturated for all $\mathbf{E}$ effective and exceptional over X .

PROPOSITION 2.3.27. *The $\mathbb{Q}$ -Cartier closure $\overline{D}$ of a $\mathbb{Q}$ -Cartier integral Weil divisor D is exceptionally saturated.*

PROOF. The statement is equivalent to the well known elementary fact that, for all models $f: Y \rightarrow X$ over X ,

$$f_* \mathcal{O}_Y(\lceil f^*(D) + \sum a_i E_i \rceil) = \mathcal{O}_X(D)$$

if all E_i are exceptional and all $a_i \geq 0$. This is really easy to show: First of all, for all a_i , it is clear that $f_* \mathcal{O}_Y(\lceil f^*(D) + \sum a_i E_i \rceil) \subset \mathcal{O}_X(D)$; the other inclusion is also obvious, since, when the a_i are positive, the proper transform $D' \leq \lceil f^*(D) + \sum a_i E_i \rceil$. $\square$

In practise, when we want to verify that a given divisor is saturated, we first check that saturation holds on a particular model Y of X . As a second step, we use the next lemma to show that saturation holds on all higher models $Y' \rightarrow Y$. The assertion is technical, but, as we shall see, the conditions are often easily satisfied. The “moral” is that to verify saturation it is often sufficient to verify saturation on a particular model.

LEMMA 2.3.28. *Let (X, B) be a pair of a normal variety X and a $\mathbb{Q}$ -divisor $B \subset X$ where, as usual, we assume that $K + B$ is $\mathbb{Q}$ -Cartier. Let $\mathbf{D}$ be a b-divisor on X .*

Let now $Y \rightarrow X$ be a model of X , and assume that the following conditions are satisfied:

- (1) *Y is nonsingular and the support of $\mathbf{D}_Y + \mathbf{A}_Y$ is a simple normal crossing divisor,*
- (2) *$\mathbf{D} = \overline{\mathbf{D}_Y}$.*

With these assumptions, saturation holds on Y if and only if saturation holds on any higher model $f: Y' \rightarrow Y$. That is,

$$\text{Mob}[\mathbf{D}_Y + \mathbf{A}_Y] \leq \mathbf{D}_Y \quad \text{if and only if} \quad \text{Mob}[\mathbf{D}_{Y'} + \mathbf{A}_{Y'}] \leq \mathbf{D}_{Y'}.$$

In particular, if saturation holds on Y , then $\mathbf{D}$ is saturated. More precisely, saturation holds on all models $g: Y' \rightarrow Y$ higher than Y .

PROOF. The proof is very similar to the proof of Lemma 2.3.14; indeed, by condition 2, we may write:

$$\begin{aligned} K_{Y'} &= g^*(K_Y - \mathbf{A}_Y - \mathbf{D}_Y) + \mathbf{A}_{Y'} + \mathbf{D}_{Y'} = \\ &= g^*(K_Y + \{-\mathbf{A}_Y - \mathbf{D}_Y\}) - g^*[\mathbf{A}_Y + \mathbf{D}_Y] + \mathbf{A}_{Y'} + \mathbf{D}_{Y'}. \end{aligned}$$

By condition 1, the pair $(Y, \{-\mathbf{A}_Y - \mathbf{D}_Y\})$ is klt; it follows that

$$[\mathbf{D}_{Y'} + \mathbf{A}_{Y'}] = f^*[\mathbf{D}_Y + \mathbf{A}_Y] + E$$

where E is f -exceptional and effective. As we already noted, under these circumstances:

$$f_*\mathcal{O}_{Y'}([\mathbf{D}_{Y'} + \mathbf{A}_{Y'}]) = \mathcal{O}_Y([\mathbf{D}_Y + \mathbf{A}_Y]).$$

This formula easily implies that saturation holds on Y if and only if it holds on Y' . $\square$

LEMMA 2.3.29. *Let $(X, S + B)$ be a plt pair with discrepancy divisor $\mathbf{A} = \mathbf{A}(X, S + B)$. If a b-divisor $\mathbf{D}$ is exceptionally saturated, then it is $\mathbf{A}' = \mathbf{A} + S$ -saturated.*

PROOF. If a divisor appears in $\mathbf{A} + S$ with positive coefficient, it is exceptional. $\square$

2.3.4. Mobile b-divisors.

DEFINITION 2.3.30. An integral b-divisor $\mathbf{D}$ is *mobile* if there is a model $f: Y \rightarrow X$ such that

- (1) the linear system $|\mathbf{D}_Y|$ is free on Y , and
- (2) $\mathbf{D} = \overline{\mathbf{D}_Y}$ is the $\mathbb{Q}$ -Cartier closure of $\mathbf{D}_Y$.

REMARK 2.3.31. A mobile b-divisor is not necessarily effective. In this chapter all mobile b-divisors are effective, but it is useful, sometimes, to have the extra flexibility.

REMARK 2.3.32. If $\mathbf{D}$ is mobile and $Y \rightarrow X$ is a model over X , the following are equivalent:

- (1) the linear system $|H^0(Y, \mathbf{D})|$ is free on Y , and
- (2) $\mathbf{D} = \overline{\mathbf{D}_Y}$.

DEFINITION 2.3.33. A b-divisor $\mathbf{D}$ is *b-semiample* (*b-nef*) if there is a model $f: Y \rightarrow X$ of X such that $\mathbf{D} = \overline{\mathbf{D}_Y}$ where $\mathbf{D}_Y$ is a semiample (nef) $\mathbb{Q}$ -Cartier divisor.

DEFINITION 2.3.34. Let D be a $\mathbb{Q}$ -divisor on a variety X and

$$(0) \neq V \subset H^0(X, D) = \{f \mid \operatorname{div} f + D \geq 0\} \subset k(X)$$

a vector subspace; the *mobile b-part of D with respect to V* is the divisor

$$\mathbf{Mob}_V D = \sum m_E E \quad \text{where} \quad m_E = - \inf_{f \in V} \operatorname{mult}_E f$$

and we sum over all geometric valuations with centre on X . When $V = H^0(X, D)$ we simply speak of *the mobile b-part* $\mathbf{Mob} D$ of D . The mobile b-part with respect to V is a mobile b-divisor.

REMARK 2.3.35. If D is integral $\mathbb{Q}$ -Cartier and $f: Y \rightarrow X$ is a model, then

$$(\mathbf{Mob}_V D)_Y = f^* D - \operatorname{Fix} f^* |V|.$$

If, in addition, $V = H^0(X, D)$, then $(\mathbf{Mob} D)_Y = \operatorname{Mob} f^* D = \operatorname{Mob}[f^* D]$.

REMARK 2.3.36. If $V_1 \subset H^0(X, D_1)$ and $V_2 \subset H^0(X, D_2)$, then

$$\mathbf{Mob}_{V_1 \otimes V_2}(D_1 + D_2) = \mathbf{Mob}_{V_1} D_1 + \mathbf{Mob}_{V_2} D_2.$$

However, in general,

$$\mathbf{Mob}(D_1 + D_2) \geq \mathbf{Mob} D_1 + \mathbf{Mob} D_2.$$

($H^0(X, D_1 + D_2)$ is not, in general, generated by $H^0(X, D_1) \otimes H^0(X, D_2)$.)

LEMMA 2.3.37. *The mobile b-part $\mathbf{Mob} D$ of an integral Weil divisor D is exceptionally saturated.*

PROOF. This is a simple exercise in unravelling the definition. I show, more generally, that if $f: Y \rightarrow X$ is any model, then for all f -exceptional divisors $E \geq 0$:

$$\operatorname{Mob}[(\mathbf{Mob} D)_Y + E] \leq (\mathbf{Mob} D)_Y.$$

This follows from the definitions and the fact that, denoting by $D' \subset Y$ the proper transform, $H^0(Y, D' + E) = H^0(X, D)$. $\square$

REMARK 2.3.38. A mobile b-divisor $\mathbf{M}$ on X is not necessarily exceptionally saturated over X (it always is over a suitable model $Y \rightarrow X$). For instance, consider $X = \mathbb{P}^2$, and let $f: Y \rightarrow X$ be the blow up of a point $P \in X$. Let $L \subset X$ be a line through P and $L' \subset Y$ the proper transform. Then $\mathbf{M} = \overline{L'}$ is a mobile b-divisor which is not exceptionally saturated over X ; indeed $\mathbf{M}_Y + E = f^* L$ and

$$\operatorname{Mob}(\mathbf{M}_Y + E) = \mathbf{M}_Y + E \not\leq \mathbf{M}_Y.$$

(Of course $\mathbf{M} \neq \mathbf{Mob} L$.)

2.3.5. Restriction and saturation. I state a fundamental result on saturation: under natural conditions, exceptional saturation on X implies canonical saturation on a divisor $S \subset X$. The proof is a simple application of the “X-method”¹.

DEFINITION 2.3.39. Let $\mathbf{D}$ be a $\mathbb{Q}$ -Cartier $\mathbf{b}$ -divisor on X and $S \subset X$ an irreducible normal subvariety of codimension 1 not contained in the support of $\mathbf{D}_X$. I define the *restriction* $\mathbf{D}^0 = \text{res}_S \mathbf{D}$ of $\mathbf{D}$ to S as follows. Pick a model $f: Y \rightarrow X$ such that $\mathbf{D} = \overline{\mathbf{D}_Y}$; let $S' \subset Y$ be the proper transform. I define

$$\text{res}_S \mathbf{D} = \overline{\mathbf{D}_{Y|S'}}$$

where $\mathbf{D}_{Y|S'}$ is the ordinary restriction of divisors. (Strictly speaking, $\overline{\mathbf{D}_{Y|S'}}$ is a $\mathbf{b}$ -divisor on S' ; as already noted, $\mathbf{b}$ -divisors on S' are canonically identified with $\mathbf{b}$ -divisors on S via push forward.) It is easy to see that the restriction does not depend on the choice of the model $Y \rightarrow X$.

REMARK 2.3.40. Restriction is additive and monotone, in the sense that

- (1) $(\mathbf{D}_1 + \mathbf{D}_2)^0 = \mathbf{D}_1^0 + \mathbf{D}_2^0$, and
- (2) if $\mathbf{D}_1 \geq \mathbf{D}_2$, then $\mathbf{D}_1^0 \geq \mathbf{D}_2^0$.

REMARK 2.3.41. The restriction of a mobile $\mathbf{b}$ -divisor is a mobile $\mathbf{b}$ -divisor.

REMARK 2.3.42. Restriction of mobile $\mathbf{b}$ -divisors is compatible with restriction of rational functions in the following sense. Let $\mathbf{M}$ be a mobile $\mathbf{b}$ -divisor on X and $V = H^0(X, \mathbf{M}) \subset k(X)$. We can recover $\mathbf{M}$ from the formula:

$$\text{mult}_E \mathbf{M} = - \inf_{f \in V} \text{mult}_E f$$

for all geometric valuations E with centre on X . If $S \subset X$ is a normal irreducible subvariety of codimension 1 not contained in the support of $\mathbf{M}$, then $V \subset \mathcal{O}_{X,S}$ consists of rational functions which are regular at the generic point of S ; therefore, it makes sense to restrict these rational functions to S . If $V^0 = \text{Im}(\text{res}: V \rightarrow k(S))$, then

$$\text{mult}_F \text{res}_S \mathbf{M} = - \inf_{f \in V^0} \text{mult}_F f$$

for all geometric valuations F of $k(S)$ with centre on S . The proof is a simple exercise.

LEMMA 2.3.43. *Let $(X, S+B)$ be a relative weak Fano pair, that is, $-(K+S+B)$ is nef and big relative to some contraction $X \rightarrow Z$ to an affine variety Z . Let $\mathbf{M}$ be a mobile $\mathbf{b}$ -divisor on X , and assume that S is not contained in the support of $\mathbf{M}_X$.*

Let, as usual, $\mathbf{A} = \mathbf{A}(X, S+B)$ denote the discrepancy $\mathbf{b}$ -divisor. Assume that $\mathbf{M}$ is $\mathbf{A}' = \mathbf{A} + S$ -saturated. (By Lemma 2.3.29, this holds if the pair $(X, S+B)$ is plt and the $\mathbf{b}$ -divisor $\mathbf{M}$ is exceptionally saturated.)

Then the mobile restriction $\mathbf{M}^0 = \text{res}_S \mathbf{M}$ is canonically saturated; in other words, it is $\mathbf{A}^0 = \mathbf{A}(S, \text{Diff } B)$ -saturated.

REMARK 2.3.44. In applications in this chapter, the pair $(X, S+B)$ is always plt, but we are not assuming this in the statement.

¹This is how graduate students at Tokyo Daigaku call the “Kawamata Technique”, invented by Kawamata in his proof of the cone, contraction and rationality Theorems.

PROOF. Let $f: Y \rightarrow X$ be a log resolution, F_i the f -exceptional divisors and B'_j the strict transforms of the components of B . We assume, as we may, that $Y \rightarrow X$ is high enough that:

- (1) the union of the support of $\mathbf{A}'_Y$, $\mathbf{M}_Y$ and the f -exceptional divisors is simple normal crossing,
- (2) $\mathbf{M}_Y$ is free and $\mathbf{M} = \overline{\mathbf{M}_Y}$, and
- (3) the defining property of the saturation of $\mathbf{M}$ holds, that is,

$$\text{Mob}[\mathbf{M}_Y + \mathbf{A}'_Y] \leq \mathbf{M}_Y,$$

where $\mathbf{A}' = \mathbf{A} + S$.

If we write

$$K_Y = f^*(K_X + S + B) - S' - \sum b_j B'_j + \sum a_i F_i$$

on Y , then, by definition

$$\mathbf{A}_Y = -S' - \sum b_j B'_j + \sum a_i F_i.$$

We want to show that $\mathbf{M}^0$ is saturated. For this purpose, we first check that the saturation property holds on the model at hand $S' \rightarrow S$. For ease of notation, let us write $\mathbf{A}^0 = \mathbf{A}(S, \text{Diff } B)$. This is a b-divisor on S ; it should not be confused with the restriction of $\mathbf{A}$; indeed, $\mathbf{A}$ is not b- $\mathbb{Q}$ -Cartier, therefore restriction does not make sense in this context. The adjunction formula states precisely that $\mathbf{A}_{S'}^0 = \mathbf{A}'_{Y|S'}$. We are interested in comparing $\text{Mob}[\mathbf{M}_{Y|S'} + \mathbf{A}_{S'}^0]$ with $\mathbf{M}_{Y|S'}$. The punchline is this: by the vanishing theorem of Kawamata and Viehweg,

$$H^1(Y, [\mathbf{M}_Y + \mathbf{A}_Y]) = H^1(Y, K_Y + [-f^*(K_X + S + B) + \mathbf{M}_Y]) = (0)$$

(note that here we are using our assumption that the pair $(X, S+B)$ is weak Fano); therefore, the natural restriction map

$$H^0(Y, [\mathbf{M}_Y + \mathbf{A}'_Y]) \rightarrow H^0(S', [\mathbf{M}_{Y|S'} + \mathbf{A}_{S'}^0])$$

is surjective. In other words, $[\mathbf{M}_{Y|S'} + \mathbf{A}_{S'}^0]$ contains no new sections, and the statement follows.

It is a consequence of Lemma 2.3.28 that saturation holds on models $S'' \rightarrow S'$ higher than S' . $\square$

2.3.6. Pbd-algebras: terminology and first properties. Fix a normal variety X . In what follows I always assume that X admits a proper birational contraction $f: X \rightarrow Z$ to an affine variety Z .

CONVENTION 2.3.45. In this subsection, and from now on in this chapter, whenever working with a b-divisor $\mathbf{D}$, I always tacitly assume that the sheaf $\mathcal{O}_X(\mathbf{D})$ is coherent.

DEFINITION 2.3.46. (1) A sequence $\mathbf{D}_\bullet = \{\mathbf{D}_i\}$ of b-divisors is *convex* if $\mathbf{D}_1 > 0$ and

$$\mathbf{D}_{i+j} \geq \frac{i}{i+j} \mathbf{D}_i + \frac{j}{i+j} \mathbf{D}_j \quad \text{for all positive integers } i, j.$$

- (2) We say that $\mathbf{D}_\bullet$ is *bounded* if there is a (ordinary) divisor D on X such that all $\mathbf{D}_i \leq \overline{D}$.

REMARK 2.3.47. Note that $\mathbf{D}_\bullet$ is “increasing” in the sense that $\mathbf{D}_i \leq \mathbf{D}_k$ when i divides k . If $\mathbf{D}_\bullet$ is bounded, convexity implies that the limit

$$\mathbf{D} = \lim_{i \rightarrow \infty} \mathbf{D}_i = \sup \mathbf{D}_i \in \mathbf{Div} X \otimes \mathbb{R}$$

exists as a b-divisor with real coefficients.

REMARK 2.3.48. We are interested in situations where $\mathbf{D}_i = (1/i)\mathbf{M}_i$ arises from a sequence $\mathbf{M}_\bullet$ of integral mobile b-divisors $\mathbf{M}_i$ on X . Then, $\mathbf{D}_\bullet$ is convex if and only if $\mathbf{M}_\bullet$ is *sub-additive*, that is, $\mathbf{M}_1 > 0$ and $\mathbf{M}_{i+j} \geq \mathbf{M}_i + \mathbf{M}_j$ (for all i, j). It is a matter of preference whether we work with the convex sequence $\mathbf{D}_\bullet$ or the sub-additive sequence $\mathbf{M}_\bullet$. From the point of view of taking the limit as $i \rightarrow \infty$, as we do below, it is natural to give prominence to $\mathbf{D}_\bullet$.

DEFINITION 2.3.49. A *pbd-algebra* is the function algebra $R = R(X, \mathbf{D}_\bullet)$ naturally associated to a convex sequence $\mathbf{D}_\bullet$ of b-divisors. By definition, the i th graded piece of R is

$$R_i = H^0(X, \mathcal{O}_X(i\mathbf{D}_i)).$$

Note that, by definition of the sheaf $\mathcal{O}_X(i\mathbf{D}_i)$, R_i is equipped with a natural inclusion $R_i \subset k(X)$, and the convexity of $\mathbf{D}_\bullet$ ensures that $R_i R_j \subset R_{i+j}$; the product in the algebra is inherited from the product in $k(X)$. Thus, a pbd-algebra is a function algebra. The sequence $\mathbf{D}_\bullet$ is called the *characteristic sequence* of the pbd-algebra. We say that the algebra is *bounded* if it is bounded as a function algebra. This is equivalent to saying that the characteristic sequence is bounded. When $\mathbf{D}_i = (1/i)\mathbf{M}_i$ where $\mathbf{M}_\bullet$ is a sub-additive sequence of mobile divisors, we say that $\mathbf{M}_\bullet$ is the *mobile sequence* of the pbd-algebra.

REMARK 2.3.50. In the definition, “pbd” stands for *pseudo b-divisorial*, as opposed to “genuine” b-divisorial algebras of the form $\oplus_i H^0(X, \mathcal{O}_X(i\mathbf{D}))$ for a fixed b-divisor $\mathbf{D}$.

REMARK 2.3.51. When X is affine we omit “ $H^0(X, -)$ ” from the notation. In the general case, there is a proper birational morphism $f: X \rightarrow Z$ to an affine variety Z and $R(X, \mathbf{D}_\bullet) = R(Z, \mathbf{D}_\bullet)$. For some purposes, we can work with $Z, \mathbf{D}_\bullet$ instead of $X, \mathbf{D}_\bullet$, and thus in practise we can assume that X is affine.

LEMMA 2.3.52. Assume X admits a proper birational contraction $f: X \rightarrow Z$ to an affine variety Z . If $R = R(X, \mathbf{D}_\bullet)$ is a pbd-algebra on X , then there is a sub-additive sequence $\mathbf{M}_\bullet$ of mobile b-divisors on X , as in Remark 2.3.48, such that $\mathbf{D}_i = (1/i)\mathbf{M}_i$. In other words, every pbd-algebra on X arises from a mobile sequence.

PROOF. Working with $Z, \mathbf{D}_\bullet$ in place of $X, \mathbf{D}_\bullet$, we may assume that X is affine. Consider the subspace $V_i = H^0(X, i\mathbf{D}_i) \subset H^0(X, i\mathbf{D}_{iX})$. Then let

$$\mathbf{M}_i = \text{Mob}_{V_i} i\mathbf{D}_{iX}$$

be the mobile part, with respect to V_i , of the trace $i\mathbf{D}_{iX}$. It is a tautology that $H^0(X, \mathbf{M}_i) = H^0(X, i\mathbf{D}_i)$. $\square$

2.3.7. The limiting criterion. We state the basic criterion for finite generation of a pbd-algebra.

LEMMA 2.3.53 (Limiting criterion). *Assume that X admits a proper birational contraction $f: X \rightarrow Z$ to an affine variety Z . A pbd-algebra $R = R(X, \mathbf{D}_\bullet)$ is finitely generated if and only if there is an integer i_0 such that $\mathbf{D}_{i_0 i} = \mathbf{D}_{i_0}$ for all i .*

PROOF. By the preceding lemma, under our assumptions on X , we may assume that $\mathbf{M}_i = i\mathbf{D}_i$ is mobile. Assuming $\mathbf{D}_{i_0 i} = \mathbf{D}_{i_0}$ for all i , we show that R is finitely generated and leave the opposite, easier, implication to the reader. Passing to a truncation, we may assume that $i_0 = 1$. Then $R = \oplus_i H^0(X, i\mathbf{M}_1)$ is the b-divisorial algebra associated to the mobile b-divisor $\mathbf{M}_1$. Let $Y \rightarrow X$ be a model such that $\mathbf{M}_{1Y}$ is free and $\mathbf{M}_1 = \overline{\mathbf{M}_{1Y}}$. Then $R = R(Y, \mathbf{M}_{1Y})$ is a divisorial algebra on Y associated to a base point free divisor, hence it is finitely generated. $\square$

2.3.8. Function algebras and pbd-algebras.

REMARK 2.3.54. If $X \rightarrow Z$ is a pl flipping contraction and $R = R(X, K+S+B)$, the restricted algebra $\text{res}_S R$ is a function algebra. It is not a pbd-algebra.

LEMMA 2.3.55. *Let $V = \oplus V_i$ be a function algebra on X . There is a pbd-algebra $V \subset R^V = R(X, \mathbf{D}_\bullet)$, which is integral over V . In particular*

- (1) *V is bounded if and only if R^V is bounded, and*
- (2) *V is finitely generated if and only if R^V is finitely generated.*

PROOF. The construction of R^V is very natural; the verification that it has the required properties is easy. Working with $Z, f_*\mathbf{D}_\bullet$ in place of $X, \mathbf{D}_\bullet$, we may assume that X is affine.

By definition of a function algebra, $V_i \subset k(X)$ is a finitely generated (coherent) $\mathcal{O}_X$ -submodule (fractional ideal) for all i . Clearly, $V_i \subset H^0(X, \mathbf{M}_i)$ where $\mathbf{M}_i$ is the mobile b-divisor defined by:

$$\text{mult}_E \mathbf{M}_i = - \min_{\varphi \in V_i} \text{mult}_E \varphi.$$

Because $V_i V_j \subset V_{i+j}$, it follows that $\mathbf{M}_i + \mathbf{M}_j \leq \mathbf{M}_{i+j}$ for all i, j . Multiplying by a suitable rational function we may assume that $\mathcal{O}_X \subset V_1$ or, in other words, that the mobile b-divisor $\mathbf{M}_1 \geq 0$ is effective. We take $\mathbf{D}_i = (1/i)\mathbf{M}_i$ and $R^V = R(X, \mathbf{D}_\bullet)$.

It is easy to see that V is bounded if and only if R^V is bounded; we outline the proof that V is finitely generated if and only if R^V is finitely generated.

First, I claim that the extension of algebras

$$V = \oplus V_i \subset \oplus H^0(X, \mathbf{M}_i) = R^V$$

is integral. More precisely, I show that for all i :

$$\oplus_{j \geq 0} V_i^j \subset \oplus_{j \geq 0} H^0(X, j\mathbf{M}_i)$$

is an integral extension of algebras. This in turn is an easy consequence of the yoga of b-divisors and basic facts on coherent sheaves. Indeed, let $f: Y \rightarrow X$ be a model such that $\mathbf{M}_{iY}$ is free and $\mathbf{M}_i = \overline{\mathbf{M}_{iY}}$. Then $H^0(X, j\mathbf{M}_i) = H^0(Y, j\mathbf{M}_{iY})$ and the claim follows from [Har77, Chapter II, Theorem 5.19 and Exercise 5.14].

By construction, V and R^V are function algebras with the same field of fractions. By E. Noether's theorem on the finiteness of the integral closure, if V is finitely generated, then R^V is finitely generated.

Assume now that R^V is finitely generated. Passing to a truncation, we may assume that R^V is generated by R_1^V . By what we said, R^V is integral over the finitely generated algebra $V' = \oplus_{j \geq 0} V_1^j$. It follows that R^V is a finite module over

V' . Because $V' \subset V \subset R^V$, V also is a finite module over V' , hence V is a finitely generated algebra. $\square$

REMARK 2.3.56. The algebra R^V constructed in Lemma 2.3.55 is not necessarily integrally closed.

2.3.9. The finite generation conjecture. Let R be a pbd-algebra. We want natural conditions on R that ensure that R is finitely generated. An obvious condition, which is easily satisfied in all cases of interest to us, is the boundedness of R . I now introduce a much more subtle condition.

DEFINITION 2.3.57. A positive convex sequence $\mathbf{D}_\bullet$ of b-divisors is *$\mathbf{C}$ -asymptotically saturated* (*$\mathbf{C}$ -a-saturated* for short) if for all i, j :

$$\text{Mob}[j\mathbf{D}_i Y + \mathbf{C}_Y] \leq j\mathbf{D}_j Y$$

on high models $Y \rightarrow X$. To spell this out, there are models $Y(i, j) \rightarrow X$, depending on i, j , such that the inequality holds on all models $Y \rightarrow Y(i, j) \rightarrow X$.

In the case when $\mathbf{C} = \mathbf{A} = \mathbf{A}(X, B)$ is the discrepancy of a klt pair (X, B) , I say that the sequence is *canonically asymptotically saturated*, or simply *canonically a-saturated*. A pbd-algebra is canonically a-saturated if the characteristic sequence is canonically a-saturated.

In practise, in order to make use of canonical a-saturation, it is necessary to construct a model Y , independent of i, j , where a-saturation holds “uniformly”. The existence of this model is highly nontrivial and our construction only works in the case of surfaces.

DEFINITION 2.3.58. A positive convex sequence $\mathbf{D}_\bullet$ of b-divisors is *uniformly $\mathbf{C}$ -a-saturated* if there is a model X' of X such that for all i, j :

$$\text{Mob}[j\mathbf{D}_i Y + \mathbf{C}_Y] \leq j\mathbf{D}_j Y$$

on models $Y \rightarrow X'$ higher than X' . In this case, I say that saturation holds uniformly on X' .

When $\mathbf{C} = \mathbf{A} = \mathbf{A}(X, B)$ is the discrepancy of a klt pair (X, B) , I say that the sequence is *uniformly canonically a-saturated*. A pbd-algebra is uniformly canonically a-saturated if the characteristic sequence is uniformly canonically a-saturated.

REMARK 2.3.59. The requirement gets stronger as $i \rightarrow \infty$. In practise, I only use the following consequence of uniform asymptotic saturation. If $\mathbf{D}_\bullet$ is bounded, $\mathbf{D} = \lim \mathbf{D}_i = \sup \mathbf{D}_i$, and saturation holds uniformly on Y , then

$$\text{Mob}[j\mathbf{D}_Y + \mathbf{C}_Y] \leq j\mathbf{D}_j Y \leq j\mathbf{D}_Y$$

for all j .

DEFINITION 2.3.60. A *Shokurov algebra* (*Fano graded algebra*, or *FGA algebra* in Shokurov’s own terminology) is a bounded canonically a-saturated pbd-algebra.

FINITE GENERATION CONJECTURE 2.3.61. *Let (X, B) be a klt pair, and $f: X \rightarrow Z$ a birational weak Fano contraction to an affine variety Z . Recall what this means: $f_*\mathcal{O}_X = \mathcal{O}_Z$ and $-(K + B)$ is nef and big over Z . All Shokurov algebras on X are finitely generated.*

- REMARK 2.3.62. (1) Below, we prove the conjecture when $\dim X \leq 2$. The conjecture is not known to be true if $\dim X = 3$.
- (2) The assertion can be generalised in various ways. To identify the most general circumstances under which the conjecture can be hoped to hold is a fundamental open problem with applications to existence of flips. It is natural to wish for a more general statement containing the finite generation of the canonical algebra of a klt pair as a special case. We prove in Subsection 2.4.1 that our version of the conjecture implies the existence of pl flips.

2.3.10. A Shokurov algebra on a curve is finitely generated. We prove the 1-dimensional case of Conjecture 2.3.61; this gives good evidence that we are on the right track.

Here X is an affine algebraic curve and $B = \sum b_m P_m$ is a klt divisor, that is, $0 \leq b_m < 1$. On a curve all b-divisors are usual divisors. The discrepancy divisor is $A = \sum a_m P_m$, where

$$-1 < a_m = -b_m \leq 0.$$

Let $D_\bullet$ be the characteristic sequence; it is a positive convex bounded canonically a-saturated sequence of divisors on X . We can write

$$D_i = \sum d_{m,i} P_m.$$

Let us take the limit

$$D = \lim_{i \rightarrow \infty} D_i = \sum (\lim_{i \rightarrow \infty} d_{m,i}) P_m = \sum d_m P_m \in \text{Div } X \otimes \mathbb{R}$$

as a divisor with a priori real coefficients; the proof concentrates on showing that, actually, D has rational coefficients. Note that the boundedness assumption implies that only finitely many d_m can be nonzero. On a curve, we need not worry about high models; a-saturation means:

$$\lceil j d_{m,i} + a_m \rceil \leq j d_{m,j}.$$

Passing to the limit as $i \rightarrow \infty$, we get

$$\lceil j d_m + a_m \rceil \leq j d_{m,j} \leq j d_m.$$

By Lemma 2.3.63, d_m is rational. Note that the lemma crucially needs that $K_X + B$ is klt, that is, $-1 < a_m$, rather than just $-1 \leq a_m$. Now, taking j divisible enough to make $j d_m$ integral, we obtain

$$j d_m = \lceil j d_m + a_m \rceil \leq j d_{m,j} \leq j d_m,$$

i. e., $d_{m,j} = d_m$. After a truncation, we have $d_{m,j} = d_m$ for all j, m . Finite generation follows from the limiting criterion (Lemma 2.3.53). $\square$

LEMMA 2.3.63. *If $-1 < a \leq 0$ and $\lceil j d + a \rceil \leq j d$ for all j , then d is rational.*

PROOF. Assume by contradiction that d is irrational. The cyclic subsemigroup $A \subset \mathbb{R}/\mathbb{Z}$ generated by d is infinite, therefore it is dense. In other words, the fractional parts $\{j d\}$, as j is a positive integer, are dense in the open interval $(0, 1)$. For some j , $\{j d\} + a > 0$, and then $\lceil j d + a \rceil > j d$, a contradiction. $\square$

2.4. Finite generation on surfaces and existence of 3-fold flips

2.4.1. The finite generation conjecture implies existence of pl flips. In this subsection, we show that the finite generation conjecture implies the existence of pl flips. The rest of the chapter is devoted to proving the finite generation conjecture in the case $\dim X = 2$.

NOTATION AND CONVENTIONS 2.4.1. In this subsection, $(X, S+B)$ is a plt pair and $f: X \rightarrow Z$ is a pl flipping contraction. Fix a mobile anti-ample Cartier divisor M on X with support not containing S ; write $\mathbf{M}_i = \mathbf{Mob} iM$ and $\mathbf{D}_i = (1/i)\mathbf{M}_i$. By Remark 2.3.36, $\mathbf{M}_\bullet$ is subadditive and $\mathbf{D}_\bullet$ is positive convex. Denote by

$$R = R(X, M) = \oplus_i H^0(X, iM) = R(X, \mathbf{D}_\bullet)$$

the associated pbd-algebra. Denote by $R^0 = \oplus_i R_i^0 = \text{res}_S R$ the restricted algebra. Recall that, by definition,

$$R_i^0 = \text{Im}(\text{res}: H^0(X, iM) \rightarrow k(S)).$$

Also, write $\mathbf{M}_i^0 = \text{res}_S \mathbf{M}_i = \text{res}_S \mathbf{Mob} iM$, set $\mathbf{D}_i^0 = (1/i)\mathbf{M}_i^0$ and denote

$$R_S = R(S, \mathbf{D}_\bullet)$$

(by Remark 2.3.40 $\mathbf{M}_\bullet^0$ is subadditive and $\mathbf{D}_\bullet^0$ is positive convex).

LEMMA 2.4.2. $R_S = R^V$ is the algebra constructed in Lemma 2.3.55 starting with $V = R^0$. In particular, $R^0 \subset R_S$ is an integral extension and R^0 is finitely generated if and only if R_S is finitely generated.

PROOF. The statement follows from Remark 2.3.42 and Lemma 2.3.55. $\square$

LEMMA 2.4.3. R_S is a Shokurov algebra.

PROPOSITION 2.4.4. The finite generation conjecture in dimension $n-1$ implies existence of pl flips in dimension n .

PROOF. We know from Lemma 2.3.6 that the flip exists if the restricted algebra R^0 is finitely generated. By Lemma 2.4.2, R^0 is finitely generated if and only if R_S is finitely generated. Since R_S is a Shokurov algebra and $S \rightarrow f(S)$ is a weak Fano contraction, finite generation of R_S follows from the finite generation conjecture. $\square$

PROOF OF LEMMA 2.4.3. It is clear from the construction that R_S is a bounded pbd-algebra; we need to show that R_S is canonically a-saturated. By Lemma 2.3.37, $\mathbf{Mob} iM$ is exceptionally saturated, hence Lemma 2.3.43 applies, and it states that all $\mathbf{M}_i^0$ are canonically saturated. This almost says that the algebra is canonically a-saturated. To show the statement, we need to go back to the proof of Lemma 2.3.43 and make the necessary modifications. This is straightforward, apart from having to deal with a more complicated notation.

For ease of notation, write $\mathbf{M}_i = \mathbf{Mob} iM$; by construction, and because M is chosen to be Cartier,

$$\mathbf{M}_{iY} = \mathbf{Mob} f^* iM$$

on models $f: Y \rightarrow X$.

To check asymptotic saturation, fix a pair of integers i, j and let $f: Y \rightarrow X$ be a log resolution, F_k the f -exceptional divisors and B'_m the strict transforms of the components of B . We assume, as we may, that $Y \rightarrow X$ is high enough that

- (1) $\mathbf{M}_{iY}$ is free and $\mathbf{M}_i = \overline{\mathbf{M}_{iY}}$,
- (2) ditto for $\mathbf{M}_j$.

The model Y may depend on i, j . If we write

$$K_Y = f^*(K_X + S + B) - S' - \sum b_j B'_j + \sum a_i F_i$$

on Y , then, by definition:

$$\mathbf{A}_Y = -S' - \sum b_j B'_j + \sum a_i F_i.$$

We want to show that $R_S = R(S, \mathbf{M}_\bullet^0)$ is canonically a-saturated. Because asymptotic saturation is a property of the characteristic sequence, it is convenient at this point to introduce the b-divisors $\mathbf{D}_i = (1/i)\mathbf{M}_i$. We first check that the saturation property relating $j\mathbf{D}_i^0$ and $j\mathbf{D}_j^0$ holds on the model $S' \rightarrow S$. For ease of notation, let us write $\mathbf{A}^0 = \mathbf{A}(S, \text{Diff } B)$; the adjunction formula says that $\mathbf{A}_{S'}^0 = \mathbf{A}'_{Y|S'}$. We are interested in comparing $\text{Mob}[j\mathbf{D}_{iY|S'} + \mathbf{A}_{S'}^0]$ with $j\mathbf{D}_{jY|S'}$. By the vanishing theorem of Kawamata and Viehweg,

$$H^1(Y, [(j\mathbf{D}_i + \mathbf{A})_Y]) = (0);$$

therefore, the natural restriction map

$$H^0(Y, [(j\mathbf{D}_i + \mathbf{A})_Y]) \rightarrow H^0(S', [(j\mathbf{D}_i^0 + \mathbf{A}^0)_{S'}])$$

is surjective. To prove the statement, it is enough to show that

$$\text{Mob}[(j\mathbf{D}_i + \mathbf{A})_Y] \leq j\mathbf{D}_{jY}.$$

(This is a kind of “exceptional asymptotic saturation” property on X .) By construction, $\mathbf{M}_{iY} = \text{Mob } f^*iM$, hence what we want is equivalent to

$$\text{Mob}[(j/i) \text{Mob}(f^*iM) + \mathbf{A}'_Y] \leq \text{Mob}(f^*jM).$$

This is easy: if a divisor appears in $\mathbf{A}'_Y$ with positive coefficient, it is exceptional; therefore, the claim follows from the (ordinary) exceptional saturation property of Proposition 2.3.27.

It follows from Lemma 2.3.28 that the asymptotic saturation property for i, j holds on all models $S'' \rightarrow S'$ higher than S' . This finishes the proof. $\square$

2.4.2. Linear systems on surfaces. Let (X, B) be a 2-dimensional terminal pair. This means that X is a nonsingular surface and $B = \sum b_i B_i$ is a divisor on X such that $\text{mult}_x B = \sum b_i \text{mult}_x B_i < 1$ at every point $x \in X$. Let $f: X \rightarrow Z$ be a birational weak Fano contraction to an affine surface Z , that is, $-(K_X + B)$ is f -nef. The main result of this subsection, Theorem 2.4.6 below, states that, if $\mathbf{M}$ is a saturated mobile divisor on X , then $|H^0(X, \mathbf{M})|$ is a free linear system on X .

The proof is a simple application of the X-method. It is important to understand that the result, and its proof, only holds in dimension 2. The key point is that a mobile divisor on a surface is nef. This is not true in higher dimensions and a naïve generalisation of the statement does not hold. To find a useful analog in higher dimensions is a fundamental open problem.

If $\mathbf{M}$ is a mobile b-divisor on a variety X , it is natural to ask under what reasonable conditions $\mathbf{M} = \overline{\mathbf{M}_X}$, that is, in Shokurov’s terminology, $\mathbf{M}$ “descends” to X . Our Theorem 2.4.6 is a kind of answer to this question in the case of surfaces.

I now explain the relevance of all this to the finite generation conjecture. Let (X, B) be a 2-dimensional klt pair, $f: X \rightarrow Z$ a birational weak Fano contraction to

an affine surface Z , and $R = R(X, \mathbf{D}_\bullet)$ a Shokurov algebra with characteristic system $\mathbf{D}_\bullet$. To show that R is finitely generated, we try to adapt the one-dimensional argument. The first very serious difficulty is to find a model Y where all the asymptotic saturations

$$\text{Mob}[j\mathbf{D}_{iY} + \mathbf{A}_Y] \leq j\mathbf{D}_{jY}$$

hold independent of i, j . In the last part of this subsection we recall the (well known) construction of the terminal model $\varphi: (X', B') \rightarrow (X, B)$; this is a terminal pair (X', B') which is crepant over (X, B) , that is, $K_{X'} + B' = \varphi^*(K_X + B)$. Corollary 2.4.9 states that asymptotic saturation holds uniformly on models higher than a log resolution $Y \rightarrow X'$.

DEFINITION 2.4.5. Given a b-divisor $\mathbf{D}$ on X , we say that $\mathbf{D}$ *descends* to X , if $\mathbf{D} = \overline{\mathbf{D}_X}$.

Surfaces are very special because mobile divisors on a surface are nef. This is the basis of the following theorem, which is the main result of this subsection.

THEOREM 2.4.6. *Let (X, B) be a 2-dimensional terminal pair. (This means that X is nonsingular, and $0 \leq B = \sum b_i B_i$ is an effective $\mathbb{Q}$ -divisor on X such that $\text{mult}_x B < 1$ at all points $x \in X$.) Let $f: X \rightarrow Z$ be a birational weak Fano contraction to an affine variety Z , in other words, $-(K + B)$ is nef relative to f . If $\mathbf{M}$ is a mobile canonically saturated b-divisor on X , then $\mathbf{M}$ descends to X .*

PROOF. Let $f: Y \rightarrow X$ be a high enough log resolution of (X, B) such that

- (1) canonical saturation holds on Y , and
- (2) $\mathbf{M} = \overline{\mathbf{M}_Y}$ and, therefore, $|\mathbf{M}_Y| = |H^0(Y, \mathbf{M})|$ is free.

Claim. The divisor $E = \lceil \mathbf{A}_Y \rceil$ is integral, f -exceptional, and:

- (1) every f -exceptional divisor appears in E with > 0 coefficient, that is, the support of E is all of the exceptional set,
- (2) $H^1(Y, E) = (0)$.

To prove the claim, write

$$K_Y = f^*(K_X + B) - B' + \sum a_i E_i$$

where the discrepancy

$$\mathbf{A}_Y = -B' + \sum a_i E_i$$

has every exceptional $a_i > 0$. In particular, $E = \lceil \mathbf{A}_Y \rceil$ satisfies conclusion 1. Note that

$$-f^*(K_X + B) = -K_Y + \mathbf{A}_Y$$

is nef and big, hence, by the vanishing theorem of Kawamata and Viehweg, $H^1(Y, \lceil \mathbf{A}_Y \rceil) = H^1(Y, E) = (0)$, that is, conclusion 2 also holds. This shows the claim.

We are assuming that $\mathbf{M}$ is saturated; this means that

$$E = \text{Bs } |\mathbf{M}_Y + E|.$$

If $M_Y \in |\mathbf{M}_Y|$ is a general member, vanishing ensures that the restriction map

$$H^0(Y, \mathbf{M}_Y + E) \rightarrow H^0(M_Y, (\mathbf{M}_Y + E)|_{M_Y})$$

is surjective, therefore $E \cap M_Y = \text{Bs } |H^0(M_Y, (\mathbf{M}_Y + E)|_{M_Y})|$. But M_Y is an affine curve, hence every complete linear system on M_Y is base point free; therefore, $E \cap M_Y = \emptyset$. Because the support of E is all of the exceptional set, this says that M_Y avoids the exceptional set altogether; consequently, $\mathbf{M}_Y = f^*\mathbf{M}_X$. $\square$

We conclude this subsection with an application to the asymptotic saturation property on surfaces. Before we do that, we need to recall the following result.

THEOREM 2.4.7. *Let (X, B) be a klt pair. Assume that the klt minimal model program holds in dimension $\dim X$. Then there is a pair (X', B') and a projective birational morphism $\varphi: X' \rightarrow X$ where:*

- (1) *the pair (X', B') has terminal singularities, and*
- (2) *$K_{X'} + B' = \varphi^*(K_X + B)$, that is (X', B') is crepant over (X, B) .*

We say that (X', B') is a terminal model of (X, B) .

REMARK 2.4.8. In the case of surfaces, the terminal model is unique.

PROOF. This is a standard result going back to Kawamata [Kaw92]; I give a sketch of the proof. Let $f: Y \rightarrow X$ be a log resolution; write

$$K_Y + B_+ - B_- = f^*(K_X + B)$$

where B_+, B_- are effective and B_- is exceptional. By blowing up further, I can assume that B_+ contains all the divisors with negative discrepancy for the pair (X, B) . The sought-for pair (X', B') is the end product of the minimal model program for $K_Y + B_+$ over X . The assertion is a consequence of the following remarks:

- (1) $K_Y + B_+ \sim_f B_-$, therefore no divisorial component of B_+ is contracted by the minimal model program.
- (2) The program terminates at $\varphi: (X', B') \rightarrow (X, B)$ where

$$K_{X'} + B' = \varphi^*(K_X + B) + (\text{effective \& exceptional})$$

is nef. By the negativity of contractions, $K_{X'} + B' = \varphi^*(K_X + B)$.

- (3) We have just seen that (X', B') is crepant over (X, B) . The pair (X', B') has terminal singularities, because, by 1, X' contains all the divisors with nonpositive discrepancy for the pair (X, B) .

□

COROLLARY 2.4.9. *Let (X, B) be a klt surface, Z and an affine surface, $f: X \rightarrow Z$ a birational weak Fano contraction and $R = R(X, \mathbf{M}_\bullet)$ a Shokurov algebra with mobile system $\mathbf{M}_\bullet$. If $\varphi: (X', B') \rightarrow (X, B)$ is the terminal model, then all $\mathbf{M}_i$ descend to X' . If $G = \sum G_j$ is a divisor on X' containing the support of all the $\mathbf{M}_i|_{X'}$, and $X'' \rightarrow X'$ is a log resolution of $(X', B' + G)$, then canonical a -saturation holds uniformly on models $f: Y \rightarrow X''$ higher than X'' , that is, on all such models:*

$$\text{Mob}[j\mathbf{D}_{iY} + \mathbf{A}_Y] \leq j\mathbf{D}_{jY}.$$

PROOF. The main assertion follows from Lemma 2.3.28. □

2.4.3. The finite generation conjecture on surfaces. In this subsection we prove the finite generation conjecture on surfaces. As we know, this implies existence of 3-fold flips.

THEOREM 2.4.10. *Let (X, B) be a klt pair, where X is a surface and $f: X \rightarrow Z$ is a birational weak Fano contraction to an affine variety Z . Recall what this means: $f_*\mathcal{O}_X = \mathcal{O}_Z$ and $-(K + B)$ is nef and big over Z . All Shokurov algebras on X are finitely generated.*

The rest of this subsection is devoted to the proof of the theorem. Most of the proof works in all dimensions; below I make a note of the two places where I use the fact that X is a surface.

Set up and notation for the proof. I fix some objects and notation for use in the proof.

Let $R = R(X, \mathbf{D}_\bullet)$ be a Shokurov algebra on X with characteristic system $\mathbf{D}_\bullet$ and mobile system $\mathbf{M}_\bullet$, $\mathbf{M}_i = i\mathbf{D}_i$. Let $f: (X', B') \rightarrow (X, B)$ be the terminal model of (X, B) as in Theorem 2.4.7. Because $-(K' + B') = -f^*(K + B)$, $X' \rightarrow Z$ is still a weak Fano contraction. It is clear that $R = R(X', \mathbf{D}_\bullet)$ is also a Shokurov algebra on X' . Hence, by passing to the terminal model, **we may assume that (X, B) is a terminal pair.**

As part of the requirements of a Shokurov algebra, R is bounded; in particular, there is a divisor $G = \sum G_j$ on X such that $\text{Supp } \mathbf{D}_{iX} \subset G$ for all i and the system has a limit

$$\mathbf{D} = \lim_{i \rightarrow \infty} \mathbf{D}_i \in \text{Div } X \otimes \mathbb{R}$$

as a b-divisor with (possibly) real coefficients, and $\text{Supp } \mathbf{D}_X \subset G$.

Our aim is to show that $\mathbf{D}$ is a b-divisor with rational coefficients and, finally, that $\mathbf{D} = \mathbf{D}_m$ for some (large) m and hence for all m large and divisible.

$\mathbf{D}_X$ is semiample. For the argument below to work, it is crucial to show that $\mathbf{D}_X$ is a semiample divisor. This is the first place where we use that X is a surface. It is likely that, with some work, this step can be generalised to higher dimensions.

LEMMA 2.4.11. *The divisor $\mathbf{D}_X$ is semiample.*

PROOF. The $\mathbf{M}_i$ are mobile; therefore, **because X is a surface**, they are nef. Hence, $\mathbf{D}_X = \lim \mathbf{D}_{iX}$ is also nef. The Mori cone of X is finite rational polyhedral, because $f: X \rightarrow Z$ is a weak Fano contraction. The dual cone of nef divisors is generated by the semiample divisors supporting the contractions of its extremal faces; hence, all nef divisors on X are semiample. $\square$

Diophantine approximation. We work with the integral lattice $N_{\mathbb{Z}}^1 = \oplus \mathbb{Z}[G_j] \subset \text{Div } X$ and the vector spaces $N_{\mathbb{Q}}^1 = N^1 \otimes \mathbb{Q}$, $N_{\mathbb{R}}^1 = N^1 \otimes \mathbb{R}$. The reader must realise that these are spaces of divisors, not divisors up to linear equivalence. Because $\mathbf{D}_X$ is semiample, we can choose effective base point free divisors $P_k \in N_{\mathbb{Z}}^1$ such that $\mathbf{D}_X$ is in the cone

$$\mathcal{P} = \sum \mathbb{R}_+[P_k] \subset \sum \mathbb{R}_+[G_j] \subset N_{\mathbb{R}}^1$$

generated by the P_k . We state an elementary lemma in the style of Diophantine approximation.

LEMMA 2.4.12. *If $\mathbf{D}_X$ is not rational, then, for all $\varepsilon > 0$, there exist a positive integer m and a divisor $M \in N_{\mathbb{Z}}^1$, such that:*

- (1) *The linear system $|M|$ is free.*
- (2) *$\|m\mathbf{D}_X - M\| < \varepsilon$, where the norm is the sup norm with respect to the basis $\{G_j\}$ of N^1 .*
- (3) *The divisor $m\mathbf{D}_X - M$ is not effective.*

PROOF. We work abstractly in the integral lattice $L_{\mathbb{Z}}$ generated by the $\{P_k\}$ and the vector spaces $L_{\mathbb{Q}}$, $L_{\mathbb{R}}$. By means of the basis $\{P_k\}$, we identify these spaces with $\mathbb{Z}^l$, $\mathbb{Q}^l$, $\mathbb{R}^l$. Write $\mathbf{D}_X$ as a row vector:

$$\mathbf{D}_X = \mathbf{d} = (d_1, \dots, d_l) \in L_{\mathbb{R}}, \quad \text{all } d_i > 0$$

where the d_i are not all rational. Let $A \subset L_{\mathbb{R}}/L_{\mathbb{Z}}$ be the cyclic subsemigroup generated by $\mathbf{d}$; our assumption means exactly that A is an infinite semigroup, hence its closure $\overline{A}$ is a Lie subgroup of dimension > 0 . Note that $\overline{A}$ is a group even though A is only a semigroup. Denote by $\overline{A}_0$ the connected component of the identity of $\overline{A}$. The inverse image $V \subset L_{\mathbb{R}}$ of $\overline{A}_0$ under the quotient map $L_{\mathbb{R}} \rightarrow L_{\mathbb{R}}/L_{\mathbb{Z}}$ is a vector subspace of strictly positive dimension not contained in the orthant $\mathcal{C} = L_{\mathbb{R}} \cap \sum \mathbb{R}_+[G_j]$ of effective divisors (indeed, the orthant contains no straight lines). This means that, for all $\varepsilon > 0$, we can find a positive integer multiple $m\mathbf{d}$ of $\mathbf{d}$, and an integer vector $\mathbf{m} = (m_1, \dots, m_l) \in L_{\mathbb{Z}}$, such that

- (1) $\|m\mathbf{d} - \mathbf{m}\| < \varepsilon$, and
- (2) $m\mathbf{d} - \mathbf{m} \notin \mathcal{C}$.

Now set $M = (m_1, \dots, m_l)$; if $\varepsilon > 0$ is sufficiently small, then all the $m_i > 0$, hence $M \in \mathcal{P}$ is free. $\square$

Nonvanishing. Choose a (small) rational number $\gamma > 0$ such that the pair $(X, B + \gamma G)$ has klt singularities. If $\mathbf{A} = \mathbf{A}(X, B)$ is the discrepancy of the pair (X, B) , this means that $\text{mult}_S(\mathbf{A} - \gamma \overline{G}) > -1$ for all prime divisors S with centre on X , or, equivalently, that $\lceil \mathbf{A} - \gamma \overline{G} \rceil \geq 0$.

NONVANISHING LEMMA 2.4.13. *Assume that $\mathbf{D}_X$ is nonrational, and let M be as in Lemma 2.4.12. If $\varepsilon < \gamma$, then on every model $f: Y \rightarrow X$ of X*

$$\text{Mob}[m\mathbf{D}_Y + \mathbf{A}_Y] \geq (\overline{M})_Y = f^*M.$$

PROOF. By Corollary 2.4.9, $\mathbf{D}_i = \overline{\mathbf{D}_{iX}}$ for all i , hence also $\mathbf{D} = \overline{\mathbf{D}_X}$ descends to X . Recall that the proof of Corollary 2.4.9 uses uniform a-saturation and hence **needs** $\dim X = 2$ **in an essential way**. Because $\mathbf{D}$ descends to X , $\mathbf{D}_Y = f^*\mathbf{D}_X$ and, denoting $F = m\mathbf{D}_X - M$, I may write

$$m\mathbf{D}_Y + \mathbf{A}_Y = f^*(m\mathbf{D}_X) + \mathbf{A}_Y = f^*M + f^*F + \mathbf{A}_Y > f^*M - \varepsilon f^*G + \mathbf{A}_Y,$$

where I used $f^*F > -\varepsilon f^*G$, which follows from $\|F\| < \varepsilon$ (Lemma 2.4.12). Rounding up and taking mobile parts we obtain

$$\text{Mob}[m\mathbf{D}_Y + \mathbf{A}_Y] \geq \text{Mob}(f^*M + \lceil \mathbf{A}_Y - \gamma f^*G \rceil) \geq \text{Mob } f^*M = f^*M.$$

$\square$

$\mathbf{D}$ is rational. We show that $\mathbf{D}$ is a rational b-divisor. The main ingredient of the proof is the uniform a-saturation of the system $\mathbf{D}_\bullet$. **This needs** $\dim X = 2$ **in an essential way**.

LEMMA 2.4.14. *The b-divisor $\mathbf{D}_X$ is rational.*

PROOF. Recall that we are assuming, as we may, that (X, B) is a terminal pair. Corollary 2.4.9 states that all $\mathbf{D}_i$, and therefore also $\mathbf{D}$, descend to X . In particular, to show that $\mathbf{D}$ is rational, it is sufficient to show that $\mathbf{D}_X$ is rational. Assume, by contradiction, that $\mathbf{D}_X$ is not rational. Let $Y \rightarrow X$ be a log resolution. Corollary 2.4.9 also states that a-saturation holds uniformly on Y ; in particular

$$\text{Mob}[m\mathbf{D}_Y + \mathbf{A}_Y] \leq m\mathbf{D}_Y.$$

The nonvanishing lemma states that

$$(\overline{M})_Y = f^*M \leq \text{Mob}[m\mathbf{D}_Y + \mathbf{A}_Y].$$

Combining this with saturation we get that

$$(\overline{M})_Y \leq m\mathbf{D}_Y$$

and $m\mathbf{D}_X - M$ is effective, which contradicts the property of M stated in Lemma 2.4.12. $\square$

$\mathbf{D}_\bullet$ is eventually constant.

LEMMA 2.4.15. *The characteristic system is eventually constant, that is, $\mathbf{D} = \mathbf{D}_m$ for some m .*

Finite generation follows immediately from the limiting criterion (Lemma 2.3.53).

COROLLARY 2.4.16. *The finite generation conjecture holds in dimension 2.* $\square$

PROOF OF LEMMA 2.4.15. Let $m > 0$ be an integer such that $m\mathbf{D}_X$ is integral, then apply the proof of the nonvanishing lemma with $M = m\mathbf{D}_X$, and $0 = F = m\mathbf{D}_X - M$. As before we obtain

$$(\overline{M})_Y = f^*M \leq \text{Mob}[m\mathbf{D}_Y + \mathbf{A}_Y].$$

Use now saturation in the form

$$\text{Mob}[m\mathbf{D}_Y + \mathbf{A}_Y] \leq m\mathbf{D}_{mY}.$$

The argument no longer leads to a contradiction; instead, it shows that $\mathbf{D}_m = \mathbf{D}$ and $m\mathbf{D} = \overline{M}$. $\square$

2.5. Acknowledgements

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It is a pleasure to state once again that the ideas here are due entirely to V.V. Shokurov.

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CHAPTER 3

What is log terminal?

OSAMU FUJINO

3.1. What is log terminal?

This chapter is a guide to the world of *log terminal singularities*. The main purpose is to attract the reader's attention to the subtleties of the various kinds of log terminal singularities. Needless to say, my opinion is not necessarily the best. We hope that this chapter will help the reader understand the definition of *log terminal*. Almost all the results in this chapter are known to experts, and perhaps only to them. Note that this chapter is not self-contained. For a systematic treatments of singularities in the log MMP, *see*, for example, [KM98, Section 2.3]. We assume that the reader is familiar with the basic properties of singularities of pairs.

In the log MMP, there are too many variants of *log terminal*. This sometimes causes trouble when we treat log terminal singularities. We have at least four standard references on the log MMP: [KMM87, FA92, KM98, Mat02]. It is unpleasant that each of these standard references adopted different definitions of log terminal and even of *log resolutions*. Historically, Shokurov introduced various kinds of log terminal singularities in his famous paper [Sho92, §1]. However, we do not mention [Sho92] for simplicity. We only treat the above four standard references. Before we come to the subject, we note:

REMARK 3.1.1. In [Mat02, Chapter 4], Matsuki explains various kinds of singularities in details. Unfortunately he made the mistake of applying Theorem 3.5.1 to normal crossing divisors, whilst it is only valid with simple normal crossing divisors. Accordingly, when we read [Mat02] we have to replace *normal crossings* with *simple normal crossings* in the definition of dlt and so forth. See Definitions 3.2.10, 3.7.1, Remarks 3.7.6, 3.10.7, and (2'') of [Mat02, Definition 4-3-2].

We summarize the contents of this chapter: Sections 3.2 and 3.3 contain some preliminaries. We recall well-known definitions and fix some notation. In Section 3.4, we define the notion of *divisorially log terminal singularities*, which is one of the most important notions of log terminal singularities. In Section 3.5, we treat Szabó's resolution lemma, which is very important in the log MMP. Section 3.6 was suggested by Mori. Here, we show by example that Szabó's resolution lemma is not true for normal crossing divisors by using the Whitney umbrella. Section 3.7 deals with *log resolutions*. Here, we explain subtleties of various kinds of log terminal singularities. In Section 3.8, we collect examples to help the reader understand singularities of pairs. In Section 3.9, we describe the *adjunction formula* for dlt pairs, which plays an important role in the log MMP. Finally, Section 3.10 collects some miscellaneous comments.

NOTATION 3.1.2. The set of integers (resp. rational numbers, real numbers) is denoted by $\mathbb{Z}$ (resp. $\mathbb{Q}$, $\mathbb{R}$). We will work over an algebraically closed field k of characteristic zero; my favorite is $k = \mathbb{C}$.

3.2. Preliminaries on $\mathbb{Q}$ -divisors

Before we introduce singularities of pairs, let us recall the basic definitions about $\mathbb{Q}$ -divisors.

DEFINITION 3.2.1 ($\mathbb{Q}$ -Cartier divisor). Let $D = \sum d_i D_i$ be a $\mathbb{Q}$ -divisor on a normal variety X , that is, $d_i \in \mathbb{Q}$ and D_i is a prime divisor on X for every i . Then D is $\mathbb{Q}$ -Cartier if there exists a positive integer m such that mD is a Cartier divisor.

DEFINITION 3.2.2 (Boundary and subboundary). Let $D = \sum d_i D_i$ be a $\mathbb{Q}$ -divisor on a normal variety X , where $d_i \in \mathbb{Q}$ and D_i are distinct prime Weil divisors. If $0 \leq d_i \leq 1$ (resp. $d_i \leq 1$) for every i , then we call D a *boundary* (resp. *subboundary*).

$\mathbb{Q}$ -factoriality often plays a crucial role in the log MMP.

DEFINITION 3.2.3 ($\mathbb{Q}$ -factoriality). A normal variety X is said to be $\mathbb{Q}$ -factorial if every prime divisor D on X is $\mathbb{Q}$ -Cartier.

We give one example to understand $\mathbb{Q}$ -factoriality.

EXAMPLE 3.2.4 (cf. [Kaw88, p.140]). We consider

$$X := \{(x, y, z, w) \in \mathbb{C}^4 \mid xy + zw + z^3 + w^3 = 0\}.$$

CLAIM 3.2.5. *The variety X is $\mathbb{Q}$ -factorial. More precisely, X is factorial, that is,*

$$R = \mathbb{C}[x, y, z, w]/(xy + zw + z^3 + w^3)$$

is a UFD.

PROOF. By Nagata's lemma (see [Mum99, p.196]), it is sufficient to check that $x \cdot R$ is a prime ideal of R and $R[1/x]$ is a UFD. This is an easy exercise. $\square$

Note that $\mathbb{Q}$ -factoriality is not a local condition in the analytic topology.

CLAIM 3.2.6. *Let X^{an} be the underlying analytic space of X . Then X^{an} is not analytically $\mathbb{Q}$ -factorial at $(0, 0, 0, 0)$.*

PROOF. We consider a germ of X^{an} around the origin. Then X^{an} is local analytically isomorphic to $(xy - uv = 0) \subset \mathbb{C}^4$. So, X^{an} is not $\mathbb{Q}$ -factorial since the two divisors $(x = u = 0)$ and $(y = v = 0)$ intersect at a single point. Note that two $\mathbb{Q}$ -Cartier divisors must intersect each other in codimension one. $\square$

We recall an important property of $\mathbb{Q}$ -factorial varieties, which is much more useful than one might first expect. For the proof, see [Kol96].

PROPOSITION 3.2.7 (cf. [Kol96, VI.1, 1.5 Theorem]). *Let $f: X \rightarrow Y$ be a birational morphism between normal varieties. Assume that Y is $\mathbb{Q}$ -factorial. Then the exceptional locus $\text{Exc}(f)$ is of pure codimension one in X .*

We give the next definition for the reader's convenience. We only use the round down of $\mathbb{Q}$ -divisors in this chapter.

DEFINITION 3.2.8 (Operations on $\mathbb{Q}$ -divisors). Let $D = \sum d_i D_i$ be a $\mathbb{Q}$ -divisor on a normal variety X , where d_i are rational numbers and D_i are distinct prime Weil divisors. We define

$$\begin{aligned} \lfloor D \rfloor &= \sum \lfloor d_i \rfloor D_i, \text{ the round down of } D, \\ \lceil D \rceil &= \sum \lceil d_i \rceil D_i = -\lfloor -D \rfloor, \text{ the round up of } D, \\ \{D\} &= \sum \{d_i\} D_i = D - \lfloor D \rfloor, \text{ the fractional part of } D, \end{aligned}$$

where for $r \in \mathbb{R}$, we define $\lfloor r \rfloor = \max\{t \in \mathbb{Z} \mid t \leq r\}$.

REMARK 3.2.9. In the literature, for example [KMM87], sometimes $\lfloor D \rfloor$ (resp. $\langle D \rangle$) denotes $\lfloor D \rfloor$ (resp. $\{D\}$). The round down $\lfloor D \rfloor$ is sometimes called the *integral part* of D .

We define (simple) normal crossing divisors, which will play an important role in the following sections.

DEFINITION 3.2.10 (Normal crossing and simple normal crossing). Let X be a smooth variety. A reduced effective divisor D is said to be a *simple normal crossing divisor* (resp. *normal crossing divisor*) if for each closed point p of X , a local defining equation of D at p can be written as $f = z_1 \cdots z_{j_p}$ in $\mathcal{O}_{X,p}$ (resp. $\widehat{\mathcal{O}}_{X,p}$), where $\{z_1, \dots, z_{j_p}\}$ is a part of a regular system of parameters.

REMARK 3.2.11. The notion of normal crossing divisor is local for the étale topology (cf. [Art69, Section 2]). When $k = \mathbb{C}$, it is also local for the analytic topology. On the other hand, the notion of simple normal crossing divisor is not local for the étale topology.

REMARK 3.2.12. Let D be a normal crossing divisor. Then D is a simple normal crossing divisor if and only if each irreducible component of D is smooth.

REMARK 3.2.13. Some authorities use the word *normal crossing* to represent *simple normal crossing*. For example, a normal crossing divisor in [BEVU05] is a simple normal crossing divisor in our sense, see [BEVU05, Definition 2.1]. We recommend that the reader check the definition of (simple) normal crossing divisors whenever he reads a paper on the log MMP.

3.3. Singularities of pairs

In this section, we quickly review the definitions of singularities which we use in the log MMP. For details, see, for example, [KM98, §2.3]. First, we define the canonical divisor.

DEFINITION 3.3.1 (Canonical divisor). Let X be a normal variety with $\dim X = n$. The *canonical divisor* K_X is any Weil divisor whose restriction to the smooth part of X is a divisor of a regular n -form. The reflexive sheaf of rank one $\omega_X = \mathcal{O}_X(K_X)$ corresponding to K_X is called the *canonical sheaf*.

Next, let us recall the various definitions of singularities of pairs.

DEFINITION 3.3.2 (Discrepancies and singularities of pairs). Let X be a normal variety and $D = \sum d_i D_i$ a $\mathbb{Q}$ -divisor on X , where D_i are distinct and irreducible

such that $K_X + D$ is $\mathbb{Q}$ -Cartier. Let $f: Y \rightarrow X$ be a proper birational morphism from a normal variety Y . Then we can write

$$K_Y = f^*(K_X + D) + \sum a(E, D)E,$$

where the sum runs over all the distinct prime divisors $E \subset Y$, and $a(E, D) \in \mathbb{Q}$; $a(E, D)$ is called the *discrepancy* of E with respect to (X, D) . We define

$$\text{discrep}(X, D) := \inf_E \{a(E, D) \mid E \text{ is exceptional over } X\}.$$

From now on, we assume that D is a boundary. We say that (X, D) is

$$\begin{cases} \text{terminal} \\ \text{canonical} \\ \text{klt} \\ \text{plt} \\ \text{lc} \end{cases} \quad \text{if} \quad \text{discrep}(X, D) \quad \begin{cases} > 0, \\ \geq 0, \\ > -1 \text{ and } [D] = 0, \\ > -1, \\ \geq -1. \end{cases}$$

Here klt is an abbreviation for *Kawamata log terminal*, plt for *purely log terminal*, and lc for *log-canonical*.

REMARK 3.3.3. In [KM98, Definition 2.34], D is not a *boundary* but only a *subboundary*. In some of the literature and elsewhere in this book, (X, D) is called *sub lc* (resp. *sub plt*, *sub klt*) if $\text{discrep}(X, D) \geq -1$ (resp. > -1 , etc.) and D is only a *subboundary*.

REMARK 3.3.4 (Log discrepancies). We put $a_\ell(E, X, D) = 1 + a(E, D)$ and call it the *log discrepancy*. We define

$$\text{logdiscrep}(X, D) = 1 + \text{discrep}(X, D).$$

In some formulas, log discrepancies behave much better than discrepancies. However, we do not use log discrepancies in this chapter or elsewhere in this book.

3.4. Divisorially log terminal

Let X be a smooth variety and D a reduced simple normal crossing divisor on X . Then (X, D) is lc. Furthermore, it is not difficult to see that (X, D) is plt if and only if every connected component of D is irreducible. We would like to define some kind of log terminal singularities that contain the above pair (X, D) . So, we need a new notion of *log terminal*.

DEFINITION 3.4.1 (Divisorially log terminal). Let (X, D) be a pair where X is a normal variety and D is a boundary. Assume that $K_X + D$ is $\mathbb{Q}$ -Cartier. We say that (X, D) is *dlt* or *divisorially log terminal* if and only if there is a closed subset $Z \subset X$ such that

- (1) $X \setminus Z$ is smooth and $D|_{X \setminus Z}$ is a simple normal crossing divisor.
- (2) If $f: Y \rightarrow X$ is birational and $E \subset Y$ is an irreducible divisor with centre $c_X E \in Z$, then $a(E, D) > -1$.

So, the following example is obvious.

EXAMPLE 3.4.2. If X is a smooth variety and D is a reduced simple normal crossing divisor on X , then the pair (X, D) is dlt.

The above definition of dlt is [KM98, Definition 2.37], which is useful for many applications. However, it has a quite different flavor from the other definitions of *log terminal singularities*. We will explain the relationship between the definition of dlt and the other definitions of log terminal singularities in the following sections.

3.5. Resolution lemma

In my opinion, the most useful notion of log terminal singularities is *divisorially log terminal* (dlt, for short), which was introduced by Shokurov, *see* [FA92, 2.13.3]. We defined it in Definition 3.4.1 above. By Szabó's work [Sza94], the notion of dlt coincides with that of *weakly Kawamata log terminal* (wklt, for short). For the definition of wklt, *see* [FA92, 2.13.4]. This fact is non-trivial and based on deep results about the desingularization theorem. For the details, *see* the original fundamental paper [Sza94]. The key result is Szabó's resolution lemma [Sza94, Resolution Lemma]. The following is a weak version of the resolution lemma, but it contains the essential part of Szabó's result and is sufficient for applications. For the precise statement, *see* [Sza94, Resolution Lemma] or [BEVU05, Section 7]. By combining Theorem 3.5.1 with the usual desingularization arguments, we can recover the original resolution lemma without any difficulties. Explicitly, first we use Hironaka's desingularization theorem suitably, next we apply Theorem 3.5.1 below, and then we can recover Szabó's results. The details are left to the reader as an easy exercise, *see the proof of the resolution lemma* in [Sza94]. Note Example 3.5.4 below.

THEOREM 3.5.1. *Let X be a smooth variety and D a reduced divisor. Then there exists a proper birational morphism $f: Y \rightarrow X$ with the following properties:*

- (1) *f is a composition of blow ups of smooth subvarieties,*
- (2) *Y is smooth,*
- (3) *$f_*^{-1}D \cup \text{Exc}(f)$ is a simple normal crossing divisor, where $f_*^{-1}D$ is the strict transform of D on Y , and*
- (4) *f is an isomorphism over U , where U is the largest open set of X such that the restriction $D|_U$ is a simple normal crossing divisor on U .*

Note that f is projective and the exceptional locus $\text{Exc}(f)$ is of pure codimension one in Y since f is a composition of blowing ups.

REMARK 3.5.2. Recently, this was reproved by the new canonical desingularization algorithm. *See* [BEVU05, Theorem 7.11]. Note that in [BEVU05] normal crossing means simple normal crossing in our sense. *See* Remark 3.2.13 and Remark 3.7.4 below.

REMARK 3.5.3. Szabó's results depend on Hironaka's paper [Hir64], which is very hard to read. We recommend that the reader consult [BEVU05] for proofs. Now there are many papers on resolution of singularities; I do not know which is the best.

The following example shows that Szabó's resolution lemma, and Theorem 3.5.1, are not true if we replace simple normal crossing with normal crossing. We will treat this example in detail in the next section.

EXAMPLE 3.5.4. Let $X := \mathbb{C}^3$ and D the Whitney umbrella, that is, $W = (x^2 - zy^2 = 0)$. Then W is a normal crossing divisor outside the origin. In this

case, we can not make W a normal crossing divisor only by blowing ups of smooth subvarieties over the origin.

SKETCH OF THE PROOF. This is an exercise of how to calculate blow ups of smooth centres. If we blow up W finitely many times along smooth subvarieties over the origin, then we will find that the strict transform of W always has a pinch point, where a pinch point means a singular point that is local analytically isomorphic to $0 \in (x^2 - zy^2 = 0) \subset \mathbb{C}^3$. $\square$

Theorem 3.5.1 and Hironaka's desingularization imply the following corollary. It is useful for proving relative vanishing theorems and so on; see also Remark 3.6.11 below.

COROLLARY 3.5.5. *Let X be a non-complete smooth variety and D a simple normal crossing divisor on X . Then, there exists a compactification $\overline{X}$ of X and a simple normal crossing divisor $\overline{D}$ on $\overline{X}$ such that $\overline{D}|_X = D$. Furthermore, if X is quasi-projective, then we can make $\overline{X}$ projective.*

3.6. Whitney umbrella

We will work over $k = \mathbb{C}$ throughout this Section. First, we define *normal crossing varieties*.

DEFINITION 3.6.1 (Normal crossing variety). Let X be a variety. We say that X is *normal crossing* at x if and only if

$$\hat{\mathcal{O}}_{X,x} \simeq \mathbb{C}[[x_1, x_2, \dots, x_l]] / (x_1 x_2 \cdots x_k)$$

for some $k \leq l$. If X is normal crossing at every point, we call X a *normal crossing variety*.

REMARK 3.6.2. It is obvious that a normal crossing divisor (see Definition 3.2.10) is a normal crossing variety. By [Art69, Corollary 2.6], X is normal crossing at x if and only if $x \in X$ is locally isomorphic to $0 \in (x_1 x_2 \cdots x_k = 0) \subset \mathbb{C}^l$ for the étale (or classical) topology. So, let U be a small open neighborhood (in the classical topology) of X around x and U' the normalization of U . Then each irreducible component V of U' is smooth and $V \rightarrow U$ is an embedding.

Next, we introduce the notion of *WU singularities*.

DEFINITION 3.6.3 (WU singularity). Let X be a variety and x a closed point of X , and $p: X' \rightarrow X$ the normalization. If there exist a smooth irreducible curve $C' \subset X'$ and a point

$$x' \in \overline{C' \times_X C'} \setminus \overline{\Delta_{C'}} \cap \Delta_{C'} \cap p^{-1}(x),$$

where $\Delta_{C'}$ is the diagonal of $C' \times_X C'$, then we say that X has a WU singularity at x , where WU is an abbreviation of Whitney Umbrella.

EXAMPLE 3.6.4. Let $W = (x^2 - zy^2 = 0) \subset \mathbb{C}^3$ be the Whitney umbrella. Then the normalization of W is $\mathbb{C}^2 = \text{Spec } \mathbb{C}[u, v]$ such that the normalization map $\mathbb{C}^2 \rightarrow W$ is given by $(u, v) \mapsto (uv, u, v^2)$. Therefore, the line $(u = 0) \subset \mathbb{C}^2$ maps onto $(x = y = 0) \subset W$; thus, the origin is a WU singularity. Note that W is normal crossing outside the origin.

We give one more example.

EXAMPLE 3.6.5. Let $V = (z^3 - x^2yz - x^4 = 0) \subset \mathbb{C}^3$. Then V is singular along the y -axis. By blowing up $\mathbb{C}^3$ along the y -axis, we obtain the normalization $p: V' \rightarrow V$. Note that V' is smooth and that there is a distinguished irreducible smooth curve C' on V' which double covers the y -axis. It can be checked easily that the origin $(0, 0, 0)$ is a WU singularity of V .

REMARK 3.6.6. Let $x \in X$ be a WU singularity. We shrink X around x (in the classical topology). Then there exists an isomorphism $\sigma: C' \rightarrow C'$ of finite order such that $\sigma \neq \text{id}_{C'}$, $\sigma(x') = x'$, and $p = p \circ \sigma$ on C' . When X is the Whitney umbrella, σ corresponds to the graph $\overline{C'} \times_X C' \setminus \Delta_{C'}$ and the order of σ is two.

LEMMA 3.6.7. *Let $x \in X$ be a WU singularity. Then X is not normal crossing at x .*

PROOF. Assume that X is normal crossing at x . Let X'_1 be the irreducible component of X' containing C' . Since $X'_1 \rightarrow X$ is injective in a neighborhood of x' , $C' \times_X C' = \Delta_{C'}$ near x' . This is a contradiction. $\square$

The following theorem is the main theorem of this section.

THEOREM 3.6.8. *Let $x \in X$ be a WU singularity and $f: Y \rightarrow X$ be a proper birational morphism such that $f: f^{-1}(X \setminus \{x\}) \rightarrow X \setminus \{x\}$ is an isomorphism. Then Y has a WU singularity.*

PROOF. Let C', x' be as in Definition 3.6.3, σ as in Remark 3.6.6. Let $q: Y' \rightarrow Y$ be the normalization. Then there exists a proper birational morphism $f': Y' \rightarrow X'$. By assumption, $Y \rightarrow X$ is an isomorphism over $p(C') \setminus \{x\}$. Thus $Y' \rightarrow X'$ is an isomorphism over $C' \setminus p^{-1}(x)$. The embedding $C' \subset X'$ induces an embedding $C' \subset Y'$, and $p = p \circ \sigma$ implies $q = q \circ \sigma$. Therefore, Definition 3.6.3 implies that $q(x') \in Y$ is a WU singularity. $\square$

PROPOSITION 3.6.9. *Let $x \in X$ be a WU singularity and Z a normal crossing variety. Then there are no proper birational morphisms $g: X \rightarrow Z$ such that $p(C') \not\subset \text{Exc}(g)$, where p, C' are as in Definition 3.6.3.*

PROOF. Assume that there exists a proper birational morphism as above. We put $C := g(p(C'))$. Then the mapping degree of $C' \rightarrow C$ is greater than one by the definition of WU singularities. On the other hand, $C' \rightarrow C$ factors through the normalization Z' of Z . Thus, the mapping degree of $C' \rightarrow C$ is one. This is a contradiction. $\square$

The next corollary follows from Theorem 3.6.8 and Proposition 3.6.9.

COROLLARY 3.6.10. *There are no proper birational maps (that is, birational maps such that the first and the second projections from the graph are proper) between the Whitney umbrella W and a normal crossing variety V that induce $W \setminus \{0\} \simeq V \setminus E$, where E is a closed subset of V .*

Therefore, we obtain

REMARK 3.6.11. Corollary 3.5.5 does not hold for normal crossing divisors.

3.7. What is a log resolution?

We often use the words *good resolution* or *log resolution* without defining them precisely. This sometimes causes some serious problems. We will give our definition of log resolution later, see Definition 3.7.3. Let us recall another definition of dlt, which is equivalent to Definition 3.4.1. We do not prove the equivalence of Definition 3.4.1 and Definition 3.7.1 in this chapter. Note that it is an easy consequence of Theorem 3.5.1. For the details, see [Sza94, Divisorially Log Terminal Theorem].

DEFINITION 3.7.1 (Divisorially log terminal). Let X be a normal variety and D a boundary on X such that $K_X + D$ is $\mathbb{Q}$ -Cartier. If there exists a *log resolution* $f: Y \rightarrow X$ such that $a(E, D) > -1$ for every f -exceptional divisor E . Then we say that (X, D) is *dlt* or *divisorially log terminal*.

3.7.2. There are three questions about the above definition.

- Is f projective?
- Is the exceptional locus $\text{Exc}(f)$ of pure codimension one?
- Is $\text{Exc}(f) \cup \text{Supp}(f_*^{-1}D)$ a *simple normal crossing* divisor or only a *normal crossing* divisor?

In [KM98, Notation 0.4 (10)], they require that $\text{Exc}(f)$ is of pure codimension one and $\text{Exc}(f) \cup \text{Supp}(f_*^{-1}D)$ is a simple normal crossing divisor. We note that, in [FA92, 2.9 Definition], $\text{Exc}(f)$ is not necessarily of pure codimension one. So, the definition of lt in [FA92, 2.13.1] is the same as Definition 3.7.1 above, but lt in the sense of [FA92] is different from dlt. See Remark 3.7.5 and Examples 3.8.3 and 3.9.3 below. The difference lies in the definition of log resolution! Our definition of log resolution is the same as [KM98, Notation 0.4 (10)]. By Hironaka, log resolutions exist for varieties over a field of characteristic zero, see [BEVU05].

DEFINITION 3.7.3 (Log resolution). Let X be a variety and D a $\mathbb{Q}$ -divisor on X . A *log resolution* of (X, D) is a proper birational morphism $f: Y \rightarrow X$ such that Y is smooth, $\text{Exc}(f)$ is a divisor and $\text{Exc}(f) \cup \text{Supp}(f_*^{-1}D)$ is a simple normal crossing divisor.

REMARK 3.7.4. In the definition of log resolution in [BEVU05, Definition 7.10], they do not require that the exceptional locus $\text{Exc}(\mu)$ is of pure codimension one. However, if μ is a composition of blowing ups, then $\text{Exc}(\mu)$ is always of pure codimension one.

REMARK 3.7.5 (lt in the sense of [FA92]). If we do not assume that $\text{Exc}(f)$ is a divisor in Definition 3.7.3, then Definition 3.7.1 is the definition of lt in the sense of [FA92], see [FA92, 2.13.1].

REMARK 3.7.6 (Local in the analytic topology?). We assume that $k = \mathbb{C}$. Then the notion of terminal, canonical, klt, plt, and lc, is not only local in the Zariski topology, but also local in the analytic topology; for the precise statement, see [Mat02, Proposition 4-4-4]. However, the notion of dlt is not local in the analytic topology. This is because the notion of simple normal crossing divisors is not local in the analytic topology. So, [Mat02, Exercise 4-4-5] is incorrect. To obtain an analytically local notion of log terminal singularities, we must remove the word “simple” from Definition 3.4.1 (2). However, this new notion of log terminal singularities seems to be useless. Consider the pair $(\mathbb{C}^3, W)$, where W is the Whitney umbrella, see Section 3.5 and 3.6.

We note that, by Szabó's resolution lemma, we do not need the projectivity of f in the definition of dlt. It is because the log resolution f in Definition 3.7.1 can be taken to be a composition of blowing ups by Hironaka's desingularization and theorem 3.5.1; see also Definition 3.4.1, [KM98, Proposition 2.40 and Theorem 2.44], and [Sza94, Divisorially Log Terminal Theorem]. We summarize:

PROPOSITION 3.7.7. *The log resolution f in Definition 3.7.1 can be taken to be a composition of blow ups of smooth centres. In particular, it may always be assumed that there exists an effective f -anti-ample divisor whose support coincides with $\text{Exc}(f)$; thus, the notion of dlt coincides with that of wklt, see [FA92, 2.13.4].*

So, we can omit the notion of wklt in the log MMP. In [KMM87], they adopted normal crossing divisors instead of simple normal crossing divisors. So there is a difference between wklt and *weak log-terminal*. We note that any wklt singularity is a weak log-terminal singularity in the sense of [KMM87, Definition 0-2-10 (2)], but the converse does not always hold. See Section 3.8, especially, Example 3.8.1. In my experience, dlt, which is equivalent to wklt, is easy to treat and useful for inductive arguments, but weak log-terminal is very difficult to use. We think that [KM98, Corollary 5.50] makes dlt useful. For the usefulness of dlt, see [Fuj00a, Fuj00b, Fuj01] and Chapter 4. See also Example 3.8.1, Remark 3.8.2, and Section 3.9. We summarize:

CONCLUSION 3.7.8. *The notion of dlt coincides with that of wklt by [Sza94], see Proposition 3.7.7. In particular, a dlt singularity is automatically a weak log-terminal singularity in the sense of [KMM87]. Therefore, we can freely apply the results that were proved for weak log-terminal pairs in [KMM87] to dlt pairs. We note*

$$\text{klt} \implies \text{plt} \implies \text{dlt} \iff \text{wklt} \implies \text{weak log-terminal} \implies \text{lc}.$$

For other characterizations of dlt, see [Sza94, Divisorially Log Terminal Theorem], which is an exercise on Theorem 3.5.1. See also [KM98, Proposition 2.40, and Theorem 2.44]. The following proposition highlights a useful property of dlt singularities.

PROPOSITION 3.7.9. [KM98, Proposition 5.51] *Let (X, D) be a dlt pair. Then every connected component of $\lfloor D \rfloor$ is irreducible (resp. $\lfloor D \rfloor = 0$) if and only if (X, D) is plt (resp. klt).*

Thus, dlt is a natural generalization of plt.

CONCLUSION 3.7.10. *Lt in the sense of [FA92] seems to be useless. Examples 3.8.4 and 3.9.3 imply that the existence of a small resolution causes many unexpected phenomena. We note that if the varieties are $\mathbb{Q}$ -factorial, then there are no small resolutions by Proposition 3.2.7. Therefore, $\mathbb{Q}$ -factorial lt in the sense of [FA92] is equivalent to $\mathbb{Q}$ -factorial dlt.*

3.8. Examples

In this section, we collect some examples. The following example says that weak log-terminal is not necessarily wklt. We omit the definition of weak log-terminal since we do not use it in this chapter, see [KMM87, Definition 0-2-10].

EXAMPLE 3.8.1 (Simple normal crossing vs. normal crossing). Let X be a smooth surface and D a nodal curve on X . Then the pair (X, D) is not wklt but it is weak log-terminal.

The next fact is crucial for inductive arguments.

REMARK 3.8.2. Let (X, D) be a dlt (resp. weak log-terminal) pair and S an irreducible component of $\lfloor D \rfloor$. Then $(S, \text{Diff}(D - S))$ is dlt (resp. not necessarily weak log-terminal), where the $\mathbb{Q}$ -divisor $\text{Diff}(D - S)$ on S is defined by the following equation:

$$(K_X + D)|_S = K_S + \text{Diff}(D - S).$$

This is a so-called *adjunction formula*.

We will treat the adjunction formula for dlt pairs in detail in Section 3.9. In Example 3.8.1, $S = \lfloor D \rfloor$ is not normal. This makes weak log-terminal difficult to use for inductive arguments. The next example explains that we have to assume that $\text{Exc}(f)$ is a divisor in Definition 3.7.1.

EXAMPLE 3.8.3 (Small resolution). Let $X := (xy - uv = 0) \subset \mathbb{C}^4$. It is well-known that X is a toric variety. We take the torus invariant divisor D , the complement of the big torus. Then (X, D) is not dlt but it is lt in the sense of [FA92]; in particular, it is lc. Note that there is a small resolution.

The following is a variant of the above example.

EXAMPLE 3.8.4. [FA92, 17.5.2 Example] Let $X = (xy - uv = 0) \subset \mathbb{C}^4$ and

$$D = (x = u = 0) + (y = v = 0) + \frac{1}{2} \sum_{i=1}^4 (x + 2^i u = y + 2^{-i} v = 0).$$

If we put

$$F = \sum_{i=1}^2 (x + 2^i u = 0) + \sum_{i=3}^4 (y + 2^{-i} v = 0),$$

then $2D = F \cap X$. Thus, $2(K_X + D)$ is Cartier since X is Gorenstein. We can check that (X, D) is lt in the sense of [FA92] by blowing up $\mathbb{C}^4$ along the ideal (x, u) . In particular, (X, D) is lc. The divisor $\lfloor D \rfloor$ is two planes intersecting at a single point. Thus it is not S_2 . So, (X, D) is not dlt. See Remark 3.8.5 below.

REMARK 3.8.5. If (X, D) is dlt, then $\lfloor D \rfloor$ is seminormal and S_2 by [FA92, 17.5 Corollary].

REMARK 3.8.6. Example 3.8.4 says that [FA92, 16.9.1] is wrong. The problem is that S does not necessarily satisfy Serre's condition S_2 .

3.9. Adjunction for dlt pairs

To treat pairs effectively, we have to understand *adjunction*. Adjunction is explained nicely in [FA92, Chapter 16]. We recommend it to the reader. In this section, we treat the adjunction formula only for dlt pairs. Let us recall the definition log canonical centre, or LC centre.

DEFINITION 3.9.1 (Log canonical centre). Let (X, D) be a log canonical pair. A subvariety W of X is said to be a *log canonical centre*, or *LC centre*, for the pair (X, D) , if there exists a proper birational morphism from a normal variety $\mu: Y \rightarrow X$ and a prime divisor E on Y with discrepancy $a(E, D) \leq -1$ such that $\mu(E) = W$.

The next proposition is adjunction for a higher codimensional centre of log canonical singularities of a dlt pair. We use it in Chapter 4. For the definition of the *different* Diff , see [FA92, 16.6 Proposition].

PROPOSITION 3.9.2 (Adjunction for dlt pairs). *Let (X, D) be a dlt pair. We put $S = \lfloor D \rfloor$ and let $S = \sum_{i \in I} S_i$ be the irreducible decomposition of S . Then, W is a LC centre for the pair (X, D) with $\text{codim}_X W = k$ if and only if W is an irreducible component of $S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_k}$ for some $\{i_1, i_2, \dots, i_k\} \subset I$. By adjunction, we obtain*

$$K_{S_{i_1}} + \text{Diff}(D - S_{i_1}) = (K_X + D)|_{S_{i_1}},$$

and $(S_{i_1}, \text{Diff}(D - S_{i_1}))$ is dlt. Note that S_{i_1} is normal, W is a LC centre for the pair $(S_{i_1}, \text{Diff}(D - S_{i_1}))$, $S_{i_j}|_{S_{i_1}}$ is a reduced part of $\text{Diff}(D - S_{i_1})$ for $2 \leq j \leq k$, and W is an irreducible component of $(S_{i_2}|_{S_{i_1}}) \cap (S_{i_3}|_{S_{i_1}}) \cap \cdots \cap (S_{i_k}|_{S_{i_1}})$. By applying adjunction k times, we obtain a $\mathbb{Q}$ -divisor Δ on W such that

$$(K_X + D)|_W = K_W + \Delta$$

and (W, Δ) is dlt.

SKETCH OF PROOF. Note that S_{i_1} is normal by [KM98, Corollary 5.52], and [FA92, Theorem 17.2] and Definition 3.4.1 imply that $(S_{i_1}, \text{Diff}(D - S_{i_1}))$ is dlt. The other statements are obvious. $\square$

The above proposition is one more reason why dlt is more valuable than other flavours of log terminal singularities.

EXAMPLE 3.9.3. Let (X, D) be as in Example 3.8.4. Recall that (X, D) is lt in the sense of [FA92] but not dlt. It is not difficult to see that the LC centres for the pair (X, D) are as follows: the origin $(0, 0, 0, 0) \in X$ and the two Weil divisors $(x = u = 0)$ and $(y = v = 0)$ on X ; therefore, there are no one dimensional LC centres.

The final proposition easily follows from the above Proposition 3.9.2; it will play a crucial role in the proof of the special termination, see Chapter 4.

PROPOSITION 3.9.4. *Let (X, D) be as in Proposition 3.9.2. We write $D = \sum d_j D_j$, where $d_j \in \mathbb{Q}$ and D_j is a prime divisor on X . Let P be a divisor on W . Then the coefficient of P is 0, 1, or $1 - \frac{1}{m} + \sum \frac{r_j d_j}{m}$ for suitable non-negative integers r_j s and positive integer m . Note that the coefficient of P is 1 if and only if P is a LC centre for the pair (X, D) .*

SKETCH OF PROOF. Apply [FA92, Lemma 16.7] k times as in Proposition 3.9.2 and then apply [FA92, Lemma 7.4.3]. $\square$

3.10. Miscellaneous comments

In this section, we collect some comments.

3.10.1 ($\mathbb{R}$ -divisors). In the previous sections, we only use $\mathbb{Q}$ -divisors for simplicity. We note that almost all the definitions and results can be generalized to $\mathbb{R}$ -divisors with a little effort. In Shokurov's construction of pl flips [Sho03], $\mathbb{R}$ -divisors appear naturally and are indispensable. We do not pursue $\mathbb{R}$ -generalizations here; however, the reader who understands the results in this chapter, will have no difficulty working out the natural generalisations to $\mathcal{R}$ -divisors.

3.10.2 (Comments on the four standard references). We give miscellaneous comments on the four standard references.

- [KMM87] is the oldest standard reference of the log MMP. The notion of *log-terminal* in [KMM87, Definition 0-2-10] is equivalent to that of klt. We make a remark.

REMARK 3.10.3. Theorem 6-1-6 in [KMM87] is [Kaw85, Theorem 4.3]. We use the same notation as in the proof of Theorem 4.3 in [Kaw85]. By [Kaw85, Theorem 3.2], $'E_1^{p,q} \rightarrow ''E_1^{p,q}$ are zero for all p and q . This just implies that

$$\mathrm{Gr}^p H^{p+q}(X, \mathcal{O}_X(-[L])) \rightarrow \mathrm{Gr}^p H^{p+q}(D, \mathcal{O}_D(-[L]))$$

are zero for all p and q . Kawamata points out that we need one more Hodge theoretic argument to conclude that

$$H^i(X, \mathcal{O}_X(-[L])) \rightarrow H^i(D, \mathcal{O}_D(-[L]))$$

are zero for all i .

- [FA92] is the only standard reference that treats $\mathbb{R}$ -divisors and *differents*; see [FA92, Chapters 2 and 16]. In Chapter 2, five flavours of log terminal singularities, that is, klt, plt, dlt, wklt, and lt, were introduced following [Sho92]. Alexeev pointed out that [FA92, 4.12.2.1] is wrong; the following is a counterexample to [FA92, 4.12.1.3, 4.12.2.1].

EXAMPLE 3.10.4. Let $X = \mathbb{P}^2$, $B = \frac{2}{3}L$, where L is a line on X . Let P be any point on L . First, blow up X at P . Then we obtain an exceptional divisor E_P such that $a(E_P, B) = \frac{1}{3}$. Let L' be the strict transform of L . Next, take a blow-up at $L' \cap E_P$. Then we obtain an exceptional divisor F_P whose discrepancy $a(F_P, B) = \frac{2}{3}$. On the other hand, it is easy to see that $\mathrm{discrep}(X, B) = \frac{1}{3}$. Thus, $\min\{1, 1 + \mathrm{discrep}(X, B)\} = 1$.

REMARK 3.10.5. By this example, [Fuj04, Lemma 2.1] which is the same as [FA92, 4.12.2.1], is incorrect. For details on the discrepancy lemma, see [Fuj05a].

- [KM98] seems to be the best standard reference for singularities of pairs in the log MMP. In the definitions of singularities of pairs, they assume that D is only a subboundary, see [KM98, Definition 2.34] and Remark 3.3.3. One must be aware of this fact.

Consider the definition of lt in [KM98, Definition 2.34 (3)]. If $D = 0$ in Definition 3.3.2, then the notions klt, plt, and dlt coincide (see also Proposition 3.7.9) and they say that X has log terminal (abbreviated to lt) singularities.

REMARK 3.10.6. There is an error in [KM98, Lemma 5.17 (2)]. We can construct a counterexample easily. We put $X = \mathbb{P}^2$, $\Delta =$ a line on X , and $|H| = |\mathcal{O}_X(1)|$. Then we have

$$-1 = \text{discrep}(X, \Delta + H_g) \neq \min\{0, \text{discrep}(X, \Delta)\} = 0,$$

since $\text{discrep}(X, \Delta) = 0$.

- The latest standard reference [Mat02] explains singularities in detail; see [Mat02, Chapter 4]; however, as we pointed out before, see Remark 3.1.1, Matsuki made a mistake.

In the definition of lt , see [Mat02, Definition 4-3-2], he required that the resolution is projective. So, lt in [Mat02] is slightly different from lt in [FA92]. See Conclusion 3.7.10 above.

REMARK 3.10.7 (Comment by Matsuki). On page 178, line 8–9, “by blowing up only over the locus where $\sigma^{-1}(D) \cup \text{Exc}(\sigma)$ is not a normal crossing divisor, we obtain...” is incorrect. See Example 3.5.4 and Section 3.6.

REMARK 3.10.8 (Toric Mori theory). In [KMM87, §5-2] and [Mat02, Chapter 14], toric varieties are investigated from the Mori theoretic viewpoint. Toric Mori theory originates from Reid’s beautiful paper [Rei83]. Chapter 14 in [Mat02] corrects some minor errors in [Rei83]. In [KMM87] and [Mat02], toric Mori theory is formulated for toric projective morphism $f: X \rightarrow S$. We note that X is always assumed to be complete; therefore, the statement at the end of [Mat02, Proposition 14-1-5] is nonsense. Matsuki wrote: “In the relative setting for statement (ii), such a vector v'_i may not exist at all. If that is the case, then the two $(n-1)$ -dimensional cones $w_{i,n}$ and $w_{i,n+1}$ are on the boundary of Δ .” However, Δ has no boundary since Δ is a complete fan in [Mat02]. For the details of toric Mori theory for the case when X is not complete, see [FS04, Fuj, Sat05]. The example in [Mat02, Remark 14-2-7(ii)] is wrong; the original statement in [KMM87] is true. The statement in [Mat02, Corollary 14-2-2] contains a minor error; for details, see [Fuj, Remark 3.3, Example 3.4].

CONCLUSION 3.10.9. *Some care should be exercised when using the various notions of log terminal; we recommend that the reader check the definitions and conventions very carefully.*

3.11. Acknowledgements

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CHAPTER 4

Special termination and reduction to pl flips

OSAMU FUJINO

4.1. Introduction

This chapter is a supplement to [Sho03, Section 2]. First, we give a simple proof of special termination modulo the log MMP for lower dimensional varieties, Theorem 4.2.1. Special termination claims that the flipping locus is disjoint from the reduced part of the boundary after finitely many flips. It is repeatedly used in Shokurov's proof of pl flips [Sho03]. Next, we explain the reduction theorem: Theorem 4.3.7. Roughly speaking, the existence of pl flips and special termination imply the existence of all log flips. The reduction theorem is well-known to experts, cf. [FA92, Chapter 18]; it grew out of [Sho92].

Let us recall the two main conjectures in the log MMP.

CONJECTURE 4.1.1 ((Log) Flip Conjecture I: The existence of a (log) flip). *Let $\varphi: (X, B) \rightarrow W$ be an extremal flipping contraction of an n -dimensional pair, that is,*

- (1) φ is small projective and φ has only connected fibers,
- (2) $-(K_X + B)$ is φ -ample,
- (3) $\rho(X/W) = 1$, and
- (4) X is $\mathbb{Q}$ -factorial.

Then there should be a diagram:

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ & \searrow & \swarrow \\ & W & \end{array}$$

which satisfies the following conditions:

- (i) X^+ is a normal variety,
- (ii) $\varphi^+: X^+ \rightarrow W$ is small projective, and
- (iii) $K_{X^+} + B^+$ is φ^+ -ample, where B^+ is the strict transform of B .

Note that to prove Conjecture 4.1.1 we can assume that B is a $\mathbb{Q}$ -divisor, by perturbing B slightly.

CONJECTURE 4.1.2 ((Log) Flip Conjecture II: Termination of a sequence of (log) flips). *A sequence of (log) flips*

$$(X, B) =: (X_0, B_0) \dashrightarrow (X_1, B_1) \dashrightarrow (X_2, B_2) \dashrightarrow \cdots$$

terminates after finitely many steps. Namely, there does not exist an infinite sequence of (log) flips.

In this chapter, we sometimes write as follows: *Assume the log MMP for $\mathbb{Q}$ -factorial dlt (resp. klt) n -folds.* This means that the log flip conjectures I and II hold for n -dimensional dlt (resp. klt) pairs. For the details of the log MMP, see [KM98]. Note that in this chapter we run the log MMP only for birational morphisms. Namely, we apply the log MMP to some pair (X, B) over Y , where $f: X \rightarrow Y$ is a projective birational morphism.

We summarize the contents of this chapter: In Section 4.2, we give a simple proof of special termination. In Section 4.3, we explain the reduction theorem. This section is essentially the same as [FA92, Chapter 18]. Finally, in Section 4.4, we give a remark on the log MMP for non- $\mathbb{Q}$ -factorial varieties.

NOTATION 4.1.3. We use the basic notation and definitions in [KM98] freely; see also Chapter 3. We will work over an algebraically closed field k throughout this chapter; my favorite is $k = \mathbb{C}$.

4.2. Special termination

Special termination is in [Sho03, Theorem 2.3]. Shokurov gave a sketch of a proof in dimension four in [Sho03, Section 2]. Here, we give a simple proof, which is based on the ideas of [FA92, Chapter 7]. Note that [FA92, Chapter 7] grew out of [Sho92]. The key point of our proof is the *adjunction formula* for dlt pairs, which is explained in Section 3.9. Let us state the main theorem of this section.

THEOREM 4.2.1 (Special Termination). *We assume that the log MMP for $\mathbb{Q}$ -factorial dlt pairs holds in dimension $\leq n - 1$. Let X be a normal n -fold and B an effective $\mathbb{R}$ -divisor such that (X, B) is dlt. Assume that X is $\mathbb{Q}$ -factorial. Consider a sequence of log flips starting from $(X, B) = (X_0, B_0)$:*

$$(X_0, B_0) \dashrightarrow (X_1, B_1) \dashrightarrow (X_2, B_2) \dashrightarrow \cdots,$$

where $\phi_i: X_i \rightarrow Z_i$ is a contraction of an extremal ray R_i with $(K_{X_i} + B_i) \cdot R_i < 0$, and $\phi_i^+: X_i^+ = X_{i+1} \rightarrow Z_i$ is the log flip. Then, after finitely many flips, the flipping locus (and thus the flipped locus) is disjoint from $[B_i]$.

REMARK 4.2.2. If B is a $\mathbb{Q}$ -divisor in Theorem 4.2.1, then the log flip Conjectures I and II for $\mathbb{Q}$ -divisors are sufficient for the proof of the Theorem. This is because $\mathcal{S}(\mathbf{b}) \subset \mathbb{Q}$ (see Definition 4.2.7 below). We note that when we use special termination in Section 4.3 and [Fuj05b], B is a $\mathbb{Q}$ -divisor. If B is not a $\mathbb{Q}$ -divisor, then we need the log flip Conjecture II for $\mathbb{R}$ -divisors. For the details, see [Sho96, 5.2 Theorem].

First, we recall the definition of *flipping* and *flipped* curves.

DEFINITION 4.2.3. A curve C on X_i is called *flipping* (resp. *flipped*) if $\phi_i(C)$ (resp. $\phi_{i-1}^+(C)$) is a point.

We quickly review *adjunction* for dlt pairs. For the details, see Section 3.9.

PROPOSITION 4.2.4 (cf. Proposition 3.9.2). *Let (X, B) be a dlt pair such that $[B] = \sum_{i \in I} D_i$, where D_i is a prime divisor on X for every i . Then S is a LC centre of the pair (X, B) with $\text{codim}_X S = k$ if and only if S is an irreducible component of $D_{i_1} \cap D_{i_2} \cap \cdots \cap D_{i_k}$ for some $\{i_1, i_2, \dots, i_k\} \subset I$. Let S be a LC centre of the pair (X, B) . Then (S, B_S) is also dlt, where $K_S + B_S = (K_X + B)|_S$. Note that B_S is defined by applying adjunction k times repeatedly.*

DEFINITION 4.2.5. A morphism $\varphi: (X, B) \rightarrow (X', B')$ of two log pairs is called an *isomorphism of log pairs* if φ is an isomorphism and $\varphi_*(B) = B'$.

We need the following definition since the restriction of a log flip to a higher codimensional LC centre is not necessarily a log flip.

DEFINITION 4.2.6. Let $f: V \rightarrow W$ be a birational contraction with $\dim V \geq 2$. We say that f is type (S) if f is an isomorphism in codimension one. We say that f is type (D) if f contracts at least one divisor. Let

$$V \xrightarrow{f} W \xleftarrow{g} U$$

be a pair of birational contractions. We call this type (SD) if f is type (S) and g is type (D) . We define (SS) , (DS) , and (DD) similarly.

DEFINITION 4.2.7. Let $B = \sum b_j B^j$ be the irreducible decomposition of an $\mathbb{R}$ -divisor B . Let $\mathbf{b}$ be the set $\{b_j\}$. We define

$$\mathcal{S}(\mathbf{b}) := \left\{ 1 - \frac{1}{m} + \sum \frac{r_j b_j}{m} \mid m \in \mathbb{Z}_{>0}, r_j \in \mathbb{Z}_{\geq 0} \right\}.$$

Let P be a prime divisor on S . Then the coefficient of P in $\{B_S\}$ is an element of $\mathcal{S}(\mathbf{b})$. See Proposition 3.9.4. Before we give the definition of the *difficulty*, let us recall the following useful [FA92, 7.4.4 Lemma]. The proof is obvious.

LEMMA 4.2.8 (cf. [FA92, 7.4.4 Lemma]). *Fix a sequence of numbers $0 < b_j \leq 1$ and $c > 0$. Then there are only finitely many possible values $m \in \mathbb{Z}_{>0}$ and $r_j \in \mathbb{Z}_{\geq 0}$ such that*

$$1 - \frac{1}{m} + \sum_j \frac{r_j b_j}{m} \leq 1 - c.$$

DEFINITION 4.2.9 ([FA92, 7.5.1 Definition]). Let S be a LC centre of the dlt pair (X, B) . We define

$$d_{\mathbf{b}}(S, B_S) := \sum_{\alpha \in \mathcal{S}(\mathbf{b})} \# \left\{ E \mid a(E, S, B_S) < -\alpha \text{ and } c_S(E) \notin \lfloor B_S \rfloor \right\}.$$

This is a precise version of the *difficulty*. It is obvious that $d_{\mathbf{b}}(S, B_S) < \infty$ by Lemma 4.2.8. We note that $(U, B_S|_U)$ is klt, where $U = S \setminus \lfloor B_S \rfloor$.

Let us start the proof of Theorem 4.2.1.

PROOF OF Theorem 4.2.1.

STEP 1. After finitely many flips, the flipping locus contains no LC centres.

PROOF. We note that the number of LC centres is finite. If the flipping locus contains a LC centre, then the number of LC centres decreases by [FA92, 2.28]. $\square$

So we can assume that the flipping locus contains no LC centres of the pair (X_i, B_i) for every i . By this assumption, $\varphi_i: X_i \dashrightarrow X_{i+1}$ induces a birational map $\varphi_{i|S_i}: S_i \dashrightarrow S_{i+1}$, where S_i is a LC centre of (X_i, B_i) and S_{i+1} is the corresponding LC centre of (X_{i+1}, B_{i+1}) . We will omit the subscript $|S_i$ if there is no danger of confusion. Before we go to the next step, we prove the following Lemma.

LEMMA 4.2.10. *By adjunction, we have*

$$a(E, S_i, B_{S_i}) \leq a(E, S_{i+1}, B_{S_{i+1}}),$$

for every valuation E . In particular,

$$\text{totaldiscrep}(S_i, B_{S_i}) \leq \text{totaldiscrep}(S_{i+1}, B_{S_{i+1}})$$

for every i .

SKETCH OF THE PROOF. By the resolution Lemma 3.5, we can find a common log resolution

$$\begin{array}{ccc} & Y & \\ \swarrow & & \searrow \\ X_i & \dashrightarrow & X_{i+1} \end{array}$$

such that $Y \rightarrow X_i$ and $Y \rightarrow X_{i+1}$ are isomorphisms over the generic points of all LC centres. We note that $X_i \dashrightarrow X_{i+1}$ is an isomorphism at the generic point of every LC centres. Apply the negativity Lemma to the flipping diagram $X_i \rightarrow Z_i \leftarrow X_{i+1}$ and compare discrepancies. Then, by restricting to S_i and S_{i+1} , we obtain the desired inequalities of discrepancies. $\square$

STEP 2. Assume that $\varphi_i: X_i \dashrightarrow X_{i+1}$ induces an isomorphism of log pairs, for every $(d-1)$ -dimensional LC centre for every i . Then, after finitely many flips, φ_i induces an isomorphism of log pairs, for every d -dimensional LC centre.

REMARK 4.2.11. The above statement is slightly weaker than Shokurov's claim (B_d). See the proof of special termination 2.3 in [Sho03].

REMARK 4.2.12. It is obvious that φ_i induces an isomorphism of log pairs for every 0-dimensional LC centre. When $d = 1$, Step 2 is obvious by Lemmas 4.2.8 and 4.2.10.

So we can assume that $d \geq 2$.

REMARK 4.2.13. Let (S_i, B_{S_i}) be a LC centre. Assume that $\varphi_i: (S_i, B_{S_i}) \rightarrow (S_{i+1}, B_{S_{i+1}})$ is an isomorphism of log pairs. Then S_i contains no flipping curves and S_{i+1} contains no flipped curves. This is obvious by applying the negativity Lemma to $S_i \rightarrow T_i \leftarrow S_{i+1}$, where T_i is the normalization of $\phi_i(S_i)$.

PROPOSITION 4.2.14. *The inequality $d_{\mathbf{b}}(S_i, B_{S_i}) \geq d_{\mathbf{b}}(S_{i+1}, B_{S_{i+1}})$ holds. Moreover, if $S_i \rightarrow T_i \leftarrow S_i^+ = S_{i+1}$ is type (SD) or (DD), then $d_{\mathbf{b}}(S_i, B_{S_i}) > d_{\mathbf{b}}(S_{i+1}, B_{S_{i+1}})$, where T_i is the normalization of $\phi_i(S_i)$. Note that there exists a $\phi_i^+|_{S_{i+1}}$ -exceptional divisor E on S_{i+1} . By adjunction and the negativity Lemma,*

$$a(E, S_i, B_{S_i}) < a(E, S_{i+1}, B_{S_{i+1}}) = -\alpha$$

for some $\alpha \in \mathcal{S}(\mathbf{b})$. Therefore, after finitely many flips, $S_i \rightarrow T_i \leftarrow S_{i+1}$ is type (SS) or (DS).

PROOF. See [FA92, 7.5.3 Lemma, 7.4.3 Lemma]. We note that φ_i is an isomorphism of log pairs on $\lfloor B_{S_i} \rfloor$ by assumption. Therefore,

$$c_{S_i}(E) \subset \lfloor B_{S_i} \rfloor \text{ if and only if } c_{S_{i+1}}(E) \subset \lfloor B_{S_{i+1}} \rfloor.$$

More precisely, if $c_{S_i}(E)$ (resp. $c_{S_{i+1}}(E)$) is contained in $\lfloor B_{S_i} \rfloor$ (resp. $\lfloor B_{S_{i+1}} \rfloor$), then φ_i is an isomorphism at the generic point of $c_{S_i}(E)$ (resp. $c_{S_{i+1}}(E)$) by the negativity lemma. Therefore, we obtain $d_{\mathbf{b}}(S_i, B_{S_i}) \geq d_{\mathbf{b}}(S_{i+1}, B_{S_{i+1}})$, by Lemma 4.2.10. $\square$

So we can assume that every step is type (SS) or (DS) by shifting the index i .

LEMMA 4.2.15. *By shifting the index i , we can assume that $a(E, S_i, B_{S_i}) = a(E, S_{i+1}, B_{S_{i+1}})$ for every i if E is a divisor on both S_i and S_{i+1} .*

PROOF. By Lemma 4.2.10, we have $a(v, S_i, B_{S_i}) \leq a(v, S_{i+1}, B_{S_{i+1}})$ for every valuation v . We note that the coefficient of E is $-a(E, S_i, B_{S_i}) \geq 0$ and that $-a(E, S_i, B_{S_i}) = 1$ or $-a(E, S_i, B_{S_i}) \in \mathcal{S}(\mathbf{b})$. Thus, Lemma 4.2.8 implies that $-a(E, S_i, B_{S_i})$ becomes stationary after finitely many steps. $\square$

Let $f: S_0^0 \rightarrow S_0$ be a $\mathbb{Q}$ -factorial dlt model, that is, $(S_0^0, B_{S_0^0})$ is $\mathbb{Q}$ -factorial and dlt such that $K_{S_0^0} + B_{S_0^0} = f^*(K_{S_0} + B_{S_0})$. Note that we need the log MMP in dimension d to construct a dlt model. Applying the log MMP to $S_0^0 \rightarrow T_0$, we obtain a sequence of divisorial contractions and log flips over T_0

$$S_0^0 \dashrightarrow S_0^1 \dashrightarrow \cdots,$$

and finally a relative log minimal model $S_0^{k_0}$. Since $S_1 \rightarrow T_0$ is the log canonical model of $S_0^0 \rightarrow S_0 \rightarrow T_0$, we have a unique natural morphism $g: S_0^{k_0} \rightarrow S_1$ (see [FA92, 2.22 Theorem]). We note that $K_{S_0^{k_0}} + B_{S_0^{k_0}} = g^*(K_{S_1} + B_{S_1})$. Applying the log MMP to $S_1^0 := S_0^{k_0} \rightarrow S_1 \rightarrow T_1$ over T_1 , we obtain a sequence

$$S_1^0 \dashrightarrow \cdots \dashrightarrow S_1^{k_1} \rightarrow S_2$$

for the same reason, where $S_1^{k_1}$ is a relative log minimal model of $S_1^0 \rightarrow S_1 \rightarrow T_1$. Run the log MMP to $S_2^0 := S_1^{k_1} \rightarrow S_2 \rightarrow T_2$. Repeating this procedure, we obtain a sequence of log flips and divisorial contractions. This sequence terminates by the log MMP in dimension d .

LEMMA 4.2.16. *If $S_i \rightarrow T_i$ or $S_{i+1} \rightarrow T_i$ is not an isomorphism, then S_i^0 is not isomorphic to $S_i^{k_i}$ over T_i .*

PROOF. If $S_i \rightarrow T_i$ is not an isomorphism, then $K_{S_i^0} + B_{S_i^0}$ is not nef over T_i . So, S_i^0 is not isomorphic to $S_i^{k_i}$ over T_i . If $S_i \rightarrow T_i$ is an isomorphism, then $K_{S_i^0} + B_{S_i^0}$ is nef over T_i and $S_i^{k_i} = S_i^0$. In particular, S_{i+1} is isomorphic to $S_i \simeq T_i$. $\square$

Thus we obtain the required results.

REMARK 4.2.17. In Step 2, we obtain no information about flipping curves which are not contained in $\lfloor B_i \rfloor$ but which intersect $\lfloor B_i \rfloor$.

STEP 3. After finitely many flips, we can assume that $\lfloor B_i \rfloor$ contains no flipping curves and no flipped curves by Step 2. If the flipping locus intersects $\lfloor B_i \rfloor$, then there exists a flipping curve C such that $C \cdot \lfloor B_i \rfloor > 0$. Note that X_i is $\mathbb{Q}$ -factorial. Then $\lfloor B_{i+1} \rfloor$ intersects every flipped curve negatively. So $\lfloor B_{i+1} \rfloor$ contains a flipped curve. This is a contradiction.

Therefore, we finished the proof of Theorem 4.2.1. $\square$

REMARK 4.2.18. Our proof heavily relies on the adjunction formula for higher codimensional LC centres of a dlt pair. It is treated in Section 3.9. In the final step (Step 3), $\mathbb{Q}$ -factoriality plays a crucial role. As explained in Chapter 3, $\mathbb{Q}$ -factoriality and the notion of dlt are not analytically local.

REMARK 4.2.19. For recent developments in the termination of 4-fold log flips, see [Fuj04, Fuj05a, Fuj05b].

4.3. Reduction theorem

In this section, we prove the reduction Theorem [Sho03, Reduction Theorem 1.2]. It says that the existence of pl flips and the special termination imply the existence of all log flips. Here is the definition of a (*elementary*) *pre limiting contraction*.

DEFINITION 4.3.1 (Pre limiting contractions). We call $f: (X, D) \rightarrow Z$ a *pre limiting contraction* (*pl contraction*, for short) if

- (1) (X, D) is a dlt pair,
- (2) f is small and $-(K_X + D)$ is f -ample, and
- (3) there exists an irreducible component $S \subset \lfloor D \rfloor$ such that S is f -negative.

Furthermore, if the above f satisfies

- (4) $\rho(X/Z) = 1$, and
- (5) X is $\mathbb{Q}$ -factorial,

then $f: (X, D) \rightarrow Z$ is called an *elementary pre limiting contraction* (*elementary pl contraction*, for short).

CAUTION 4.3.2. *I do not know what is the best definition of a (elementary) pre limiting contraction. Compare Definition 4.3.1 with [Sho03, 1.1] and [FA92, 18.6 Definition]. We adopt the above Definition in this chapter. The reader should check the Definition of pl contractions himself, when he reads other papers.*

The following is the definition of *log flips* in this section, which is much more general than log flips in Conjecture 4.1.1.

DEFINITION 4.3.3 (Log flips). By a *log flip* of f we mean the $(K_X + D)$ -flip of a contraction $f: (X, D) \rightarrow Z$ assuming that

- (a) (X, D) is klt,
- (b) f is small,
- (c) $-(K_X + D)$ is f -nef, and
- (d) D is a $\mathbb{Q}$ -divisor.

A $(K_X + D)$ -flip of f is a *log canonical model* $f^+: (X^+, D^+) \rightarrow Z$ of (X, D) over Z , that is, a diagram

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ & \searrow & \swarrow \\ & Z & \end{array}$$

which satisfies the following conditions:

- (i) X^+ is a normal variety,
- (ii) $f^+: X^+ \rightarrow Z$ is small, projective, and
- (iii) $K_{X^+} + D^+$ is f^+ -ample, where D^+ is the strict transform of D .

Note that if a log canonical model exists then it is unique.

REMARK 4.3.4. For the definitions of *log minimal models* and *log canonical models*, see [KMM87, Definition 3.50]. There, they omit “log” for simplicity. So, a log canonical (resp. log minimal) model is called a *canonical* (resp. *minimal*) model in [KMM87].

Let us introduce the notion of *PL-flips*.

DEFINITION 4.3.5 (PL-flips). A (*elementary*) *pl-flip* is the flip of f , where f is a (elementary) pl contraction as in Definition 4.3.1. Note that if the flip exists then it is unique up to isomorphism over Z .

We will use the next definition in the proof of the reduction theorem.

DEFINITION 4.3.6 (Birational transform). Let $f: X \dashrightarrow Y$ be a birational map. Let $\{E_i\}$ be the set of exceptional divisors of f^{-1} and D an $\mathbb{R}$ -divisor on X . The *birational transform* of D is defined as

$$D_Y := f_*D + \sum E_i.$$

The following is the main theorem of this section. This is essentially the same as [FA92, Chapter 18].

THEOREM 4.3.7 (Reduction Theorem). *Log flips exist in dimension n provided that:*

$$\begin{array}{ll} (PLF)_n^{el} & \text{elementary pl-flips exist in dimension } n, \text{ and} \\ (ST)_n & \text{special termination holds in dimension } n. \end{array}$$

PROOF. Let (X, D) be a klt pair and let $f: X \rightarrow Z$ be a contraction as in Definition 4.3.3. We define $T := f(\text{Exc}(f)) \subset Z$. We may assume that Z is affine without loss of generality.

STEP 1. Let H' be a Cartier divisor on Z such that

- (i) $H := f^*H' = f_*^{-1}H'$ contains $\text{Exc}(f)$.
- (ii) H' is reduced and contains $\text{Sing}(Z)$ and the singular locus of $\text{Supp } f(D)$.
- (iii) Fix a resolution $\pi: Z' \rightarrow Z$. Let $F_j \subset Z'$ be divisors that generate $N^1(Z'/Z)$. We assume that H' contains $\pi(F_j)$ for every j . (This usually implies that H' is reducible.) We note that we can assume that $\text{Supp } \pi(F_j)$ contains no irreducible components of $\text{Supp } f(D)$ for every j without loss of generality. Therefore, we can assume that H and D have no common irreducible components.

The main consequence of the last assumption is the following:

- (iv) Let $h: Y \rightarrow Z$ be any proper birational morphism such that Y is $\mathbb{Q}$ -factorial. Then the irreducible components of the proper transform of H' and the h -exceptional divisors generate $N^1(Y/Z)$.

STEP 2. By Hironaka's desingularization theorem, there is a projective log resolution $h: Y \rightarrow X \rightarrow Z$ for $(X, D + H)$, which is an isomorphism over $Z \setminus H'$.

Then $K_Y + (D + H)_Y$ is a $\mathbb{Q}$ -factorial dlt pair, where $(D + H)_Y$ is the birational transform of $D + H$ (see Definition 4.3.6). Observe that h^*H' contains $h^{-1}(T)$ and h^*H' contains all h -exceptional divisors.

STEP 3. Run the log MMP with respect to $K_Y + (D + H)_Y$ over Z . We successively construct objects $(h_i: Y_i \rightarrow Z, (D + H)_{Y_i})$ such that $\lfloor (D + H)_{Y_i} \rfloor$ contains the support of h_i^*H' , and every flipping curve for h_i is contained in $\text{Supp } h_i^*H'$. If C_i is a flipping curve, then $C_i \subset h_i^*H'$ and $C_i \cdot h_i^*H' = 0$. By Step 1 (iv) and Step 2, there is an irreducible component $F_i \subset h_i^*H'$ such that $C_i \cdot F_i \neq 0$. Thus a suitable irreducible component of h_i^*H' intersects C_i negatively. This means that the only flips that we need are elementary pl-flips. By special termination, we end up with a $\mathbb{Q}$ -factorial dlt pair $\bar{h}: (\bar{Y}, (D + H)_{\bar{Y}}) \rightarrow Z$ such that $K_{\bar{Y}} + (D + H)_{\bar{Y}}$ is $\bar{h}$ -nef.

STEP 4 (cf. [KMM87, Theorem 7.44]). This step is called “subtracting H ”. It is independent of the other steps. So we use different notation throughout Step 4. Of course, we assume $(PLF)_n^{el}$ and $(ST)_n$ throughout this step.

THEOREM 4.3.8 (Subtraction Theorem). *Let $(X, S+B+H)$ be an n -dimensional $\mathbb{Q}$ -factorial dlt pair with effective $\mathbb{Q}$ -divisors S , B , and H such that $[S] = S$, $[B] = 0$. Let $f: X \rightarrow Y$ be a projective birational morphism. Assume the following:*

- (i) $H \equiv_f -\sum b_j S_j$, where $b_j \in \mathbb{Q}_{\geq 0}$, and S_j is an irreducible component of S for every j .
- (ii) $K_X + S + B + H$ is f -nef.

Then $(X, S+B)$ has a log minimal model over Y .

PROOF. We give a proof in the form of several lemmas by running the log MMP over Y guided by H . The notation and the assumptions of Theorem 4.3.8 are assumed in these lemmas. $\square$

LEMMA 4.3.9. *There exists a rational number $\lambda \in [0, 1]$ such that*

- (1) $K_X + S + B + \lambda H$ is f -nef, and
- (2) if $\lambda > 0$, then there exists a $(K_X + S + B)$ -negative extremal ray R over Y such that $R \cdot (K_X + S + B + \lambda H) = 0$.

PROOF. This follows from the Cone Theorem. See, for example, [KMM87, Complement 3.6]. We note that [KMM87, §3.1] assumes that the pair has only klt singularities. However, the Rationality Theorem holds for dlt pairs. Therefore, [KMM87, Complement 3.6] is true for dlt pairs. See [KMM87, Theorem 3.15, Remark 3.16]. $\square$

If $\lambda = 0$, then the Theorem is proved. Therefore, we assume that $\lambda > 0$ and let $\phi: X \rightarrow V$ be the contraction of R .

LEMMA 4.3.10. *If ϕ contracts a divisor E , then conditions (i) and (ii) in Theorem 4.3.8 above, still hold if we replace $f: X \rightarrow Y$ with $V \rightarrow Y$ and B, S, H with $\phi_*B, \phi_*S, \lambda\phi_*H$.*

PROOF. This is obvious. $\square$

LEMMA 4.3.11. *If ϕ is a flipping contraction, then ϕ is an elementary pl contraction (see Definition 4.3.1). If $p: X \dashrightarrow X^+$ is the flip of ϕ , then conditions (i) and (ii) above, still hold if we replace $f: X \rightarrow Y$ with $f^+: X^+ \rightarrow Y$ and B, S, H with $p_*B, p_*S, \lambda p_*H$.*

PROOF. One has to prove that ϕ is an elementary pl contraction. By hypothesis $R \cdot (K_X + S + B + \lambda H) = 0$ and $R \cdot (K_X + S + B) < 0$, thus one sees $R \cdot H > 0$. Hence by condition (i), there exists j_0 such that $R \cdot S_{j_0} < 0$. The latter part is obvious. $\square$

LEMMA 4.3.12. *We can apply the above procedure to the new set up in cases Lemma 4.3.10 and Lemma 4.3.11 if $\lambda \neq 0$. After repeating this finitely many times, λ becomes 0, and one obtain a log minimal model of $(X, S+B)$ over Y . In particular, Theorem 4.3.8 holds.*

PROOF. It is obvious that Lemma 4.3.10 does not occur infinitely many times. The flip in Lemma 4.3.11 is a $(K_X + S + B)$ -flip where the flipping curve is contained in S . Hence there cannot be an infinite sequence of such flips by special termination (see Theorem 4.2.1). The end product is a log minimal model. $\square$

STEP 5. We go back to the original setting. Apply *Theorem 4.3.8* to $\bar{h}: (\bar{Y}, (D + H)_{\bar{Y}}) \rightarrow Z$, which was obtained in *Step 3*. More precisely, we put $f = \bar{h}$, $X = \bar{Y}$, $Y = Z$, $S + B + H = (D + H)_{\bar{Y}}$, $B = \{(D + H)_{\bar{Y}}\}$, and H is the strict transform of H' , and apply *Theorem 4.3.8*. Then we obtain

$$\tilde{h}: (\tilde{Y}, D_{\tilde{Y}}) \rightarrow Z$$

such that $\tilde{Y}$ is $\mathbb{Q}$ -factorial, $K_{\tilde{Y}} + D_{\tilde{Y}}$ is dlt and $\tilde{h}$ -nef. By the negativity lemma ([KMM87, Lemma 3.38]), we can easily check that $\tilde{h}$ is small and $(\tilde{Y}, D_{\tilde{Y}})$ is klt. This is a log minimal model of (X, D) over Z .

STEP 6. By the base point free Theorem over Z , we obtain the log canonical model of the pair (X, D) over Z , which is the required flip.

Therefore, we have finished the proof of the reduction Theorem. $\square$

COROLLARY 4.3.13. *In dimension $n \leq 4$, $(PLF)_n^{el}$ implies the existence of all log flips.*

PROOF. Special termination $(ST)_n$ holds if $n \leq 4$, since the log MMP is true in dimension ≤ 3 . Thus, this corollary is obvious by Theorem 4.3.7. $\square$

4.4. A remark on the log MMP

In this Section, we explain the log MMP for non- $\mathbb{Q}$ -factorial varieties. This result is used in Shokurov's original construction of pl flips [Sho03] but it is not needed in this book. We present the result here because we believe that it is of independent interest.

For simplicity, we treat only klt pairs and $\mathbb{Q}$ -divisors in this Section.

THEOREM 4.4.1 (Log MMP for non- $\mathbb{Q}$ -factorial varieties). *Assume that the log MMP holds for $\mathbb{Q}$ -factorial klt pairs in dimension n . Then the following modified version of the log MMP works for (not necessarily $\mathbb{Q}$ -factorial) klt pairs in dimension n .*

PROOF AND EXPLANATION. Let us start with a projective morphism $f: X \rightarrow Y$, where $X_0 := X$ is a (not necessarily $\mathbb{Q}$ -factorial) normal variety, and a $\mathbb{Q}$ -divisor $D_0 := D$ on X such that (X, D) is klt. The aim is to set up a recursive procedure which creates intermediate morphisms $f_i: X_i \rightarrow Y$ and divisors D_i . After finitely many steps, we obtain a final object $\tilde{f}: \tilde{X} \rightarrow Y$ and $\tilde{D}$. Assume that we have already constructed $f_i: X_i \rightarrow Y$ and D_i with the following properties:

- (i) f_i is projective,
- (ii) D_i is a $\mathbb{Q}$ -divisor on X_i ,
- (iii) (X_i, D_i) is klt.

If $K_{X_i} + D_i$ is f_i -nef, then we set $\tilde{X} := X_i$ and $\tilde{D} := D_i$. Assume that $K_{X_i} + D_i$ is not f_i -nef. Then we can take a $(K_{X_i} + D_i)$ -negative extremal ray R (or, more generally, a $(K_{X_i} + D_i)$ -negative extremal face F) of $\overline{NE}(X_i/Y)$. Thus we have a contraction morphism $\varphi: X_i \rightarrow W_i$ over Y with respect to R (or, more generally,

with respect to F). If $\dim W_i < \dim X_i$ (in which case we call φ a *Fano contraction*), then we set $\tilde{X} := X_i$ and $\tilde{D} := D_i$ and stop the process. If φ is birational, then we put

$$X_{i+1} := \text{Proj}_{W_i} \bigoplus_{m \geq 0} \varphi_* \mathcal{O}_{X_i}(m(K_{X_i} + D_i)),$$

$D_{i+1} :=$ the strict transform of $\varphi_* D_i$ on X_{i+1} and repeat this process. We note that (X_{i+1}, D_{i+1}) is the log canonical model of (X_i, D_i) over W_i and that the existence of log canonical models follows from the log MMP for $\mathbb{Q}$ -factorial klt n -folds. If $K_{W_i} + \varphi_* D_i$ is $\mathbb{Q}$ -Cartier, then $X_{i+1} \simeq W_i$. So, this process coincides with the usual one if the varieties X_i are $\mathbb{Q}$ -factorial. It is not difficult to see that $X_i \rightarrow W_i \leftarrow X_{i+1}$ is of type (DS) or (SS) (for the definitions of (DS) and (SS) , see Definition 4.2.6). So, this process always terminates by the same arguments as in Step 2 of the proof of Theorem 4.2.1 in Section 4.2. $\square$

We give one Example of 3-dimensional non- $\mathbb{Q}$ -factorial terminal flips. The reader can find various examples of non- $\mathbb{Q}$ -factorial contractions in [Fuj, Section 4].

EXAMPLE 4.4.2 (3-dimensional non- $\mathbb{Q}$ -factorial terminal flip). Let e_1, e_2, e_3 form the usual basis of $\mathbb{Z}^3$, and let e_4 be given by

$$e_1 + e_3 = e_2 + e_4,$$

that is, $e_4 = (1, -1, 1)$. We put $e_5 = (a, 1, -r) \in \mathbb{Z}^3$, where $0 < a < r$ and $\gcd(r, a) = 1$. We consider the following fans:

$$\begin{aligned} \Delta_X &= \{\langle e_1, e_2, e_3, e_4 \rangle, \langle e_1, e_2, e_5 \rangle, \text{ and their faces} \}, \\ \Delta_W &= \{\langle e_1, e_2, e_3, e_4, e_5 \rangle, \text{ and its faces} \}, \text{ and} \\ \Delta_{X^+} &= \{\langle e_1, e_4, e_5 \rangle, \langle e_2, e_3, e_5 \rangle, \langle e_3, e_4, e_5 \rangle, \text{ and their faces} \}. \end{aligned}$$

We put $X := X(\Delta_X)$, $X^+ := X(\Delta_{X^+})$, and $W := X(\Delta_W)$. Then we have a commutative diagram of toric varieties:

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ & \searrow & \swarrow \\ & W & \end{array}$$

such that

- (i) $\varphi: X \rightarrow W$ and $\varphi^+: X^+ \rightarrow W$ are small projective toric morphisms,
- (ii) $\rho(X/W) = 1$ and $\rho(X^+/W) = 2$,
- (iii) both X and X^+ have only terminal singularities,
- (iv) $-K_X$ is φ -ample and K_{X^+} is φ^+ -ample, and
- (v) X is not $\mathbb{Q}$ -factorial, but X^+ is $\mathbb{Q}$ -factorial,

Thus, this diagram is a terminal flip. Note that the ampleness of $-K_X$ (resp. K_{X^+}) follows from the convexity (resp. concavity) of the roofs of the maximal cones in Δ_X (resp. Δ_{X^+}). The figure below should help to understand this Example.

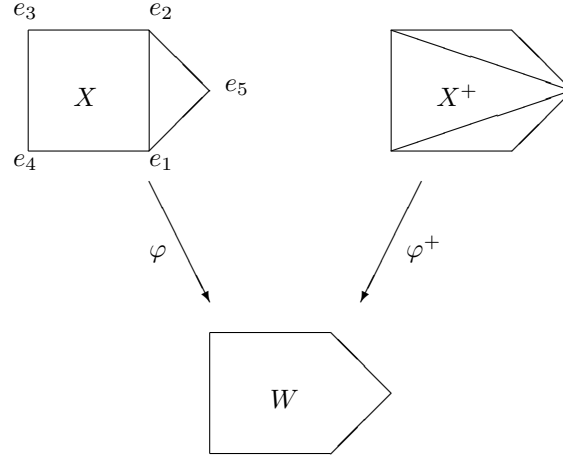


FIGURE 1

One can check the following properties:

- (1) X has one ODP and one quotient singularity,
- (2) the flipping locus is $\mathbb{P}^1$ and it passes through the singular points of X , and
- (3) the flipped locus is $\mathbb{P}^1 \cup \mathbb{P}^1$ and these two $\mathbb{P}^1$ s intersect each other at the singular point of X^+ .

This example implies that the relative Picard number may increase after a flip when X is not $\mathbb{Q}$ -factorial. So, we do not use the Picard number directly to prove the termination of the log MMP.

4.5. Acknowledgements

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CHAPTER 5

Extension theorems and the existence of flips

CHRISTOPHER HACON AND JAMES MCKERNAN

5.1. Introduction

The purpose of this chapter is to give a proof of the following

THEOREM 5.1.1. *Assume termination of flips in dimension $n - 1$. Then flips exist in dimension n .*

Since termination of flips holds in dimension 3, it follows that

COROLLARY 5.1.2. *Flips exist in dimension four.*

The chapter is organized as follows:

In the rest of §5.1, we will explain some necessary background and give a sketch of the main arguments in the proof of Theorem 5.1.1. In §5.2 we recall the applications of the MMP required for the proof of Theorem 5.1.1. In §5.3, we give a criterion for certain algebras to be finitely generated. In §5.4, we explain the extension result of [HM06] and its refinements from [HM] and then we give the proof of Theorem 5.1.1.

5.1.1. The conjectures of the MMP. Throughout this section we consider a n -dimensional klt pair (X, Δ) where X is $\mathbb{Q}$ -factorial and Δ is a $\mathbb{Q}$ -divisor. Of course one could also consider log canonical or divisorial log terminal pairs and $\mathbb{R}$ -divisors instead of $\mathbb{Q}$ -divisors. The main conjecture of the MMP is the following:

CONJECTURE 5.1.3 (Minimal model conjecture). *Let (X, Δ) be a klt $\mathbb{Q}$ -factorial pair of dimension n , where Δ is a $\mathbb{Q}$ -divisor, and let $f: X \rightarrow S$ be a projective morphism. Then there exists a klt pair (X', Δ') and a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X' \\ & \searrow f & \swarrow f' \\ & S & \end{array},$$

such that f' is projective, ϕ^{-1} has no exceptional divisors, $\Delta' = \phi_* \Delta$, $a(E, K_X + \Delta) \leq a(E, K_{X'} + \Delta')$ for every ϕ -exceptional divisor E and either:

- (1) $K_{X'} + \Delta'$ is f' -nef, or
- (2) there is a non-birational contraction of relative Picard number 1, $g: X' \rightarrow S'$ over S such that $-(K_{X'} + \Delta')$ is g -ample.

It is expected that the pair (X', Δ') may be constructed via a finite number of well understood intermediate steps. If $K_X + \Delta$ is f -nef, we are in case 1. Otherwise,

by the Cone Theorem, there is a contraction morphism $g: X \rightarrow Z$, over S , of relative Picard number one such that $-(K_X + \Delta)$ is g -ample. If $\dim X > \dim Z$, we have case 2. Therefore we may assume that g is birational. If g contracts a divisor in X , then we say that g is a *divisorial contraction* and we may replace (X, Δ) by $(Z, g_*\Delta)$. Notice that with each divisorial contraction the rank of the Picard group drops by 1 and hence one can perform this procedure only finitely many times. The remaining case is when g is *small*, i.e. when the exceptional set has codimension at least 2 in X . In this case, Z is no longer $\mathbb{Q}$ -factorial, so we can not replace X by Z . Instead, we try to construct the flip:

DEFINITION 5.1.4. Let $g: X \rightarrow Z$ be a small projective morphism of normal varieties of relative Picard number 1. If D is any $\mathbb{Q}$ -divisor such that $-D$ is g -ample, then the *flip* of g (if it exists) is a small projective birational morphism $g^+: X^+ \rightarrow Z$ such that X^+ is normal and $D^+ = (g^+)^{-1}_* g_*(D)$ is g^+ -ample.

If the flip of g exists, it is unique and it is given by

$$X^+ = \text{Proj}_Z \mathfrak{R}$$

where

$$\mathfrak{R} = R(X, D) = \bigoplus_{n \in \mathbb{N}} g_* \mathcal{O}_X(nD).$$

It is too much to expect that the flip exists for an arbitrary choice of D , however we have the following:

CONJECTURE 5.1.5 (Existence of Flips). *Let (X, Δ) be a klt $\mathbb{Q}$ -factorial pair of dimension n , where Δ is a $\mathbb{Q}$ -divisor. Let $g: X \rightarrow Z$ be a flipping contraction, so that $-(K_X + \Delta)$ is relatively ample, and g is a small contraction of relative Picard number one.*

Then the flip $g^+: X^+ \rightarrow Z$ of g exists.

We remark that (X^+, Δ^+) is also klt and X^+ is $\mathbb{Q}$ -factorial.

Assuming Conjecture 5.1.5, we then replace (X, Δ) by (X^+, Δ^+) . This “improves” the situation by replacing $K_X + \Delta$ -negative curves by $K_{X^+} + \Delta^+$ -positive curves. For this procedure to eventually lead us to a minimal model, i.e. to a solution of Conjecture 5.1.3, we need the following:

CONJECTURE 5.1.6 (Termination of Flips). *Let (X, Δ) be a klt $\mathbb{Q}$ -factorial pair of dimension n , where Δ is a $\mathbb{Q}$ -divisor.*

There is no infinite sequence of $(K_X + \Delta)$ -flips.

For us the statement that the MMP holds in dimension n means that we are assuming Conjectures 5.1.5 _{n} and 5.1.6 _{n} . It is clear that these imply Conjecture 5.1.3 _{n} , but the converse seems to be unknown.

For completeness we also mention the following closely related conjecture, which together with Conjectures 5.1.5 and 5.1.6 is considered one of the most important conjectures of the minimal model program.

CONJECTURE 5.1.7 (Abundance). *Let (X, Δ) be a klt pair, where X is $\mathbb{Q}$ -factorial and Δ is a $\mathbb{Q}$ -divisor, and let $f: X \rightarrow S$ be a projective morphism, where S is affine and normal.*

If $K_X + \Delta$ is nef, then it is semiample.

5.1.2. Previous results. In dimension less than or equal to two, the situation is well understood, Conjectures 5.1.5 and 5.1.6 are trivially true as there are no flips.

In higher dimensions, the situation is as follows: 3-fold flips for terminal X and $\Delta = 0$ were constructed by Mori in his famous paper [Mor88]. Conjecture 5.1.5₃ was then proved by Shokurov and Kollár [Sho92, FA92]. Conjecture 5.1.5₄ was established much more recently by Shokurov [Sho03].

Conjecture 5.1.6₃ was proved by Kawamata [Kaw92]. The statement of Conjecture 5.1.6₃ for $\mathbb{R}$ -divisors is due to Shokurov [Sho96].

Shokurov has also established a framework for proving Conjectures 5.1.5 and 5.1.6 inductively. Conjecture 5.1.5 _{n} is reduced to the existence of pl-flips and hence to a question about the finite generation of a certain algebra on a special kind of $(n-1)$ -dimensional variety. He also shows that Conjecture 5.1.6 _{n} follows from two conjectures on the behavior of the log discrepancy of pairs (X, Δ) of dimension n (namely acc for the set of log discrepancies, whenever the coefficients of Δ are confined to belong to a set of real numbers which satisfies dcc, and semicontinuity of the log discrepancy). Finally, Birkar, in a very recent preprint, [Bir], has reduced Conjecture 5.1.6 _{n} , in the case when $K_X + \Delta$ has non-negative Kodaira dimension, to acc for the log canonical thresholds and the existence of the MMP in dimension $n-1$.

5.1.3. Sketch of the proof. Our proof of Theorem 5.1.1 follows the general strategy of [Sho03]. The first key step was already established in [Sho92], see also [FA92] and Chapter 4. In fact it suffices to prove the existence of pl-flips:

DEFINITION 5.1.8. We call a morphism $f: X \rightarrow Z$ of normal varieties, where Z is affine, a *pl flipping contraction* if

- (1) f is a small birational contraction of relative Picard number one,
- (2) X is $\mathbb{Q}$ -factorial and Δ is a $\mathbb{Q}$ -divisor,
- (3) $K_X + \Delta$ is purely log terminal, where $S = \lfloor \Delta \rfloor$ is irreducible, and
- (4) $-(K_X + \Delta)$ and $-S$ are ample.

We remark here that this definition is more restrictive than the usual definition of a pl flipping contraction: we have assumed that $\lfloor \Delta \rfloor$ is irreducible (compare to Chapter 4). The point here is that if $\lfloor \Delta \rfloor$ is allowed to be reducible and S is an irreducible f -negative component contained in $\lfloor \Delta \rfloor$, then for some rational number $0 < \epsilon \ll 1$ we let $\Delta' = \Delta - \epsilon(\Delta - S)$ and we have that $f: (X, \Delta') \rightarrow Z$ is a pl flip and $\lfloor \Delta' \rfloor$ is irreducible.

Shokurov has proved the following [Sho92, FA92], Chapter 4:

THEOREM 5.1.9. *To prove Theorem 5.1.1 it suffices to construct the flip of a pl flipping contraction.*

The main advantage is that this allows us to restrict to S , and then to proceed by induction on the dimension. If $\Delta = S + B$, then by adjunction we may write

$$(K_X + S + B)|_S = K_S + B',$$

where B' is effective and $K_S + B'$ is klt. Recall that we must show that the ring $\mathfrak{R} = R(X, K_X + S + B)$ is finitely generated. Therefore, we consider the *restricted algebra* $\mathfrak{R}_S$ given by the image of the restriction map

$$\bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}_X(m(K_X + S + B))) \rightarrow \bigoplus_{m \in \mathbb{N}} H^0(S, \mathcal{O}_S(m(K_S + B'))).$$

Shokurov has shown:

THEOREM 5.1.10. *$\mathfrak{R}$ is finitely generated if and only if $\mathfrak{R}_S$ is finitely generated.*

If $\mathfrak{R}_S = \bigoplus_{m \in \mathbb{N}} H^0(S, \mathcal{O}_S(m(K_S + B')))$, then Theorem 5.1.1 easily follows by induction. This is unfortunately too much to expect. Shokurov shows that the algebra $\mathfrak{R}_S$ satisfies two key properties, namely boundedness and a-saturation. He conjectures that any such algebra on a variety admitting a weak Fano contraction is finitely generated. He proves this conjecture in dimension 2 and this gives a very clear and conceptually satisfying proof of Conjecture 5.1.5₃, see Chapter 2. In higher dimension, this strategy seems very hard to implement. Instead we show that the restricted algebra satisfies a much stronger property which we explain below.

The problem is birational in nature so that we may replace X by a model Y and S by its proper transform T and then prove that the corresponding restricted algebra $\mathfrak{R}_T$ is finitely generated. So we consider $\mu : Y \rightarrow X$ an appropriate log resolution of (X, Δ) and we write $K_Y + \Gamma = \mu^*(K_X + \Delta) + E$ where $\mu_*\Gamma = \Delta$ and E is exceptional. Assume that $k(K_X + \Delta)$ is Cartier, then $H^0(X, \mathcal{O}_X(mk(K_X + \Delta))) \cong H^0(Y, \mathcal{O}_Y(mk(K_Y + \Gamma)))$. If we let $G_m = (1/mk)\text{Fix}(mk(K_Y + \Gamma)) \wedge \Gamma$ be the biggest divisor contained both in $\text{Fix}(mk(K_Y + \Gamma))$ and Γ , then $H^0(Y, \mathcal{O}_Y(mk(K_Y + \Gamma))) \cong H^0(Y, \mathcal{O}_Y(mk(K_Y + \Gamma - G_m)))$. Showing that $\mathfrak{R}_T$ is finitely generated is then equivalent to showing that $\bigoplus_{m \in \mathbb{N}} H^0(T, \mathcal{O}_T(mk(K_T + \Theta_m)))$ is finitely generated, where $\Theta_m = (\Gamma - T - G_m)|_T$. It turns out that we may choose T not depending on m and Y depending on m so that the homomorphism

$$H^0(Y, \mathcal{O}_Y(mk(K_Y + \Gamma - G_m))) \rightarrow H^0(T, \mathcal{O}_T(mk(K_T + \Theta_m)))$$

is surjective. If we could show that $\Theta_m = \Theta$ does not depend on m , then finite generation of $\mathfrak{R}_T$ would be equivalent to finite generation of $R(T, K_T + \Theta)$ which easily follows as we are assuming the MMP in dimension $n - 1 = \dim T$. Unluckily, this does not follow directly from the construction, however we are able to deduce that the restricted algebra $\mathfrak{R}_T$ satisfies several important properties that we now explain:

After an appropriate choice of an integer $k > 0$ and of a model T of S , we can assume that

$$(\mathfrak{R}_T)_{(k)} = \bigoplus_{m \in \mathbb{N}} H^0(T, \mathcal{O}_T(mk(K_T + \Theta_m)))$$

is an *adjoint algebra*. This means that

- (1) $\Theta_\bullet$ is a convex sequence of effective divisors that is

$$i\Theta_i + j\Theta_j \leq (i + j)\Theta_{i+j}$$

which admits a limit Θ .

- (2) (T, Θ) is klt. Notice that since we are taking a limit, we must consider a divisor Θ with real coefficients.
 (3) If $M_m := \text{Mob}(mk(K_T + \Theta_m))$ and $D_m = M_m/m$, then

$$D = \lim D_m \in \text{Div}_{\mathbb{R}}(T)$$

is semiample.

- (4) $D_\bullet$ is saturated, that is there exists a $\mathbb{Q}$ -divisor F with $\lceil F \rceil \geq 0$ such that

$$\text{Mob}[\lceil jD_i + F \rceil] \leq jD_j \quad \forall i \geq j \gg 0.$$

It turns out that using ideas of Shokurov, it is easy to show that any adjoint algebra is finitely generated.

The main idea to show that $\mathfrak{R}_T$ is an adjoint algebra, is to use the extension result proved in [HM06] and refined in [HM]. These extension results rely on the techniques of Siu [Siu98] and Kawamata [Kaw99]. For the convenience of the reader, we include a self contained treatment in section 5.4. The statement of this extension result is quite technical, and one of the main difficulties is to show that we may find birational modifications Y and T of X and S that satisfy all of the necessary hypothesis.

The key property that must be satisfied is that the base locus of $mk(K_Y + \Gamma)$ should not contain any intersection of components of $\text{Supp}(\Gamma)$. It is easy to see that the base locus of $mk(K_Y + \Gamma)$ does not contain T . We may also assume that Γ is a divisor with simple normal crossings support such that the components of $\Gamma - T$ are disjoint. Therefore, we must simply ensure that the base locus of $mk(K_Y + \Gamma)$ does not contain any component of $\Gamma - T$ nor any component of $(\Gamma - T) \cap T$. By canceling common components of $\Gamma - T$ and $(1/mk)\text{Fix}(mk(K_Y + \Gamma))$ we obtain a $\mathbb{Q}$ -divisor Γ_m such that $mk\Gamma_m$ is integral and the base locus of $mk(K_Y + \Gamma_m)$ does not contain any components of $\Gamma - T$. If the base locus of $mk(K_Y + \Gamma_m)$ contains a component C of $(\Gamma - T) \cap T$, then we replace Y by its blow up along a C . Then the corresponding exceptional divisor is contained in both Γ and the the base locus of $mk(K_Y + \Gamma_m)$. Therefore, we may cancel an appropriate multiple of this divisor from Γ as above. Repeating this procedure finitely many times we have that the base locus of $mk(K_Y + \Gamma_m)$ does not contain any intersection of components of $\text{Supp}(\Gamma_m)$. Notice that this procedure produces a variety $Y = Y_m$ depending on m , but it does not change the divisor T .

We then define the $\mathbb{Q}$ -divisors $\Theta_m = (\Gamma_m - T)|_T$ and we must check that $\bigoplus_{m \in \mathbb{N}} H^0(T, \mathcal{O}_T(mk(K_T + \Theta_m)))$ is an adjoint algebra. Properties 1 and 2 follow easily from the construction. Property 3, is also relatively straightforward since as a consequence of the MMP in dimension $n - 1 = \dim T$, we may assume that (after passing to a subsequence), the pairs (T, Θ_m) have a common minimal model (cf. Section 5.2). Property 4, is more delicate. Roughly speaking, we would like to deduce this property, by comparing the linear series $|mk(K_T + \Theta_m)|$ and $|mk(K_Y + \Gamma_m)|$. Since $\text{Mob}|mk(K_Y + \Gamma_m)| = \text{Mob}|mk\mu^*(K_X + \Delta)|$, it is easy to see that a similar saturation property holds for $\text{Mob}|mk(K_Y + \Gamma_m)|$ provided that F is μ -exceptional. We would like to use Kawamata-Viehweg vanishing to compare these linear series on Y and on T . Unluckily, we may not assume that $\text{Mob}(mk(K_T + \Theta_m))$ is base point free, however there exists a fixed integer $s > 0$ such that $\text{Mob}(msk(K_T + \Theta_m))$ is base point free for all $m > 0$. We may then also assume that (for an appropriate choice of $Y = Y_m$) $Q_m = \text{Mob}(msk(K_{Y_m} + \Gamma_m))$ is free. It is at this point that we make use of the fact that $-(K_X + \Delta)$ is ample so that by Kawamata-Viehweg vanishing, the homomorphism

$$H^0(Y, \mathcal{O}_Y(K_Y + T + \lceil (j/i)Q_i - \mu^*(K_X + \Delta) \rceil)) \rightarrow H^0(T, \mathcal{O}_T(\lceil (j/i)Q_i|_T + F|_T))$$

is surjective where $F = K_Y + T - \mu^*(K_X + \Delta)$ and $F|_T = K_T - \mu^*(K_S + B')$. Therefore, one sees that

$$(\text{Mob}(\lceil (j/i)Q_i + F \rceil))|_T \geq \text{Mob}(\lceil (j/i)Q_i|_T + F|_T).$$

It is not hard to see that since F is exceptional, then $\text{Mob}(jsk(K_Y + \Gamma))|_T \geq (\text{Mob}(\lceil (j/i)Q_i + F \rceil))|_T$ and that for an appropriate choice of Y , we have $M_{js} \geq$

$\text{Mob}(j sk(K_Y + \Gamma))|_T$. It also follows easily that if s divides i , then $\text{Mob}(\lceil (j/i)Q_i \rceil_T + F|_T) \geq \text{Mob}(\lceil (js/i)M_i + F|_T \rceil)$. Therefore, condition 4 also holds (for all $i, j > 0$ divisible by s).

5.1.4. Notation and conventions. We work over the field of complex numbers $\mathbb{C}$. A $\mathbb{Q}$ -Cartier divisor D on a normal variety X is *nef* if $D \cdot C \geq 0$ for any curve $C \subset X$. We say that two $\mathbb{Q}$ -divisors D_1, D_2 are *$\mathbb{Q}$ -linearly equivalent* ($D_1 \sim_{\mathbb{Q}} D_2$) if there exists an integer $m > 0$ such that mD_i are linearly equivalent. We say that a $\mathbb{Q}$ -Weil divisor D is *big* if we may find an ample divisor A and an effective divisor B , such that $D \sim_{\mathbb{Q}} A + B$.

A *log pair* (X, Δ) is a normal variety X and an effective $\mathbb{Q}$ -Weil divisor Δ such that $K_X + \Delta$ is $\mathbb{Q}$ -Cartier. We say that a log pair (X, Δ) is *log smooth*, if X is smooth and the support of Δ is a divisor with simple normal crossings. A projective morphism $g: Y \rightarrow X$ is a *log resolution* of the pair (X, Δ) if Y is smooth and $g^{-1}(\Delta) \cup \{\text{exceptional set of } g\}$ is a divisor with normal crossings support. We write $g^*(K_X + \Delta) = K_Y + \Gamma$ and $\Gamma = \sum a_i \Gamma_i$ where Γ_i are distinct reduced irreducible divisors. The *log discrepancy* of Γ_i is $1 - a_i$ where $a_i = -a(\Gamma_i, K_X + \Delta)$. The *locus of log canonical singularities of the pair* (X, Δ) , denoted $\text{nklt}(X, \Delta)$, is equal to the image of those components of Γ of coefficient at least one (equivalently log discrepancy at most zero). The pair (X, Δ) is *klt (kawamata log terminal)* if for every (equivalently for one) log resolution $g: Y \rightarrow X$ as above, the coefficients of Γ are strictly less than one, that is $a_i < 1$ for all i . Equivalently, the pair (X, Δ) is klt if the locus of log canonical singularities is empty. We say that the pair (X, Δ) is *purely log terminal* if the log discrepancy of any exceptional divisor is greater than zero. A *log canonical place* for the pair (X, Δ) is an exceptional divisor of log discrepancy at most zero and a *log canonical centre* is the image of a log canonical place.

We will also write

$$K_Y + \Gamma = g^*(K_X + \Delta) + E,$$

where Γ and E are effective, with no common components, $g_*\Gamma = \Delta$ and E is g -exceptional. Note that this decomposition is unique.

Note that the group of Weil divisors with rational or real coefficients forms a vector space, with a canonical basis given by the prime divisors. If A and B are two $\mathbb{R}$ -divisors, then we let $[A, B]$ denote the line segment

$$\{\lambda A + \mu B \mid \lambda + \mu = 1, \lambda \geq 0, \mu > 0\}.$$

Given an $\mathbb{R}$ -divisor, $\|D\|$ denotes the sup norm with respect to this basis. We say that D' is *sufficiently close to* D if there is a finite dimensional vector space V such that D and $D' \in V$ and D' belongs to a sufficiently small ball of radius $\delta > 0$ about D ,

$$\|D - D'\| < \delta.$$

We recall some definitions involving divisors with real coefficients:

DEFINITION 5.1.11. Let X be a variety.

- (1) An *$\mathbb{R}$ -Weil divisor* D is an $\mathbb{R}$ -linear combination of prime divisors.
- (2) Two $\mathbb{R}$ -divisors D and D' are *$\mathbb{R}$ -linearly equivalent* if their difference is an $\mathbb{R}$ -linear combination of principal divisors.
- (3) An *$\mathbb{R}$ -Cartier divisor* D is an $\mathbb{R}$ -linear combination of Cartier divisors.

- (4) An $\mathbb{R}$ -Cartier divisor D is *ample* if it is $\mathbb{R}$ -linearly equivalent to $\sum a_i A_i$ with A_i ample Cartier divisors, $a_i \in \mathbb{R}_{\geq 0}$ and some $a_i > 0$.
- (5) An $\mathbb{R}$ -divisor D is *effective* if it is a positive real linear combination of prime divisors.
- (6) An $\mathbb{R}$ -Cartier divisor D is *big* if it is $\mathbb{R}$ -linearly equivalent to the sum of an ample divisor and an effective divisor.
- (7) An $\mathbb{R}$ -Cartier divisor D is *semiample* if there is a contraction $\pi: X \rightarrow Y$ such that D is linearly equivalent to the pullback of an ample divisor.

Note that we may pullback $\mathbb{R}$ -Cartier divisors, so that we may define the various flavors of log terminal and log canonical in the obvious way.

DEFINITION 5.1.12. Let B be an integral divisor on X such that $|B| \neq \emptyset$. Let $F = \text{Fix} B$ be the *fixed part* of the linear system $|B|$, and set $M = B - F$. We may write

$$|B| = |M| + F.$$

We call $M = \text{Mob } B$ the *mobile part* of B , and we call $B = M + F$ the *decomposition* of B into its mobile and fixed part. We say that a divisor is *mobile* if the fixed part is empty.

We will need the following lemmas:

LEMMA 5.1.13. Let $T \subset Y$ be a smooth divisor in a smooth variety, B be an integral divisor on Y such that $T \not\subset \text{Supp}(B)$, $|B|_T \neq \emptyset$ and $H^0(Y, \mathcal{O}_Y(B)) \rightarrow H^0(T, \mathcal{O}_T(B))$ is surjective. Then

$$\text{Mob}(B|_T) \leq (\text{Mob}(B))|_T.$$

PROOF. We write $B = M + F$ and $B|_T = M_T + F_T$ for the decompositions in to mobile and fixed parts. Since any divisor in $|B|_T$ is the restriction of a divisor in $|B|$, then $F_T \geq F|_T$ and so

$$(\text{Mob}(B))|_T = M|_T = M_T + F_T - F|_T \geq M_T = \text{Mob}(B|_T).$$

□

LEMMA 5.1.14. Let $f: Y \rightarrow X$ be a birational morphism of smooth varieties, $T \subset X$ a smooth divisor such that $(f^{-1})_* T \rightarrow T$ is an isomorphism and B be an integral divisor on X such that $|B| \neq \emptyset$ and $T \not\subset \text{Supp}(B)$. Then

$$(\text{Mob}(f^*B))|_T \leq (\text{Mob}(B))|_T.$$

PROOF. We write $B = M + F$ and $f^*B = M' + F'$ for the decompositions in to mobile and fixed parts. Since every section of $f^*\mathcal{O}_Y(B)$ is obtained by pulling back a section of $\mathcal{O}_Y(B)$, one has that $F' \geq f^*F$ and so that identifying T and $(f^{-1})_* T$ then

$$\begin{aligned} (\text{Mob}(B))|_T &= M|_T = (f^*M)|_T = (f^*B - f^*F)|_T \geq \\ &= (f^*B - F')|_T = M'|_T = (\text{Mob}(f^*B))|_T. \end{aligned}$$

□

LEMMA 5.1.15. Let $T \subset Y$ be a smooth divisor in a smooth variety, B be an integral divisor on Y such that $T \not\subset \text{Supp}(B)$, $|B|_T \neq \emptyset$, $\text{Fix}(B|_T)$ has simple normal crossings and $H^0(Y, \mathcal{O}_Y(B)) \rightarrow H^0(T, \mathcal{O}_T(B))$ is surjective. Then, there exists a birational map $\mu: Y' \rightarrow Y$ given by a finite sequence of blow ups along the

(strict transforms of the) irreducible components of $\text{Fix}(B|_T)$ such that (identifying T with its strict transform) we have $\text{Mob}(\mu^*B)|_T = \text{Mob}(B|_T)$.

PROOF. If $\text{Fix}(B)|_T = \text{Fix}(B|_T)$ we are done. The components of $\text{Fix}(B|_T) - \text{Fix}(B)|_T$ are contained in the base locus of $\text{Mob } B$ so that if $\mu_1 : Y_1 \rightarrow Y$ is the blow up of Y along a component C of $\text{Fix}(B|_T)$, then the exceptional divisor is contained in the base locus of $\mu_1^* \text{Mob } B$. It follows that (identifying T with its strict transform), we have $\text{Fix}(\mu_1^*B|_T) = \text{Fix}(B|_T)$ so that

$$0 \leq \text{Fix}(\mu_1^*B|_T) - \text{Fix}(\mu_1^*B)|_T \leq \text{Fix}(B|_T) - \text{Fix}(B)|_T - C.$$

It is clear that after repeating this procedure finitely many times, we obtain a birational map $\mu : Y' \rightarrow Y$ such that $\text{Fix}(\mu^*B|_T) - \text{Fix}(\mu^*B)|_T = 0$. $\square$

5.2. The real minimal model program

We begin by recalling the following well known small generalisation of the base point free theorem [KMM87, Theorem 7.1]:

THEOREM 5.2.1 (Base Point Free Theorem). *Let (X, Δ) be a $\mathbb{Q}$ -factorial klt pair, where Δ is an $\mathbb{R}$ -divisor. Let $f : X \rightarrow Z$ be a projective morphism, where Z is affine and normal, and let D be a nef $\mathbb{R}$ -divisor, such that $aD - (K_X + \Delta)$ is nef and big, for some positive real number a . Then D is semiample.*

ASSUMPTION 5.2.2. *Conjectures 5.1.5 and 5.1.6 hold. That is, if (X, Δ) is a klt $\mathbb{Q}$ -factorial pair of dimension n , where Δ is a $\mathbb{Q}$ -divisor, then $K_X + \Delta$ flips exist and there is no infinite sequence of $K_X + \Delta$ flips.*

We will use the following result which cf. 2.1-2.3 of [KMM94].

LEMMA 5.2.3. *Let (X, Δ) be a $\mathbb{Q}$ -factorial pair with klt singularities, $\Delta \in \text{Div}_{\mathbb{Q}}(X)$ and $f : X \rightarrow Z$ a projective morphism to a normal variety. Let W be an effective $\mathbb{Q}$ -divisor such that $K_X + \Delta$ is not relatively nef, but $K_X + \Delta + W$ is relatively nef. Then there is a $(K_X + \Delta)$ extremal ray R over Z and a rational number $0 < \lambda \leq 1$ such that $K_X + \Delta + \lambda W$ is relatively nef but trivial on R .*

LEMMA 5.2.4. *Let X be a $\mathbb{Q}$ -factorial normal variety, $f : X \rightarrow Z$ a projective morphism to a normal variety, $\Delta \in \text{Div}_{\mathbb{Q}}(X)$ an effective divisor, $A \in \text{Div}_{\mathbb{Q}}(X)$ an ample divisor and $\alpha \in \mathbb{R}_{>0}$ such that $(X, \Delta + \alpha A)$ is klt and $\kappa(K_X + \Delta + \alpha A) \geq 0$ for all rational numbers $a > \alpha$. If Assumption 5.2.2 holds, then there exists a rational map $\phi : X \dashrightarrow Y$ over Z given by a finite sequence of $(K_X + \Delta + \alpha A)$ -negative divisorial contractions and flips over Z such that $K_Y + \phi_*(\Delta + \alpha A)$ is relatively nef.*

PROOF. Let $\Delta_t = \Delta + tA$. Fix $t > \alpha$ such that $K_X + \Delta_t$ is nef. If $K_X + \Delta_\alpha$ is not nef, then $K_X + \Delta_{\alpha'}$ is not nef for some rational number $\alpha < \alpha' < t$. Therefore, there is a $(K_X + \Delta_{\alpha'})$ -extremal ray R and a rational number $\alpha' < t' \leq t$ such that $K_X + \Delta_{t'}$ is nef and $(K_X + \Delta_{t'}) \cdot R = 0$. It is easy to see that R is extremal for $K_X + \Delta$ and $K_X + \Delta_\alpha$. Since $K_X + \Delta_{t'}$ is big, R defines a divisorial contraction or a flip $\phi : X \dashrightarrow X'$. We replace t by t' , X by X' , Δ by $\phi_*\Delta$ and A by ϕ_*A . Then A is no longer ample, but $K_X + \Delta_t$ is nef and so we may repeat the above procedure. Since each birational map produced as above, is a step of the $K_X + \Delta$ MMP, it follows that this procedure terminates after a finite number of steps. The lemma now follows easily. $\square$

PROPOSITION 5.2.5. *Let X be a $\mathbb{Q}$ -factorial normal variety, $f: X \rightarrow Z$ be a projective morphism to a normal variety, $\Delta \in \text{Div}_{\mathbb{R}}(X)$ an effective divisor such that (X, Δ) is klt and $K_X + \Delta$ is relatively big. If Assumption 5.2.2 holds, then there exists a rational map $\phi: X \dashrightarrow Y$ over Z given by a finite sequence of $K_X + \Delta$ divisorial contractions over Z and flips over Z such that $K_Y + \phi_*\Delta$ is relatively nef and big.*

PROOF. Since $K_X + \Delta$ is big, we have that $K_X + \Delta \sim_{\mathbb{R}} A + E$ where A is an ample $\mathbb{R}$ -divisor and E is an effective $\mathbb{R}$ -divisor. For $0 < \epsilon \ll 1$, we have that $K_X + \Delta + \epsilon(A + E)$ is klt. We let $\Delta' = \Delta + \epsilon(A + E)$, then the $(K_X + \Delta')$ -MMP is equivalent to the $(K_X + \Delta)$ -MMP.

It is easy to see that we may choose $\Delta'' \in \text{Div}_{\mathbb{Q}}(X)$ with the same support as $\Delta + E$ such that $\|\Delta + \epsilon E - \Delta''\| \ll 1$. In particular, we may assume that $A'' = \Delta + \epsilon E - \Delta'' + \epsilon A$ is ample. We then apply Lemma 5.2.4 to the pair $(X, \Delta'' + A'')$ and the proposition follows. $\square$

THEOREM 5.2.6. *Let (X, Δ) be a $\mathbb{Q}$ -factorial klt pair of dimension n , such that $K_X + \Delta$ is $\mathbb{R}$ -Cartier and big. Let $f: X \rightarrow Z$ be any projective morphism, where Z is normal. Fix a finite dimensional vector subspace V of the space of $\mathbb{R}$ -divisors containing Δ .*

If Assumption 5.2.2 holds in dimension n , then there are finitely many birational maps $\psi_i: X \dashrightarrow W_i$, $1 \leq i \leq l$ over Z , such that for every divisor $\Theta \in V$ sufficiently close to Δ , there is an integer $1 \leq i \leq l$ with the following properties:

- (1) ψ_i is the composition of a sequence of $(K_X + \Theta)$ -negative divisorial contractions and birational maps over Z , which are isomorphisms in codimension two,
- (2) W_i is $\mathbb{Q}$ -factorial, and
- (3) $K_{W_i} + \psi_{i*}\Theta$ is relatively semiample.

Moreover, there is a positive integer s such that

- (4) *if $r(K_X + \Theta)$ is integral then $sr(K_{W_i} + \psi_{i*}\Theta)$ is base point free.*

PROOF. If Θ is sufficiently close to Δ , then $K_X + \Theta$ is big, so that by Theorem 5.2.1, (3) is equivalent to the weaker condition that $K_{W_i} + \psi_{i*}\Theta$ is nef. We let $\phi: X \dashrightarrow Y$ be the rational map over Z defined in Proposition 5.2.5. Then $K_Y + \phi_*\Delta$ is relatively nef and big. It is easy to see that we may replace (X, Δ) by $(Y, \phi_*\Delta)$ and hence we may assume that $K_X + \Delta$ is relatively nef and big.

By Theorem 5.2.1 $K_X + \Delta$ is relatively semiample. Let $\psi: X \rightarrow W$ be the corresponding contraction over Z . Then there is an ample $\mathbb{R}$ -divisor H on W such that $K_X + \Delta = \psi^*H$. Thus, if Θ is sufficiently close to Δ , writing

$$K_X + \Theta = K_X + \Delta + (\Theta - \Delta) = \psi^*H + (\Theta - \Delta),$$

one sees that if $K_X + \Theta$ is relatively nef over W , then it is relatively nef over Z .

So, replacing Z by W , we may assume that f is birational and $K_X + \Delta$ is relatively $\mathbb{R}$ -linearly equivalent to zero. Let B be the closure in V of a ball with radius δ centred at Δ . If δ is sufficiently small, then for every $\Theta \in B$, $K_X + \Theta$ is klt. Pick Θ a point of the boundary of B . Since $K_X + \Delta$ is relatively $\mathbb{R}$ -linearly equivalent to zero, note that for every curve C (contracted by $X \rightarrow Z$),

$$(K_X + \Theta) \cdot C < 0 \quad \text{iff} \quad (K_X + \Theta') \cdot C < 0, \quad \forall \Theta' \in (\Delta, \Theta].$$

In particular every step of the $(K_X + \Theta)$ -MMP over Z is a step of $(K_X + \Theta')$ -MMP over Z , for every $\Theta' \in (\Delta, \Theta]$. Since we are assuming existence and termination of flips, we have a birational map $\psi: X \dashrightarrow W$ over Z , such that $K_W + \psi_*\Theta$ is nef, and it is clear that $K_W + \psi_*\Theta'$ is nef, for every $\Theta' \in (\Delta, \Theta]$.

At this point we want to proceed by induction on the dimension of B . To this end, note that as B is compact and Δ is arbitrary, our result is equivalent to proving that (3) holds in B . By what we just said, this is equivalent to proving that (3) holds on the boundary of B , which is a compact polyhedral cone (since we are working in the sup norm) and we are done by induction on the dimension of B .

We now prove (4). As W_i has rational singularities and it is $\mathbb{Q}$ -factorial, it follows that the group of Weil divisors modulo Cartier divisors is a finite group. Thus there is a fixed positive integer s_i such that if $r(K_X + \Theta)$ is integral, then $s_i r(K_{W_i} + \psi_{i*}\Theta)$ is Cartier. By Kollár's effective base point free theorem, [Kol93], there is then a positive integer M such that $M s_i r(K_{W_i} + \psi_{i*}\Theta)$ is base point free. If we set s to be M times the least common multiple of the s_i , then this is (4). $\square$

The key consequence of Theorem 5.2.6 is:

COROLLARY 5.2.7. *Let (X, Δ) be a klt $\mathbb{Q}$ -factorial pair of dimension n , where $K_X + \Delta$ is a big $\mathbb{R}$ -divisor. Let $f: X \rightarrow Z$ be a projective morphism to a normal affine variety.*

Fix a finite dimensional vector subspace V of the space of $\mathbb{R}$ -divisors containing Δ . If $K_X + \Delta$ is relatively big, and Assumption 5.2.2 holds in dimension n , then there is a smooth model $g: Y \rightarrow X$, a simple normal crossings divisor $\Gamma \subset Y$ and a positive integer s , such that if $\pi: Y \rightarrow Z$ is the composition of f and g then:

- (1) *If the divisor Θ in V is sufficiently close to Δ , and if r is a positive integer such that $r(K_X + \Theta)$ is integral, then the moving part of $g^*(rs(K_X + \Theta))$ is base point free and the union of its fixed part with $g^*\Theta$ is contained in the support of Γ .*
- (2) *If k is a positive integer, $\Theta_\bullet$ is a convex sequence with limit Δ , $mk(K_X + \Theta_m)$ is integral, $P_m = \text{Mob}(m s k g^*(K_X + \Theta_m))$ and $D_m = P_m/m$, then $D = \lim D_m$ exists and is semiample.*

PROOF. Let $\psi_i: X \dashrightarrow W_i$ be the models, whose existence is guaranteed by Theorem 5.2.6, and let $g: Y \rightarrow X$ be any birational morphism which resolves the indeterminacy of ψ_i , for $1 \leq i \leq l$. Let $\phi_i: Y \rightarrow W_i$ be the induced birational morphisms, so that we have commutative diagrams

$$\begin{array}{ccc} & Y & \\ g \swarrow & & \searrow \phi_i \\ X & \dashrightarrow^{\psi_i} & W_i. \end{array}$$

We may assume that Y is a log resolution of $(W_i, (\psi_i)_*\Theta)$ for $1 \leq i \leq l$.

Let Θ be sufficiently close to Δ . Then for some i , $K_{W_i} + \psi_{i*}\Theta$ is semiample. Suppressing the index i , we may write

$$g^*(K_X + \Theta) = \phi^*(K_W + \psi_*\Theta) + E,$$

where E is effective and ϕ -exceptional.

Suppose that $r(K_X + \Theta)$ is integral. Then the moving part of $g^*(rs(K_X + \Theta))$ is equal to the moving part of $\phi^*(rs(K_W + \psi_*\Theta))$, and we can apply Theorem 5.2.6

to conclude that there is a fixed s such that the moving part of $g^*(rs(K_X + \Theta))$ is base point free and (1) follows.

For (2), notice that by passing to a subsequence, we may assume that there is a fixed integer i such that

$$P_m = \phi_i^*(msk(K_{W_i} + (\psi_i)_*\Theta_m))$$

and so that $D = sk\phi_i^*(K_{W_i} + (\psi_i)_*\Theta)$ which is nef and hence semiample. $\square$

5.3. Finite generation

5.3.1. Generalities on Finite generation. In this section we give some of the basic definitions and results concerning finite generation

Let $f: X \rightarrow Z$ be a projective morphism of normal varieties, where Z is affine. Let A be the coordinate ring of Z . Recall Definition 2.3.1 that a *function algebra* on X is a graded subalgebra

$$\mathfrak{R} = \bigoplus_{i \in \mathbb{N}} \mathfrak{R}_i$$

of $k(X)[T]$ such that $\mathfrak{R}_0 = A$ and each $\mathfrak{R}_i$ is a coherent A -module. We will refer to a function algebra simply as a *graded A -algebra*.

DEFINITION 5.3.1. Let $\mathfrak{R}$ be any graded A -algebra. A *truncation* of $\mathfrak{R}$ is any A -algebra of the form

$$\mathfrak{R}_{(d)} = \bigoplus_{m \in \mathbb{N}} \mathfrak{R}_{md},$$

for a positive integer $d > 0$.

Then, it is known from Chapter 2 that:

LEMMA 5.3.2. $\mathfrak{R}$ is finitely generated if and only if there is a positive integer d such that $\mathfrak{R}_{(d)}$ is finitely generated.

DEFINITION 5.3.3. A *divisorial algebra* is a graded A -algebra of the form

$$\bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}_X(mB))$$

(or equivalently of the form $\bigoplus_{m \in \mathbb{N}} f_*\mathcal{O}_X(mB)$) where B is an integral Weil divisor on X .

We have:

LEMMA 5.3.4. Let X be a normal variety and let $\mathfrak{R}$ and $\mathfrak{R}'$ be two divisorial algebras associated to divisors D and D' .

If there exist positive integers a and a' such that $aD \sim a'D'$ then $\mathfrak{R}$ is finitely generated iff $\mathfrak{R}'$ is finitely generated.

PROOF. Clear, since $\mathfrak{R}$ and $\mathfrak{R}'$ have the same truncation. $\square$

We want to restrict a divisorial algebra to a prime divisor S :

LEMMA 5.3.5. Let S be a reduced irreducible divisor on X . If the algebra $\mathfrak{R}$ is finitely generated then so is the restricted algebra $\mathfrak{R}_S$. Conversely, if S is linearly equivalent to a positive rational multiple of B , where B is an effective divisor which does not contain S and the restricted algebra $\mathfrak{R}_S$ is finitely generated, then so is $\mathfrak{R}$.

PROOF. See Section 2.3.1. $\square$

The restricted algebra is not necessarily divisorial. However we will show that if $X \rightarrow Z$ is a pl flipping contraction, then on an appropriate model of X , the corresponding algebra satisfies several key properties (it is in fact an adjoint algebra in the sense of Definition 5.3.10 below).

DEFINITION 5.3.6. We say that a sequence of divisors $B_\bullet$ is *additive* if

$$B_i + B_j \leq B_{i+j},$$

we say that it is *convex* if

$$\frac{i}{i+j}B_i + \frac{j}{i+j}B_j \leq B_{i+j},$$

and we say that it is *bounded* if there is a divisor B such that

$$B_i \leq B.$$

We will also say that a sequence $B_\bullet$ is *eventually convex* if there is an integer m_0 such that

$$\frac{i}{i+j}B_i + \frac{j}{i+j}B_j \leq B_{i+j} \quad \forall i, j \geq m_0.$$

It is clear that any bounded (eventually) convex sequence admits a limit. When it is clear from the context, we will often abuse the above definitions and refer to an eventually convex sequence simply as a convex sequence.

DEFINITION 5.3.7. Let $\mathfrak{R}$ be the graded A -algebra associated to the additive sequence $B_\bullet$. Let

$$B_m = M_m + F_m,$$

be the decomposition of B_m into its mobile and fixed parts. The sequence of divisors $M_\bullet$ is called the *mobile sequence* and the sequence of $\mathbb{Q}$ -divisors $D_\bullet$ given by

$$D_i = \frac{M_i}{i},$$

is called the *characteristic sequence*.

Clearly the mobile sequence is additive and the characteristic sequence is convex. The key point is that finite generation of an algebra only depends on the mobile part in each degree, even up to a birational map:

LEMMA 5.3.8. *Let $\mathfrak{R}$ be the graded A -algebra on X associated to an additive sequence $B_\bullet$. Let $g: Y \rightarrow X$ be any birational morphism and let $\mathfrak{R}'$ be the graded A -algebra on Y associated to an additive sequence $B'_\bullet$.*

*If the mobile part of g^*B_i is equal to the mobile part of B'_i then $\mathfrak{R}$ is finitely generated iff $\mathfrak{R}'$ is finitely generated.*

PROOF. Clear. □

LEMMA 5.3.9. *Let $\mathfrak{R}$ be the graded A -algebra on X associated to an additive sequence $B_\bullet$ and let D be the limit of the characteristic sequence.*

If D is semi-ample and $D = D_k$ for some positive integer k then $\mathfrak{R}$ is finitely generated.

PROOF. Passing to a truncation, we may assume that D is free and $D = D_1$. But then

$$mD = mD_1 = mM_1 \leq M_m = mD_m \leq mD,$$

and so $D = D_m$, for all positive integers m . Let $h: X \rightarrow W$ be the contraction over Z associated to M_1 , so that $M_1 = h^*H$, for some very ample divisor on W . We have $g^*M_m = mg^*M_1 = h^*(mH)$ and so the algebra $\mathfrak{R}$ is nothing more than the coordinate ring of W under the embedding of W in $\mathbb{P}^n$ given by H , which is easily seen to be finitely generated by Serre vanishing. $\square$

5.3.2. Adjoint Algebras. The results in this section are due to Shokurov [Sho03] and are explained in Chapter 2. We restate these results in a convenient form, without the use of $\mathbb{Q}$ -divisors.

DEFINITION 5.3.10. Let Y be a smooth variety, Z an affine variety and $\pi: Y \rightarrow Z$ be a projective morphism. An *adjoint algebra* is an algebra of the form

$$\mathfrak{R} = \bigoplus_{m \in \mathbb{N}} H^0(Y, \mathcal{O}_Y(B_m))$$

where $B_\bullet$ is an additive sequence such that

- (1) there exists an integer $k > 0$ such that $B_m = mk(K_Y + \Delta_m)$ where Δ_m is an effective, bounded and eventually convex sequence of $\mathbb{Q}$ -divisors on Y with limit $\Delta \in \text{Div}_{\mathbb{R}}(Y)$ such that (Y, Δ) is klt,
- (2) let $M_m = \text{Mob}(B_m)$ be the mobile sequence and $D_m = M_m/m$ the characteristic sequence. Then D_m is saturated, that is there exists a $\mathbb{Q}$ -divisor F on Y with $\lceil F \rceil \geq 0$ such that

$$\text{Mob}(\lceil jD_i + F \rceil) \leq jD_j$$

for all $i \geq j \gg 0$,

- (3) $D = \lim D_m$ is semiample.

LEMMA 5.3.11 (Diophantine Approximation). *Let Y be a smooth variety and let $\pi: Y \rightarrow Z$ be a projective morphism, where Z is affine and normal. Let D be a semiample $\mathbb{R}$ -divisor on Y . Let $\epsilon > 0$ be a positive rational number.*

Then there is an integral divisor M and a positive integer m such that

- (1) M is base point free,
- (2) $\|mD - M\| < \epsilon$, and
- (3) If $mD \geq M$ then $mD = M$.

PROOF. Immediate from Lemma 2.4.12. $\square$

THEOREM 5.3.12. *Let Y be a smooth variety and $\pi: Y \rightarrow Z$ a projective morphism, where Z is affine and normal. Let $\mathfrak{R}$ be an adjoint algebra on Y .*

Then $\mathfrak{R}$ is finitely generated.

PROOF. Let $D_i = (1/i)M_i$ and $D = \lim D_i$. If $G = \text{Supp } D$, then there exists a number $\epsilon > 0$ such that $\lceil F - \epsilon G \rceil \geq 0$. By Lemma 5.3.11, there is a positive integer m , an integral divisor M such that $\|mD - M\| \leq \epsilon$ and if $mD \geq M$, then $mD = M$. Since

$$mD + F = M + (mD - M) + F \geq M + F - \epsilon G,$$

we have that

$$\text{Mob}(\lceil mD + F \rceil) \geq M.$$

Since the sequence D_i is saturated, we have that for all $i \geq m \gg 0$,

$$\text{Mob}(\lceil mD_i + F \rceil) \leq mD_m.$$

Taking the limit as i goes to infinity, we have

$$M \leq \text{Mob}(\lceil mD + F \rceil) \leq mD_m \leq mD.$$

It follows that these inequalities are all equalities and so $D = D_m$. The assertion now follows from Lemma 5.3.9. $\square$

5.4. Multiplier ideal sheaves and extension results

5.4.1. Definition and first properties of multiplier ideal sheaves. We begin by recalling the usual definition of multiplier ideal sheaves and some properties that will be needed later on:

DEFINITION 5.4.1. Let Y be a smooth variety and D an effective $\mathbb{Q}$ -divisor and $\mu: W \rightarrow Y$ a log resolution of the pair (Y, D) . The *multiplier ideal sheaf* of D is defined as

$$\mathcal{J}(Y, D) = \mathcal{J}(D) = \mu_* \mathcal{O}_W(K_{W/Y} - \lfloor \mu^* D \rfloor).$$

Note that the pair (Y, D) is klt iff the multiplier ideal sheaf is equal to $\mathcal{O}_Y$. Another key property of multiplier ideal sheaves is that they are independent of the log resolution. Multiplier ideal sheaves have the following basic property [Tak06, Example 2.2 (1)]:

LEMMA 5.4.2. *Let (Y, Δ) be a klt pair, where Y is a smooth variety, and let D be any effective $\mathbb{Q}$ -divisor. Let $\sigma \in H^0(Y, L)$ be any section of a line bundle L , with zero locus $\Sigma \subset Y$.*

If $D - \Sigma \leq \Delta$ then $\sigma \in H^0(Y, L \otimes \mathcal{J}(D))$.

PROOF. Let $\mu: W \rightarrow Y$ be a log resolution of the pair $(Y, D + \Delta)$. As Σ is integral

$$\lfloor \mu^* D \rfloor - \mu^* \Sigma \leq \lfloor \mu^* \Delta \rfloor,$$

and as the pair (Y, Δ) is klt,

$$K_{W/Y} - \lfloor \mu^* \Delta \rfloor \geq 0.$$

Thus

$$\begin{aligned} \mu^* \sigma &\in H^0(W, \mu^* L(-\mu^* \Sigma)) \subset \\ &\subset H^0(W, \mu^* L(-\mu^* \Sigma + K_{W/Y} - \lfloor \mu^* \Delta \rfloor)) \subset \\ &\subset H^0(W, \mu^* L(K_{W/Y} - \lfloor \mu^* D \rfloor)). \end{aligned}$$

Pushing forward via μ , we get

$$\sigma \in H^0(Y, L \otimes \mathcal{J}(D)).$$

$\square$

We also have the following important vanishing result, which is an easy consequence of Kawamata-Viehweg vanishing:

THEOREM 5.4.3. (Nadel Vanishing) *Let Y be a smooth variety and let Δ be an effective divisor. Let $\pi: Y \rightarrow Z$ be any projective morphism and let N be any integral divisor such that $N - \Delta$ is relatively big and nef.*

Then

$$R^i \pi_* (\mathcal{O}_Y(K_Y + N) \otimes \mathcal{J}(\Delta)) = 0, \quad \text{for } i > 0.$$

COROLLARY 5.4.4. *Let Y be a smooth variety and let $\Delta = \sum \delta_i \Delta_i$ be a $\mathbb{Q}$ -divisor with simple normal crossings support and $0 \leq \delta_i \leq 1$. Let $\pi: Y \rightarrow Z$ be any projective morphism and let N be any integral divisor such that $N - \Delta$ is relatively big and nef and its restriction to any intersection $\Delta_{i_0} \cap \dots \cap \Delta_{i_t}$ of components of $\lfloor \Delta \rfloor$ is relatively big.*

Then

$$R^i \pi_* \mathcal{O}_Y(K_Y + N) = 0, \quad \text{for } i > 0.$$

PROOF. We proceed by induction on the number of components of $\lfloor \Delta \rfloor$ and on the dimension of Y . If $\lfloor \Delta \rfloor = 0$, the claim follows from Nadel's vanishing theorem above. Therefore, we may assume that there is an irreducible divisor $S \subset \lfloor \Delta \rfloor$. We consider the short exact sequence

$$0 \rightarrow \mathcal{O}_Y(K_Y + N - S) \rightarrow \mathcal{O}_Y(K_Y + N) \rightarrow \mathcal{O}_S(K_S + (N - S)|_S) \rightarrow 0.$$

By induction, one sees that $R^i \pi_* \mathcal{O}_Y(K_Y + N - S) = 0$ and $R^i \pi_* \mathcal{O}_S(K_S + (N - S)|_S) = 0$ for $i > 0$ and hence $R^i \pi_* \mathcal{O}_Y(K_Y + N) = 0$ as required. $\square$

We now wish to generalise the definition of multiplier ideal sheaves to the setting of log pairs. To this end, we fix some notation:

ASSUMPTION 5.4.5. *We consider a smooth log pair (Y, Δ) where every component of Δ has coefficient one. Given a log resolution $\mu: W \rightarrow Y$, we will write*

$$\mu^*(K_Y + \Delta) = K_W + \Theta - E$$

where Θ and E are effective divisors with no common components and $\mu_ \Theta = \Delta$.*

Note that W is smooth, $\Theta + E$ has simple normal crossings support and that E is μ -exceptional.

DEFINITION 5.4.6. Let L be a line bundle on Y and $V \subset H^0(Y, L)$ be a non-zero vector subspace such that there is a divisor $D \in |V|$ which does not contain any log canonical centre of (Y, Δ) (equivalently D does not contain any subvariety obtained by intersecting components of Γ). Pick a log resolution of $|V|$, i.e. a resolution $\mu: W \rightarrow Y$ such that $\mu^*|V| = |V'| + F$ where $|V'|$ is base point free and $F + \Theta + \text{Exc}(\mu)$ has simple normal crossings support. For any rational number $c > 0$, the *multiplier ideal sheaf* $\mathcal{J}_{\Delta, c \cdot |V|}$ is defined by the following formula

$$\mathcal{J}_{\Delta, c \cdot |V|} = \mu_* \mathcal{O}_W(K_{W/Y} + \Theta - \mu^* \Delta - \lfloor cF \rfloor).$$

REMARK 5.4.7. When $\Delta = 0$, this is the usual definition of multiplier ideal sheaf i.e. $\mathcal{J}_{0, c \cdot |V|} = \mathcal{J}(Y, c \cdot |V|)$.

REMARK 5.4.8. Note that any component of the strict transform of Δ is automatically a component of Θ . By the Resolution Lemma of [Sza94], we may in fact choose a resolution which is a sequence of blow ups along smooth centres not containing any of the log canonical centres of (Y, Δ) , and therefore we may assume in this case that $\Theta = (\mu^{-1})_* \Delta$.

PROPOSITION 5.4.9. *The multiplier ideal sheaf $\mathcal{J}_{\Delta, c \cdot |V|}$ (when defined) does not depend on the choice of the log resolution $\mu: W \rightarrow Y$.*

PROOF. This is an easy consequence of [HM06, Lemma 3.4]. $\square$

LEMMA 5.4.10. *For any divisor $0 \leq \Delta' \leq \Delta$, we have*

$$\mathcal{J}_{\Delta, c \cdot |V|} \subset \mathcal{J}_{\Delta', c \cdot |V|}.$$

In particular $\mathcal{J}_{\Delta, c \cdot |V|} \subset \mathcal{J}(Y, c \cdot |V|)$.

PROOF. Let F be an exceptional divisor extracted by μ . Then the coefficient of F in

$$(K_W + \Theta) - \mu^*(K_Y + \Delta),$$

is either the log discrepancy minus one, whenever the log discrepancy is at least one, or it is zero. In particular the coefficient of F is an increasing function of the log discrepancy of F . It follows easily that letting $K_W + \Theta' - E' = \mu^*(K_Y + \Delta')$ where Θ' and E' are effective and have no common components, then

$$(K_W + \Theta) - \mu^*(K_Y + \Delta) \leq (K_W + \Theta') - \mu^*(K_Y + \Delta'),$$

which in turn gives the inclusion of ideal sheaves. $\square$

LEMMA 5.4.11. *If L is a divisor such that for some $p > 0$, no log canonical centre of (Y, Δ) is contained in the base locus of $|pL|$. Then*

$$\mathcal{J}_{\Delta, \frac{c}{p} \cdot |pL|} \subset \mathcal{J}_{\Delta, \frac{c}{pk} \cdot |pkL|}$$

holds for every integer $k > 0$.

PROOF. This is analogous to [Laz04, Lemma 11.1.1]. $\square$

DEFINITION 5.4.12. If L is a divisor such that for some $m > 0$, no log canonical centre of (Y, Δ) is contained in the base locus of $|mL|$. The *asymptotic multiplier ideal sheaf* $\mathcal{J}_{\Delta, c \cdot ||L||}$ associated to Y , Δ , c and L is then defined as the unique maximal member of the family of ideals $\{\mathcal{J}_{\Delta, \frac{c}{p} \cdot |pL|}\}_{p \in m\mathbb{N}}$ and so

$$\mathcal{J}_{\Delta, c \cdot ||L||} = \mathcal{J}_{\Delta, \frac{c}{p} \cdot |pL|} \quad \text{for all } p \text{ sufficiently divisible.}$$

It is easy to see that $\mathcal{J}_{\Delta, c \cdot ||L||}$ does not depend on m . We now fix some further notation:

- ASSUMPTION 5.4.13. (1) $T \subset Y$ is a smooth divisor in a smooth variety,
 (2) Δ a reduced divisor on Y not containing T such that $(Y, T + \Delta)$ is log smooth,
 (3) given a log resolution $\mu: W \rightarrow Y$, we will now write

$$\mu^*(K_Y + T + \Delta) = K_W + T' + \Theta - E$$

where T' is the strict transform of T , Θ and E are effective, $T' + \Theta$ and E have no common components and $\mu_*\Theta = \Delta$,

- (4) Z is an affine normal variety, and $\pi: Y \rightarrow Z$ is a projective morphism.

Since we are assuming that Z is affine, for any coherent sheaf $\mathcal{F}$ on Y , we may identify $\pi_*\mathcal{F}$ with $H^0(Y, \mathcal{F})$. We will denote restriction to T via subscripts. For example $D_T = D|_T$ and if T is not contained in the base locus of $|L|$, then $|L|_T \subset |L_T|$ denotes the restriction of the linear series $|L|$ to T .

DEFINITION 5.4.14. If L is a divisor, such that for some $m > 0$, no log canonical centre of $(Y, T + \Delta)$ is contained in the base locus of $|mL|$. The *asymptotic multiplier ideal sheaf*

$$\mathcal{J}_{\Delta_T, c \cdot \|L\|_T} \subset \mathcal{O}_T$$

associated to Δ , c and L is then defined as the unique maximal member of the family of ideals $\{\mathcal{J}_{\Delta_T, \frac{c}{p} \cdot \|pL\|_T}\}_{p \in \mathbb{N}}$.

LEMMA 5.4.15. *If L is a divisor such that for some $m > 0$, no log canonical centre of $(Y, T + \Delta)$ is contained in the base locus of $|mL|$, then*

$$\mathcal{J}_{\Delta_T, \|L\|_T} \subset \mathcal{J}_{\Delta_T, \|L_T\|}.$$

PROOF. This follows easily from the inclusion of linear series $|pL|_T \subset |pL_T|$. $\square$

LEMMA 5.4.16. *Let L be a divisor such that for some $m > 0$, no log canonical centre of $(Y, T + \Delta)$ is contained in the base locus of $|mL|$. Then*

- (1) $\mathcal{J}_{\Delta, c_1 \cdot \|L\|} \subset \mathcal{J}_{\Delta, c_2 \cdot \|L\|}$, for any rational numbers $c_1 \geq c_2$,
- (2) $\mathcal{J}_{\Delta, c \cdot \|L\|} \subset \mathcal{J}_{\Delta, c \cdot \|L+H\|}$, for any semiample divisor H ,
- (3) $\text{Im}(\pi_* \mathcal{O}_Y(L) \rightarrow \pi_* \mathcal{O}_T(L)) \subset \pi_* \mathcal{J}_{\Delta_T, \|L\|_T}(L_T)$.

PROOF. (1) and (2) are clear. We may assume that $\mathcal{J}_{\Delta_T, \|L\|_T} = \mathcal{J}_{\Delta_T, \frac{1}{k} \|kL\|_T}$ and that $\mu: W \rightarrow Y$ is chosen so that $\mu^*|kL| = |V_k| + F_k$ and $\mu^*|L| = |V| + F$ where $F + F_k + (\mu^{-1})_*T + \text{Exc}(\mu)$ has simple normal crossing support and $|V|, |V_k|$ are base point free. Since $kF \geq F_k$ one sees that there are inclusions

$$\mathcal{O}_W(\mu^*L - F) \subset \mathcal{O}_W(\mu^*L - \lfloor F_k/k \rfloor) \subset$$

$$\mathcal{O}_W(\mu^*L - \lfloor F_k/k \rfloor + E) \subset \mathcal{O}_W(\mu^*L + E).$$

We now push forward via $\pi \circ \mu$. Since $|V| = \mu^*|L| - F$ and E is effective and μ -exceptional, both the left and right hand sides push forward to $\pi_* \mathcal{O}_Y(L)$. So the image of $\pi_* \mathcal{O}_Y(L)$ is equal to the image of

$$(\pi \circ \mu)_* \mathcal{O}_W(\mu^*L + E - \lfloor F_k/k \rfloor).$$

Since $E_{T'} = K_{T'}/T + \Theta_{T'} - \mu^* \Delta_T$, one sees that the image of $\pi_* \mathcal{O}_Y(L)$ is contained in

$$(\pi \circ \mu)_* \mathcal{O}_{T'}(\mu^*L_T + E_{T'} - \lfloor F_k/k \rfloor) = \pi_* \mathcal{J}_{\Delta_T, \|L\|_T}(L_T).$$

$\square$

LEMMA 5.4.17. *Suppose that L is a divisor on Y such that*

- (1) $L \equiv A + B$ where A is an ample $\mathbb{Q}$ -divisor, and B is an effective $\mathbb{Q}$ -divisor,
- (2) there exists an integer $m > 0$ such that the base locus of $|mB|$ contains no log canonical centres of $(Y, T + \Delta)$.

Then

$$\begin{aligned} \pi_* \mathcal{J}_{\Delta_T, \|L\|_T}(K_T + \Delta_T + L_T) &\subset \\ &\subset \text{Im}(\pi_* \mathcal{O}_Y(K_Y + T + \Delta + L) \rightarrow \pi_* \mathcal{O}_T(K_T + \Delta_T + L_T)). \end{aligned}$$

PROOF. For any k sufficiently big and divisible we consider $\mu: W \rightarrow Y$ a log resolution of $|kL|$ so that in particular $\mu^*[kL] = |V_k| + F_k$ where $|V_k|$ is base point free. We write $\mu^*(K_Y + \Delta + T) = K_W + T' + \Theta - E$ where $T' := (\mu^{-1})_*T$. We have

$$K_{T'} + \Theta_{T'} - E_{T'} = \mu^*(K_T + \Delta_T)$$

where $\Theta_{T'}, E_{T'}$ are effective with no common components. Therefore

$$\mathcal{J}_{\Delta_T, ||L||_T} = \mu_* \mathcal{O}_{T'}(E_{T'} - \lfloor F_k/k \rfloor).$$

There is a short exact sequence

$$\begin{aligned} 0 \rightarrow \mathcal{O}_W(K_W + \Theta + \mu^*L - \lfloor F_k/k \rfloor) \rightarrow \\ \rightarrow \mathcal{O}_W(K_W + T' + \Theta + \mu^*L - \lfloor F_k/k \rfloor) \rightarrow \\ \rightarrow \mathcal{O}_{T'}(K_{T'} + \Theta_{T'} + (\mu^*L - \lfloor F_k/k \rfloor)_{T'}) \rightarrow 0. \end{aligned}$$

CLAIM 5.4.18. *If k is sufficiently big and divisible, then*

$$R^1(\pi \circ \mu)_*(\mathcal{O}_W(K_W + \Theta + \mu^*L - \lfloor F_k/k \rfloor)) = 0.$$

PROOF OF THE CLAIM. As in (5.4.8), we may assume that $\Theta = (\mu^{-1})_*\Delta$. Recall that $\mu^*[kL] = |V_k| + F_k$ and $|V_k|$ is base point free. We may assume that $F_k \leq k\mu^*B$ so that F_k and Θ have simple normal crossings support and no common component and $V_k \geq k\mu^*A$. Therefore V_k/k is relatively nef and big and V_k/k restricted to any intersection $\Theta_{i_0} \cap \dots \cap \Theta_{i_t}$ of components of Θ is relatively big. The assertion now follows by applying Corollary 5.4.4 with $N = \Theta + \mu^*L - \lfloor F_k/k \rfloor$ and $\Delta = \Theta + \{F_k/k\}$ so that $N - \Delta \sim_{\mathbb{Q}} V_k/k$. $\square$

Since $\mu_*\mathcal{O}_W(K_W + T' + \Theta + \mu^*L) = \mathcal{O}_Y(K_Y + T + \Delta + L)$, we have a surjective homomorphism

$$\begin{aligned} \pi_*\mathcal{O}_Y(K_Y + T + \Delta + L) \supset (\pi \circ \mu)_*\mathcal{O}_W(K_W + T' + \Theta + \mu^*L - \lfloor \frac{F_k}{k} \rfloor) \\ \rightarrow (\pi \circ \mu)_*\mathcal{O}_{T'}(K_{T'} + \Theta + \mu^*L - \lfloor \frac{F_k}{k} \rfloor) = \pi_*\mathcal{J}_{\Delta_T, ||L||_T}(K_T + \Delta_T + L_T), \end{aligned}$$

whence the assertion. $\square$

5.4.2. Extending Sections.

THEOREM 5.4.19. *Let $T \subset Y$ be a smooth divisor in a smooth variety. Let H be a sufficiently very ample divisor and set $A = (\dim Y + 1)H$. Assume that*

- (1) *B is an effective $\mathbb{Q}$ -divisor such that $T + B$ has simple normal crossings support and $(Y, T + B)$ is log canonical,*
- (2) *k is a positive integer such that kB is integral,*
- (3) *Given $L = k(K_Y + T + B)$, then there exists an integer $p > 0$ such that no log canonical centre of $(Y, T + \lceil B \rceil)$ is contained in the base locus of $|pL|$.*

Then

$$\mathcal{J}_{||mL_T + H_T||} \subset \mathcal{J}_{\lceil B_T \rceil, ||mL + H + A||_T}, \quad \forall m > 0.$$

PROOF. We follow the argument of [HM06] which in turn is based on [Kaw99]. We proceed by induction on m .

By assumption $mL_T + H_T$ is big, and there exists an integer $p > 0$ such that no log canonical centre of $(Y, T + \lceil B \rceil)$ is contained in the base locus of $|p(mL + H + A)|$ (for any $m > 0$). Therefore, both sides of the proposed inclusion are well-defined.

If $m = 0$, the result is clear. Assume the result for m . We have that

$$B = \sum b_j B_j \quad \text{where} \quad 0 < b_j = \frac{\beta_j}{k} \leq 1.$$

We define

$$\Delta^i = \sum_j \delta_i(j) B_j \quad \text{where} \quad \delta_i(j) = \begin{cases} 0 & \text{if } 1 \leq i \leq k - \beta_j, \\ 1 & \text{if } k - \beta_j < i \leq k + 1. \end{cases}$$

With this choice of Δ^i , we have

$$0 \leq \Delta^1 \leq \Delta^2 \leq \Delta^3 \leq \dots \leq \Delta^k = \Delta^{k+1} = \lceil B \rceil,$$

and every coefficient of Δ^i is one. Set

$$D_i = K_Y + T + \Delta^i \quad \text{and} \quad D_{\leq i} = \sum_{j \leq i} D_j.$$

With this choice of D_i , we have $L = D_{\leq k}$. Possibly replacing H by a multiple, we may assume that $H_i = D_{\leq i-1} + \Delta^i + H$ and $A_i = H + D_{\leq i} + A$ are ample, for $1 \leq i \leq k$. We are now going to prove, by induction on i , that

$$\mathcal{J}_{\lceil mL_T + H_T \rceil} \subset \mathcal{J}_{\Delta_T^{i+1}, \lceil mL + D_{\leq i} + H + A \rceil_T} \quad \text{where } 0 \leq i \leq k,$$

where we adopt the convention that $D_{\leq 0} = 0$.

Since $\Delta^1 \leq \lceil B \rceil$, it follows by Lemma 5.4.10 that

$$\mathcal{J}_{\lceil B_T \rceil, \lceil mL + H + A \rceil_T} \subset \mathcal{J}_{\Delta_T^1, \lceil mL + H + A \rceil_T},$$

and so the case $i = 0$ follows from the inclusion

$$\mathcal{J}_{\lceil mL_T + H_T \rceil} \subset \mathcal{J}_{\lceil B_T \rceil, \lceil mL + H + A \rceil_T},$$

which we are assuming by induction on m . Assume the result up to $i - 1$. Then

$$\begin{aligned} & H^0(T, \mathcal{J}_{\lceil mL_T + H_T \rceil}(mL + D_{\leq i} + H + A)) \\ & \subset H^0(T, \mathcal{J}_{\Delta_T^i, \lceil mL + D_{\leq i-1} + H + A \rceil_T}(mL + D_{\leq i} + H + A)) \\ & = H^0(T, \mathcal{J}_{\Delta_T^i, \lceil mL + D_{\leq i-1} + H + A \rceil_T}(mL + D_{\leq i-1} + H + A + K_T + \Delta_T^i)) \\ & \subset \text{Im}\left(H^0(Y, \mathcal{O}_Y(mL + D_{\leq i} + H + A)) \rightarrow H^0(T, \mathcal{O}_T(mL + D_{\leq i} + H + A))\right) \\ & \subset H^0(T, \mathcal{J}_{\Delta_T^{i+1}, \lceil mL + D_{\leq i} + H + A \rceil_T}(mL + D_{\leq i} + H + A)), \end{aligned}$$

where we use induction to get the inclusion of the first line in the second line, Lemma 5.4.17 to get the inclusion of the third line in the fourth line (recall that $D_{\leq i-1} + H + A = A_{i-1}$ is ample and there is an integer $p > 0$ such that the base locus of pmL contains no log canonical centre of $(Y, T + \Delta^i)$), and (3) of Lemma 5.4.16 to get the inclusion of the fourth line in the fifth line. By an easy generalisation of [Kaw99, Lemma 3.2], the coherent sheaf

$$\mathcal{J}_{\lceil mL_T + H_T \rceil}(mL + D_{\leq i} + H + A) = \mathcal{J}_{\lceil mL_T + H_T \rceil}(mL + H_i + A + K_T),$$

is generated by global sections; this implies that

$$\mathcal{J}_{||mL_T+H_T||} \subset \mathcal{J}_{\Delta_T^{i+1}, ||mL+D_{\leq i}+H+A||_T}.$$

This completes the induction on i . It follows that

$$\mathcal{J}_{||mL_T+H_T||} \subset \mathcal{J}_{\lceil B_T \rceil, ||(m+1)L+H+A||_T}.$$

But then by (1) and (2) of Lemma 5.4.16, one sees that

$$\mathcal{J}_{||(m+1)L_T+H_T||} \subset \mathcal{J}_{||mL_T+H_T||} \subset \mathcal{J}_{\lceil B_T \rceil, ||(m+1)L+H+A||_T},$$

and this completes the induction on m and the proof. $\square$

COROLLARY 5.4.20. *With the same notation and assumptions of Theorem 5.4.19. For any positive integer m , the image of the natural homomorphism*

$$H^0(Y, \mathcal{O}_Y(mL + H + A)) \rightarrow H^0(T, \mathcal{O}_T(mL + H + A))$$

contains the image of $H^0(T, \mathcal{O}_T(mL + H))$ considered as a sub-vector space of $H^0(T, \mathcal{O}_T(mL + H + A))$ by the inclusion induced by any section in $H^0(Y, \mathcal{O}_Y(A))$ not vanishing along T .

PROOF. We fix a section $\sigma \in H^0(Y, \mathcal{O}_Y(A))$ not vanishing on T . Then we have

$$H^0(T, \mathcal{J}_{||mL_T+H_T||}(mL + H)) \cdot \sigma \subset H^0(T, \mathcal{J}_{||mL_T+H_T||}(mL + H + A)).$$

As we have seen in the proof of Theorem 5.4.19, $H^0(T, \mathcal{J}_{||mL_T+H_T||}(mL + H + A))$ is contained in the image of $H^0(Y, \mathcal{O}_Y(mL + H + A))$. The assertion now follows since

$$H^0(T, \mathcal{O}_T(mL_T + H_T)) = H^0(T, \mathcal{J}_{||mL_T+H_T||}(mL + H)).$$

$\square$

THEOREM 5.4.21. *Let $T \subset Y$ be a smooth divisor in a smooth variety and let $\pi: Y \rightarrow Z$ be a projective morphism, where Z is normal and affine. Let m be a positive integer, and let L be a Cartier divisor on Y , such that $L \sim_{\mathbb{Q}} m(K_Y + T + B)$. Assume that*

- (1) *T is not contained in the support of B , $(Y, T + B)$ is log smooth and $\lfloor B \rfloor = 0$ so that $(Y, T + B)$ is purely log terminal,*
- (2) *$B \sim_{\mathbb{Q}} A + C$, where A is an ample $\mathbb{Q}$ -divisor and C is an effective $\mathbb{Q}$ -divisor, which does not contain T ,*
- (3) *there exists an integer $p > 0$ such that no log canonical centre of $(Y, T + \lceil B \rceil)$ is contained in the base locus of $|pL|$.*

Then the natural restriction homomorphism

$$H^0(Y, \mathcal{O}_Y(L)) \rightarrow H^0(T, \mathcal{O}_T(L)),$$

is surjective.

PROOF. By Corollary 5.4.20, one sees that there exists a sufficiently ample divisor H not containing T such that, for all $l > 0$ sufficiently divisible, the natural homomorphism

$$H^0(Y, \mathcal{O}_Y(lL + H)) \rightarrow H^0(T, \mathcal{O}_T(lL + H))$$

contains the image of $H^0(T, \mathcal{O}_T(lL))$, considered as a subspace of $H^0(T, \mathcal{O}_T(lL + H))$ by the inclusion induced by H_T .

For any sufficiently small rational number $\epsilon > 0$, we have that $K_Y + T + (1 - \epsilon)B + \epsilon A + \epsilon C$ is purely log terminal. Let $A' = \epsilon A$, and $B' = (1 - \epsilon)B + \epsilon C$, then $A' + B' \sim_{\mathbb{Q}} B$ and $K_T + A'_T + B'_T$ is klt.

Fix a non-zero section

$$\sigma \in H^0(T, \mathcal{O}_T(L)).$$

Let Σ be the zero locus of σ . As observed above, we may find a divisor $G^l \sim lL + H$, such that

$$G_T^l = l\Sigma + H_T.$$

If we set

$$N = L - K_Y - T \quad \text{and} \quad \Theta = \frac{m-1}{ml}G^l + B',$$

then

$$N \sim_{\mathbb{Q}} (m-1)(K_Y + T + B) + A' + B'.$$

Since

$$N - \Theta \sim_{\mathbb{Q}} A' + B' - \frac{m-1}{ml}H - B' = A' - \frac{m-1}{ml}H,$$

is ample for sufficiently large number l , it follows that,

$$H^1(Y, \mathcal{O}_Y(L - T) \otimes \mathcal{J}(\Theta)) = H^1(Y, \mathcal{O}_Y(K_Y + N) \otimes \mathcal{J}(\Theta)) = 0,$$

by Nadel's vanishing Theorem 5.4.3. Consider now $\mu: W \rightarrow Y$, a log resolution of $(Y, T + \Theta)$ and let $T' = (\mu^{-1})^*T$. We have a short exact sequence

$$\begin{aligned} 0 \rightarrow \mathcal{O}_W(K_{W/Y} - \lrcorner \mu^* \Theta \lrcorner - \mu^* T) \rightarrow \\ \rightarrow \mathcal{O}_W(K_{W/Y} - \lrcorner \mu^* \Theta \lrcorner + T' - \mu^* T) \rightarrow \\ \rightarrow \mathcal{O}_{T'}(K_{T'/T} - \lrcorner \mu^* \Theta \lrcorner) \rightarrow 0. \end{aligned}$$

Pushing forward, since $R^1 \mu_* \mathcal{O}_W(K_{W/Y} - \lrcorner \mu^* \Theta \lrcorner) = 0$, we get the short exact sequence

$$0 \rightarrow \mathcal{J}(Y, \Theta)(-T) \rightarrow \text{adj}(Y, T; \Theta) \rightarrow \mathcal{J}(T, \Theta_T) \rightarrow 0$$

where $\text{adj}(Y, T; \Theta) = \mu_* \mathcal{O}_W(K_{W/Y} - \lrcorner \mu^* \Theta \lrcorner + T' - \mu^* T) \subset \mathcal{O}_Y$. Then

$$H^0(Y, \mathcal{O}_Y(L) \otimes \text{adj}(Y, T; \Theta)) \rightarrow H^0(T, \mathcal{O}_T(L) \otimes \mathcal{J}(T, \Theta_T))$$

is surjective. Now

$$\Theta_T - \Sigma = B'_T + \frac{m-1}{ml}(l\Sigma + H_T) - \Sigma \leq B'_T + \frac{m-1}{ml}H_T.$$

Since (T, B'_T) is klt, then

$$\left(T, B'_T + \frac{m-1}{ml}H_T\right),$$

is klt for l sufficiently large. But then

$$\sigma \in H^0(T, \mathcal{O}_T(L) \otimes \mathcal{J}(\Theta_T)),$$

by Lemma 5.4.2 and so σ lifts to $H^0(Y, \mathcal{O}_Y(L))$. $\square$

This next result shows that if the components of B are disjoint, then one can substantially weaken the hypothesis that no log canonical centre of $(Y, T + \lceil B \rceil)$ is contained in the base locus of $|L|$.

COROLLARY 5.4.22. *Let $T \subset Y$ be a smooth divisor in a smooth variety and let $g: Y \rightarrow X$ and $f: X \rightarrow Z$ be projective morphisms of normal varieties and $\pi: Y \rightarrow Z$ the composition of f and g . Assume that Z is affine g is birational and T is not g -exceptional. Let m be a positive integer such that $m(K_Y + T + B)$ is Cartier. Assume that*

- (1) *T is not contained in the support of B , $(Y, T + B)$ is log smooth and $\lfloor B \rfloor = 0$ so that $(Y, T + B)$ is purely log terminal,*
- (2) *$B \geq g^*A$, where mA is a very ample divisor on X ,*
- (3) *there exists a divisor $G \in |m(K_Y + T + B)|$ such that G and $T + B$ have no common component, and*
- (4) *no two components of $B - g^*A$ intersect.*

Then we may find a birational map $h: Y' \rightarrow Y$ such that

- (i) *h is given by a sequence of blow ups along smooth codimension 2 subvarieties corresponding to the intersection of divisors of log discrepancy 0 for $(Y, \lfloor T + B - g^*A \rfloor)$. In particular, in a neighbourhood of T these are just the components of $T \cap \lfloor B - g^*A \rfloor$ (and their strict transforms) so that if $T' = (h^{-1})_*T$ denotes the strict transform of T , then $h: T' \rightarrow T$ is an isomorphism,*
- (ii) *if we write $K_{Y'} + \Gamma' = h^*(K_Y + T + B) + E'$ where Γ' and E' are effective with no common component $h_*\Gamma' = T + B$ and E' is exceptional, then there exists a $\mathbb{Q}$ -divisor $T' \leq \Gamma \leq \Gamma'$ such that $m\Gamma$ is integral and identifying T' with T and setting $(\Gamma - T)_T = \Theta$, one has that the restriction homomorphism*

$$H^0(Y', \mathcal{O}_{Y'}(m(K_{Y'} + \Gamma))) \rightarrow H^0(T, \mathcal{O}_T(m(K_T + \Theta)))$$

is surjective for any integer $l > 0$, and

- (iii) *the natural homomorphism*

$$H^0(Y', \mathcal{O}_{Y'}(m(K_{Y'} + \Gamma))) \rightarrow H^0(Y, \mathcal{O}_Y(m(K_Y + T + B)))$$

is an isomorphism.

PROOF. We are free to replace mA by a linearly equivalent divisor. Therefore, we may assume that mA is a very general divisor in $|mA|$. Since no two components of $B - g^*A$ intersect, the only possible log canonical centres of $(Y, T + \lfloor B \rfloor)$ contained in G , are the components of $T \cap \lfloor B - g^*A \rfloor$.

We consider now a birational morphism $h: Y' \rightarrow Y$ given by a sequence of blow ups with smooth codimension 2 centres equal to the irreducible components of $T \cap \lfloor B - g^*A \rfloor$ that are contained in G . Since the only such centres are divisors in T , it follows that $h|_{T'}: T' \rightarrow T$ is an isomorphism. We may write

$$K_{Y'} + \Gamma' = h^*(K_Y + T + B),$$

where, by an easy log discrepancy computation, Γ' is effective and contains T' . Note that $m(K_{Y'} + \Gamma')$ is integral and that $G' = h^*G \in |m(K_{Y'} + \Gamma')|$. Let

$$m(K_{Y'} + \Gamma') = N + F',$$

be the decomposition of $m(K_{Y'} + \Gamma')$ into its moving and fixed parts.

CLAIM 5.4.23. *After finitely many blow ups along smooth codimension 1 subvarieties of T' given by the (strict transforms of the) irreducible components of $T \cap \lfloor B - g^*A \rfloor$, we may assume that in a neighbourhood of T' , the base locus*

of N does not contain any log canonical centre of $K_{Y'} + [\Gamma' - h^*g^*A]$ and if a component F_0 of F' contains a log canonical centre of $K_{Y'} + [\Gamma' - h^*g^*A]$, then $F_0 \subset \text{Supp}(\Gamma' - h^*g^*A)$.

PROOF. Let $C_1, \dots, C_r$ be the codimension 1 smooth divisors in T given by $T \cap \text{Supp}(B - g^*A)$. Let $h : Y' \rightarrow Y$ be a morphism obtained by blowing up Y along a sequence of subvarieties C_i and their strict transforms. Then the strict transform of $T \subset Y$ is isomorphic to T and so we again denote it by T . Similarly for the strict transform of each $C_i \subset T$. It is also easy to see that h induces an isomorphism of log pairs between $(T', (\Gamma' - T')|_{T'})$ and $(T, (\Gamma - T)|_T)$.

Suppose that C_i is contained in $\text{Bs}(\text{Mob}(m(K_Y + T + B)))$ with multiplicity n , then we blow up along C_i . We denote this morphism also by $h : Y' \rightarrow Y$ and we let E be the corresponding exceptional divisor. Then $N = |h^*\text{Mob}(m(K_Y + T + B)) - nE|$ has no fixed divisors and $\text{mult}_{C_i}|N| \leq n$ (here, as above $N = \text{Mob}(m(K_{Y'} + \Gamma'))$). Repeating this procedure finitely many times, we may assume that $C_i \not\subset \text{Bs}(N)$ for all i (this can be seen by a computation on a surface given by the intersection of $\dim Y' - 2$ general very ample divisors). Further blowing up along the C_i 's, we may assume that $\Gamma' - h^*g^*A + F'$ has simple normal crossings at a general point of each C_i (this can also be checked by restricting to a surface given by the intersection of $\dim Y' - 2$ general very ample divisors). Since each C_i is given by the intersection of 2 divisors in $\text{Supp}(\Gamma' - h^*g^*A)$, then, if a component F_0 of F' contains C_i , it must be contained in $\text{Supp}(\Gamma' - h^*g^*A)$. $\square$

The only other log canonical centres of $K_{Y'} + [\Gamma' - h^*g^*A]$ are smooth codimension 2 subvarieties given by the intersection of 2 components of $[\Gamma' - T' - h^*g^*A]$. These centres do not intersect T' . After repeatedly blowing up such centres (as in the proof of 5.4.26), we may assume that all the codimension 2 log canonical centres of $K_{Y'} + [\Gamma' - h^*g^*A]$ are contained in T' .

Cancelling common components of F' and Γ' , we may therefore find divisors F and Γ , with no common components, such that

$$m(K_{Y'} + \Gamma) = N + F,$$

is the decomposition of $m(K_{Y'} + \Gamma)$ into its moving and fixed parts. It is clear that F contains no log canonical centres of $(Y', [\Gamma])$.

Since A is ample, there is an effective and $g \circ h$ -exceptional divisor H such that $h^*g^*A - H$ is ample. In this case

$$\Gamma - T' \sim_{\mathbb{Q}} (h^*g^*A - H) + (\Gamma - T' - h^*g^*A + H) = A' + C.$$

(ii) now follows from Theorem 5.4.21. Since $0 \leq (\Gamma' - \Gamma) \leq F'$, there is a natural identification

$$H^0(Y', \mathcal{O}_{Y'}(m(K_{Y'} + \Gamma))) \cong H^0(Y', \mathcal{O}_{Y'}(m(K_{Y'} + \Gamma'))),$$

and so (iii) also follows. $\square$

In the next lemma, we continue to assume the notation and the hypothesis of Corollary 5.4.22.

LEMMA 5.4.24. *Let $\mu : \bar{Y} \rightarrow Y$ be any birational morphism such that $\mu^*(K_Y + T + B) = K_{\bar{Y}} + \bar{T} + \bar{B} - \bar{F}$ where $\bar{T} = (\mu^{-1})_*T$, $\bar{Y}$ is smooth, $\bar{T} + \bar{B}$, $\bar{F}$ are effective with simple normal crossings support and no common components, $\mu_*\bar{B} = B$ and $\bar{F}$ is μ -exceptional. Then*

- (1) After cancelling common components of $m\bar{B}$ and $\text{Fix}(m(K_{\bar{Y}} + \bar{T} + \bar{B}))$ and further blowing up $\bar{Y}$ along smooth centres contained in $\bar{Y} - \bar{T}$, the hypothesis of Corollary 5.4.22 hold for $\bar{g} = g \circ \mu : \bar{Y} \rightarrow X$, $\bar{T}$ and $\bar{B}$ (in place of $g : Y \rightarrow X$, T and B) so that we may define $\bar{h} : \bar{Y}' \rightarrow \bar{Y}$ and $\bar{\Theta}$ analogously to $h : Y' \rightarrow Y$ and Θ in Corollary 5.4.22.
- (2) $\bar{\Theta} = (\mu|_{\bar{T}}^{-1})_* \Theta$ so that

$$K_{\bar{T}} + \bar{\Theta} = (\mu|_{\bar{T}})^*(K_T + \Theta) + \bar{F}'$$

where $\bar{F}'$ is an effective and $\mu|_{\bar{T}}$ -exceptional $\mathbb{Q}$ -divisor.

PROOF. 1 and 2 of Corollary 5.4.22 are clearly satisfied by $\bar{g} = g \circ \mu : \bar{Y} \rightarrow X$, $\bar{T}$ and $\bar{B}$. 3 is satisfied once we cancel common components of $m\bar{B}$ and $\text{Fix}(m(K_{\bar{Y}} + \bar{T} + \bar{B}))$. For 4 of Corollary 5.4.22, notice that the components of $(\bar{B} - \bar{g}^*A) \cap \bar{T}$ are the divisors in $\bar{T}$ of log discrepancy less than one for $(T, B - A)$ and hence these components are disjoint. It is then clear that 4 holds in a neighbourhood of $\bar{T}$. It suffices now to repeatedly blow up $\bar{Y}$ along intersections of the various components of $\bar{B} - \bar{g}^*A$.

In the definition of Θ , we pick an appropriate birational map $h : Y' \rightarrow Y$ and after cancelling common components of Γ' and $\frac{1}{m}\text{Fix}(m(K_{Y'} + \Gamma'))$ we obtain a divisor Γ such that $(\Gamma - T)|_T = \Theta$. It is clear that Θ is supported on the components C_i of $(\Gamma - T)|_T$ i.e. on those (finitely many) divisors of log-discrepancy less than one for $(T, (\Gamma - T)|_T)$. $\bar{\Theta}$ is also supported on the divisors in $\bar{T}$ of log-discrepancy less than one for $(T, (\Gamma - T)|_T)$, i.e. on the strict transforms $\bar{C}_i$ of C_i . Therefore, to show that $\bar{\Theta} = (\mu|_{\bar{T}}^{-1})_* \Theta$, it suffices to check that the multiplicity of Θ and $\bar{\Theta}$ along each of the components C_i and $\bar{C}_i$ coincides. To see this, notice that passing to a common resolution, we may reduce to the case that $\mu : \bar{Y} \rightarrow Y$ is induced by a sequence of blow ups along smooth centres. Each time we blow up along a centre not containing any C_i , we have an isomorphism in a neighbourhood of the generic points of each C_i and hence the coefficients of Θ are unchanged. If we blow up along a centre containing some C_i , then we may in fact assume that we are blowing up along C_i and an easy computation shows that in this case the coefficients of Θ are also unchanged.

Since (T, Θ) is terminal, one sees that the divisor $\bar{F}'$ defined above is effective and $\mu|_{\bar{T}}$ -exceptional. $\square$

COROLLARY 5.4.25. *The divisor Θ defined in Corollary 5.4.22 does not depend on the choice of the morphism $h : Y' \rightarrow Y$.*

PROOF. Immediate from Lemma 5.4.24. $\square$

The next result is quite standard. It shows that the assumption that the components of B are disjoint is not as restrictive as one might think. For any divisor $B = \sum b_i B_i$, we let $\langle B \rangle = \sum a_i B_i$ where $a_i = b_i$ if $0 < b_i < 1$ and $a_i = 0$ otherwise.

LEMMA 5.4.26. *Let (X, Δ) be a log pair. We may find a log resolution*

$$g : Y \rightarrow X,$$

with the following properties. Suppose that we write

$$K_Y + \Gamma = g^*(K_X + \Delta) + E,$$

where Γ and E are effective, with no common components, $g_*\Gamma = \Delta$, and E is exceptional.

Then no two components of $\langle \Gamma \rangle$ intersect.

PROOF. This is well known and easy to prove, see for example [HM, Lemma 6.7]. $\square$

5.4.3. The Restricted Algebra is an Adjoint Algebra. The main result of this section is the following:

THEOREM 5.4.27. *Let (X, Δ) be a $\mathbb{Q}$ -factorial log pair of dimension n and let $f: X \rightarrow Z$ be a projective morphism, where Z is affine and normal. Let k be a positive integer such that $D = k(K_X + \Delta)$ is Cartier, and let $\mathfrak{R}$ be the divisorial algebra associated to D . Assume that*

- (1) $K_X + \Delta$ is purely log terminal,
- (2) $S = \lfloor \Delta \rfloor$ is irreducible,
- (3) $S \not\subset \text{Bs}|D|$,
- (4) $\Delta - S \geq A$, where kA is very ample, and
- (5) $-(K_X + \Delta)$ is ample.

If the MMP holds in dimension $n-1$ (cf. Assumption 5.2.2), then the restricted algebra $\mathfrak{R}_S$ is finitely generated.

Recall that by definition the *restricted algebra* $\mathfrak{R}_S$ is the image of the homomorphism

$$\bigoplus_{m \in \mathbb{N}} f_* \mathcal{O}_X(mD) \rightarrow \bigoplus_{m \in \mathbb{N}} f_* \mathcal{O}_S(mD).$$

The only interesting case of Theorem 5.4.27 is when f is birational, since otherwise the condition that $-(K_X + \Delta)$ is ample implies that $\kappa(X, K_X + \Delta) = -\infty$.

We note that Theorem 5.4.27 implies Theorem 5.1.1.

LEMMA 5.4.28. *Theorem 5.4.27_n implies Theorem 5.1.1_n.*

PROOF. By Theorem 5.1.9 it suffices to prove the existence of pl flips. Since Z is affine and f is small, it follows that S is mobile. By Theorem 5.1.10 it follows that it suffices to prove that the restricted algebra is finitely generated. Hence it suffices to prove that a pl flip satisfies the hypothesis of Theorem 5.4.27. Properties (1-2) and (5) are automatic and (3) follows as S is mobile. $\Delta - S$ is automatically big, as f is birational, and so $\Delta - S \sim_{\mathbb{Q}} A + B$, where A is ample, and B is effective. As S is mobile, we may assume that B does not contain S . Replace $\Delta - S$ by $(1 - \epsilon)(\Delta - S) + \epsilon(A + B)$ for some rational number $0 < \epsilon \ll 1$, and replace k by a sufficiently divisible multiple. $\square$

PROOF OF THEOREM 5.4.27. As remarked above, we may assume that f is birational. Let $g: Y \rightarrow X$ be any morphism, whose existence is guaranteed by Lemma 5.4.26. We may write

$$K_Y + \Gamma = g^*(K_X + \Delta) + E,$$

where Γ and E are effective, with no common components, $g_*\Gamma = \Delta$, E is exceptional and no two components of $\langle \Gamma \rangle$ intersect. Since $k(K_X + \Delta)$ is Cartier, $k(K_Y + \Gamma)$ and kE are integral. Let T be the strict transform of S and let π be the composition of f and g . Let

$$mk(K_Y + \Gamma) = N_m + G_m,$$

be the decomposition of $mk(K_Y + \Gamma)$ into its moving and fixed parts. By assumption, T is not a component of G_m . We may assume that kA is a very general divisor in $|kA|$ so that in particular g^*A and the strict transform of A are equal.

Cancelling common components of Γ and G_m , we may find divisors $T + g^*A \leq \Gamma_m^0 \leq \Gamma$ and G_m^0 , with no common components, such that

$$mk(K_Y + \Gamma_m^0) = N_m + G_m^0.$$

Set $\Theta = (\Gamma - T)|_T$. It is now easy to see that the hypothesis of Corollary 5.4.22 are satisfied (with $B = \Gamma_m^0 - T$) and so, for all $m > 0$, there is a birational map $h_m : Y_m \rightarrow Y$ which is a log resolution of (Y, Γ) such that

- (i) In a neighbourhood of T , h is given by a sequence of blow ups along smooth divisors in T so that if $T_m = (h_m^{-1})_*T$ denotes the strict transform of T , then $h_m|_{T_m} : T_m \rightarrow T$ is an isomorphism.
- (ii) If we write $K_{Y_m} + \Gamma'_m = h_m^*(K_Y + \Gamma) + E_m$, where Γ'_m, E_m are effective with no common components, $(h_m)_*\Gamma'_m = \Gamma$ and E_m is exceptional, then there exists a divisor Γ_m such that $T_m \leq \Gamma_m \leq \Gamma'_m$ and identifying T_m with T , and setting $(\Gamma_m - T)|_T = \Theta_m$, one has that the restriction homomorphisms

$$H^0(Y_m, \mathcal{O}_{Y_m}(mlk(K_{Y_m} + \Gamma_m))) \rightarrow H^0(T, \mathcal{O}_T(mlk(K_T + \Theta_m)))$$

are surjective for all integers $l > 0$.

- (iii) The natural homomorphism

$$H^0(Y_m, \mathcal{O}_{Y_m}(mk(K_{Y_m} + \Gamma_m))) \rightarrow H^0(Y, \mathcal{O}_Y(mk(K_Y + \Gamma)))$$

is an isomorphism.

The sequence Θ_m is possibly not convex, because it is obtained by canceling common components of Γ'_m and of the fixed part of $mk(K_{Y_m} + \Gamma'_m)$ (and then restricting to T). The problem occurs when the coefficient of a component of $mk\Gamma'_m$ goes from being smaller than the coefficient of the corresponding divisor in the fixed part of $mk(K_{Y_m} + \Gamma'_m)$ to being bigger. Since there are only finitely many components of Θ_m , it is clear that for m sufficiently divisible, the sequence Θ_m is convex. Since (a truncation of) $\Theta_\bullet$ is a convex sequence with $\Theta_m \leq \Theta$, the limit Θ' of the sequence $\Theta_\bullet$ exists and $K_T + \Theta'$ is klt. Therefore (1) of Definition 5.3.10 is satisfied.

CLAIM 5.4.29. *We may also assume that for all $m > 0$, $P_m = \text{Mob}(msk(K_T + \Theta_m))$ is base point free, $\lim(P_m/m)$ is semiample and*

$$\Phi = \text{Supp}(\Theta) + \text{Fix}(msk(K_T + \Theta_m))$$

is contained in a divisor with simple normal crossings support (not depending on m).

PROOF. Let s be the integer and $\mu : \bar{T} \rightarrow T$ be the morphism whose existence is guaranteed by Corollary 5.2.7. Therefore, $\text{Mob}(msk\mu^*(K_T + \Theta_m))$ is base point free for all m sufficiently divisible, $\lim(1/m) \text{Mob}(msk\mu^*(K_T + \Theta_m))$ is semiample and $\text{Fix}(msk\mu^*(K_T + \Theta_m))$ is contained in a simple normal crossings divisor (which does not depend on m). We may write $K_{\bar{T}} + \bar{\Theta}_m = \mu^*(K_T + \Theta_m) + F_m$ where $\bar{\Theta}_m = (\mu^{-1})_*\Theta_m$ and F_m is effective and exceptional. Clearly $\text{Mob}(msk(K_{\bar{T}} + \bar{\Theta}_m))$ is also base point free and $\lim(1/m) \text{Mob}(msk(K_{\bar{T}} + \bar{\Theta}_m))$ is also semiample. We may assume that $\mu : \bar{T} \rightarrow T$ is induced by a birational morphism $\mu : \bar{Y} \rightarrow Y$ which is a log resolution of (Y, Γ) .

We now replace Y, T by $\bar{Y}, \bar{T}$. By Lemma 5.4.24, we may assume that (i), (ii) and (iii) continue to hold. $\square$

Notice that up to this point, we may assume that each morphism $h_m : Y_m \rightarrow Y$ is obtained by a sequence of blow ups along smooth codimension 2 subvarieties obtained by intersecting divisors of log discrepancy 0 for $(Y, [\Gamma - g^*A])$. Since properties (i), (ii) and (iii) continue to hold after further blowing up along codimension 2 subvarieties as above, we may therefore assume that for all $m' \geq m$ the morphism $h_{m'}$ factors through h_m .

We now proceed to show that we may assume that after replacing Y_m by appropriate models obtained by repeated blow ups of Y along components of Φ and along smooth centres disjoint from T , the following properties also holds:

(iv)

$$Q_m = \text{Mob}(\text{msk}(K_{Y_m} + \Gamma_m)) \text{ is free,}$$

$$Q_m|_T = P_m = \text{Mob}(\text{msk}(K_T + \Theta_m)),$$

(v) for all $m > 0$ we have

$$(\text{Mob}(\text{msk}(K_{Y_{ms}} + \Gamma_{ms})))|_T = \text{Mob}(\text{msk}(K_T + \Theta_{ms})),$$

(vi) for all $m' \geq m \geq 0$, $h_{m'}$ factors as $h_m \circ h_{m',m}$ where $h_{m',m} : Y_{m'} \rightarrow Y_m$ is a morphism which in a neighbourhood of T is induced by a finite sequence of blow ups along the components of the simple normal crossing divisor Φ . (Here, $Y_0 = Y$ and $h_{m',0} = h_{m'}$).

In a neighbourhood of T , we may assume that (up to this point) h_m is given by $\text{Bl}_{C_{i_m}} \circ \dots \circ \text{Bl}_{C_1}$, a sequence of blow ups along the components C_i of $([\Gamma - g^*A]) \cap T$. Since $([\Gamma - g^*A]) \cap T \subset \Phi$ and Φ is a simple normal crossings divisor in T , replacing each $h_{m,m-1} = \text{Bl}_{C_{i_m}} \circ \dots \circ \text{Bl}_{C_{i_{m-1}}}$ by $\text{Bl}_{D_{j_m}} \circ \dots \circ \text{Bl}_{D_{j_{m-1}+1}} \circ \text{Bl}_{C_{i_m}} \circ \dots \circ \text{Bl}_{C_{i_{m-1}}}$, where the D_{j_t} are components of Φ , we may still assume that properties (i), (ii) and (iii) continue to hold. We are hence free to replace each morphism $h_{m,m-1}$ by its composition with further blow ups along components of Φ .

Notice that

$$\text{Fix}(\text{msk}(K_T + \Theta_m)) \subset \Phi$$

and so by (ii) and Lemma 5.1.15, after further blowing up Y_{m-1} along the components of Φ , we may assume that $Q_m|_T = P_m$. It is easy to see that

$$\text{Fix}(\text{msk}(K_T + \Theta_{ms})) \subset \text{Fix}(\text{msk}(K_T + \Theta_m)) + \text{Supp}(\Theta_1) \subset \Phi$$

and hence (after further blowing up along components of Φ) we may also assume that

$$(\text{Mob}(\text{msk}(K_{Y_{ms}} + \Gamma_{ms})))|_T = \text{Mob}(\text{msk}(K_T + \Theta_{ms})).$$

Further blow ups along smooth centres not intersecting T will ensure that $|Q_m|$ is base point free.

LEMMA 5.4.30. *Let $M_m = \text{Mob}(\text{msk}(K_T + \Theta_m))$, then*

$$\frac{P_{ms}}{ms} \geq \frac{M_{ms}}{m} \geq \frac{P_m}{m} \geq \frac{sM_m}{m}.$$

PROOF. This follows easily from the inclusion of linear series

$$\begin{aligned} |msk(K_T + \Theta_{ms})|^{\times s} &\subset |ms^2k(K_T + \Theta_{ms})|, \\ |msk(K_T + \Theta_m)| + msk(\Theta_{ms} - \Theta_m) &\subset |msk(K_T + \Theta_{ms})|, \quad \text{and} \\ |mk(K_T + \Theta_m)|^{\times s} &\subset |msk(K_T + \Theta_m)|. \end{aligned}$$

□

Therefore, by Claim 5.4.29, $D = \lim(M_i/i) = \lim(P_i/si)$ is semiample and so 3 of Definition 5.3.10 is satisfied. We have already seen that 1 of Definition 5.3.10 holds. In Lemma 5.4.31 below, we will show that a truncation of the characteristic sequence of $\mathfrak{R}_T = \bigoplus \pi_* \mathcal{O}_T(M_m)$ is saturated. Therefore, 2 of Definition 5.3.10 is also satisfied (by a truncation of $\mathfrak{R}_T$), and so by Lemma 5.3.2 and Theorem 5.3.12, $\mathfrak{R}_T$ is finitely generated. Therefore, as $\mathfrak{R}_S$ is isomorphic to $\mathfrak{R}_T$, it is also finitely generated. □

It suffices therefore to prove the following:

LEMMA 5.4.31. *There exists a divisor F_T on T such that $\lceil F_T \rceil \geq 0$ and*

$$\text{Mob}(\lceil (js/is)M_{is} + F_T \rceil) \leq M_{js} \quad \forall i \geq j > 0.$$

In particular, the truncation of the restricted algebra $(\mathfrak{R}_T)_{(s)}$ is saturated with respect to F_T .

PROOF. We write $g_m = g \circ h_m: Y_m \rightarrow X$ and we set

$$K_{Y_m} + \Gamma'_m = g_m^*(K_X + \Delta) + E'_m.$$

Let $F_m = E'_m - \Gamma'_m + T$, then writing $K_S + \Delta_S = (K_X + \Delta)|_S$, one sees that

$$K_T + (\Gamma'_m - T)|_T = g_m^*(K_S + \Delta_S) + E'_m|_T.$$

An easy log discrepancy computation shows that $E'_m|_T$ is exceptional and so the above expression is unique. Therefore, $F_m|_T$ is independent of m and so we denote it by F_T . By assumption (X, Δ) is purely log terminal and so (S, Δ_S) is klt i.e.

$$\lceil F_T \rceil \geq 0.$$

We will need the following two claims:

CLAIM 5.4.32. *The natural restriction map,*

$$H^0(Y_i, \mathcal{O}_{Y_i}(\lceil (j/i)Q_i + F_i \rceil)) \rightarrow H^0(T, \mathcal{O}_T(\lceil (j/i)P_i + F_T \rceil)),$$

is surjective, for any positive integers i and j . In particular, by Lemma 5.1.13, we have that

$$(\text{Mob}(\lceil (j/i)Q_i + F_i \rceil))|_T \geq \text{Mob}(\lceil (j/i)P_i + F_T \rceil).$$

PROOF. Recall that by (iv) Q_i is base point free and $Q_i|_T = P_i$. Considering the short exact sequence

$$\begin{aligned} 0 \rightarrow \mathcal{O}_{Y_i}(\lceil (j/i)Q_i + F_i \rceil - T) &\rightarrow \mathcal{O}_{Y_i}(\lceil (j/i)Q_i + F_i \rceil) \\ &\rightarrow \mathcal{O}_T(\lceil (j/i)P_i + F_T \rceil) \rightarrow 0, \end{aligned}$$

it follows that the obstruction to the surjectivity of the restriction map above is given by,

$$\begin{aligned} H^1(Y_i, \mathcal{O}_{Y_i}(\lceil (j/i)Q_i + (F_i - T) \rceil)) \\ = H^1(Y_i, \mathcal{O}_{Y_i}(K_{Y_i} + \lceil g_i^*(-(K_X + \Delta)) + (j/i)Q_i \rceil)) \end{aligned}$$

which vanishes by Kawamata-Viehweg vanishing, as $g_i^*(-(K_X + \Delta))$, is big and nef and $(j/i)Q_i$ is base point free and hence nef. $\square$

CLAIM 5.4.33. *For every pair of positive integers i and j we have*

$$\text{Mob}(\lceil (j/i)Q_i + F_i \rceil) \leq \text{Mob}(j\text{sk}g_i^*(K_X + \Delta)).$$

PROOF. Since E'_i is g_i -exceptional, we have

$$\begin{aligned} Q_i &= \text{Mob}(\text{isk}(K_{Y_i} + \Gamma_i)) \leq \text{Mob}(\text{isk}(K_{Y_i} + \Gamma'_i)) = \\ &= \text{Mob}(\text{isk}g_i^*(K_X + \Delta) + \text{isk}E'_i) = \text{Mob}(\text{isk}g_i^*(K_X + \Delta)) \leq \\ &\leq g_i^*(\text{isk}(K_X + \Delta)). \end{aligned}$$

Therefore,

$$\begin{aligned} \text{Mob}(\lceil (j/i)Q_i + F_i \rceil) &\leq \text{Mob}(\lceil j\text{sk}g_i^*(K_X + \Delta) + F_i \rceil) \\ &= \text{Mob}(j\text{sk}g_i^*(K_X + \Delta) + \lceil F_i \rceil) = \text{Mob}(j\text{sk}g_i^*(K_X + \Delta)) \end{aligned}$$

where we used the fact that $\lceil F_i \rceil$ is g_i -exceptional. $\square$

Therefore, since by (iii) $\text{Mob}(j\text{sk}g_{j_s}^*(K_X + \Delta)) = \text{Mob}(j\text{sk}(K_{Y_{j_s}} + \Gamma_{j_s}))$, by (v) we have that $M_{j_s} = \text{Mob}(j\text{sk}g_{j_s}^*(K_X + \Delta))|_T$. Since for any $i \geq j$, g_{is} factors through g_{j_s} , by Lemma 5.1.14, one sees that

$$\text{Mob}(j\text{sk}g_{j_s}^*(K_X + \Delta))|_T \geq \text{Mob}(j\text{sk}g_{is}^*(K_X + \Delta))|_T.$$

Therefore, by Lemma 5.4.33, Corollary 5.4.32 and Lemma 5.4.30 we have that

$$\begin{aligned} M_{j_s} &\geq (\text{Mob}(j\text{sk}g_{is}^*(K_X + \Delta)))|_T \geq \text{Mob}(\lceil (j/is)Q_{is} + F_{is} \rceil)|_T \geq \\ &\geq \text{Mob}(\lceil (j/is)P_{is} + F_T \rceil) \geq \text{Mob}(\lceil (js/is)M_{is} + F_T \rceil) \end{aligned}$$

and Lemma 5.4.31 follows. $\square$

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CHAPTER 6

Saturated mobile $\mathbf{b}$ -divisors on weak del Pezzo klt surfaces

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6.1. Introduction

In this chapter, we study saturated mobile $\mathbf{b}$ -divisors on del Pezzo surfaces. Saturation is a key concept introduced by Shokurov; for instance, the finite generation conjecture implies existence of flips and it states that a canonically $\mathbf{a}$ -saturated algebra on a relative Fano pair is finitely generated.

Here, we study the simplest nontrivial examples of saturation, in the hope to build an intuitive understanding of what the condition means. Our main result is generalised in the CCS conjecture, which is the subject of Chapter 7. The examples show that the notion of saturation for mobile $\mathbf{b}$ -divisors is very subtle and, even in the surface case, it is far from straightforward.

DEFINITION 6.1.1. Let (X, B) be a klt pair; as usual, we denote by $\mathbf{A} = \mathbf{A}(X, B)$ the discrepancy $\mathbf{b}$ -divisor of the pair (X, B) . An effective $\mathbf{b}$ -divisor $\mathbf{D}$ on X is $\mathbf{A}$ -saturated, or *canonically saturated*, if $\mathbf{D}_Y$ is $\mathbf{A}_Y$ -saturated on high models $Y \rightarrow X$ of X .

To spell this out, there exists a model $Y \rightarrow X$ such that

$$\mathrm{Mob}[\mathbf{D}_{Y'} + \mathbf{A}_{Y'}] \leq \mathbf{D}_{Y'}$$

for all models $Y' \rightarrow Y$. If this formula holds on some model $W \rightarrow X$, we say that $\mathbf{A}$ -saturation holds on W .

DEFINITION 6.1.2. A *relative weak del Pezzo pair* is a pair (X, B) of a surface X and a $\mathbb{Q}$ -divisor $B \subset X$, together with a proper morphism $f: X \rightarrow Z$ such that $-(K_X + B)$ is f -nef and f -big.

The main result of this chapter is the following:

THEOREM 6.1.3. *Let $(X, B) \rightarrow Z$ be a relative weak del Pezzo klt surface pair. Denote by $(X', B') \rightarrow (X, B)$ the terminal model. There are:*

- (1) *A finite set of Z -morphisms $\varphi_i: X' \rightarrow T_i$ where T_i/Z is either a normal surface $/Z$ or $\mathbb{P}^1/Z$,*
- (2) *(when $Z = \{\mathrm{pt}\}$ only) finitely many normal surfaces Y_j together with projective birational morphisms $h_j: Y_j \rightarrow X$ and elliptic fibrations $\chi_j: Y_j \rightarrow \mathbb{P}^1$,*

such that the following holds: If $\mathbf{M}$ is any mobile $\mathbf{A}$ -saturated $\mathbf{b}$ -divisor on X , then either

- (a) $\mathbf{M}$ descends to $(X', B') \rightarrow (X, B)$ and there is an index i such that $\mathbf{M}_{X'} = \varphi_i^*(\text{ample divisor on } T_i)$, or
- (b) For some j , $\mathbf{M}$ descends to Y_j and $\mathbf{M}_{Y_j} = \chi_j^*(\text{ample divisor on } \mathbb{P}^1)$, or
- (c) X is a proper surface and $-(K_X + B) \cdot \mathbf{M}_X \leq 1$.

REMARK 6.1.4. (1) In Case (b), $\mathbf{M}$ can never descend to X' ; indeed, if $\chi: X' \rightarrow \Delta$ is an elliptic fibration, then $-(K + B)$ can not be ample along the fibres of χ .

- (2) With more work, it is possible to show that the b-divisors in Case (c) of the theorem form, in a natural way, a *bounded family* $\mathfrak{F}$ of b-divisors. We do not pursue this here; in particular, we do not give a precise definition of a bounded family of b-divisors. By contrast, it is clear that the b-divisors in Case (a) and Case (b) of the theorem do not form a bounded family.

In §6.3, we show that Theorem 6.1.3 is a consequence of the following, which is proved in §6.4.

THEOREM 6.1.5. *Let (X, B) be weak del Pezzo terminal surface pair. Then there are:*

- (1) *finitely many conic fibrations $\varphi_i: X \rightarrow \mathbb{P}^1$, and*
- (2) *finitely many normal surfaces Y_j , together with projective birational morphisms $h_j: Y_j \rightarrow X$ and elliptic fibrations $\chi_j: Y_j \rightarrow \mathbb{P}^1$,*

such that: If $\mathbf{M}$ is a mobile $\mathbf{A}$ -saturated b-divisor and $|\mathbf{M}|$ is composed with a pencil, then either

- (a) *$\mathbf{M}$ descends to X and, for some i , $\mathbf{M}_X = \varphi_i^*(\text{ample divisor on } \mathbb{P}^1)$, or*
- (b) *For some j , $\mathbf{M}$ descends to Y_j and $\mathbf{M}_{Y_j} = \chi_j^*(\text{ample divisor on } \mathbb{P}^1)$.*

The proof is in §6.4.

6.2. Example

We give an important xample of $\mathbf{A}$ -saturated b-divisor.

EXAMPLE 6.2.1. Let X be a del Pezzo surface with canonical singularities and degree $1 \leq d \leq 9$. Let $P \in X$ be a nonsingular point and assume that $P \in F_1, F_2 \in |-K_X|$ are nonsingular at P and are tangent to order d at P :

$$(F_1 \cdot F_2)_P = dP$$

It is easy to find this configuration on any del Pezzo surface.

We work with a general member $B_0 \in |F_1, F_2|$ and set $B = bB_0$ for $1 - 1/d \leq b < 1$; it is clear that (X, B) is a klt del Pezzo surface.

Let $f: Y \rightarrow X$ be the sequence of blow ups of infinitely near points starting with $P_0 = P \in X$:

$$Y = Y_d = \text{Bl}_{P_{d-1}} Y_{d-1} \rightarrow \cdots \rightarrow Y_2 = \text{Bl}_{P_1} Y_1 \rightarrow Y_1 = \text{Bl}_{P_0} X \rightarrow X$$

which resolves the base locus of the linear system $|F_1, F_2|$; define

$$\mathbf{M} = \overline{F_1 - P_0 - \cdots - P_{d-1}}.$$

We claim that $\mathbf{M}$ is $\mathbf{A}$ -saturated. Indeed, by construction, $\mathbf{M}$ descends to Y . Denoting by $E_i \subset Y$ the exceptional divisor above the point P_{i-1} , we have

$$[\mathbf{A}(X, B)_Y] = E_1 + \cdots + E_d.$$

It is easy to see that $\mathbf{M}$ is $\mathbf{A}$ -saturated; indeed

$$\begin{aligned} \text{Mob}(\mathbf{M}_Y + [\mathbf{A}(X, B)_Y]) &= \text{Mob}(\mathbf{M}_Y + E_1 + \cdots + E_d) = \\ &= \text{Mob}(f^*(F_1) - P_1 - \cdots - P_{d-1}) = f^*(F_1) - P_0 - \cdots - P_{d-1}. \end{aligned}$$

6.3. Preliminaries

In this section, we show that Theorem 6.1.5 implies Theorem 6.1.3.

PROPOSITION 6.3.1. *Let $(X, B) \rightarrow Z$ be a relative weak del Pezzo terminal pair and $\mathbf{M}$ a mobile $\mathbf{A}$ -saturated b-divisor on X . Then either*

- (a) $\mathbf{M}$ descends to X , or
- (b) $|H^0(X, \mathbf{M})|$ defines an elliptic fibration $g: Y \rightarrow \Delta$. If $F \subset Y$ is a fibre, then $[\mathbf{A}_Y] \cdot F = 1$. In particular, there is a unique component E of $[\mathbf{A}_Y]$ which meets F , $a(E, B) \leq 1$, and the set-theoretic base locus of $|H^0(X, \mathbf{M})|$ consists of the single point $P = c_X E \in X$. Or
- (c) X is a proper surface and $-(K_X + B) \cdot \mathbf{M}_X \leq 1$.

REMARK 6.3.2. (1) The proof of the proposition shows that, if $\mathbf{M}$ is (composed with) a pencil, then Case (a) or (b) holds.
 (2) It is natural to want to study Case (b) in greater detail; we do this in Lemma 6.4.5 in the next section, after additional preliminaries.
 (3) With more work, it is possible to show that the b-divisors in Case (c) of the theorem form, in a natural way, a *bounded family* $\mathfrak{F}$ of b-divisors. We do not pursue this here; in particular, we do not give a precise definition of a bounded family of b-divisors.

EXAMPLE 6.3.3. We give an example of Case (c). Let $E \subset X = \mathbb{P}^2$ be a nonsingular cubic; fix a positive integer $d \geq 4$. There are finitely many points p with the property that $\mathcal{O}_E(3dp) \simeq \mathcal{O}_E(d)$, but this does not hold for any smaller multiple. Pick anyone of these points $P \in E$, and let z be the length $3d$ curvilinear scheme with support P contained in E , $P \in z \subset E$. Let $\pi: Y \rightarrow X$ be the birational map determined by z , so that π is a sequence of $3d$ blow ups, with centre along E and its strict transforms.

Let C be the general element of the linear system of plane curves of degree which contain z . By our assumption on P , C is reduced and irreducible. We have

$$K_C = (K_X + C)|_C = C|_C - E|_C = C|_C - 3dP,$$

so that $C|_C = K_C + 3dP$. Let D be the strict transform of C in Y , and let $\mathbf{M} = \overline{D}$, considered as a b-divisor on X . By construction $D|_D = K_D$. Then $\mathbf{M}$ is mobile and big. As before, $\mathbf{M}$ is $\mathbf{A}(X, B)$ -saturated where $B = bE$, for any $1 - 1/(3d) \leq b \leq 1$. On the other hand, visibly $\mathbf{M}$ does not descend to X . Note that:

$$-(K_X + B) \cdot \mathbf{M}_X \leq \left(3 + 3 \frac{1 - 3d}{3d}\right)d = 1.$$

PROOF OF PROPOSITION 6.3.1. The statement extracts the part of Theorem 2.4.6 that still works; the proof is very similar. Let $f: Y \rightarrow X$ be a high enough log resolution of (X, B) such that

- $\mathbf{A}$ -saturation holds on Y , and
- $\mathbf{M}$ descends to Y , that is, $|\mathbf{M}_Y| = |H^0(Y, \mathbf{M})|$ is free.

CLAIM 6.3.4. *The divisor $E = [\mathbf{A}_Y]$ is integral, f -exceptional, and:*

- (1) every f -exceptional divisor appears in E with > 0 coefficient, that is, the support of E is all of the exceptional set,
- (2) $H^1(Y, E) = (0)$.

The first part of the claim is obvious; to prove the second part, note that

$$-f^*(K_X + B) = -K_Y + \mathbf{A}_Y$$

is nef and big, hence $H^1(Y, \lceil \mathbf{A}_Y \rceil) = H^1(Y, E) = (0)$. We are assuming that $\mathbf{M}$ is $\mathbf{A}$ -saturated; in particular, this implies that

$$E = \text{Bs} |\mathbf{M}_Y + E|.$$

A general member M_Y of $|\mathbf{M}_Y|$ is connected unless possibly in the case where $|\mathbf{M}_Y|$ is composed with a pencil. If we write $M_Y = \coprod_{i=0}^k M_i$, where the M_i are the connected components, then vanishing ensures that the restriction map

$$H^0(Y, \mathbf{M}_Y + E) \rightarrow H^0(M_0, (\mathbf{M}_Y + E)|_{M_0})$$

is surjective, therefore $E \cap M_0 = \text{Bs} |H^0(M_0, (\mathbf{M}_Y + E)|_{M_0})|$.

If M_0 is an affine curve, this implies that $E \cap M_0 = \emptyset$ and $\mathbf{M}$ descends to X , as in Theorem 2.4.6. (This happens, for instance, if $X \rightarrow Z$ is birational; here, however, we are not assuming that $X \rightarrow Z$ is birational.) From now on, we assume that M_0 is a (nonsingular) proper curve. We have

$$\begin{aligned} (\mathbf{M}_Y + E)|_{M_0} &= (M_0 + \lceil \mathbf{A}_Y \rceil)|_{M_0} = \\ &= (M_0 + K_Y + \lceil -f^*(K_X + B) \rceil)|_{M_0} = K_{M_0} + D \end{aligned}$$

where $D = \lceil -f^*(K_X + B) \rceil|_{M_0}$ is a divisor of degree $\deg D > 0$. Indeed, because $-(K_X + B)$ is nef and big, $-f^*(K_X + B) \cdot C > 0$ for all but finitely many curves $C_i \subset Y$ for which $-f^*(K_X + B) \cdot C_i = 0$. Because M_0 is a mobile curve, M_0 is not one of the C_i .

Elementary and well known properties of linear series on nonsingular proper algebraic curves imply that:

- Either:** $|H^0(M_0, (\mathbf{M}_Y + E)|_{M_0})|$ is base point free, and then $\mathbf{M}$ descends to X ,
Or: $\text{Bs} |H^0(M_0, (\mathbf{M}_Y + E)|_{M_0})| = E|_{M_0} = P \in M_0$ consists of a single point and $\deg D = 1$.

In the first case, we are in Case (a) of the statement. In the second case, we consider two subcases.

If $|\mathbf{M}_Y|$ is (composed with) a pencil, then it defines a fibration $g: Y \rightarrow \Delta$ to a curve. It follows that $\mathbf{M}_Y|_{M_0} = 0$, hence

$$K_{M_0} + D = (\mathbf{M}_Y + E)|_{M_0} = E|_{M_0}$$

has degree 1, which implies that $K_{M_0} = 0$, M_0 is a curve of genus 1 and we are in Case (b) of the statement.

Otherwise, $|\mathbf{M}_Y|$ is not (composed with) a pencil, $M_Y = M_0$ is connected, and then we are in Case (c) of the statement. $\square$

THEOREM 6.1.5 IMPLIES THEOREM 6.1.3. Because $-(K + B)$ is nef and big, the Mori cone $\overline{\text{NE}}(X'/Z)$ is a finitely generated polyhedron; in addition, every extremal ray is generated by a rational curve on X . Therefore the set of Z -morphisms $\varphi: X' \rightarrow T$ with $\varphi_* \mathcal{O}_{X'} = \mathcal{O}_T$ is finite. In (1), we take this finite set.

When $Z = \{\text{pt}\}$, in (2) we take the finite set of Theorem 6.1.5(2).

With these choices, we show that the statement holds. Let $\mathbf{M}$ be a mobile and $\mathbf{A}$ -saturated b-divisor. If we are in Case (a) of Proposition 6.3.1, then $\mathbf{M}$ descends to X' , hence $|\mathbf{M}|$ defines a morphism; the Stein factorisation of this morphism is one of the $\varphi_i: X' \rightarrow T_i$. If we are in Case (b) of Proposition 6.3.1, then the statement follows from the corresponding bit in Theorem 6.1.5. If we are in Case (c) of Proposition 6.3.1, the statement is made in Proposition 6.3.1. $\square$

6.4. Mobile b-divisors of Iitaka dimension one

The aim of this section is to prove Theorem 6.1.5. The key result is Lemma 6.4.5, where we study in detail Case (b) of Proposition 6.3.1.

DEFINITION 6.4.1. A mobile b-divisor $\mathbf{M}$ on a variety X is *indecomposable* if a general member of $|\mathbf{M}|$ is irreducible. Otherwise we say that $\mathbf{M}$ is *decomposable*.

REMARK 6.4.2. (1) If $\mathbf{M}$ is decomposable, then $|\mathbf{M}|$ is composed with a pencil. If $\mathbf{M}$ descends to Y , then we can write $\mathbf{M}_Y = \sum_{i=0}^k M_i$ where the M_i move in a connected, possibly irrational, pencil. If X is a rational surface, then the pencil is rational. In this case, we can write

$$\mathbf{M} = \sum_{i=0}^k \mathbf{M}_i$$

where $\mathbf{M}_{iY} = M_i$, and the $\mathbf{M}_i$ are indecomposable and all linearly equivalent to $\mathbf{M}_0$.

(2) If $\mathbf{M}$ is $\mathbf{A}$ -saturated, then $\mathbf{M}_0$ is also $\mathbf{A}$ -saturated. Thus, in the proof of Theorem 6.1.5, we can and do restrict ourselves to indecomposable b-divisors.

LEMMA 6.4.3. Let X be a nonsingular surface, $B = bB_0$ where $B_0 \subset X$ is a curve and $0 \leq b < 1$, and let $P_0 = P \in B_0$ be a nonsingular point. Let $f: Y \rightarrow X$ be a birational morphism, which is the composition of k blowups of infinitely near points

$$Y = Y_k = \text{Bl}_{P_{k-1}} Y_{k-1} \rightarrow \cdots \rightarrow Y_2 = \text{Bl}_{P_1} Y_1 \rightarrow Y_1 = \text{Bl}_{P_0} X \rightarrow X.$$

Denote by $E \subset Y$ the last exceptional divisor extracted by f . If $a(E, B) \leq 1$, then either

- (1) $k = 1$, or
- (2) $k > 1$, the centre of each blow up lies on the strict transform of B_0 , and $b \geq 1 - 1/k$.

REMARK 6.4.4. When $b = 0$, $B = \emptyset$ and we conclude that $k = 1$.

PROOF. The statement is obvious if $k = 1$. If $k > 1$, we claim that the centre of each blow up lies on the strict transform of B_0 and

$$a(E, B) = k(1 - b).$$

Denote by $f_{k-1}: Y_{k-1} \rightarrow X$ the composition of the first $k - 1$ blowups and by $E_{k-1} \subset Y_{k-1}$ the last exceptional divisor extracted by f_{k-1} . If $a(E_{k-1}, B) = a_{k-1}$, then the coefficient of E_{k-1} in $f_{k-1}^*(K + B)$ is $-a_{k-1}$, hence

$$1 \geq a(E, B) \geq 1 - b + a_{k-1} > a_{k-1}.$$

It follows that $1 \geq a_{k-1}$. By induction, we may assume that the centres of the first $k-1$ blowups lie on the strict transform of B_0 and that

$$a_{k-1} = (k-1)(1-b).$$

If P_{k-1} is not on the strict transform of B_0 , then $a(E, B) \geq 1 + a_{k-1} > 1$, a contradiction. Therefore, P_{k-1} is on the strict transform of B_0 and

$$a(E, B) = 1 - b + a_{k-1} = k(1-b)$$

as was to be shown. Finally, $1 \geq k(1-b)$ implies $b \geq (k-1)/k$. $\square$

In the following lemma we study in detail Case (b) of Proposition 6.3.1.

LEMMA 6.4.5. *Let (X, B) be weak del Pezzo terminal surface pair and $\mathbf{M}$ an indecomposable $\mathbf{A}$ -saturated mobile b -divisor on X . Assume that we are in Case (b) of Proposition 6.3.1, that is, $\mathbf{M}$ does not descend to X and $|\mathbf{M}|$ is a pencil of elliptic curves. The following hold:*

1. *Let $P = \text{Bs } |H^0(X, \mathbf{M})| \in X$ be the base locus of $|H^0(X, \mathbf{M})|$ (by Proposition 6.3.1 this base locus consists of a single point). Then a general member $F \in |H^0(X, \mathbf{M})|$ is nonsingular at P and, if $F_1, F_2 \in |H^0(X, \mathbf{M})|$, then $F_1 \cdot F_2 = dP$ where $d = F^2$.*

2. *The divisor $\mathbf{M}_X$ is eventually free and defines a birational morphism*

$$\varphi: X \rightarrow \bar{X} = \text{Proj } \bigoplus_{n \geq 0} H^0(X, n\mathbf{M}_X)$$

where

- (1) $\mathbf{M}_{\bar{X}} \in |-K_{\bar{X}}|$,
- (2) $\bar{X}$ is a del Pezzo surface with canonical singularities and degree $K_{\bar{X}}^2 = d$,
- (3) if $\bar{B} = \varphi(B)$, then $(\bar{X}, \bar{B})$ is a klt del Pezzo pair.

3. *Write $B = \sum b_i B_i$; if $P \in B_0$ is a nonsingular point of $\text{Supp } B$, then $b_0 \geq 1 - 1/d$. If, in addition, $d > 1$, then $F \cdot B_0 = dP$ and $\bar{B}_0 = \varphi(B_0) \in |-K_{\bar{X}}|$; in particular, $\bar{B}_0$ is a curve of arithmetic genus 1.*

REMARK 6.4.6. In **2**, we are not saying that, as a b -divisor on $\bar{X}$, $\mathbf{M}$ is $\mathbf{A}(\bar{X}, \bar{B})$ -saturated.

PROOF. The first statement follows immediately from Lemma 6.4.3.

Because $\mathbf{M}$ is mobile and X is a surface, it follows that $\mathbf{M}_X$ is nef. The base point free theorem implies that $\mathbf{M}_X$ is eventually free. By the first statement, a general member $F \in |\mathbf{M}| \subset |\mathbf{M}_X|$ is a nonsingular elliptic curve. Consider the exact sequence

$$0 \rightarrow \mathcal{O}_X(K_X) \rightarrow \mathcal{O}_X(K_X + F) \rightarrow \mathcal{O}_F \rightarrow 0.$$

Since X is a rational surface, $H^0(X, K_X) = H^1(X, K_X) = (0)$; from the exact sequence, we then get $H^0(X, K_X + F) = H^0(\mathcal{O}_F) = \mathbb{C}$. This shows that $K_X + F \sim E \geq 0$ is linearly equivalent to an effective divisor which is exceptional over $\bar{X}$; hence, $\varphi(F) \in |-K_{\bar{X}}|$ and $\bar{X}$ is a Gorenstein del Pezzo surface.

The divisor $-(K + B)$ is nef and big; therefore, we can write

$$(K + B) - \sum a_i E_i = \varphi^*(K_{\bar{X}} + \bar{B})$$

where the E_i are exceptional and all $a_i \geq 0$. It follows that $(\bar{X}, \bar{B})$ is a klt del Pezzo pair. In particular $\bar{X}$ has rational Gorenstein, hence canonical singularities. This shows the second statement.

By Lemma 6.4.3, $b_0 \geq 1 - 1/d$ and $(B_0 \cdot F)_P = dP$, while it is still possible that $F \cdot B_0 > d$.

CLAIM 6.4.7. *If $d \geq 2$, then either $B_0 \cdot F = d$ or $B_0 \cdot F = d + 1$.*

Later we show that in the first case $\bar{B}_0 = \varphi(B_0) \in |-K_{\bar{X}}|$, and that the second case does not occur. To show the claim, note that

$$0 < -(K + B) \cdot F \leq -(K + b_0 B_0) \cdot F = d - b_0 B_0 \cdot F \leq d - \frac{d-1}{d} B_0 \cdot F,$$

thus $(d-1)B_0 \cdot F < d^2$ and the claim follows.

Denote by $Y \rightarrow X$ a resolution of the base locus of $|\mathbf{M}|$ and by $g: Y \rightarrow \Delta$ the associated elliptic fibration.

If $B_0 \cdot F = d$, the strict transform $B'_0 \subset Y$ is contained in a fibre. Since $B_0 \cdot F = d$, $\bar{B}_0 = \varphi(B_0) \in |-K_{\bar{X}}|$ and then $\bar{B}_0$ is a curve of arithmetic genus 1.

In the remaining part of the proof we derive a contradiction from the assumption that $d \geq 2$ and $B_0 \cdot F = d + 1$. The strict transform $B'_0 \subset Y$ is a section of $g: Y \rightarrow \mathbb{P}^1$; hence $B'_0 = B_0 = \mathbb{P}^1$. The linear system

$$|\mathbf{M}|_{|B_0} - dP$$

is a pencil of divisors of degree 1; therefore, $\varphi(B_0) = \bar{B}_0 = \mathbb{P}^1$. The morphism $\varphi: X \rightarrow \bar{X}$ factors through the minimal resolution $\tilde{X} \rightarrow \bar{X}$ of $\bar{X}$. Denote by $\psi: X \rightarrow \tilde{X}$ the induced morphism, write $\tilde{B} = \psi(B)$, and $\tilde{B}_0 = \psi(B_0)$; it is easy to see, as above, that $(\tilde{X}, \tilde{B})$ is a weak del Pezzo klt surface pair.

CLAIM 6.4.8. $H^1(\tilde{X}, -K_{\tilde{X}} - \tilde{B}_0) = (0)$.

To show the claim, write $\tilde{B} = b\tilde{B}_0 + C + D$ where

$$C = \sum c_i C_i \text{ with all } c_i \geq 1/2 \quad \text{and} \quad D = \sum d_i D_i \text{ with all } d_i < 1/2.$$

By the vanishing theorem of Kawamata and Viehweg,

$$(0) = H^i(\tilde{X}, -K_{\tilde{X}} + \lceil -2\tilde{B} \rceil) = H^i(\tilde{X}, -\tilde{B}_0 - \lceil C \rceil)$$

for $i > 0$. We have an exact sequence:

$$\begin{aligned} (0) = H^1(\tilde{X}, -K_{\tilde{X}} - \tilde{B}_0 - \lceil C \rceil) &\rightarrow H^1(\tilde{X}, -K_{\tilde{X}} - \tilde{B}_0) \rightarrow \\ &\rightarrow H^1(\lceil C \rceil, (-K_{\tilde{X}} - \tilde{B}_0)|_{\lceil C \rceil}). \end{aligned}$$

The claim follows once we show that the group on the right vanishes. First of all $-K_{\tilde{X}}$ is nef and big ($\tilde{X} \rightarrow \bar{X}$ is the minimal resolution of a del Pezzo surface with canonical singularities); it follows that $-K_{\tilde{X}} \cdot C_i = 0$ for all i , for otherwise $-K_{\tilde{X}} \cdot C_i > 0$ for some i and then we would have:

$$\begin{aligned} 0 < -K_{\tilde{X}} \cdot (-K_{\tilde{X}} - \tilde{B}) &= -K_{\tilde{X}} \cdot (-K_{\tilde{X}} - b\tilde{B}_0 - C - D) \leq \\ &\leq -K_{\tilde{X}} \cdot \left(-K_{\tilde{X}} - \frac{d-1}{d}\tilde{B}_0 - \frac{1}{2}\sum C_i \right) = \\ &= d - \frac{d^2-1}{d} - \frac{1}{2}\sum (-K_{\tilde{X}}) \cdot C_i \leq \frac{1}{d} - \frac{1}{2}, \end{aligned}$$

which is a contradiction if $d \geq 2$. It follows that $\lceil C \rceil$ is a disjoint union of reduced ADE cycles. If G is a connected component of $\lceil C \rceil$, then $\tilde{B}_0|_G$ is nef and, because

$B_0 = \tilde{B}_0 = \overline{B}_0 = \mathbb{P}^1$, necessarily $\tilde{B}_0 \cdot G \leq 1$, hence $H^1(G, \tilde{B}_{0|G}) = (0)$. We conclude that

$$H^1(\lceil C \rceil, (-K_{\tilde{X}} - \tilde{B}_0)|_{\lceil C \rceil}) = H^1(\lceil C \rceil, -\tilde{B}_{0|\lceil C \rceil}) = (0)$$

and the claim follows.

It follows from the claim that the restriction

$$H^0(\tilde{X}, -K_{\tilde{X}}) \rightarrow H^0(\tilde{B}_0, -K_{\tilde{X}|\tilde{B}_0})$$

is surjective. This is a contradiction, because the group on the left has dimension $d + 1$, while the group on the right has dimension $d + 2$. This contradiction shows that the second case of Claim 6.4.7 does not occur and thus completes the proof. $\square$

COROLLARY 6.4.9. *Let (X, B) be a weak del Pezzo terminal pair. The set of nonsingular points $P \in \text{Supp } B$ such that*

- (1) $P = \text{Bs } |\mathbf{M}|$ for some $\mathbf{M}$ as in Case (b) of Proposition 6.3.1, and
- (2) in the notation of Lemma 6.4.5, $d \geq 2$

is finite.

PROOF. Because B has a finite number of components, it is sufficient to show that there are finitely many such points on each component of B . Let B_0 be a component of B and $P, Q \in B_0$ points with the stated property. Then

$$P - Q \in \text{Pic } \overline{B}_0$$

is a d -torsion point; by Lemma 6.4.5(3), if $d \geq 2$, then $\overline{B}_0$ is a curve of arithmetic genus 1, hence the d -torsion of $\text{Pic } \overline{B}_0$ is finite. $\square$

PROOF OF THEOREM 6.1.5. We construct a finite set $\mathfrak{F}$ of surfaces Y , birational morphisms $Y \rightarrow X$ and fibrations $Y \rightarrow \mathbb{P}^1$. We take $\mathfrak{F} = \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \mathfrak{F}_3 \cup \mathfrak{F}_4$ where $\mathfrak{F}_i$ are defined as follows:

First, $\mathfrak{F}_1$ is the set of all morphisms $\varphi: X \rightarrow \mathbb{P}^1$ with $\varphi_* \mathcal{O}_X = \mathcal{O}_{\mathbb{P}^1}$; this set is finite because the Mori cone $\overline{\text{NE}} X$ is polyhedral.

Consider now the finite set $\mathfrak{F}_2$ of birational morphisms $\varphi: X \rightarrow \overline{X}$ where $\overline{X}$ is a del Pezzo surface with canonical singularities and degree $d = 1$, and φ is an isomorphism above the base point P of $|-K_{\overline{X}}|$. To an element of $\mathfrak{F}_2$ we associate the surface $Y \rightarrow X$ obtained by blowing up the unique point above P , and we let $Y \rightarrow \mathbb{P}^1$ be the obvious elliptic fibration.

By Corollary 6.4.9, there are only finitely many nonsingular points $P \in \text{Supp } B$ that can be the base point of a mobile $\mathbf{A}$ -saturated b-divisor $\mathbf{M}$ on X with the properties stated there; $\mathfrak{F}_3$ is the set of surfaces $Y \rightarrow X$ obtained by blowing up d points above P in the strict transform of B .

Finally, $\mathfrak{F}_4$ is the set of surfaces $E \subset Y \rightarrow P \in X$ which extract all valuations E as in Case (b) of Proposition 6.3.1 with centre a singular point of $\text{Supp } B$. This set is finite since, for every point $P \in X$, there are finitely many valuations E such that $a(E, B) \leq 1$.

Let now $\mathbf{M}$ be a mobile $\mathbf{A}$ -saturated b-divisor on X and assume that $\mathbf{M}$ is indecomposable and $|\mathbf{M}|$ is composed with a pencil. If $\mathbf{M}$ descends to X , then it appears in $\mathfrak{F}_1$. Otherwise, we are in Case (b) of Proposition 6.3.1 and either $d = 1$, or $d \geq 2$. If $d = 1$, by Lemma 6.4.5, $\mathbf{M}$ is listed in $\mathfrak{F}_2$. If $d \geq 2$, then either P is a nonsingular point of $\text{Supp } B$, in which case $\mathbf{M}$ is listed in $\mathfrak{F}_3$, or P is a singular point of $\text{Supp } B$ and in this case $\mathbf{M}$ is listed in $\mathfrak{F}_4$. $\square$

CHAPTER 7

Confined divisors

JAMES M^cKERNAN

7.1. Introduction

To any graded functional algebra $R \subset k(X)[t]$ on a normal variety X , one can associate a sequence of mobile b-divisors $\mathbf{M}_\bullet$ which satisfy an obvious additivity relation

$$\mathbf{M}_i + \mathbf{M}_j \leq \mathbf{M}_{i+j}.$$

This sequence gives rise to the characteristic system $\mathbf{D}_\bullet$, defined by

$$\mathbf{D}_i = \frac{\mathbf{M}_i}{i},$$

which in turn gives rise to an integral extension of graded algebras $R \subset R(X, \mathbf{D}_\bullet)$; the algebra $R(X, \mathbf{D}_\bullet)$ is known as a pbd-algebra.

Shokurov isolates two properties of pbd-algebras:

- (1) *boundedness* $\mathbf{D}_i \leq \mathbf{D}$, $\mathbf{D}$ fixed,
- (2) *asymptotic canonical saturation* there is a natural number d such that

$$\text{Mob}[j\mathbf{D}_i + \mathbf{A}] \leq j\mathbf{D}_j,$$

whenever d divides i and j ,

which conjecturally imply that the given algebra is finitely generated:

CONJECTURE 7.1.1 (FGA). *Suppose that we are given a pbd-algebra $R = R(X, \mathbf{D}_\bullet)$, a morphism $X \rightarrow Z$ and a boundary Δ , where*

- (1) $K_X + \Delta$ is kawamata log terminal,
- (2) $X \rightarrow Z$ is proper, and Z is affine,
- (3) $-(K_X + \Delta)$ is relatively big and nef,
- (4) $\mathbf{D}_\bullet$ is bounded, and
- (5) $\mathbf{D}_\bullet$ is asymptotically canonically saturated.

Then R is finitely generated.

In fact any pbd-algebra which satisfies (1-5) is called a Shokurov algebra, so that 7.1.1 is equivalent to the conjecture that every Shokurov algebra is finitely generated.

To prove 7.1.1 the key point is to prove that the characteristic system stabilises. In this chapter, among other things, we prove $(\text{FGA})_{\leq 2}$; at the moment, we are unable to prove $(\text{FGA})_3$.

Shokurov is able to state a general conjecture, the CCS conjecture, whose statement is independent of the existence of the ambient variety X . $(\text{CCS})_n$ implies $(\text{FGA})_n$ which in turn implies flips exist in dimension $n + 1$.

DEFINITION 7.1.2. Let $(X, B = B_X)$ be a pair of a normal variety and a $\mathbb{Q}$ -divisor B , where we always assume that $K_X + B$ is $\mathbb{Q}$ -Cartier. A *crepant model* of (X, B_X) is a pair (Y, B_Y) , together with a birational map $\pi : X \dashrightarrow Y$ such that for some commutative diagram

$$\begin{array}{ccc} & Z & \\ p \swarrow & & \searrow q \\ X & \dashrightarrow \pi \dashrightarrow & Y. \end{array}$$

we have

$$p^*(K_X + B_X) = K_Z + B_Z = q^*(K_Y + B_Y).$$

In particular note that the discrepancy b -divisor is invariant up to a choice of crepant model.

7.2. Diophantine Approximation and Descent

In this section we recall some of the arguments that follow §2.4.3. The idea there was to recycle the one dimensional case. The problem in higher dimensions is that $\mathbf{M}_i X$ need not be base point free. Of course for each i we can find Y_i so that $\mathbf{M}_i Y_i$ is base point free. The problem is to find one model Y that works universally for all i .

In dimension two the choice of this model presents no problems; pick Y to be the unique terminal model of the klt pair (X, B) . In higher dimensions the choice of Y is more problematic. As usual, write $\mathbf{D}_i = (1/i)\mathbf{M}_i$, $\mathbf{D} = \lim \mathbf{D}_i$ (assuming that it exists). It turns out that a minimal reasonable requirement is that $\mathbf{D}_Y$ is nef; in practice we require even more, namely that $\mathbf{D}_Y$ is semiample.

We first recall the limiting criterion, see Section 2.3.7.

THEOREM 7.2.1. (*Limiting Criterion*) Let X be a normal variety and let $X \rightarrow Z$ be a proper birational morphism to an affine variety.

Then the pbd-algebra $R = R(X, \mathbf{D}_\bullet)$ is finitely generated iff there is an integer k such that the subsequence $\mathbf{D}_{k\bullet}$ is constant.

DEFINITION 7.2.2. Let $\mathbf{D}$ be a b -divisor. The *obstruction divisor* $\mathbf{O} = \mathbf{O}(X, \mathbf{D})$ for some model X is

$$\mathbf{O} = \overline{\mathbf{D}}_X - \mathbf{D}.$$

REMARK 7.2.3. The obstruction divisor is exceptional over X . It follows that the obstruction divisor only depends on the numerical class of $\mathbf{D}$. Note also that the obstruction divisor is effective provided that $\mathbf{D}$ is b -nef.

THEOREM 7.2.4. Let (X, B) be a klt pair and let $\pi : X \rightarrow Z$ be a proper birational morphism to the affine variety Z . Let $R = R(X, \mathbf{D}_\bullet)$ be a pbd-algebra on X , and let $\mathbf{D} = \lim_i \mathbf{D}_i$.

Suppose that we may find a variety Y/Z , such that

- (1) $\mathbf{D}_Y$ is semiample.
- (2) There is a sequence of positive integers $r_1, r_2, \dots$, such that

$$[\mathbf{A}(X, B) - r_i \mathbf{O}_i] \geq 0 \quad \text{where} \quad \lim_{i \rightarrow \infty} r_i = \infty \quad \text{and} \quad \mathbf{O}_i = \mathbf{O}(Y, \mathbf{D}_i).$$

Then R is finitely generated, $\mathbf{D}$ descends to Y and $\mathbf{D}_i = \mathbf{D}$, for i large enough.

PROOF OF 7.2.4. We follow the proof of the Nonvanishing Lemma closely. By assumption there is a fixed effective divisor G on Y such that the support of $\mathbf{D}_{iY}$ is contained in G . Pick a rational number $\gamma > 0$ such that the pair $(Y, B_Y + \gamma G)$ is klt. Then for any rational number $0 < \epsilon \leq \gamma$, we have $\lceil \mathbf{A}(Y, B_Y + \epsilon G) \rceil \geq 0$.

By Lemma 2.4.12, there is a positive integer m and an integral divisor M on Y with the following properties,

- (i) the linear system $|M|$ is base point free,
- (ii) $\|m\mathbf{D}_Y - M\| < \epsilon/4$, and
- (iii) If $m\mathbf{D}_Y - M$ is effective then $m\mathbf{D}_Y = M$.

Now pick $i \gg m$, such that

$$\|m\mathbf{D}_{iY} - m\mathbf{D}_Y\| < \epsilon/4.$$

If we set $F = m\mathbf{D}_{iY} - M$, the triangle inequality yields

$$\|F\| \leq \|m\mathbf{D}_{iY} - m\mathbf{D}_Y\| + \|m\mathbf{D}_Y - M\| \leq \epsilon/2.$$

Hence for every model $g: W \rightarrow Y$ we have

$$\begin{aligned} m\mathbf{D}_{iW} &= g^*(m\mathbf{D}_{iY}) - m\mathbf{O}_{iW} = g^*M + g^*F - m\mathbf{O}_{iW} \geq \\ &\quad g^*M + 1/2g^*(-\epsilon G) - m\mathbf{O}_{iW}. \end{aligned}$$

Thus

$$\begin{aligned} m\mathbf{D}_{iW} + \mathbf{A}(Y, B_Y) &\geq \\ &\quad g^*M + 1/2\mathbf{A}(X, B + \epsilon G)_W + 1/2(\mathbf{A}(X, B)_W - 2m\mathbf{O}_{iW}). \end{aligned}$$

Since $i \gg m$, we may assume that $r_i > 2m$. Rounding up, and taking the mobile part, we then get $\text{Mob}[m\mathbf{D}_{iW} + \mathbf{A}_W] \geq g^*M$.

On the other hand, by asymptotic saturation, we may find a good resolution $g: W \rightarrow Y$, such that $\text{Mob}[m\mathbf{D}_{iW} + \mathbf{A}_W] \leq m\mathbf{D}_{mW}$. But then $g^*M \leq m\mathbf{D}_{mW}$ so that, pushing down to Y , $M \leq m\mathbf{D}_{mW}$. It follows by (iii) that $\mathbf{D}_i$ is eventually constant, and we must have equality in the inequalities above, so that $\mathbf{D}_i = \overline{M} = \mathbf{D}$. $\square$

Thus there is now a clear strategy to prove $(\text{FGA})_n$. Find sufficient conditions to guarantee the existence of a model satisfying (1) and (2) of 7.2.4. In the next section we focus on condition (2), since this condition is quite novel. Shokurov's clever idea is to drop any reference to the sequence of b-divisors $\mathbf{D}_\bullet$ and work instead with general canonically saturated b-divisors.

7.3. Confined b-divisors

DEFINITION 7.3.1. We will say that the pair (X, Δ) is *canonical* if every exceptional divisor has discrepancy ≥ 0 and if the coefficient of every component of Δ is ≤ 1 .

DEFINITION 7.3.2. Let X be a variety and let B be a divisor such that $K_X + B$ is terminal. Let $\mathfrak{F}$ be a set of divisors. We say that $\mathfrak{F}$ is *canonically confined by c* if for every $G \in \mathfrak{F}$, $K_X + B + cG$ is canonical.

We say that $\mathfrak{F}$ is *canonically confined by c up to linear equivalence* if for every $G' \in \mathfrak{F}$ there exists $G \in |G'|$ such that $K_X + B + cG$ is canonical.

There are similar definitions, where canonical is replaced by log canonical. There are similar notions for b-divisors.

Note the following subtlety:

EXAMPLE 7.3.3. Linear equivalence of b-divisors makes sense. We define $|\mathbf{M}|$ to be the linear system of all effective b-divisors linearly equivalent to $\mathbf{M}$. We have

$$|\mathbf{M}|_X \subset |\mathbf{M}_X|,$$

and in general the inequality is strict. For example let L be a line in $X = \mathbb{P}^2$ and let $\mathbf{M}$ be the b-divisor given by the single prime divisor L . Then $\mathbf{M}_X = L$ so that $|\mathbf{M}_X|$ is the complete linear system of lines in $\mathbb{P}^2$. However $|\mathbf{M}|_X = \{L\}$. A more interesting example arises if we take $Y \rightarrow X$ the blow up of $\mathbb{P}^2$ at a point $p \in L$, M the strict transform of L and we set $\mathbf{M} = \overline{M}$. In this case $|\mathbf{M}_X|$ is again the complete linear system of lines in $\mathbb{P}^2$ and $|\mathbf{M}|_X$ is the linear system of lines through p . The whole point is that $\mathbf{M}$ is not exceptionally saturated. Indeed

$$\text{Mob}(\mathbf{M}_Y + E) = \text{Mob}(M + E) = M + E > M = \mathbf{M}_Y.$$

LEMMA 7.3.4. *Suppose that we have a sequence of b-divisors $\mathbf{D}_\bullet$ on $(X/Z, B)$. Suppose that the set $\mathfrak{F} = \{\mathbf{M}_i\}$ is log canonically confined by c up to linear equivalence on a crepant model (Y, B_Y) .*

Then (2) of 7.2.4 holds.

PROOF. By assumption, for every i , we may find $\mathbf{G}_i \in |\mathbf{M}_i|$ such that $(Y, B + c\mathbf{G}_i)$ is klt, that is $\lceil \mathbf{A}(X, B + c\mathbf{G}_i) \rceil \geq 0$, where G_i is the trace of $\mathbf{G}_i$ on Y . Hence

$$\begin{aligned} \lceil \mathbf{A}(X, B) - ci\mathbf{O}(\mathbf{D}_i) \rceil &= \lceil \mathbf{A}(X, B) - c\mathbf{O}(\mathbf{M}_i) \rceil = \lceil \mathbf{A}(Y, B_Y) - c\mathbf{O}(\mathbf{G}_i) \rceil \\ &= \lceil c\mathbf{G}_i + \mathbf{A}(Y, B_Y) - c\overline{G}_i \rceil = \lceil c\mathbf{G}_i + \mathbf{A}(Y, B_Y + cG_i) \rceil \geq 0. \end{aligned}$$

Now set $r_i = ci$. □

Thus we would like to find sufficient conditions ensuring that a set of b-divisors is log canonically confined up to linear equivalence.

LEMMA 7.3.5. *Let (X, Δ) be a pair, $\pi: Y \rightarrow X$ a birational morphism and E an exceptional divisor extracted by π . Let D be any $\mathbb{Q}$ -Cartier divisor. Let us write*

$$K_Y + \Gamma = \pi^*(K_X + \Delta), \quad \tilde{D} + bE = \pi^*D, \quad \text{and} \quad K_Y + \Gamma' = \pi^*(K_X + \Delta + aD).$$

If the coefficient of E in Γ is e and in Γ' is e' , then $e' = e + ab$.

PROOF. Clear. □

LEMMA 7.3.6. *Let $\mathfrak{F}$ be a set of divisors.*

- (1) *If $\mathfrak{F}$ is canonically confined then it is log canonically confined. The converse holds provided that $K_X + \Delta$ is terminal.*
- (2) *$\mathfrak{F}$ is log canonically confined iff the log canonical threshold of the elements of $\mathfrak{F}$ is bounded away from zero.*
- (3) *Suppose that X is smooth and the pair (X, Δ) is terminal. Then $\mathfrak{F}$ is canonically confined iff the multiplicity of the elements of $\mathfrak{F}$ is bounded.*
- (4) *If $\mathfrak{F}$ forms a bounded set (in the sense of moduli) then $\mathfrak{F}$ is canonically confined.*
- (5) *If every element of $\mathfrak{F}$ is base point free then $\mathfrak{F}$ is canonically confined up to linear equivalence.*

PROOF. We first prove (3). Suppose that X is smooth. It is clear by Lemma 7.3.5 that if $\mathfrak{F}$ is log canonically confined, then the multiplicity of the elements of $\mathfrak{F}$ is bounded. Now suppose that the pair (X, Δ) is terminal. Then the log discrepancy is at least $1 + \epsilon$, for some $\epsilon > 0$. If the multiplicity is bounded, then we may pick c so that for every $G \in \mathfrak{F}$, the multiplicity of cG is at most $c\epsilon$. It follows by Lemma 7.3.5 again, and an obvious induction, that $(X, \Delta + cG)$ is canonical.

Now we turn to (1). One direction is clear. So suppose that $\mathfrak{F}$ is log canonically confined by c . Pick a good resolution $\pi: Y \rightarrow X$; write

$$\pi^*(K + \Delta) = K_Y + \Delta' + E$$

where $\Delta' = \pi_*^{-1}\Delta$ (by construction, $K_Y + \Delta' + E$ is a terminal sub-boundary) and let $\mathfrak{F}^* = \{\pi^*G \mid G \in \mathfrak{F}\}$. By construction, for every $G^* \in \mathfrak{F}^*$, $K_Y + \Delta' + E + cG^*$ is a log canonical sub-boundary. It follows that $\mathfrak{F}^*$ has bounded multiplicities; therefore, by what we just said, $\mathfrak{F}^*$ is canonically confined on the pair (Y, Δ') ; from this it follows that $\mathfrak{F}$ is canonically confined on (X, Δ) .

The other cases are easy. $\square$

There are similar notions for b-divisors and linear equivalence, whenever appropriate. In particular (5) holds for b-divisors associated to base point free divisors.

(4) and (5) in conjunction present a strategy to prove that a set $\mathfrak{F}$ of b-divisors is log canonically confined up to linear equivalence; prove that the subset of b-divisors that are not base point free is bounded. In practice it turns out that, at least for a surface, it is easier to prove this on a crepant terminal pair. For this reason Shokurov prefers to work with the notion of canonically confined.

7.4. The CCS conjecture

In this section we state a version of the CCS conjecture, and we develop some general techniques to attack this conjecture. Unfortunately it is not clear how to formulate the CCS conjecture. Clearly we want a conjecture that is strong enough to imply the FGA conjecture. On the other hand, we want a statement which does indeed hold, and which someone might hopefully prove in the future.

With this said, in the interests of pedagogy, we give the easiest version of the CCS conjecture to state, and we have limited ourselves to stating only one version. Anyone with a serious interest in this topic is well advised to consult the paper of Shokurov [Sho03], which by way of contrast states many different conjectures, which are all variations on the following theme:

CONJECTURE 7.4.1 (CCS). *Suppose that $K_X + B$ is klt and $-(K_X + B)$ is nef and big over Z (respectively fix a rational map $X \dashrightarrow T$ over Z , with normalized graph*

$$\begin{array}{ccc} & \Gamma & \\ p \swarrow & & \searrow q \\ X & \text{-----} & T \end{array}$$

such that $q_\mathcal{O}_\Gamma = \mathcal{O}_T$). Then there is a constant $c > 0$ and a bounded family (respectively finite set)*

$$\{(Y_t, T_t) \mid t \in U\},$$

with the following properties:

For every $t \in U$, (Y_t, B_t) is a crepant $\mathbb{Q}$ -factorial model of X and there are contraction morphisms $\psi_t: Y_t \rightarrow T_t$, $\phi_t: T_t \rightarrow Z$, where the relative cone of nef divisors of ϕ_t is spanned by a finite number of semiample divisors. Moreover for every mobile and $\mathbf{A} = \mathbf{A}(X, B)$ -saturated b -divisor $\mathbf{M}$ (respectively convex sequence $\mathbf{M}_\bullet$ of mobile and $\mathbf{A}$ -saturated b -divisors, where $\mathbf{M} = \mathbf{M}_i$), we may find $t \in U$ with the following properties

- (1) We may find $G \in |\mathbf{M}|_{Y_t}$ such that $(Y_t, B_{Y_t} + cG)$ is log canonical,
- (2) $\mathbf{M}_Y = \psi_t^* M$ where M is a big and nef divisor on T_t .

Note that we do not require that B_t is effective in the CCS conjecture (although see the remark below).

THEOREM 7.4.2. *$(\text{CCS})_n$ implies $(\text{FGA})_n$.*

PROOF. We are given a convex bounded canonically asymptotically saturated sequence $\mathbf{M}_\bullet$ of mobile b -divisors and it suffices to prove that the sequence $\mathbf{D}_\bullet$ is eventually constant. By convexity, possibly up to truncation, the divisors $\mathbf{M}_i$ define the same contraction. By $(\text{CCS})_n$ we may assume that there is a fixed pair $(Y/T/Z, B_Y)$ such that (1) and (2) of Conjecture 7.4.1 holds for infinitely many $\mathbf{M}_i$. As $\mathbf{M}_i$ is pulled back from T , then so is $\mathbf{D}_Y$ and as every nef divisor on T is semiample, $\mathbf{D}_Y$ is semiample. Now apply Theorem 7.2.4. $\square$

REMARK 7.4.3. Given $\mathbf{M}$, the CCS conjecture imposes three non-trivial conditions on the model Y_t :

- (a) We may find $G \in |\mathbf{M}|_{Y_t}$ such that $(Y_t, B_{Y_t} + cG)$ is log canonical,
- (b) $\mathbf{M}_{Y_t}$ is nef, and
- (c) $\mathbf{M}_{Y_t}$ is semiample.

Now we only expect conditions (a) and (b) to hold on a relatively high model. Indeed roughly speaking one needs to eliminate any component of the base locus (certainly those of high multiplicity).

On the other hand one only expects condition (c) to hold on a relatively low model. Indeed, in practice, the way to show (c) is to show first that the cone of nef divisors of T_t/Z is spanned by a finite number of semiample divisors, and this is a very stringent condition. In practice, the most natural way to satisfy this condition is to require that there is an effective divisor Θ on T such that $-(K_T + \Theta)$ is nef and big over Z , where $K_T + \Theta$ is klt. Thus if ψ_t is birational, that is $\mathbf{M}$ is big, then we need Y_t to be Fano, or at least close to Fano.

Since we have two conflicting requirements on Y_t , the CCS conjecture seems very hard.

Here is an example to illustrate the fact that we need to work on more than one model:

EXAMPLE 7.4.4. Let $f: X \rightarrow Z$ be a small birational contraction of threefolds of relative Picard number one. Assume, for example, that both X and the exceptional locus C is smooth and that there is a surface D which is f -negative.

Consider the constant sequence of b -divisors, $\mathbf{D}_i = \overline{D}$. Then the sequence of b -divisors $\mathbf{M}_i$ is not canonically confined. Indeed suppose that it were. First note that canonically confined and log canonically confined are equivalent. Thus there would be a constant $c > 0$ and divisors $S_i \sim iD$ such that $K_X + c_i S_i$ is maximally log canonical at the generic point of C , where $c_i \geq c$.

But then by codimension two subadjunction,

$$(K_X + c_i S_i)|_C = K_C + G_i,$$

where G_i is effective. On the other hand, the degree of the left hand side goes to $-\infty$ whilst the right hand side is bounded from below, a contradiction.

On the other hand if we can find the opposite $f^+: X^+ \rightarrow Z$ of f , then the existence of these divisors on X^+ is trivial, since D^+ is ample. This shows that we need at least nef in the CCS conjecture. If f is a flopping contraction then this also shows that U is non-trivial.

7.5. The surface case

With the work done in Chapter 6, CCS_2 is a small observation:

THEOREM 7.5.1. $(\text{CCS})_{\leq 2}$ holds.

COROLLARY 7.5.2. $(\text{FGA})_{\leq 2}$ holds.

Given any valuation ν of X , consider the sequence of blow ups obtained by iteratively blowing up the centre of ν . We will call this sequence of blow ups the Zariski tower. We say that ν is a geometric valuation, if this sequence stops, in the sense that ν corresponds to a divisor. We call the first model on which the centre of ν is a divisor, the top of the tower. The number of blows ups to get X to the top is called the length of the tower.

DEFINITION 7.5.3. Let X be a variety and let $\mathfrak{V}$ be a set of geometric valuations on X . We say that $\mathfrak{V}$ forms a *bounded family*, if there is a bounded family $Y \rightarrow B$ of varieties, such that for every $\nu \in \mathfrak{V}$, there is a $t \in B$ such that Y_t is birational to X and ν is a divisorial valuation on Y_t .

REMARK 7.5.4. Note that if $\mathfrak{V}$ is a bounded set of valuations, then we may further assume that for each t , there is a birational morphism $Y_t \rightarrow X$.

LEMMA 7.5.5. Fix a log terminal pair (X, B) , where X is a surface. Fix a real number b .

Then the set of valuations of log discrepancy at most b is bounded.

PROOF. Since the set of all surfaces that can be obtained from X by a fixed number of blow ups forms a bounded family, it suffices to prove that there is a uniform bound on the length of a valuation ν of log discrepancy at most b .

Passing to a log resolution, we might as well assume that the pair (X, B) has normal crossings. Pick m such that $m(K_X + B)$ is Cartier. It follows that if ν is an algebraic valuation of log discrepancy a , then ma is an integer. In particular there are only finitely many possible values a of the log discrepancy less than b .

It suffices then, to prove that for any tower of blow ups, the log discrepancy of each successive exceptional divisor goes up strictly. So suppose that we are at some step Y of the Zariski tower and that the centre of ν on Y is q . Then at most two components of $C = B_Y$ pass through q , with coefficients $x \leq y$ say. In this case, by direct calculation, the exceptional divisor has log discrepancy

$$2 - x - y > 1 - x \geq 1 - y,$$

and we are done by induction. $\square$

REMARK 7.5.6. Note that the corresponding result in higher dimensions fails. The problem is that we have no control over the degrees of the centres of any valuation of bounded log discrepancy, so that knowing that the length of the Zariski tower is bounded is of no use.

PROOF OF THEOREM 7.5.1. Let X be a surface over Z and let (Y, B_Y) be a terminal model. We are looking for a bounded family of pairs (Y_t, T_t) , such that $\mathbf{M}_{Y_t}$ is nef and pulled back from T_t , every nef divisor on T_t is semiample, and there is a constant c such that there is an $D \in |\mathbf{M}|_{Y_t}$ with $K_{Y_t} + B_{Y_t} + cD$ canonical. We will choose models Y_t over Y .

Suppose first that $|\mathbf{M}|$ is big. In this case we may take $Y_t = Y$ and $T_t = Y$. Indeed, since $-(K_Y + B_Y)$ is nef and big over Z , it is automatic that every nef divisor is semiample, by the base point free Theorem. If $|\mathbf{M}|_Y$ is free, then we may take any $c < 1$. If $\mathbf{M}$ is big, but $|\mathbf{M}|_Y$ is not free, then, by Theorem 6.1.3, Z is a point and

$$-(K_Y + B_Y) \cdot \mathbf{M}_Y \leq 1.$$

As $|\mathbf{M}|_Y$ has no base components, it follows that $S \in |\mathbf{M}|_Y$ forms a bounded family and the existence of c is then clear.

Otherwise, by Theorem 6.1.3 again, it follows that $\mathbf{M}_Y$ defines a pencil, and there is a model $\psi: W \rightarrow Y$ and a morphism $f: W \rightarrow T$ down to a curve T , such that the only exceptional divisor E which is horizontal for f has log discrepancy at most two. As $|\mathbf{M}|_W$ is base point free, we may take any $c < 1$. As E dominates T , $T \simeq \mathbb{P}^1$ and the condition that every nef divisor on T is semiample is trivial.

It suffices to prove that we may take a bounded family of such surfaces W . We prove this in two different ways. One way is to apply Lemma 7.5.5. It follows that the valuations corresponding to E form a bounded family, and from there it is easy to see that the set of all such surfaces W forms a bounded family. On the other hand, we could appeal to the fact that $-(K_Y + B_Y) \cdot C$ is at most one, to conclude that C belongs to a bounded family. It follows that we may resolve the basis locus of the linear system associated to C using a bounded family. But this also then resolves the basis locus of $|\mathbf{M}|_X$ and so W belongs to a bounded family.

Now suppose that we fix a contraction $X \dashrightarrow T$. If this contraction is birational, then arguing as above we can work on the terminal model, and we can take U to be the single point $\{(Y, Y)\}$. Otherwise $T \simeq \mathbb{P}^1$, and again we can take U be a single point. $\square$

7.6. A strategy for the general case

We conclude the chapter by presenting a general strategy to prove the CCS conjecture, modelled on the surface case just discussed.

Suppose that we are given a pair (X, B) and an integral mobile b-divisor $\mathbf{M}$. Our aim is to control the base locus of the linear system $|\mathbf{M}|_X$. To this end, we pick a general element $S \in |\mathbf{M}|_X$. We work on some high model $\pi: Y \rightarrow X$ over X . In particular we assume that Y is smooth and $|\mathbf{M}|_Y$ is free. Let T be the divisor in $|\mathbf{M}|_Y$ lying over S .

LEMMA 7.6.1. *Let (X, B) be a log pair over Z and let $\mathbf{M}$ be a mobile b-divisor. Pick a general element $S \in |\mathbf{M}|$.*

Then we may find a sufficiently high model $\pi: Y \rightarrow X$, with the following properties

- (a) $|\mathbf{M}|_Y$ is free.
- (b) The union the support of the inverse image of B and S is a divisor with normal crossings.
- (c) The locus $V \subset X$ where π is not an isomorphism is the union of the locus where the pair (X, B) does not have normal crossings and the base locus of $|\mathbf{M}|_X$.

Further, if Y is any model which satisfies (a-c), then Y has the following additional property:

- (d) If $\mathbf{M}$ is $\mathbf{A}$ -saturated, then $\mathbf{M}_Y$ is $\mathbf{A}_Y(Y, B_Y)$ -saturated over Z .

PROOF. Let $\psi: W \rightarrow X$ be a log resolution of the pair $(X, B + S)$, which is an isomorphism outside of the locus where the pair $(X, B + S)$ does not have normal crossings. As $\mathbf{M}$ is mobile, S is certainly reduced, and the locus where $(X, B + S)$ does not have normal crossings, is contained in the locus where the pair (X, B) does not have normal crossings, union the base locus of $|\mathbf{M}|_X$. Blowing up the base locus of $|\mathbf{M}|_W$, we may assume that (a-c) hold.

By Remark 2.3.25, (d) holds. $\square$

Clearly the base locus of $|\mathbf{M}|_X$ is contained in S . As usual, we restrict to T and analyse the base locus on T . However, instead of restricting $|\mathbf{M}|_Y$ to T , we consider the restriction L of the line bundle $\mathcal{O}_Y(\mathbf{M} + \lceil \mathbf{A}_Y \rceil)$. In this case the base locus of $|\mathbf{M}|_X$ is related to the base locus of $|L|$ in a much more indirect fashion.

DEFINITION 7.6.2. Let $K_X + B$ be a pair over Z and let $\mathbf{M}$ be an integral mobile b-divisor. Pick a general element $S \in |\mathbf{M}|_X$ and pick a model $\pi: Y \rightarrow X$ satisfying (a-c) of Lemma 7.6.1. Let V be the locus where π is not an isomorphism and F the inverse image of V . Let T be the divisor in $|\mathbf{M}|_Y$ lying over S , and let L be the restriction to T of the line bundle $\mathcal{O}_Y(\mathbf{M} + \lceil \mathbf{A}_Y \rceil)$.

We say that the pair $(L, |\mathbf{M}|_X)$ is g-free if

- (1) The base locus of $|L|$ is contained in F .
- (2) Given any point $p \in V$, we may find $S \in |\mathbf{M}|_X$ such that the base locus of $|L|$ does not intersect the fibre $\pi^{-1}(p)$.

Here the g of g-free stands for generic. Note that even though g-free is a property of both L and the linear system $|\mathbf{M}|_X$, we will abuse notation and often drop the reference to the choice of S .

REMARK 7.6.3. Note that if the singular locus of X is at most zero dimensional, then then $|L|$ is g-free iff $|L|$ is free, since in this case S is disjoint from the singular locus of X . In particular this holds if X has dimension at most two, or if X is a terminal threefold. In general however, S will always intersect the singular locus of V and the inverse image of this locus is always contained in the base locus of L .

PROPOSITION 7.6.4. Suppose that $-(K_X + B)$ is nef and big over Z , where $K_X + B$ is terminal. Let $\mathbf{M}$ be an integral mobile b-divisor which is $\mathbf{A} = \mathbf{A}(X, B)$ -saturated over Z . Let $S \in |\mathbf{M}|_X$ be a general element. Pick a model $\pi: Y \rightarrow X$ satisfying (a-c) of Lemma 7.6.1.

Let $T \in |\mathbf{M}|_Y$ be the general element corresponding to S , and let L be the restriction of the line bundle $\mathcal{O}_Y(\mathbf{M}_Y + \lceil \mathbf{A}_Y \rceil)$ to T .

- (1) Let B be a component of the base locus of $|\mathbf{M}|_X$ and let E be an exceptional divisor, with centre B . Then every component E' of $E \cap T$ is a component of the base locus of $|L|$.

- (2) $|\mathbf{M}|_X$ is free iff L is g -free.
(3) The line bundle L has the form

$$c_1(L) = K_T + \Delta + D,$$

where Δ has normal crossings, every coefficient lies between zero and one, and $D = (-\pi^*(K_X + \Delta))|_T$ is the pullback of a nef and big divisor from S .

PROOF. We first prove (1). Now, as $\mathbf{M}_Y$ is $\mathbf{A}_Y$ saturated, it follows that

$$\text{Mob } |\mathbf{M}_Y + \lceil \mathbf{A}_Y \rceil| \leq |\mathbf{M}_Y|.$$

By assumption $|\mathbf{M}|_Y$ is a free linear system. As the pair (X, B) is klt, $\lceil \mathbf{A}_Y \rceil$ is effective, and as the pair (X, B) is terminal, the support of $\lceil \mathbf{A}_Y \rceil$ is precisely the exceptional locus. Thus every component of the exceptional locus is a component of the base locus of the linear system $|\mathbf{M}_Y + \lceil \mathbf{A}_Y \rceil|$.

Let B be a component of the base locus of the linear system $|\mathbf{M}|_X$, and let E be an exceptional divisor on Y with centre B . By what we just said, E is a component of the linear system $|\mathbf{M}_Y + \lceil \mathbf{A}_Y \rceil|$. Let E' be any component of $E \cap T$. Consider the restriction exact sequence,

$$0 \longrightarrow \mathcal{O}_Y(\lceil \mathbf{A}_Y \rceil) \longrightarrow \mathcal{O}_Y(T + \lceil \mathbf{A}_Y \rceil) \longrightarrow L \longrightarrow 0.$$

Taking global sections, we get a restriction map on linear systems

$$|T + \lceil \mathbf{A}_Y \rceil| \rightarrow |L|.$$

This map is surjective as

$$H^1(Y, \lceil \mathbf{A}_Y \rceil) = H^1(Y, K_Y + \lceil -\pi^*(K_X + B) \rceil) = 0,$$

by Kawamata-Viehweg vanishing. Suppose that E' is not a component of the base locus of $|L|$. Then we could find a section of L which does not vanish at any point of E' . By surjectivity, we could lift this section to Y , and it would follow that E is not a component of the base locus of $|T + \lceil \mathbf{A}_Y \rceil|$, a contradiction. Thus E' is a component of the base locus of $|T + \lceil \mathbf{A}_Y \rceil|$. Hence (1).

We turn to the proof of (2). By assumption $|\mathbf{M}|_Y$ is a free linear system. Since $\mathbf{A}_Y$ is exceptional and $\lceil \mathbf{A}_Y \rceil$ is effective, it follows that the only possibly base locus of the linear system $|\mathbf{M}_Y + \lceil \mathbf{A}_Y \rceil|$ is supported on the exceptional locus of π . We have a commutative diagram,

$$\begin{array}{ccccc} T & \hookrightarrow & Y & \longleftarrow & F \\ f \downarrow & & \downarrow \pi & & \downarrow \\ S & \hookrightarrow & X & \longleftarrow & V \end{array}$$

Suppose that $|\mathbf{M}|_X$ is free. Then the only locus where the map f is not an isomorphism, is supported in $V \cap S$. As $|\mathbf{M}|_X$ is free, given any point $p \in V$, we may always pick an S that does not contain this point, so that T will not even intersect the fibre over p .

Now suppose that the base locus of the linear system $|L|$ is supported in V , and that we may avoid the fibre over any point p of V . Suppose that $|\mathbf{M}|_X$ is not free. Pick a component B of the base locus, and a point $p \in B$. Let q be a point in the fibre over p . Then there must be an exceptional divisor E , containing q , with centre B which intersects T , since the linear system $|\mathbf{M}|_Y$ is free. But then any component E' of $E \cap T$ is a component of the base locus of $|L|$, by (1), and one of

these components contains q . But then the whole fibre over p is in the base locus of $|L|$, which contradicts our assumption that L is g -free. This proves (2).

Finally we prove (3). We have

$$\begin{aligned} c_1(L) &= (T + \mathbf{A}_Y)|_T = (T + K_Y + \lceil (-\pi^*(K_X + B)) \rceil)|_T \\ &= K_T + \lceil D \rceil = K_T + \Delta + D, \end{aligned}$$

where we applied adjunction to get from line two to line three, D is the restriction of $-(K_Y + B)$ to T and Δ is defined as the difference $\lceil D \rceil - D$. Now $-(K_X + B)$ is big and nef over Z , so that D is certainly nef over Z . However as T is the general element of a base point free linear system, it follows that D is also big over Z . Clearly the coefficients of Δ lie between zero and one. On the other hand, the support of Δ is contained in the support of the strict transform of B and the exceptional locus. As this has normal crossings in X and T is the general element of a base point free linear system, it follows that Δ has normal crossings. Hence (3). $\square$

Clearly, given Proposition 7.6.4, we should study carefully the base locus of adjoint line bundles. In general, since we are only assuming that D is big and nef, and we make no other assumption about the positivity of L , and since we want to prove that $|L|$ is close to being free, rather than some multiple of L , this is rather a tall order.

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CHAPTER 8

Kodaira's canonical bundle formula and subadjunction

JÁNOS KOLLÁR

8.1. Introduction

Let (X, Δ) be a log canonical pair. The aim of this chapter is to study the structure of the set of points where (X, Δ) is not klt.

I call this set the *non-klt locus* of (X, Δ) , but, unfortunately, in the literature the misleading name *locus of log canonical singularities* is more frequent. Thus, for any (X, Δ) , set

$$\text{nklt}(X, \Delta) = \{x \in X : (X, \Delta) \text{ is not klt at } x\}.$$

It is frequently also denoted by $LCS(X, \Delta)$.

Let $f: X' \rightarrow X$ be any log resolution of (X, Δ) and $E_1, \dots, E_m$ all the divisors on X' with discrepancy -1 . Then

$$\text{nklt}(X, \Delta) = f(E_1 + \dots + E_m),$$

thus one can effectively compute $\text{nklt}(X, \Delta)$ from any log resolution of (X, Δ) .

There are certain subvarieties of $\text{nklt}(X, \Delta)$ that are specially important. A $W \subset X$ is called a *log canonical centre* or *LC centre* if there is a log resolution $f: X' \rightarrow X$ and a divisor E with discrepancy -1 such that $f(E) = W$.

Let $f: X' \rightarrow X$ be a log resolution as above. For any $J \subset \{1, \dots, m\}$, any irreducible component of $f(\cap_{i \in J} E_i)$ is a log canonical centre. Conversely, it is easy to see that every log canonical centre of (X, Δ) is obtained this way from a fixed log resolution. In particular, the number of log canonical centres is finite.

A log canonical centre $W \subset X$ is called *exceptional* if there is a unique divisor E_W on X' with discrepancy -1 such that $f(E_W) = W$ and $f(E') \cap W = \emptyset$ for every other divisor $E' \neq E$ on X' with discrepancy -1 .

Exceptional centres seem rather special, but in almost all cases (e.g., when $(X, 0)$ is klt) the “tie breaking method” shows that every minimal log canonical centre of (X, Δ) is an exceptional log canonical centre of some other (X, Δ') .

The case when $W \subset X$ is a codimension 1 log canonical centre, (that is, when $\Delta = W + \Delta'$ for some Δ' whose support does not contain W) is described by the inversion of adjunction theorems [FA92, Sec.17]. The precise relationship is a little technical, but these theorems say, roughly, that there is an effective $\mathbb{Q}$ -divisor $\text{Diff}(\Delta')$ on W such that

- (1) $K_W + \text{Diff}(\Delta') = (K_X + \Delta)|_W$, and
- (2) $(W, \text{Diff}(\Delta'))$ is also log canonical.

Our main interest here is in understanding the higher codimension log canonical centres of (X, Δ) .

Let W be any log canonical centre of (X, Δ) . Then there is a log resolution $f: X' \rightarrow X$ and a divisor E with discrepancy -1 such that $f(E) = W$. Write

$$K_{X'} + E + \Delta' \sim_{\mathbb{Q}} f^*(K_X + \Delta).$$

By the usual adjunction formula,

$$K_E + \Delta'|_E = (K_{X'} + E + \Delta')|_E \sim_{\mathbb{Q}} f^*((K_X + \Delta)|_W). \quad (*)$$

We can thus view $f: E \rightarrow W$ as a log-analog of elliptic surfaces in that the log canonical class of $(E, \Delta'|_E)$ is a pull back of some divisor from the base W .

The generalization of Kodaira's canonical bundle formula to this setting was accomplished in [Fuj86] for surfaces and in [Kaw98] in general. Our presentation is also influenced by the papers of Ambro [Ambb, Amb04, Amb05a]. The end result we would like to get is a formula

$$K_E + \Delta'|_E = f^*(K_W + L + B), \quad (**)$$

where B is effective, (W, B) is log canonical and L is semiample.

Putting $(*)$ and $(**)$ together would give the *Subadjunction Theorem*:

- (1') $K_W + L + B = (K_X + \Delta)|_W$, and
- (2') (W, B) is also log canonical.

There are, unfortunately, 2 technical points that get in the way.

First, we should not expect L to be semiample (= the pull back of an ample divisor by a morphism), but only to be *mobile* (= the pull back of an ample divisor by a rational map). Actually, we can prove only the weaker result that L is pseudo-mobile (= numerically a limit of mobile divisors).

Second, in general $K_W + B$ is not $\mathbb{Q}$ -Cartier, thus it makes no sense to talk about it being log canonical. This forces us to introduce a slightly artificial extra divisor, but this seems to cause only minor problems in applications.

In Section 8.8 we prove Ambro's theorem that $\text{nklt}(X, \Delta)$ is seminormal and Section 8.9 gives a summary of the positivity properties of relative dualizing sheaves which are used in the proof.

Finally Section 8.10 shows how these methods imply Zhang's theorem that a log Fano variety is rationally connected.

8.2. Fujita's canonical bundle formula

Let $f: S \rightarrow C$ be a relatively minimal elliptic surface. That is, S is smooth and proper, the smooth fibers of f are all elliptic curves and no -1 -curve is contained in a fiber of f .

Basic invariants of $f: S \rightarrow C$ are

- i) the j -invariant of the smooth fibers, viewed as a map $j: C \rightarrow M_1 \cong \mathbb{P}^1$,
- ii) the set of singular fibers $B \subset C$, and
- iii) the singular fibers E_P for $P \in B$.

Kodaira's canonical bundle formula (see [Kod63, Sec.12] or [BPVdV84, Sec.IV.12]) computes the canonical class of S in terms of $\chi(\mathcal{O}_S)$, $\chi(\mathcal{O}_C)$ and the multiple fibers.

It is, however, much better to compute K_S in terms of the above 3 invariants. A formula of this type was developed by [Fuj86], building on earlier examples of [Uen73].

It turns out that we need to know only a single rational number $c(P)$ attached to the singular fibers. First it was related to the order of the monodromy around the singular fiber, but now we recognize it as the log canonical threshold of the fiber. That is

$$c(P) := \max\{c: (S, c \cdot E_P) \text{ is log canonical}\}.$$

Fujita's formula computes the canonical bundle of S in terms of these invariants:

THEOREM 8.2.1. *With the above notation,*

$$K_S \sim_{\mathbb{Q}} f^* \left(K_C + \frac{1}{12} j^* \mathcal{O}_{\mathbb{P}^1}(1) + \sum_{P \in B} (1 - c(P)) [P] \right). \quad (8.2.1.1)$$

Its main features can be summarized as follows:

- i) The canonical class K_S is the pull-back of a $\mathbb{Q}$ -divisor D on C .
- ii) D can be written as $K_C + J + B$, where
 - J is a $\mathbb{Q}$ -linear equivalence class which is the pull-back of an ample divisor from the moduli space of the smooth fibers (we call this the *moduli part* or the *j-part*), and
 - B is an effective $\mathbb{Q}$ -divisor that depends on the singular fibers only (we call this the *boundary part*).

A far reaching generalization of these observations is *Itaka's program* which asserts that the situation is similar for any algebraic fiber space $f: X \rightarrow Y$ whose generic fiber is not uniruled. See [Mor87] for a summary of the known results and an introduction to the methods.

Our first aim is to develop a generalization of Fujita's canonical bundle formula to morphisms $f: X \rightarrow Y$ whose general fiber has trivial canonical class. In fact, the resulting formula also applies to pairs (X, Δ) , as long as $K_X + \Delta$ is $\mathbb{Q}$ -linearly trivial on the generic fiber. We can even allow Δ to contain certain divisors with negative coefficient and this turns out to be crucial in many applications.

Before we start with the higher dimensional case, let us think about its general features based on further study of elliptic surfaces.

In dimensions ≥ 3 we can not achieve relative minimality easily, so it makes sense to check what happens to Fujita's formula for elliptic surfaces

$$f': S' \xrightarrow{\pi} S \xrightarrow{f} C$$

which are not relatively minimal.

The first thing we notice is that now $K_{S'}$ is not a pull-back of anything. Indeed, if $E \subset S'$ is a -1 -curve contained in a fiber of f' then $(E \cdot K_{S'}) = -1$ but E has zero intersection number with any divisor that is pulled back from C .

We can write $K_{S'} = \pi^* K_S + E$ for some effective divisor E and then the formula becomes

$$K_{S'} - E \sim_{\mathbb{Q}} (f')^* \left(K_C + \frac{1}{12} j^* \mathcal{O}_{\mathbb{P}^1}(1) + \sum_{p \in B} (1 - c(p)) [p] \right). \quad (8.2.1.2)$$

This looks quite unsatisfactory since it is not straightforward to determine E from the singular fibers. If S' is obtained from S by blowing up distinct points $p_i \in S$ with exceptional curves E_i , then $E = \sum E_i$. However, if we first blow up $p_1 \in S$ to get E_1 and then $p_2 \in E_1 \subset B_{p_1} S$ to get E_2 , then $E = E_1 + 2E_2$ and now E_1 is a -2 -curve. A typical singular fiber of f also consists of -2 -curves, thus here we treat different -2 -curves in the fibers differently.

We may decide not to deal with this precisely, and just say that in (8.2.1.2), E is some divisor whose support is in the singular fibers. In this form, we lose uniqueness. Indeed, if we decrease the coefficient of $[P]$ from $1 - c(P)$ to $1 - c(P) - d$, we can compensate for this by replacing E by $E + d \cdot f^{-1}(P)$. Thus we obtain a formula

$$K_{S'} + R_S \sim_{\mathbb{Q}} (f')^* \left(K_C + \frac{1}{12} j^* \mathcal{O}_{\mathbb{P}^1}(1) + B_C \right),$$

where B_C is some $\mathbb{Q}$ -divisor with support in B and R_S is some $\mathbb{Q}$ -divisor with support in $(f')^{-1}(B)$. Of course here B_C and R_S are interrelated, but we do not have a precise statement any longer.

Instead of looking at this as a loss, we should treat it as an opportunity. So far we tried to keep the left hand side of (8.2.1.1) to be K_S or as closely related to K_S as we could. This, however, did not work very well in the non minimal case.

Let us now shift our attention to C and try to make a good choice there. So we need to write down a divisor of the form

$$\sum_{P \in B} a(P)[P] \quad \text{for some } a(P) \in \mathbb{Q}.$$

If we pick $a(P) = 0$, then we do not see the singular fibers at all. The next best choice is $a(P) = 1$ for every P . In the elliptic surface case, this works marvelously.

THEOREM 8.2.2. *Let $f: S \rightarrow C$ be an elliptic surface, not necessarily relatively minimal. Let $B \subset C$ be a finite set containing all critical values of f . Then there is a unique $\mathbb{Q}$ -divisor R on S such that*

- (1) $K_S + R \sim_{\mathbb{Q}} f^* \left(K_C + \frac{1}{12} j^* \mathcal{O}_{\mathbb{P}^1}(1) + B \right)$,
- (2) $\text{Supp } R \subset f^{-1}(B)$,
- (3) (S, R) is lc, and
- (4) for every $P \in B$ there is a $Q \in f^{-1}(P)$ such that (S, R) is not klt at Q .

I leave it to the reader to derive this result from Theorem 8.2.1. It also follows as a very special case of Theorem 8.5.1.

The main advantage of this version is that the conditions on B and R are easy to generalize to higher dimensions. We have to focus on the log canonical condition. The advantage of the *log canonical normalization* is one of the key insights coming from the theory of singularities of pairs. (See [Kol97] or [KM98] for introductions.)

Note the very nice additional feature that we do not need to know which fibers are singular. If $f^{-1}(p)$ is a smooth fiber and we add p to B by accident, then condition (8.2.2.4) says that R must contain a divisor with coefficient 1 lying over p . Since $f^{-1}(p)$ is irreducible, we have to add $f^{-1}(p)$ to R with coefficient 1.

8.3. The general canonical bundle formula

Let X, Y be normal projective varieties and $f: X \rightarrow Y$ a dominant morphism with connected fibers. Let F be the generic fiber of f . If H is a $\mathbb{Q}$ -Cartier divisor on Y then $f^*H|_F \sim_{\mathbb{Q}} 0$.

Conversely, if G is any divisor on X such that $G|_F \sim_{\mathbb{Q}} 0$ then an easy lemma (8.3.4) shows that there is a (non unique) vertical divisor G' such that $G + G' \sim_{\mathbb{Q}} f^*H$ for some divisor H on Y .

8.3.1 (Main questions). Let us apply the above to a divisor of the form $G = K_X + R_1$, assuming as before that $(K_X + R_1)|_F = K_F + R_1|_F \sim_{\mathbb{Q}} 0$. We thus

get that there is a vertical divisor R_2 such that $K_X + R_1 + R_2 \sim_{\mathbb{Q}} f^* H_{R_1, R_2}$ for some divisor H_{R_1, R_2} depending on R_1, R_2 . We consider two closely related questions:

- (1) Given $f: X \rightarrow Y$ and R_1 , find an optimal choice of R_2 .
- (2) Given $f: X \rightarrow Y$ and R such that

$$K_X + R \sim_{\mathbb{Q}} f^* M \quad \text{for some divisor } M,$$

write $M = K_Y + J + B$ in an insightful way.

The first part is quite easy to do by following the example of Theorem 8.2.2. Remember, however, that in the surface case we needed to know the critical values of the morphism f , or, equivalently, the singular fibers of f . In higher dimensions every fiber may be singular, so a different definition is needed. It is again the log canonical condition that gives the right answer.

Basically, we would like to say that the fiber $X_y := f^{-1}(y)$ is “good” if $(X_y, R|_{X_y})$ is lc, but for technical reasons it is more convenient to formulate it slightly differently.

DEFINITION 8.3.2. Let X and Y be normal and $f: X \rightarrow Y$ a dominant projective morphism. Let R be a $\mathbb{Q}$ -divisor on X and $B \subset Y$ an irreducible divisor.

We say that f has slc (=semi log canonical) fiber over the generic point of B if no irreducible component of R dominates B and $(X, R + f^*B)$ is lc over the generic point of B .

We say that f has *slc fibers in codimension 1* over an open set $Y^0 \subset Y$ if f has slc fibers over the generic point of every prime divisor which intersects Y^0 . Note that this implies that $(X \setminus f^{-1}(Z), R)$ is lc for some closed subset $Z \subset Y^0$ of codimension ≥ 2 .

If X is smooth and R is a relative snc subboundary over an open set Y^0 , then f has slc fibers in codimension 1 over Y^0 .

If $\pi: X' \rightarrow X$ is birational and we write $K_{X'} + R' \sim_{\mathbb{Q}} \pi^*(K_X + R)$ with $\pi_* R' = R$, then f has an slc fiber over the generic point of B iff $f \circ \pi$ has an slc fiber over the generic point of B .

Thus, if (X, R) is lc then there are only finitely many divisors $B \subset Y$ such that f does not have an slc fiber over the generic point of B .

Although we do not need it, it is worth noting that by [Kaw] this is equivalent to the following 3 conditions:

- (1) f^*B is generically reduced over the generic point of B ,
- (2) no component of R dominates B , and
- (3) the generic fiber of $(f^*B, R|_{f^*B})$ is slc.

The following proposition gives a solution to (8.3.1.1). (Note that the $\mathbb{Q}$ -factoriality assumption on Y seems restrictive, but it is easy to satisfy by throwing away all the singular points of Y .)

PROPOSITION 8.3.3. Let X, Y be normal and $f: X \rightarrow Y$ a projective, dominant morphism with generic fiber F . Assume that Y is $\mathbb{Q}$ -factorial. Let R_1 be a $\mathbb{Q}$ -divisor on X and B a reduced divisor on Y . Assume furthermore that

- (1) $(K_X + R_1)|_F \sim_{\mathbb{Q}} 0$, and
- (2) f has irreducible slc fibers in codimension 1 over $Y \setminus B$.

Then there is a unique $\mathbb{Q}$ -divisor R_2 on X and a unique $\mathbb{Q}$ -linear equivalence class L on Y such that

- (3) $K_X + R_1 + R_2 \sim_{\mathbb{Q}} f^*(K_Y + L + B)$,
- (4) $\text{Supp } R_2 \subset f^{-1}(B)$,
- (5) $(X \setminus f^{-1}(Z), R_1 + R_2)$ is lc for some closed subset $Z \subset Y$ of codimension ≥ 2 ,
- (6) every irreducible component of B is dominated by a log canonical centre of $(X, R_1 + R_2)$.

PROOF. By Lemma 8.3.4, there is a vertical divisor V on X such that $K_X + R_1 + V \sim_{\mathbb{Q}} 0$. Set $M = -B$, then we can write this as

$$K_X + R_1 + V \sim_{\mathbb{Q}} f^*(M + B). \quad (8.3.3.3^*)$$

We next change V and M until we get the right form.

Let $a_i V_i$ be an irreducible summand of V such that $f(V_i) \not\subset B$. By assumption, the generic fiber of $V_i \rightarrow f(V_i)$ is irreducible of dimension $\dim X - \dim Y$, thus $W_i := f(V_i)$ is a divisor and V_i is the only irreducible component of $f^{-1}(W_i)$ which dominates W_i . Thus we can subtract $a_i f^*(W_i)$ from V and $a_i W_i$ from M while keeping (8.3.3.3*) valid. Repeating this if necessary, at the end we can assume that $f(\text{Supp } V) \subset B$ and (8.3.3.3*) holds.

We know that $(X, R_1 + V - c f^*(B))$ is lc along $f^{-1}(B)$ for $c \gg 1$. Next we can add a suitable $\sum a_i f^*(B_i)$ such that $(X, R_1 + V - c f^*(B) + \sum a_i f^*(B_i))$ is lc but not klt over an open subset of each B_i . Set $R_2 := V - c f^*(B) + \sum a_i f^*(B_i)$. We have satisfied both (8.3.3.5) and (8.3.3.6).

Finally assume that we have two such choices, giving

$$K_X + R_1 + R_2 \sim_{\mathbb{Q}} f^*(K_Y + L + B) \quad \text{and} \quad K_X + R_1 + R'_2 \sim_{\mathbb{Q}} f^*(K_Y + L' + B).$$

This gives that $R_2 - R'_2 \sim_{\mathbb{Q}} f^*(L - L')$. By (8.3.5) this implies that $R_2 - R'_2 = f^*(\sum c_i B_i)$ for some $c_i \in \mathbb{Q}$. Assume that $c_j > 0$ and let $W \subset X$ be an lc centre of $(X, R_1 + R'_2)$ dominating B_j . Then $(X, R_1 + R'_2 + c_j f^*(B_j))$ is not lc along W , but at the generic point of W it coincides with $(X, R_1 + R_2)$. Thus $c_j \leq 0$ for every j and similarly $c_j \geq 0$ for every j . Therefore $R_2 = R'_2$ and so $L \sim_{\mathbb{Q}} L'$. $\square$

LEMMA 8.3.4. *Let X, Y be projective varieties and $f: X \rightarrow Y$ a morphism with generic fiber F . Let D be a $\mathbb{Q}$ -divisor on X such that $D|_F \sim_{\mathbb{Q}} 0$.*

Then there is a vertical divisor E on X such that $D + E \sim_{\mathbb{Q}} 0$.

PROOF. By assumption there is an integer $m > 0$ and a rational function h on F such that $mD|_F = (h)_F$. One can also view h as a rational function on X , thus $E' := mD - (h)_X$ is a divisor on X which is disjoint from the generic fiber, thus vertical. Set $E := -\frac{1}{m}E'$. $\square$

LEMMA 8.3.5. *Let X, Y be projective varieties and $f: X \rightarrow Y$ a dominant morphism with connected fibers. Let D be a vertical $\mathbb{Q}$ -divisor on X such that $D \sim_{\mathbb{Q}} f^*B$ for some $\mathbb{Q}$ -divisor B .*

Then there is a $\mathbb{Q}$ -divisor B' on X such that $D = f^(B')$.*

PROOF. By assumption there is an $m > 0$ and an $h \in k(X)$ such that $mD - f^*(mB) = (h)$. Since h has neither poles nor zeros on the generic fiber F , $h|_F$ is constant, that is, $h \in f^*k(Y)$. Thus $h = f^*h'$ for some $h' \in k(Y)$ and then $mD = f^*(mB + (h'))$. $\square$

The second aim (8.3.1.2) is harder to achieve, and the precise results are somewhat technical. They are much easier to state if suitable normal crossing assumptions are satisfied.

DEFINITION 8.3.6 (Standard normal crossing assumptions). We say that $f: X \rightarrow Y$ and the divisors R, B satisfy the standard normal crossing assumptions if the following hold

- (1) X, Y are smooth,
- (2) $R + f^*B$ and B are snc divisors,
- (3) f is smooth over $Y \setminus B$, and
- (4) R is a relative snc divisor over $Y \setminus B$.

In practice, the assumptions on X and on divisors on X are completely harmless. By contrast, we have to do some work to compare our problem on Y with the corresponding problem on a blow up of Y .

We can now formulate the first version of the general Kodaira type formula. Later we also consider certain cases when R_h is not effective, which is crucial for some applications.

THEOREM 8.3.7. *Let $f: X \rightarrow Y$ and R, B satisfy the normal crossing assumptions (8.3.6). Assume that $K_X + R \sim_{\mathbb{Q}} f^*H_R$ for some $\mathbb{Q}$ -divisor H_R on Y . Let $R = R_h + R_v$ be the horizontal and vertical parts of R and assume that R_h is an effective subboundary. Then we can write*

$$K_X + R \sim_{\mathbb{Q}} f^*(K_Y + J(X/Y, R) + B_R)$$

such that the following hold.

- (1) The moduli part (or j -part) $J(X/Y, R)$ is nef and depends only on $(F, R_h|_F)$ and Y , where F is the generic fiber of f
- (2) The boundary part B_R depends only on $f: X \rightarrow Y$ and R_v . More precisely, B_R is the unique smallest $\mathbb{Q}$ -divisor supported on B such that

$$R_v + f^*(B - B_R) \leq \text{red}(f^*B).$$

Moreover,

- (3) (Y, B_R) is lc iff (X, R) is lc,
- (4) if $\lfloor R_h \rfloor = 0$ then (Y, B_R) is klt iff (X, R) is klt, and
- (5) B_i appears in B_R with nonnegative coefficient iff $f_*\mathcal{O}_X(\lceil -R_v \rceil) \not\subset \mathcal{O}_{B_i, Y}$.

PROOF. This is a special case of the general version to be established in (8.5.1). Here I only explain the condition (8.3.7.2) and show how it implies (8.3.7.3–5).

The important observation is that the condition (8.3.7.2) is a linear extension of the log canonical normalization suggested by Theorem 8.2.2.

Indeed, note first that by (8.3.7.2) $B_R = B$ iff $R_v \leq \text{red}(f^*B)$ and every irreducible component of B is dominated by an irreducible component of R_v which has coefficient 1. More generally, (X, R) is lc iff $R_v \leq \text{red}(f^*B)$ which is equivalent to $B \geq B_R$ using (8.3.7.2). Finally $B \geq B_R$ iff (Y, B_R) is lc, proving (8.3.7.3). Similarly, if $\lfloor R_h \rfloor = 0$ then (X, R) is klt iff $R_v < \text{red}(f^*B)$ which is equivalent to $B > B_R$ hence to (Y, B_R) being klt. This is (8.3.7.4).

In order to see (8.3.7.5), we can replace Y by $Y \setminus \cup_{j \neq i} B_j$. Thus we may assume that B is irreducible. Then, from (8.3.7.2) we see that $B_R < 0 \Leftrightarrow R_v + f^*B < \text{red } f^*B$. This can be rearranged to $f^*B - \text{red } f^*B < -R_v$ which is equivalent to $f^*B \leq \lceil -R_v \rceil$ and finally to $\mathcal{O}_{B, Y}(B) \subset f_*\mathcal{O}_X(\lceil -R_v \rceil)$, proving (8.3.7.5). $\square$

In connection with (8.3.7), there are several unsolved problems.

8.3.8 (Open problems). Notation and assumptions as in (8.3.7). Conjecturally, $J(X/Y, R)$ is the pull-back of an ample $\mathbb{Q}$ -divisor by a rational map. Moreover, this rational map should be the natural map of Y to a compactified moduli space of the smooth fibers.

One key problem is that it is not clear (not even conjecturally) how to compactify the moduli space of the smooth fibers.

Another difficulty, as shown to me by Ambro and Birkar, is that in the case when $(F, R|_F)$ is not klt, we may have only a semiample line bundle on the moduli space of the smooth fibers.

To see such an example, let $f: X \rightarrow \mathbb{P}^1$ be a minimal ruled surface and $E \subset X$ a section with $E^2 = -m < 0$ and typical fiber F_0 . Set $R := E + \frac{1}{n}(H_1 + \cdots + H_n)$ where $H_i \in |E + mF_0|$. Then $K_X + R \sim_{\mathbb{Q}} f^*K_{\mathbb{P}^1}$, thus $B_R = J(X/\mathbb{P}^1, R) = 0$.

It is actually pretty clear why $J(X/Y, R)$ does not see much of the moduli of $(F, R|_F)$. If $R = Z + R'$ where Z is a smooth horizontal divisor, then

$$(K_X + R)|_Z = (K_Z + Z + R')|_Z = K_Z + R'|_Z.$$

Thus, applying (8.3.7) to $f: Z \rightarrow Y$ we get that

$$K_X + R \sim_{\mathbb{Q}} f^*(K_Y + J(Z/Y, R'|_Z) + B_{R'|_Z}).$$

By repeating this procedure if necessary, we see that the log canonical case of (8.3.7) essentially follows from the klt case.

A slight problem is that I do not see how to prove that the resulting formulas for $J(X/Y, R)$ and B_R are independent of the choice of Z . For the present applications this does not matter, so the reader could skip the general case and the necessary mixed Hodge theoretic considerations.

8.4. Fiber spaces of log Kodaira dimension 0

In this section we define the j -part of the canonical bundle formula (8.3.7) in the general setting when R_h need not be effective.

DEFINITION 8.4.1. Let (X, Δ) be an lc pair, X proper and Δ effective. We say that (X, Δ) has *log Kodaira dimension 0*, denoted by $\kappa(X, \Delta) = 0$ if $h^0(X, \mathcal{O}_X(m(K_X + \Delta))) = 1$ for every sufficiently divisible positive m .

Equivalently, there is a unique, effective $\mathbb{Q}$ -divisor E such that $K_X + \Delta \sim_{\mathbb{Q}} E$.

For us it will be more convenient to set $R = \Delta - E$. Then (X, R) is lc, $K_X + R \sim_{\mathbb{Q}} 0$, but R is usually not effective.

Write $R = R_{(\geq 0)} - R_{(\leq 0)}$ as the difference of its positive and negative parts. Note that $R_{(\leq 0)} \leq E$ but the two are different if Δ and E have common components.

If $\kappa(X, \Delta) = 0$ then also $h^0(X, \mathcal{O}_X(mR_{(\leq 0)})) = 1$ for every sufficiently divisible positive m .

It turns out that instead of assuming $\kappa(X, \Delta) = 0$, we need only a weaker restriction for the canonical bundle formula to work.

DEFINITION 8.4.2. Assume that (X, R) is lc and $K_X + R \sim_{\mathbb{Q}} 0$. Define

$$p_g^+(X, R) := h^0(X, \mathcal{O}_X(\lceil R_{(\leq 0)} \rceil)).$$

Explanation. If X is smooth and $K_X \sim E$ is effective then $R = -E$. In this case $p_g^+(X, R) = h^0(X, \mathcal{O}_X(K_X))$, thus $p_g^+(X, R) = p_g(X)$, the usual geometric genus.

If $\kappa(X, \Delta) = 0$ then $h^0(X, \mathcal{O}_X(mR_{(\leq 0)})) \leq 1$ for every m . Since $\lceil R_{(\leq 0)} \rceil \leq mR_{(\leq 0)}$ for $m \gg 1$, we see that in this case $p_g^+(X, R) = 1$.

For a pair (X, Δ) one defines the log canonical ring as

$$R(X, K_X + \Delta) := \sum_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X + \lfloor m\Delta \rfloor)).$$

The reason for rounding down is that this way we do get a ring. Our definition of p_g^+ corresponds to rounding up: $h^0(X, \mathcal{O}_X(mK_X + \lceil m\Delta \rceil))$. That is why I added the superscript $^+$ to the notation. Note that

$$R^+(X, K_X + \Delta) := \sum_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X + \lceil m\Delta \rceil))$$

is a module over the log canonical ring $R(X, K_X + \Delta)$ which may be interesting to study.

LEMMA 8.4.3. *Let $g: X' \rightarrow X$ be a proper birational morphism and write*

$$K_{X'} + R' \sim_{\mathbb{Q}} g^*(K_X + R) \quad \text{with} \quad g_*R' = R.$$

Then $p_g^+(X, R) \geq p_g^+(X', R')$ and equality holds if $p_g^+(X, R) = 1$.

PROOF. The inequality follows from $g_*(R'_{(\leq 0)}) = R_{(\leq 0)}$. Since $p_g^+ \geq 1$ always, this implies the last claim. $\square$

Let $f: (X, \Delta) \rightarrow Y$ be a proper morphism whose generic fiber has log Kodaira dimension 0. As we already noted, we would like to define a compactified moduli space $\mathcal{M}$ for the fibers and a semiample $\mathbb{Q}$ -divisor $\mathcal{H}$ on $\mathcal{M}$ such that the “ j -part” of the canonical bundle formula is given by $j^*\mathcal{H}$ where $j: Y \dashrightarrow \mathcal{M}$ is the moduli map. In the case of elliptic curves $\mathcal{M} \cong \mathbb{P}^1$ and

We are, unfortunately, not able to accomplish this in general. Instead we construct $\mathbb{Q}$ -divisor classes $J(X/Y, R)$ which play the role of $j^*\mathcal{H}$ in the canonical bundle formula. These classes $J(X/Y, R)$ have all the right properties, except possibly semiampleness.

The general definition is a bit technical, so let us start with a classical case.

EXAMPLE 8.4.4. Let $f: X \rightarrow Y$ be a proper morphism of smooth varieties with generic fiber F such that $K_F \sim 0$. Then f is smooth over an open subset $Y^0 \subset Y$. Set $X^0 := f^{-1}(Y^0)$.

The key observation is that for a smooth projective variety Z , $H^0(Z, \omega_Z)$ can be identified with the bottom piece of the Hodge filtration of $H^{\dim Z}(Z, \mathbb{C})$. (See Section 8.9 for a summary of the relevant concepts and facts from Hodge theory.)

Use this for the smooth fibers of f to obtain that $f_*\omega_{X^0/Y^0}$ is the lowest piece of the Hodge filtration of the variation of Hodge structures $R^{\dim F}f_*\mathbb{C}_{X^0}$.

If $Y \setminus Y^0$ is a snc divisor and $R^{\dim F}f_*\mathbb{C}_{X^0}$ has unipotent monodromies around it, then Hodge theory gives a canonical extension of $f_*\omega_{X^0/Y^0}$ to a line bundle $\mathcal{J}$ on Y , which actually coincides with $f_*\omega_{X/Y}$, see (8.9.7). In this case we can set

$$J(X/Y) := \text{the divisor class corresponding to } \mathcal{J}.$$

There is always a generically finite morphism $\pi: Y' \rightarrow Y$ and a resolution $f': X' \rightarrow X \times_Y Y' \rightarrow Y'$ such that $f': X' \rightarrow Y'$ satisfies these conditions. Set

$$J(X/Y) := \frac{1}{\deg \pi} \pi_* J(X'/Y').$$

For $R \neq 0$ the definition follows the same pattern. First we relate $p_g^+(X, R)$ to Hodge theory and then we use the theory of variation of Hodge structures to get the right $J(X/Y, R)$ if various normal crossing assumptions hold. The general case is then defined by push forward from a normal crossing version.

8.4.5 (Hodge theoretic interpretation of $p_g^+(X, R)$). Assume that X is smooth, R is a snc divisor, (X, R) is lc and $K_X + R \sim_{\mathbb{Q}} 0$. We can uniquely write

$$R = D + \Delta - G \quad \text{where} \quad D = \lfloor R_{(\geq 0)} \rfloor, \quad G = \lceil R_{(\leq 0)} \rceil, \quad (8.4.5.1)$$

and so Δ is the fractional part of R satisfying $\lfloor \Delta \rfloor = 0$. In particular, $p_g^+(X, R) = h^0(X, \mathcal{O}_X(G))$. Since (X, R) is lc, D and Δ have no common irreducible components and D is reduced. Furthermore, (X, R) is klt iff $D = 0$.

Choose any $m > 0$ such that $m\Delta$ is an integral divisor and $m(K_X + R) \sim 0$. Setting $V = \mathcal{O}_X(G - K_X - D)$, we have an isomorphism

$$V^m \cong \mathcal{O}_X(m\Delta). \quad (8.4.5.2)$$

As in (8.9.3), the canonical section $1 \in H^0(X, \mathcal{O}_X(m\Delta))$ determines an m -sheeted cyclic cover $\pi: X' \rightarrow X$ such that

$$\pi_* \omega_{X'} \cong \sum_{i=0}^{m-1} \omega_X \otimes V^i(-\lfloor i\Delta \rfloor). \quad (8.4.5.3)$$

For $i = 1$ we get the direct summand

$$\omega_X \otimes V(-\lfloor \Delta \rfloor) = \omega_X \otimes V \cong \mathcal{O}_X(G - D). \quad (8.4.5.4)$$

First, we obtain the following:

8.4.5.5 *Claim.* If (X, R) is klt then $H^0(X, \mathcal{O}_X(G))$ is naturally a direct summand of $H^0(X', \omega_{X'})$. $\square$

Although X' has quotient singularities, we still get a pure Hodge structure on $H^i(X', \mathbb{C})$ and $H^0(X', \omega_{X'})$ is the bottom piece of the Hodge filtration for $i = \dim X$ (8.9.11). This is completely adequate for our purposes, but it is conceptually better to view this in one of the following ways.

- (6) The μ_m -Galois action of the covering $X' \rightarrow X$ also acts on $H^{\dim X}(X', \mathbb{C})$. The corresponding eigenspaces are sub Hodge structures, giving a direct sum decomposition

$$H^{\dim X}(X', \mathbb{C}) = \sum_{i=0}^{m-1} H^{\dim X}(X', \mathbb{C})^{(i)},$$

and we can choose the indexing such that $H^0(X, \omega_X \otimes V^i(-\lfloor i\Delta \rfloor))$ is naturally the bottom piece of the Hodge filtration on the i th summand.

- (7) The canonical section $1 \in H^0(X, \mathcal{O}_X(m\Delta))$ determines a flat structure of $V|_{X \setminus \Delta}$. Let $\mathbb{V}$ denote the sheaf of locally constant sections. (This can also be viewed as a sheaf associated to a representation $\pi_1(X \setminus \Delta) \rightarrow \mu_m$.) It turns out that the Hodge structure on $H^j(X, \mathbb{V})$ is pure and in fact naturally isomorphic to the above $H^j(X', \mathbb{C})^{(1)}$. Thus $H^0(X, \mathcal{O}_X(G - D))$ is naturally the bottom piece of the Hodge filtration of $H^{\dim X}(X, \mathbb{V})$. (Note that the natural map between topological and coherent cohomology

on X' is given by $H^j(X', \mathbb{C}) \rightarrow H^j(X', \mathcal{O}_{X'})$. Under the μ_m -action, one of the eigenvalues gives

$$H^j(X, \mathbb{V}^{-1}) \rightarrow H^j(X, V^{-1}).$$

For $j = \dim X$ we take the dual sequence

$$H^0(X, \omega_X \otimes V) \hookrightarrow H^{\dim X}(X, \mathbb{V})$$

to get the claimed injection. Here the first term is coming from Serre duality and the second from Poincaré duality on the $(2 \dim X)$ -dimensional space X .)

If (X, R) is lc but not klt then $D \neq 0$. Set $D' := \text{red } \pi^*(D)$. Then we get that

$$\pi_* \omega_{X'}(D') \cong \sum_{i=0}^{m-1} \omega_X \otimes V^i(D - [i\Delta]).$$

As before, we obtain that $H^0(X, \mathcal{O}_X(G))$ is naturally a direct summand of $H^0(X', \omega_{X'}(D'))$. These groups are related to mixed Hodge structures as follows. (See [GS75, Sec.5] for a very clear introduction to mixed Hodge theory on smooth (open) varieties.) As explained in (8.9.11), these results extend to $X' \setminus D'$, although the latter has quotient singularities.

- (6') The mixed Hodge structure on $H^{\dim X}(X' \setminus D', \mathbb{C})$ has only weights $\geq \dim X$ and we get a direct sum decomposition of mixed Hodge structures

$$H^{\dim X}(X' \setminus D', \mathbb{C}) = \sum_{i=0}^{m-1} H^{\dim X}(X' \setminus D', \mathbb{C})^{(i)}.$$

Furthermore, $H^0(X, \mathcal{O}_X(G))$ is naturally a direct summand of $H^0(X', \omega_{X'}(D'))$, which is the bottom piece of the Hodge filtration.

- (7') The mixed Hodge structure on $H^{\dim X}(X \setminus D, \mathbb{V})$ has only weights $\geq \dim X$ and $H^0(X, \mathcal{O}_X(G))$ is naturally isomorphic to the bottom piece of its Hodge filtration.

DEFINITION 8.4.6 (Moduli part or j -part). Let $f: (X, R) \rightarrow Y$ be a proper morphism of normal varieties with generic fiber F such that

- (1) $(F, R|_F)$ is lc, $K_F + R|_F \sim_{\mathbb{Q}} 0$ and $p_g^+(F, R|_F) = 1$.

Let $Y^0 \subset Y$ and $X^0 := f^{-1}(Y^0)$ be open subsets such that $K_{X^0} + R^0 \sim_{\mathbb{Q}} 0$ where $R^0 := R|_{X^0}$. As before, write

$$R^0 = D^0 + \Delta^0 - G^0 \quad \text{where} \quad D^0 = \lfloor R_{(\geq 0)}^0 \rfloor, \quad G^0 = \lceil R_{(\leq 0)}^0 \rceil.$$

As a first step, assume that the following additional conditions also hold:

- (2) X^0, Y^0 are smooth and
 (3) R^0 is a relative snc divisor over Y^0 .

Setting $V = \mathcal{O}_{X^0}(G^0 - K_{X^0} - D^0)$, we have an isomorphism

$$V^{\otimes m} \cong \mathcal{O}_{X^0}(m\Delta^0),$$

which defines a local system $\mathbb{V}$ on $X^0 \setminus (D^0 \cup \Delta^0)$. Assume also that

- (4) Y is smooth, $Y \setminus Y^0$ is a snc divisor and
 (5) $R^{\dim F} f_* \mathbb{V}$ has only unipotent monodromies.

Then the bottom piece of the Hodge filtration of $R^{\dim F} f_* \mathbb{V}$ has a natural extension to a line bundle $\mathcal{J}$ on Y . Set

$$J(X/Y, R) := \text{the divisor class corresponding to } \mathcal{J}.$$

If the conditions (8.4.6.2–5) are not satisfied, then take a generically finite morphism $\pi: Y' \rightarrow Y$ and a resolution $f': X' \rightarrow X \times_Y Y' \rightarrow Y'$ such that $f': X' \rightarrow Y'$ satisfies the conditions (8.4.6.1–5). Set

$$J(X/Y, R) := \frac{1}{\deg \pi} \pi_* J(X'/Y', R').$$

REMARK 8.4.7. It is not immediately clear that the above construction does not depend on the various choices made.

One of the subtle choices is the isomorphism $V^{\otimes m} \cong \mathcal{O}_{X^0}(m\Delta^0)$. Any two isomorphisms differ by an invertible element of $\mathcal{O}_{X^0}$; these are all pull backs of units on Y^0 . Depending on which unit we chose, we get completely different cyclic covers.

(For example, in the simple case when $Y^0 = \mathbb{A}^1 \setminus \{0\}$ with coordinate x , we can use either of the isomorphisms $\mathcal{O}_{Y^0}^{\otimes 2} \cong \mathcal{O}_{Y^0}$ given by $\sigma_0(1 \otimes 1) = 1$ or $\sigma_1(1 \otimes 1) = x$. The first one gives the disconnected double cover $y^2 = 1$ and the second the connected double cover $y^2 = x$.)

The divisor of a unit of $\mathcal{O}_{Y^0}$ on Y is of the form $\sum c_i B_i$, where $Y \setminus Y^0 = \sum B_i$ and by (8.9.3), the resulting cyclic cover depends only on $c_i \bmod m$. Thus, given $f: X \rightarrow Y$ and R we get only finitely many possible local systems $\mathbb{V}_j$.

Correspondingly, we get finitely many possible local systems $R^{\dim F} f_* \mathbb{V}_j$ and any two differ only by tensoring with a rank 1 local system which comes from a representation $\pi_1(Y^0) \rightarrow \mu_m$. Therefore by pulling back to a suitable étale cover of Y^0 , the local systems $R^{\dim F} f_* \mathbb{V}_j$ become isomorphic. Thus the unipotent reduction corrects for the ambiguity in the construction of $\mathbb{V}^0$.

This is, however, one of the main reasons that $J(X/Y, R)$ is defined only as a $\mathbb{Q}$ -linear equivalence class, even when there is a reasonably natural choice of an integer divisor in this equivalence class.

8.4.8 (Base change diagrams). Let $f: X \rightarrow Y$ be a proper morphism of normal varieties. Let $h: Y' \rightarrow Y$ be any dominant and generically finite map of normal varieties and choose any $h_X: X' \rightarrow X \times_Y Y'$ which is birational onto the main component with X' normal and projective. Let $\pi_X: X' \rightarrow X$ be the composite. If we are given a $\mathbb{Q}$ -divisor R on X such that $K_X + R \sim_{\mathbb{Q}} 0$ then there is a unique $\mathbb{Q}$ -divisor R' on X' such that $K_{X'} + R' \sim_{\mathbb{Q}} 0$ and $(h_X)_* R' = (\deg \pi_X) \cdot R$. We obtain a base change diagram:

$$\begin{array}{ccccc} \pi_X: X' & \rightarrow & Y' \times_Y X & \rightarrow & X \\ & f' \searrow & \downarrow & & \downarrow f \\ & & Y' & \xrightarrow{\pi} & Y \end{array}$$

The main properties of $J(X/Y, R)$ are summarized in the next result:

PROPOSITION 8.4.9. *Let $f: (X, R) \rightarrow Y$ be a proper morphism of normal varieties with generic fiber F . Assume that $(F, R|_F)$ is lc, $K_F + R|_F \sim_{\mathbb{Q}} 0$ and $p_g^+(F, R|_F) = 1$. Then the class $J(X/Y, R)$ constructed in (8.4.6) is well defined, and it has the following properties.*

- (1) (Birational invariance) $J(X/Y, R)$ depends only on $(F, R|_F)$ and Y .

(2) (*Push forward*) For any base change diagram as in (8.4.8)

$$J(X/Y, R) = \frac{1}{\deg \pi} \pi_* J(X'/Y', R').$$

(3) (*Pull back*) If X, Y, R, B satisfy the normal crossing assumptions (8.3.6) then

$$J(X'/Y', R') = \pi^* J(X/Y, R).$$

(4) (*Semipositivity*) If X, Y, R, B satisfy the normal crossing assumptions (8.3.6) then $J(X/Y, R)$ is nef.

PROOF. If a variation of Hodge structures has unipotent monodromies, then so does every pull back and the canonical extension commutes with pull backs. This shows (8.4.9.3).

The property (8.4.9.2) is part of our definition and coupled with (8.4.9.3) it shows that $J(X/Y, R)$ is well defined. (8.4.9.1) is a special case of (8.4.9.2).

Finally (8.4.9.4) again follows from Hodge theory (8.9.8). $\square$

REMARK 8.4.10. The base change conjecture [Ambb, Conj.5] says that $J(X/Y, R)$ should commute with arbitrary pull backs but it was pointed out by Prokhorov that this is too much. We expect that $J(X/Y, R)$ is a pull back of a fixed ample class by a rational map $Y \dashrightarrow \mathcal{M}$. Pull backs by rational maps do not commute with further pull backs; we get problems over the points where the map is not defined.

Thus (8.4.9.3) may be the strongest form of the base change conjecture that one can expect to hold. It is essentially equivalent to the form proved in [Amb04, Amb05a].

8.5. Kawamata's canonical bundle formula

We are now ready to prove the general Kodaira formula for fiber spaces whose generic fiber satisfies $p_g^+ = 1$.

THEOREM 8.5.1. *Let X, Y be normal projective varieties and $f: X \rightarrow Y$ a dominant morphism with generic fiber F . Let R be a $\mathbb{Q}$ -divisor on X such that $K_X + R$ is $\mathbb{Q}$ -Cartier and B a reduced divisor on Y . Assume that*

- (1) $K_X + R \sim_{\mathbb{Q}} f^*(\text{some } \mathbb{Q}\text{-Cartier divisor on } Y)$,
- (2) $p_g^+(F, R|_F) = 1$, and
- (3) f has slc fibers in codimension 1 over $Y \setminus B$.

Then one can write

$$K_X + R \sim_{\mathbb{Q}} f^*(K_Y + J(X/Y, R) + B_R), \quad \text{where}$$

- $J(X/Y, R)$ is the moduli part defined in (8.4.6). It depends only on $(F, R|_F)$ and on Y .
- B_R is the unique $\mathbb{Q}$ -divisor supported on B for which there is a codimension ≥ 2 closed subset $Z \subset Y$ such that
 - (a) $(X \setminus f^{-1}(Z), R + f^*(B - B_R))$ is lc and
 - (b) every irreducible component of B is dominated by a log canonical centre of $(X, R + f^*(B - B_R))$.

REMARK 8.5.2. 1. The formulation is slightly sloppy since $B - B_R$ need not be a $\mathbb{Q}$ -Cartier divisor, thus the pull back $f^*(B - B_R)$ need not make sense. Note, however, that we can assume that $Z \supset \text{Sing } Y$, thus we really care only about the pull back of $(B - B_R)|_{Y \setminus Z}$ which is defined.

2. In (8.3.7) we could say that B_R depends only on R_v , but in the singular case this is not quite right since a log canonical centre may be created by a complicated intersection between the vertical and the horizontal parts of R .

PROOF. We start with several reduction steps and then we compare our situation with the Hodge theoretic construction of $J(X/Y, R)$.

Step 1: Achieving normal crossing. Consider any base change diagram as in (8.4.8) with X', Y' smooth and π, π_X birational. Write $K_{X'} + R' \sim_{\mathbb{Q}} \pi_X^*(K_X + R)$ such that $\frac{1}{\deg \pi}(\pi_X)_*R' = R$. If

$$K_{X'} + R' \sim_{\mathbb{Q}} (f')^*(K_{Y'} + J(X'/Y', R') + B'_{R'})$$

then $J(X/Y, R) := \pi_*J(X'/Y', R')$ and $B_R := \pi_*B'_{R'}$ have the right properties.

In particular, in order to prove (8.5.1), we may assume from now on that the normal crossing assumptions (8.3.6) are satisfied.

Step 2: Reduction to $B_R = B$. Since Y is smooth, $B - B_R$ is $\mathbb{Q}$ -Cartier. If we replace R by $R + f^*(B - B_R)$ and B_R by $B_R + (B - B_R) = B$ then we can assume in addition that

(8.5.1.4) there is a codimension ≥ 2 subset $Z \subset Y$ such that $(X \setminus f^{-1}(Z), R)$ is lc, and

(8.5.1.5) every irreducible component of B is dominated by a log canonical centre of (X, R) .

We then need to prove that

$$K_X + R \sim_{\mathbb{Q}} f^*(K_Y + J(X/Y, R) + B).$$

Note that there is a $\mathbb{Q}$ -divisor class L such that $K_X + R \sim_{\mathbb{Q}} f^*(K_Y + L + B)$, and we only need to prove that $L = J(X/Y, R)$.

Step 3: Creating unipotent monodromies and making L integral. We can further improve the situation by using other ramified finite covers $\pi: Y' \rightarrow Y$.

Assume that Y' is smooth and there is a divisor $T \subset Y$ such that π is étale over $Y \setminus T$ and both $R + f^{-1}(B + T)$ and $\pi^{-1}(B + T)$ are snc divisors.

Let X' be a resolution of $Y' \times_Y X$ and $\pi_X: X' \rightarrow X$ the projection. As in (8.9.1.1) write

$$K_{X'} + R' = \pi_X^*(K_X + R + f^{-1}T) \quad \text{and} \quad K_{Y'} + B' = \pi^*(K_Y + B + T).$$

Set $B' := \text{red } \pi^*(B)$. Note that the assumptions (8.5.1.4–5) need not hold for $f': X' \rightarrow Y'$ and R' . That is, if $B'_i \subset B'$ is π -exceptional, then (X', R') need not be lc over the generic point of B'_i and if it is, there may not be a log canonical centre dominating B'_i . Both of these, however, are satisfied if we add $c_i(f')^*(B'_i)$ to R' for a suitable $c_i \in \mathbb{Q}$.

In view of the push forward formula (8.4.9.2), we see that if (8.5.1) holds for $f': X' \rightarrow Y'$ then it also holds for $f: X \rightarrow Y$.

We use this to make 2 improvements.

First, we reduce to the case when L is an integral divisor. Since L is a $\mathbb{Q}$ -divisor, mL is an integral divisor for some $m > 0$. By (8.9.5), we can choose $\pi: Y' \rightarrow Y$ such that $\pi^*(mL) \sim mL'$ for some integral divisor L' . Thus $\pi^*L \sim_{\mathbb{Q}} L'$ and, with a slight shift in notation, we can assume in the sequel that L is a line bundle.

Again using (8.9.10) we reduce to the case when every possible $R^{\dim F} f_* \mathbb{V}_j$ constructed in (8.4.6) has unipotent monodromies around every irreducible component of B . Since there are only finitely many possible $\mathbb{V}_j$, this can be done.

Step 4: Constructing the right cyclic cover. There is a unique way to write

$$R - f^*B = D + \Delta - G + E,$$

where Δ is effective, $[\Delta] = 0$, the divisors D, E, G are integral, effective, without common irreducible components, D is horizontal and E is vertical. Note that $D = 0$ if $(F, R|_F)$ is klt.

Pick $m > 0$ such that $m\Delta$ is an integral divisor and there is a linear equivalence

$$m(f^*(K_Y + L + B) - (K_X + D + E - G + f^*B)) \sim m\Delta.$$

Next construct a degree m cyclic cover $\pi_X: (X', \Delta') \rightarrow (X, \Delta)$ as in (8.9.3). Let $X^0 \subset X$ be as in (8.4.6). The restriction of π_X to X^0 gives one of the cyclic covers used in the construction of the local systems $R^{\dim F} f_* \mathbb{V}_j$ in (8.4.6).

By (8.9.3.2), we see that

$$(\pi_X)_* \omega_{X'} = \sum_{i=0}^{m-1} \mathcal{O}_X(K_X + i(f^*L - K_{X/Y} - D - E + G) - \lfloor i\Delta/m \rfloor).$$

Considering the $i = 1$ summand, and setting $D' := \pi_X^{-1}(D)$, we conclude that $\mathcal{O}_X(f^*(K_Y + L) + G - E)$ is a direct summand of $(\pi_X)_* \omega_{X'}(D')$. Thus

$$L \otimes f_*(\mathcal{O}_X(G - E)) \cong f_* \mathcal{O}_X(f^*L + G - E)$$

is a direct summand of $(f')_* \omega_{X'/Y}(D')$.

If $(F, R|_F)$ is klt then $D' = 0$ hence $(f')_* \omega_{X'/Y}(D') = (f')_* \omega_{X'/Y}$ is the canonical extension of the bottom piece of the Hodge filtration of $R^{\dim F} f_* \mathbb{V}_j$ by (8.9.7). If $(F, R|_F)$ is lc then $(f')_* \omega_{X'/Y}(D')$ is the canonical extension of the bottom piece of the Hodge filtration of $R^{\dim F} f_* \mathbb{V}_j$, but we need to rely on (8.9.12).

In any case, we conclude that $L \otimes f_*(\mathcal{O}_X(G - E)) = \mathcal{O}_Y(J(X/Y, R))$, which implies that $f_*(\mathcal{O}_X(G - E))$ is a line bundle.

Thus it remains to show that tensoring with $f_*(\mathcal{O}_X(G - E))$ does not change anything.

Step 5: Proving $f_(\mathcal{O}_X(G - E)) = \mathcal{O}_Y$* Since $f_*(\mathcal{O}_X(G - E))$ is a line bundle and G is effective, $f_*(\mathcal{O}_X(G - E)) \supset \mathcal{O}_{Y \setminus f(E)}$ and the two can differ only along B . Let $B_i \subset B$ be an irreducible component. Then $f_* \mathcal{O}_X(G - E) = \mathcal{O}_Y$ near the general point of B_i iff

- i) $\text{mult}_{R_{ij}}(G - E) \geq 0$ for every divisor R_{ij} dominating B_i , and
- ii) there is divisor R_{ij} such that $\text{mult}_{R_{ij}}(G - E) < \text{mult}_{R_{ij}}(f^*B_i)$.

A log canonical centre dominating B_i is an irreducible divisor R_{ij} which appears in R with coefficient 1. Thus the coefficient of R_{ij} in $G - E$ is one less than the coefficient of R_{ij} in $f^*(B)$. This proves part ii).

By (8.5.1.4), every R_{ij} appears in R with coefficient ≤ 1 and hence in $R - f^*B$ with coefficient ≤ 0 . This shows i).

Thus the two line bundles $f_* \mathcal{O}_X(G - E)$ and $\mathcal{O}_Y$ agree outside a codimension ≥ 2 subset which implies that $f_* \mathcal{O}_X(G - E) = \mathcal{O}_Y$ and hence $L = J(X/Y, R)$. $\square$

8.6. Subadjunction

We are now ready to prove Kawamata's subadjunction theorem.

THEOREM 8.6.1. [Kaw98] *Let (X, Δ) be an lc pair, Δ effective and $W \subset X$ an exceptional log canonical centre. Let H be an ample divisor on X and $\epsilon > 0$ a rational number.*

Then W is normal and there is an effective $\mathbb{Q}$ -divisor Δ_W on W such that

- (1) (W, Δ_W) is klt, and
- (2) $(K_X + \Delta + \epsilon H)|_W \sim_{\mathbb{Q}} K_W + \Delta_W$.

REMARK 8.6.2. If $J(X/Y, R)$ is semiample in (8.3.7), then the theorem can be strengthened in two ways.

1. One could also allow $\epsilon = 0$, so H would not be needed at all. Note, however, that even in this case we would not have a unique and natural choice for Δ_W . Rather, Δ_W is a sum of a part coming from B_R (this part is unique) and of another part coming from $J(X/Y, R)$ which is a general divisor in a $\mathbb{Q}$ -linear equivalence class.

2. The result would also apply to any log canonical centre with the weaker conclusion that W is semi normal and (W, Δ_W) is slc.

PROOF. Choose a log resolution $g: X' \rightarrow X$ and write $K_{X'} + E + \Delta' \sim_{\mathbb{Q}} g^*(K_X + \Delta)$ with $g_*\Delta' = \Delta$. Here E is the unique exceptional divisor with discrepancy -1 whose image dominates W . Since W is an exceptional log canonical centre, $(E, R := \Delta'|_E)$ is klt. Let $g_E: E \rightarrow W$ be the restriction and note that

$$K_E + R \sim_{\mathbb{Q}} (g_E)^*((K_X + \Delta)|_W). \quad (8.6.1.3)$$

We can further assume that there is a resolution $\pi: W' \rightarrow W$ and a divisor B' on W' such that $g_E: E \rightarrow W$ factors through $f: E \rightarrow W'$ and R, B' and f satisfy the normal crossing assumptions (8.3.6).

$$\begin{array}{ccc} X' & \supset & E \xrightarrow{f} W' \\ g \downarrow & & g_E \downarrow \swarrow \pi \\ X & \supset & W \end{array}$$

Thus (8.3.7) gives a $\mathbb{Q}$ -divisor B'_R supported on B' such that

$$K_E + R \sim_{\mathbb{Q}} f^*(K_{W'} + J(E/W', R) + B'_R), \quad (8.6.1.4)$$

$J(E/W', R)$ is nef and (W', B'_R) is klt.

Since $H|_W$ is ample, $J(E/W', R) + \epsilon\pi^*(H|_W)$ is nef and big. Any nef and big divisor can be written as an ample divisor plus an arbitrary small effective divisor (cf. [KM98, 2.61]). Thus there is an effective divisor $J_\epsilon \sim_{\mathbb{Q}} J(E/W', R) + \epsilon\pi^*(H|_W)$ such that $(W', J_\epsilon + B'_R)$ is klt.

Furthermore, comparing (8.6.1.3) and (8.6.1.4), we conclude that

$$K_{W'} + J_\epsilon + B'_R \sim_{\mathbb{Q}} \pi^*((K_X + \Delta + \epsilon H)|_W). \quad (8.6.1.5)$$

By pushing it forward, we conclude that

$$(K_X + \Delta + \epsilon H)|_W \sim_{\mathbb{Q}} K_W + \pi_*(J_\epsilon + B'_R).$$

Set $\Delta_W := \pi_*(J_\epsilon + B'_R)$.

The only remaining step is to prove that W is normal and Δ_W is effective. (In general, $J_\epsilon + B'_R$ is not effective.)

Since every prime divisor in Δ' has coefficient < 1 and $g_*\Delta' = \Delta$, we can write $\Delta' = \Delta^* - A$ where A is integral, effective, g -exceptional and Δ^* is effective with $[\Delta^*] = 0$. Pushing forward the sequence

$$0 \rightarrow \mathcal{O}_{X'}(A - E) \rightarrow \mathcal{O}_{X'}(A) \rightarrow \mathcal{O}_E(A|_E) \rightarrow 0$$

we get that

$$\mathcal{O}_X \cong g_* \mathcal{O}_{X'}(A) \rightarrow g_* \mathcal{O}_E(A|_E) \rightarrow R^1 g_* \mathcal{O}_{X'}(A - E) = 0,$$

where the Kawamata-Viehweg vanishing applies since

$$A - E \sim_{\mathbb{Q}} K_{X'} + \Delta^* - g^*(K_X + \Delta).$$

Therefore, $(g_E)_* \mathcal{O}_E(A|_E)$ is a quotient of $\mathcal{O}_X$. On the other hand, $\mathcal{O}_E(A|_E) \supset \mathcal{O}_E$ thus $(g_E)_* \mathcal{O}_E(A|_E)$ is an $\mathcal{O}_W$ -sheaf which contains $\mathcal{O}_{W^n}$ where $W^n \rightarrow W$ is the normalization. Thus $(g_E)_* \mathcal{O}_E(A|_E) = \mathcal{O}_{W^n} = \mathcal{O}_W$.

Note that $A|_E = \lceil -\Delta'|_E \rceil = \lceil -R \rceil$. Thus $(g_E)_* \mathcal{O}_E(\lceil -R \rceil) = \mathcal{O}_W$ hence Δ_W is effective by (8.3.7.5). $\square$

8.7. Log canonical purity

Let $f: X \rightarrow Y$ be a dominant, projective morphism between normal varieties. There is a subset $Z \subset Y$ of codimension at least 2 such that f is flat over $Y \setminus Z$. This observation and other methods frequently make it easy to control $f: X \rightarrow Y$ over $Y \setminus Z$, but we have very little information about what happens over Z . Exceptional divisors $E \subset X$, namely those that satisfy $f(E) \subset Z$, are particularly difficult to study.

The next result says that sometimes these exceptional divisors automatically behave the right way.

THEOREM 8.7.1. *Let $f: X \rightarrow Y$ be a dominant, projective morphism between smooth varieties. Let R be a (not necessarily effective) snc divisor. Assume that:*

- (1) *There is a simple normal crossing divisor $B \subset Y$ such that f is smooth over $Y \setminus B$ and R is a relative snc divisor over $Y \setminus B$.*
- (2) *$K_X + R \sim_{\mathbb{Q}} f^*L$ for some $\mathbb{Q}$ -divisor L on Y .*
- (3) *There is a codimension ≥ 2 closed subset $Z \subset Y$ such that $(X \setminus f^{-1}(Z), R)$ is lc.*

Then (X, R) is lc.

PROOF. Since R is a snc divisor, (X, R) is lc iff all the coefficients in R are ≤ 1 . By (3) this holds except possibly for some f -exceptional divisors. If F_1 has coefficient > 1 in R , then it also has coefficient > 1 in $R - \epsilon f^*B$ for $0 < \epsilon \ll 1$. Write

$$R - \epsilon f^*B = D + \Delta_{\epsilon} + E_{\epsilon} - G_{\epsilon},$$

where Δ_{ϵ} is effective with $\lfloor \Delta_{\epsilon} \rfloor = 0$, $D, E_{\epsilon}, G_{\epsilon}$ are effective, integral and without common irreducible components, D is horizontal and E_{ϵ} is vertical. (D does not depend on ϵ .)

All the non exceptional components in E_0 have coefficient 1 by (3). Thus, for $0 < \epsilon \ll 1$, E_{ϵ} contains only exceptional components, and $F_1 \in \text{Supp } E_{\epsilon}$. This contradicts (8.7.2). $\square$

PROPOSITION 8.7.2. *Let $f: X \rightarrow Y$ be a dominant, projective morphism between smooth varieties. Let $D + \Delta$ be an effective, snc divisor with $\lfloor \Delta \rfloor = 0$, D reduced, horizontal, and E, G effective divisors on X such that $D + E$ and G have no common irreducible components and E is vertical. Assume that:*

- (1) *There is a simple normal crossing divisor $B \subset Y$ such that f is smooth over $Y \setminus B$ and $D + \Delta$ is a relative snc divisor over $Y \setminus B$.*
- (2) *$K_X + D + \Delta \sim_{\mathbb{Q}} f^*L + G - E$ for some $\mathbb{Q}$ -divisor L on Y .*

Then $f(E) \subset Y$ has pure codimension 1.

PROOF. As in (8.5.1.Step.3) we reduce to the case when L is Cartier and set $\mathcal{L} := \mathcal{O}_Y(L)$.

Next, as in (8.9.5) and (8.9.11), there is a finite cyclic cover $\pi: X' \rightarrow X$ and an snc divisor $D' \subset X'$ such that $f^*\mathcal{L}(G - E)$ is a direct summand of $\pi_*\omega_{X'}(D')$, and $f \circ \pi: X' \rightarrow Y$ satisfies the conditions of (8.9.12).

Thus we conclude that $R^i f_*(f^*\mathcal{L}(G - E))$ is locally free for every i . Consider now the exact sequence

$$0 \rightarrow f^*\mathcal{L}(G - E) \rightarrow f^*\mathcal{L}(G) \rightarrow f^*\mathcal{L}(G)|_E \rightarrow 0,$$

and its direct image on Y

$$0 \rightarrow f_*(f^*\mathcal{L}(G - E)) \rightarrow f_*(f^*\mathcal{L}(G)) \rightarrow f_*(f^*\mathcal{L}(G)|_E) \xrightarrow{\partial} R^1 f_*(f^*\mathcal{L}(G - E)).$$

Since $\text{Supp } f_*(f^*\mathcal{L}(G)|_E) \subset \text{Supp}(f(E)) \subsetneq Y$ and $R^1 f_*(f^*\mathcal{L}(G - E))$ is torsion free, we see that ∂ is zero, hence the following sequence is exact

$$0 \rightarrow f_*(f^*\mathcal{L}(G - E)) \rightarrow f_*(f^*\mathcal{L}(G)) \rightarrow f_*(f^*\mathcal{L}(G)|_E) \rightarrow 0.$$

Note that $f_*(f^*\mathcal{L}(G))$ is torsion free since it is a push forward of a torsion free sheaf. We saw that $f_*(f^*\mathcal{L}(G - E))$ is locally free, and in particular it is S_2 . Thus it has no nontrivial extension with a sheaf whose support has codimension ≥ 2 . Thus $\text{Supp } f_*(f^*\mathcal{L}(G)|_E)$ has pure codimension 1.

On the other hand, since E and G have no common irreducible components, we have an injection

$$\mathcal{L} \otimes f_*\mathcal{O}_E \hookrightarrow \mathcal{L} \otimes f_*(\mathcal{O}_X(G)|_E) = f_*(f^*\mathcal{L}(G)|_E),$$

thus $f(E) \subset Y$ also has pure codimension 1. $\square$

8.8. Ambro's seminormality theorem

In this section we prove Ambro's theorem that $\text{nklt}(X, \Delta)$ is seminormal. Note that in general the irreducible components of $\text{nklt}(X, \Delta)$ are not normal. As simple examples one can consider the log canonical pairs $(\mathbb{C}^2, (y^2 = x^3 + x^2))$ or $(\mathbb{C}^3, (x^2 = zy^2))$.

As far as I can tell, the result does not imply that every irreducible component of $\text{nklt}(X, \Delta)$ is seminormal (cf. [Kol96, I.7.2.2.3]) but this follows from [Amb03].

The definition of seminormal schemes is recalled in (8.8.2) and their basic properties are proved in (8.8.3–8.8.6).

If (X, Δ) is dlt, then every codimension 1 irreducible component of $\text{nklt}(X, \Delta)$ is normal [KM98, 2.52].

THEOREM 8.8.1. [Amba] *Let (X, Δ) be an lc pair, Δ effective. Then $\text{nklt}(X, \Delta)$ is seminormal.*

PROOF. The argument is similar to the end of the proof of (8.6.1). Let $W := \text{nklt}(X, \Delta)$ and $h: W^{sn} \rightarrow W$ its seminormalization. Take a log resolution $f: Y \rightarrow X$ and write

$$f^*(K_X + \Delta) \sim_{\mathbb{Q}} K_Y + \Delta_Y + B - A,$$

where $[\Delta'] = 0$ and $A, B \geq 0$ are integral without common irreducible components. Note that B is reduced since (X, Δ) is lc and $W = f(B)$. B is seminormal by (8.8.5),

thus $f: B \rightarrow W$ lifts to $f^{sn}: B \rightarrow W^{sn}$ by (8.8.6). Consider the exact sequence

$$0 \rightarrow \mathcal{O}_Y(A - B) \rightarrow \mathcal{O}_Y(A) \rightarrow \mathcal{O}_B(A|_B) \rightarrow 0,$$

and its push forward

$$f_*\mathcal{O}_Y(A) \rightarrow f_*\mathcal{O}_B(A|_B) \rightarrow R^1f_*\mathcal{O}_Y(A - B).$$

Note that $f_*\mathcal{O}_Y(A) = \mathcal{O}_X$ since A is f -exceptional and $R^1f_*\mathcal{O}_Y(A - B) = 0$ since

$$A - B \sim_{\mathbb{Q}} K_Y + \Delta_Y - f^*(K_X + \Delta).$$

Thus we have a surjection $\mathcal{O}_X \rightarrow f_*\mathcal{O}_B(A|_B)$. Since $B \subset f^{-1}(W)$, we see that $f_*\mathcal{O}_B(A|_B)$ is an $\mathcal{O}_W$ -sheaf. Therefore $f_*\mathcal{O}_B(A|_B) = \mathcal{O}_W$.

On the other hand, we have containments of sheaves

$$\mathcal{O}_W = f_*\mathcal{O}_B(A|_B) \supset f_*\mathcal{O}_B = h_*f_*^{sn}\mathcal{O}_B \supset h_*\mathcal{O}_{W^{sn}} \supset \mathcal{O}_W.$$

Thus all these are equalities. In particular, $h_*\mathcal{O}_{W^{sn}} = \mathcal{O}_W$ thus $W^{sn} = W$ and so W is seminormal. $\square$

8.8.2 (seminormal schemes). A morphism of schemes $f: Y \rightarrow X$ is called a *partial seminormalization* if

- (1) Y is reduced,
- (2) f is finite with irreducible fibers and
- (3) for every point $x \in X$, the injection of the residue fields $f^*: k(x) \hookrightarrow k(f^{-1}(x))$ is an isomorphism.

In characteristic zero this is equivalent to f being finite with geometrically irreducible fibers, but in positive characteristic the latter condition allows purely inseparable maps as well.

For this notion nilpotents do not matter, thus $f: Y \rightarrow X$ is a partial seminormalization iff the corresponding morphism between the reduced schemes $f_{\text{red}}: Y \rightarrow \text{red } X$ is.

For any scheme Z , the (fiber) product $f: Y \times Z \rightarrow X \times Z$ is also a partial seminormalization. The composite (resp. fiber product) of two partial seminormalizations is again a partial seminormalization.

A scheme X over a field of characteristic 0 is called *seminormal* if every partial seminormalization $f: Y \rightarrow X$ is an isomorphism. Since $\text{red } X \rightarrow X$ is a partial seminormalization, this implies that a seminormal scheme is reduced.

LEMMA 8.8.3. *Let X be a reduced scheme whose normalization $X^n \rightarrow X$ is finite. Then X^n dominates every partial seminormalization $Y \rightarrow X$. In particular, a normal scheme is seminormal.*

PROOF. Let $f: Y \rightarrow X$ be a partial seminormalization. Then f is finite and applying (8.8.2.3) to the generic points $x \in X$ we see that f is birational over each irreducible component. Thus the normalization $X^n \rightarrow X$ dominates Y . $\square$

DEFINITION-LEMMA 8.8.4. *Let X be a reduced scheme whose normalization $X^n \rightarrow X$ is finite. Then there is a unique maximal partial seminormalization $X^{sn} \rightarrow X$ such that*

- (1) X^{sn} is seminormal, and
- (2) X^{sn} dominates every partial seminormalization $Y \rightarrow X$.

For a scheme X , we call $X^{sn} := (\text{red } X)^{sn} \rightarrow X$ the *seminormalization* of X .

PROOF. Since X^n is finite over X , there is no infinite ascending chain of schemes between X^n and X . Thus there is a (possibly non unique) maximal partial seminormalization $X^{sn} \rightarrow X$.

As we noted, partial seminormalizations are closed under composition and fiber products. Thus X^{sn} is seminormal and it dominates every partial seminormalization $Y \rightarrow X$. $\square$

It is not hard to see that if X is seminormal then every open subset of X is also seminormal. It is, however, not true that an irreducible component of a seminormal variety is also seminormal [Kol96, I.7.2.2.3].

Let $0 \in C$ be a reduced curve singularity over an algebraically closed field with normalization $n: C^n \rightarrow C$. Set $n^{-1}(0) = \{p_1, \dots, p_m\} \in C^n$. Let C^{sn} be the curve obtained from C^n by identifying the points $\{p_1, \dots, p_m\}$ while keeping the tangent directions of the m branches linearly independent. It is easy to see that $C^{sn} \rightarrow C$ is the seminormalization of C . Thus the smooth points and the ordinary nodes are the only seminormal planar curve singularities.

LEMMA 8.8.5. *Let Z be a smooth variety and $X \subset Z$ a reduced hypersurface. Then X is seminormal iff its codimension one singularities are ordinary nodes.*

PROOF. If the codimension one singularities are ordinary nodes, then X is seminormal outside a codimension 2 subset. Thus the extension $\mathcal{O}_X \hookrightarrow \mathcal{O}_{X^{sn}}$ is an isomorphism outside a codimension 2 subset. But X is S_2 , thus every such extension is an isomorphism. The converse is clear. $\square$

seminormalization is much more functorial than normalization:

LEMMA 8.8.6. *Let $f: Y \rightarrow X$ be a partial seminormalization and $g_X: Z \rightarrow X$ any morphism with Z seminormal. Then there is a (unique) lifting $g_Y: Z \rightarrow Y$ such that $g_X = f \circ g_Y$.*

PROOF. As we noted, $f_Z: Z \times_X Y \rightarrow Z \times_X X = Z$ is also a partial seminormalization, hence an isomorphism since Z is seminormal. Set $g_Y = (g_X \times_X f) \circ (f_Z)^{-1}$. $\square$

8.9. Covering tricks and Semipositivity of $f_*\omega_{X/Y}$

In this section we gather various results on the semipositivity of $f_*\omega_{X/Y}$ that we needed earlier.

8.9.1 (Ramification and log canonical pairs). Let $f: X \rightarrow Y$ be a finite, dominant morphism between normal varieties. Let $\sum B_i$ be a reduced divisor containing the branch locus. Set $\text{red } f^{-1}(B_i) = \sum_j R_{ij}$. Let e_{ij} be the ramification index of f along R_{ij} . For any $a_i \in \mathbb{Q}$, the Hurwitz formula can be written as

$$f^*(K_Y + \sum_i B_i - \sum_i a_i B_i) = K_X + \sum_{ij} R_{ij} - \sum_{ij} e_{ij} a_i R_{ij}. \quad (8.9.1.1)$$

If $\sum B_i$ is a snc divisor, then $(Y, \sum(1 - a_i)B_i)$ is lc (resp. klt) iff $a_i \geq 0$ (resp. $a_i > 0$) for every i . We see that these are equivalent to $(X, \sum(1 - e_{ij}a_i)R_{ij})$ being lc (resp. klt).

In general, we can use this formula for a log resolution of $(Y, \sum B_i)$ to conclude the following (cf. [FA92, 20.3]):

LEMMA 8.9.2. *Let $f: X \rightarrow Y$ be a finite, dominant morphism between normal varieties. Let B be a $\mathbb{Q}$ -divisor on Y such that $K_Y + B$ is $\mathbb{Q}$ -Cartier. Write $f^*(K_Y + B) = K_X + R$. Then (Y, B) is lc (resp. klt) iff (X, R) is lc (resp. klt). $\square$*

8.9.3 (Cyclic covers). (cf. [KM98, 2.49–53]) Let X be a smooth variety, L a line bundle on X and D an integral (not necessarily effective) divisor. Assume that $L^m \cong \mathcal{O}_X(D)$. Let s be any rational section of L and 1_D the constant (rational) section of $\mathcal{O}_X(D)$. Then $1_D/s^m$ is a rational function which gives a well defined element of the quotient group $k(X)^*/(k(X)^*)^m$, thus a well defined degree m field extension $k(X)(\sqrt[m]{1_D/s^m})$. Let $\pi: X' \rightarrow X$ denote the normalization of X in the field $k(X)(\sqrt[m]{1_D/s^m})$. Then

- (1) $\pi_*\mathcal{O}_{X'} = \sum_{i=0}^{m-1} L^{-i}(\lfloor iD/m \rfloor)$, and
- (2) $\pi_*\omega_{X'} = \sum_{i=0}^{m-1} \omega_X \otimes L^i(-\lfloor iD/m \rfloor)$.

In particular, if E is any integral divisor then the normalized cyclic cover obtained from $L^m \cong \mathcal{O}_X(D)$ is the same as the normalized cyclic cover obtained from $(L(E))^m \cong \mathcal{O}_X(D + mE)$.

Note that we do not assume that X is proper, so the formulas apply to any normal variety W by setting $X := W \setminus \text{Sing } W$.

8.9.4 (Ramified covering tricks). There are two very useful finite covers that can be used to handle $\mathbb{Q}$ -divisors and other torsion problems. The first, introduced by [BG71] relies on the observation that the map

$$m_n: \mathbb{P}^N \rightarrow \mathbb{P}^N : \quad m_n^*(x_i) = x_i^n$$

satisfies $m_n^*\mathcal{O}(1) = \mathcal{O}(n)$. For arbitrary X , take a general morphism $g: X \rightarrow \mathbb{P}^N$ for some N and consider the fiber product with the coordinate projection

$$m_{X,n}: X_n := X \times_{\mathbb{P}^N} \mathbb{P}^N \rightarrow X.$$

Then $m_{X,n}^*(g^*\mathcal{O}_{\mathbb{P}^N}(1))$ becomes the n th tensor power of a line bundle. A general position argument shows that X_n is smooth and one can assume that the preimage of an snc divisor $D \subset X$ is again an snc divisor $D_n \subset X_n$ (cf. [KM98, 2.67]). In this construction, the ramification divisor is in general position.

The method of [Kaw81] creates ramification along a given snc divisor. It starts with a smooth divisor $D \subset X$ and an natural number m . Pick any M such that $mM - D$ is very ample, and pick general $H_i \sim mM - D$ such that $\cap H_i = \emptyset$. Although the cyclic covers obtained from $mM \sim D + H_i$ are all singular, their composite is a smooth Abelian cover $X' \rightarrow X$ which has m -fold ramification along the divisors D and all the H_i .

The end result of these methods is the following:

PROPOSITION 8.9.5. *Let X be a smooth variety, $\sum B_i + \sum D_j$ an snc divisor, L_k Cartier divisors and m_i, c_k positive integers. Then there is a smooth variety X' and a finite morphism $\pi: X' \rightarrow X$ such that*

- (1) π is an Abelian Galois map,
- (2) $\pi^{-1}(\sum B_i + \sum D_j)$ is an snc divisor,
- (3) $\pi^*B_i = m_i \cdot \text{red } \pi^*B_i$,
- (4) π^*D is reduced, and
- (5) there are line bundles M_k such that $\pi^*L_k \cong M_k^{c_k}$.

8.9.6 (Variations of Hodge structures). (See [GS75] for a very clear introduction.) Let X, Y be smooth projective varieties and $f: X \rightarrow Y$ a morphism. Let $B \subset Y$ be a snc divisor, $R := f^{-1}(B)$ and assume that $f: X \setminus R \rightarrow Y \setminus B$ is smooth.

The cohomology groups of the fibers $X_y := f^{-1}(y)$ give local systems $R^i f_* \mathbb{C}_{X \setminus R}$ over $Y \setminus B$, which are variations of Hodge structures.

Set $n = \dim X - \dim Y$. The identifications

$$H^i(X_y, \omega_{X_y}) \cong H^{n,i}(X_y) \subset H^{n+i}(X_y, \mathbb{C})$$

give subvectorbundles

$$R^i f_* \omega_{X/Y}|_{Y \setminus B} \hookrightarrow \mathcal{O}_{Y \setminus B} \otimes_{\mathbb{C}} R^{n+i} f_* \mathbb{C}_X|_{Y \setminus B}.$$

The local system $R^i f_* \mathbb{C}_X|_{Y \setminus B}$ has a natural metric, once we fix an ample divisor H on X which gives a class $[H] \in H^2(X_y, \mathbb{C})$ for every $y \in Y \setminus B$. Basically, for $i \leq n$ we take $\alpha \wedge \beta \wedge [H]^{n-i}$ and evaluate it on the fundamental class.

A power of $\sqrt{-1}$ must be thrown in to account for two problems:

- (1) for i odd the above pairing is skew symmetric, and
- (2) non primitive classes have extra sign problems.

We get a metric which is flat but the flat structure is not unitary. By the computations of Griffiths (see [Gri70] or [Sch73, Sec.7]), this metric restricts to a metric with semipositive curvature on $R^i f_* \omega_{X/Y}|_{Y \setminus B}$. The limiting behavior of this metric near B is also understood, and one gets the first semipositivity result:

THEOREM 8.9.7. [Fuj78, Kaw81, Kol86] *Notation as above. Then*

- (1) $\mathcal{O}_{Y \setminus B} \otimes_{\mathbb{C}} R^{n+i} f_* \mathbb{C}_X|_{Y \setminus B}$ has an (upper) canonical extension to a locally free sheaf on Y .
- (2) $R^i f_* \omega_{X/Y}$ coincides with the corresponding (upper) canonical extension of the bottom piece of the Hodge filtration.
- (3) Let $g: C \rightarrow Y$ be a map from a curve to Y whose image is not contained in B . Then every quotient bundle of $g^*(R^i f_* \omega_{X/Y})$ has nonnegative degree.

The (upper) canonical extension does not commute with pull backs and the property (8.9.7.3) can definitely fail if $g(C) \subset B$, but there is a well understood remedy.

Let $\gamma_j \subset Y \setminus B$ be a small loop around an irreducible component $B_j \subset B$ with base point y_j . It is known (Borel, cf. [Sch73, 4.5]) that as we move around γ_j , the resulting linear map, $\rho_j \in GL(H^{n+i}(X_{y_j}, \mathbb{C}))$ is quasi-unipotent, that is, some power of ρ_j is unipotent. ρ_j is called the *monodromy* around B_j .

THEOREM 8.9.8. [Ste76, Kaw81, Kol86] *Notation as above. Assume that $R^i f_* \mathbb{C}_X|_{Y \setminus B}$ has unipotent monodromies around every irreducible component of B .*

- (1) *The canonical extension commutes with pull backs.*
- (2) *The locally free sheaf $R^i f_* \omega_{X/Y}$ is semi positive. That is, for every $g: C \rightarrow Y$, every quotient bundle of $g^*(R^i f_* \omega_{X/Y})$ has nonnegative degree.*

A key technical point is that as long as Y is smooth and B is a snc divisor, we can ignore what happens over any codimension 2 subset $Z \subset Y$.

THEOREM 8.9.9. [Kat71] *Notation as above. Assume that there is a codimension 2 subset $Z \subset Y$ such that $f^*(B \setminus Z)$ is a reduced snc divisor. Then $R^i f_* \mathbb{C}_X|_{Y \setminus B}$ has unipotent monodromies for every i .*

If the monodromy $\rho_j \in GL(H^i(X_{y_j}, \mathbb{C}))$ is not unipotent but $\rho_j^{m_j}$ is unipotent, then we can create unipotent monodromies by a simple base change.

PROPOSITION 8.9.10. *Notation as above. Let Y' be a smooth variety and $\tau: Y' \rightarrow Y$ a finite Galois map such that m_j divides the ramification order of τ above B_j . Let $X' \rightarrow X \times_Y Y'$ be any resolution. Then $R^i f_* \mathbb{C}_{X'}|_{Y' \setminus \tau^{-1}(B)}$ has unipotent monodromies.*

The semistable reduction theorem of Mumford [KKMSD73] says that the assumptions of (8.9.9) can be satisfied after a suitable base change $\tau: Y' \rightarrow Y$, but in practice it is much harder to achieve semistability than to get unipotent monodromies.

8.9.11 (Hodge theory and quotient singularities). The above results of Hodge theory also hold if X, Y are not smooth but have quotient singularities. These were established in [Ste76]. In our situation, however, everything can be reduced to the smooth case as follows.

For us the singular varieties appear as cyclic covers $X' \rightarrow X$ of a smooth variety X with snc ramification divisor. The covering trick (8.9.5) shows that each such X' is dominated by a smooth Abelian cover $X'' \rightarrow X$. Thus $H^i(X', \mathbb{C})$ is a direct summand of $H^i(X'', \mathbb{C})$ and one can read off the Hodge theory of X' from the Hodge theory of X'' .

We also need to study sheaves of the form $R^m f_* \omega_{X/Y}(D)$, which relate to variations of mixed Hodge structures.

THEOREM 8.9.12. *Let X, Y be smooth projective varieties and $f: X \rightarrow Y$ a morphism. Let $\{D_i : i \in I\}$ be an snc divisor on X and for every $J \subset I$ set $D_J := \cap_{i \in J} D_i$. Let $B \subset Y$ be a snc divisor, $R := f^{-1}(B)$ and assume the following:*

- (1) *For every $J \subset I$, $f: D_J \setminus R \rightarrow Y \setminus B$ is smooth, and*
- (2) *$R^m f_* \mathbb{C}_{D_J \setminus R}$ has unipotent monodromies for every m .*

Then the sheaves $R^m f_ \omega_{X/Y}(D)$ are locally free and semi positive for every m .*

PROOF. The right way to prove this is to relate the sheaves $R^m f_* \omega_{X/Y}(D)$ to the variations of mixed Hodge structures $R^m f_* \mathbb{C}_{X \setminus (D \cup R)}$, but I do not know good references.

It is easy, however, to reduce the study of $R^m f_* \omega_{X/Y}(D)$ to the sheaves $R^m f_* \omega_{D_J/Y}$. This is accomplished by repeatedly using exact sequences of the form

$$0 \rightarrow \omega_{Z/Y} \rightarrow \omega_{Z/Y}(F) \rightarrow \omega_{F/Y} \rightarrow 0,$$

where Z^n, Y^k are smooth and F is a smooth divisor on Z . If $B \subset Y$ is a snc divisor and $Z \rightarrow Y$ and $F \rightarrow Y$ are smooth over $Y \setminus B$ then by [Kol86, Thm.2.6], the sheaves $R^{m+1} f_* \omega_{Z/Y}$ (resp. $R^m f_* \omega_{F/Y}$) are locally free and they are the canonical extensions of the bottom pieces of the Hodge filtrations of $R^{m+1+n-k} f_* \mathbb{C}_{Z \setminus R}$ (resp. $R^{m+n-1-k} f_* \mathbb{C}_{F \setminus R}$). The boundary maps

$$R^{m+n-1-k} f_* \mathbb{C}_{F \setminus R} \rightarrow R^{m+1+n-k} f_* \mathbb{C}_{Z \setminus R}$$

are weight $(1, 1)$ maps of variations of Hodge structures, hence the image is a direct summand in both. This implies that the image of the boundary map

$$R^m f_* \omega_{F/Y} \rightarrow R^{m+1} f_* \omega_{Z/Y}$$

is a direct summand in both sheaves. Therefore,

$$R^m f_* \omega_{Z/Y}(F)$$

has a 2 step filtration whose graded pieces are direct summands of

$$R^m f_* \omega_{Z/Y} \quad \text{and} \quad R^m f_* \omega_{F/Y}.$$

Applying this repeatedly, we obtain that the sheaves $R^m f_* \omega_{X/Y}(D)$ have a filtration whose graded pieces are direct summands of sheaves $R^j f_* \omega_{D_J/Y}$. Thus the sheaves $R^m f_* \omega_{X/Y}(D)$ are locally free and semi positive for every m .

Note also that in the proof of (8.5.1) we are interested in a rank one summand of some $R^m f_* \omega_{X/Y}(D)$, hence we end up with a direct summand of a single $R^j f_* \omega_{D_J/Y}$ for some j and J . $\square$

8.10. Log Fano varieties

In this section we prove Zhang's theorem that log Fano varieties are rationally connected. While this is not related to subadjunction, it gives another nice application of the general Kodaira type formulas.

See [KM98] for a treatment of rationally connected varieties.

THEOREM 8.10.1. [Zha] *Let (X, Δ) be a proper, klt pair such that $\Delta \geq 0$ and $-(K_X + \Delta)$ is nef and big. Then X is rationally connected.*

The key point is the following result, which is an application of (8.5.1).

PROPOSITION 8.10.2. *Let (X, Δ) be a proper, klt pair such that $\Delta \geq 0$ and $-(K_X + \Delta)$ is nef and big. Let $f: X \dashrightarrow Z$ be a dominant rational map to a smooth variety Z with $\dim Z \geq 1$. Then Z is uniruled.*

PROOF. We can choose birational maps $\pi: X' \rightarrow X$ and $\tau: Z' \rightarrow Z$ such that X', Z' are smooth, projective, f lifts to a morphism $f': X' \rightarrow Z'$ and there is a snc divisor $B' \subset Z'$ such that f' is smooth over $Z' \setminus B'$. We can also assume that every f' exceptional divisor is also π exceptional.

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Z' \\ \pi \downarrow & & \downarrow \tau \\ X & \dashrightarrow & Z \end{array}$$

$-(K_{X'} + \Delta_1) \sim_{\mathbb{Q}} -\pi^*(K_X + \Delta)$ is also nef and big, so we can write it as $(\text{ample}) + \Delta_2$ where $(X', \Delta_1 + \Delta_2)$ is klt. By doing further blow ups if necessary, we can also achieve that there is a snc strict subboundary Δ' and an ample $\mathbb{Q}$ -divisor H' on X' such that $K_{X'} + \Delta' + H' \sim_{\mathbb{Q}} 0$ and $\pi_* \Delta'$ is effective.

Let H'_Z be a small ample $\mathbb{Q}$ -divisor such that $H'' \sim_{\mathbb{Q}} H' - (f')^* H'_Z$ is ample, effective and $\Delta' + H'' + (f')^{-1}(B')$ is a snc divisor.

If F denotes the generic fiber of f' then $p_g^+(F, (\Delta' + H'')|_F) = 1$ since $\Delta'_{\leq 0}$ is f' -exceptional. Thus we can apply (8.5.1) to

$$f': (X', \Delta' + H'') \rightarrow Z' \quad \text{and} \quad B' \subset Z'.$$

We get a divisor R' supported on $(f')^{-1}(B')$ such that $\Delta' + H'' + R'$ is a subboundary and

$$K_{X'} + \Delta' + H'' + R' \sim_{\mathbb{Q}} (f')^*(K_{Z'} + L' + B'),$$

which we rewrite as

$$K_{X'} + \Delta' + H' + R' - (f')^*(B') \sim_{\mathbb{Q}} (f')^*(K_{Z'} + L' + H'_Z). \quad (8.10.2.6)$$

Let $C \subset X$ be a general complete intersection curve. Then $(C \cdot (K_X + \Delta)) < 0$, $C \not\subset \text{Supp } \Delta$ and π^{-1} is defined along C . Set $C' := \pi^{-1}(C) \subset X'$. We claim that

$$(C' \cdot (K_{X'} + \Delta' + H' + R' - (f')^*(B'))) \leq 0.$$

Indeed, $(C' \cdot (K_{X'} + \Delta' + H')) = 0$, so the only question is what happens when we add $R' - (f')^*(B') =: \sum d_i D_i$. There are 2 cases to consider.

If $\pi(D_i)$ is a divisor in X , then D_i has a nonnegative coefficient in Δ' . Every D_i has coefficient ≤ 1 in $\Delta' + R'$, thus D_i has coefficient ≤ 1 in R' . Since every divisor in R' has coefficient at least 1 in $(f')^{-1}(B')$, we conclude that $d_i \leq 0$. So adding these $d_i D_i$ can only decrease the intersection number with C' .

The other D_j are π -exceptional and so disjoint from C' since C is general. They have no effect on the intersection number with C' . Therefore

$$(f'_* C' \cdot (K_{Z'} + L' + H'_Z)) \leq 0,$$

Here $f'_* C'$ is the image of a general complete intersection curve, and such images cover an open subset of Z' . By (8.5.1) L' is pseudo-mobile, thus $(f'_* C' \cdot L') \geq 0$ and $(f'_* C' \cdot H'_Z) > 0$ since H'_Z is ample. Hence $(f'_* C' \cdot K_{Z'}) < 0$ and thus Z' is uniruled by [MM86]. $\square$

PROOF OF (8.10.1). X is uniruled by [MM86]. Thus the MRC fibration $f: X \dashrightarrow Z$ is not birational (see [Kol96, IV.5]) and it is enough to prove that $\dim Z = 0$. If $\dim Z \geq 1$ then Z is not uniruled by [GHS03], which contradicts (8.10.2). $\square$

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CHAPTER 9

Non-klt techniques

FLORIN AMBRO

9.1. Introduction

In this note we present Shokurov's finite generation of FGA algebras in dimension one and two, in the presence of singularities worse than Kawamata log terminal (Theorem 9.3.1).

In the preliminary, we briefly recall the codimension one adjunction formula for log pairs, which is used elsewhere in this volume. We also recall the notion of discriminant of a log pair with respect to a fibration, which is used in the proof of Theorem 9.3.1. The discriminant plays an important role in higher codimensional adjunction [Kaw97, Kaw98].

9.2. Preliminary

We consider algebraic varieties defined over an algebraically closed field of characteristic zero. A log pair (X, B) is a normal variety X endowed with a $\mathbb{Q}$ -Weil divisor B , such that $K + B$ is $\mathbb{Q}$ -Cartier. Note that B may have negative coefficients. The locus where (X, B) has Kawamata log terminal singularities is an open subset of X , whose complement, the *non-klt locus*, is denoted by $\text{nklt}(X, B)$. A *fibration* is a proper surjective morphism of normal varieties $f: X \rightarrow Y$ such that $\mathcal{O}_Y = f_*\mathcal{O}_X$. We assume the reader is familiar with Shokurov's terminology of b-divisors.

9.2.1. Codimension one adjunction. The adjunction formula $(K + S)|_S = K_S$, where S is a nonsingular divisor in a nonsingular variety X , is a useful tool in the study of algebraic varieties. Its singular version was introduced by Reid, Kawamata and Shokurov. Below, we follow Shokurov [Sho93b].

Let (X, B) be a log pair, let W be a prime divisor on X such that $\text{mult}_W(B) = 1$ and let $\nu: W^\nu \rightarrow W$ be the normalization.

Let $\mu: Y \rightarrow X$ be a log resolution, write $\mu^*(K + B) = K_Y + B_Y$. Note that B_Y is a well defined $\mathbb{Q}$ -divisor, since we use the same top rational form in the definitions of both K and K_Y . We denote by E the proper transform of W on Y . Since E is nonsingular, the induced morphism $E \rightarrow W$ factors through the normalization

W^ν :

$$\begin{array}{ccccc}
 & E & \longrightarrow & Y & \\
 f \swarrow & \downarrow \mu|_E & & \downarrow \mu & \\
 W^\nu & & & & \\
 \searrow \nu & \downarrow & & \downarrow & \\
 & W & \longrightarrow & X &
 \end{array}$$

We can write $B_Y = E + B'$, where B' is an $\mathbb{Q}$ -divisor which does not contain E in its support. Define $B_E = B'|_E$. By the classical adjunction formula mentioned above, we have

$$\mu^*(K + B)|_E = (K_Y + E + B')|_E = K_E + B_E.$$

Since the above diagram commutes, we obtain $K_E + B_E = f^*\nu^*(K + B)$. Since f is birational, we infer $K_E + B_E = f^*(K_{W^\nu} + B_{W^\nu})$, where

$$B_{W^\nu} = f_*(B_E).$$

The $\mathbb{Q}$ -Weil divisor B_{W^ν} is called the *different* of (X, B) on W^ν . It is well defined, the above construction being independent of the choice of the log resolution.

PROPOSITION 9.2.1. *The following properties hold:*

- (1) (W^ν, B_{W^ν}) is a log pair and the following adjunction formula holds

$$(K + B)|_{W^\nu} = K_{W^\nu} + B_{W^\nu}.$$

- (2) B_{W^ν} is effective if B is effective in a neighborhood of W .
(3) If (X, B) has log canonical singularities near W , then (W^ν, B_{W^ν}) has log canonical singularities.
(4) Let D be a $\mathbb{Q}$ -Cartier divisor on X such that $\text{mult}_W(D) = 0$. Then $(X, B + D)$ is a log pair, $\text{mult}_W(B + D) = 0$ and

$$(B + D)_{W^\nu} = B_{W^\nu} + D|_{W^\nu}.$$

9.2.2. The discriminant of a log pair. Let $f: X \rightarrow Y$ be a fibration and let (X, B) be a log pair structure on X such that (X, B) has log canonical singularities over the generic point of Y . The singularities of the log pair (X, B) over the codimension one points of Y define a $\mathbb{Q}$ -Weil divisor B_Y on Y , called the *discriminant* of (X, B) on Y .

Let $P \subset Y$ be a prime divisor. Since Y is normal, there exists an open set $U \subset Y$ such that $U \cap P \neq \emptyset$ and $P|_U$ is a Cartier divisor. Let a_P be the largest real number t such that the log pair $(f^{-1}(U), B|_{f^{-1}(U)} + tf^*(P|_U))$ has log canonical singularities over the generic point of P . It is clear that $a_P = 1$ for all but finitely many prime divisors P of Y . The discriminant is defined by the following formula

$$B_Y = \sum_P (1 - a_P)P.$$

Let $\mu: Y' \rightarrow Y$ be a birational contraction. Letting X' be the normalization of the graph of the rational map $\mu^{-1} \circ f: X \dashrightarrow Y'$, we obtain a birational contraction

μ_X and a fiber space f' making the following diagram commute.

$$\begin{array}{ccc} X & \xleftarrow{\mu_X} & X' \\ f \downarrow & & \downarrow f' \\ Y & \xleftarrow{\mu} & Y' \end{array}$$

Let $(X', B_{X'})$ be the induced log-crepant log pair structure on X' , that is $\mu_X^*(K + B) = K_{X'} + B_{X'}$. Then $(X', B_{X'})$ has log canonical singularities over the generic point of Y' , and the discriminant of $(X', B_{X'})$ on Y' is a well defined $\mathbb{R}$ -Weil divisor $B_{Y'}$. We have

$$B_Y = \mu_*(B_{Y'}).$$

Thus, the family of $\mathbb{Q}$ -Weil divisors $\mathbf{B} = (B_{Y'})_{Y'}$ is a $\mathbb{Q}$ -b-divisor of Y , called the *discriminant $\mathbb{Q}$ -b-divisor* induced by (X, B) via f . By construction, $\mathbf{B}$ depends only on the discrepancy $\mathbb{Q}$ -b-divisor $\mathbf{A}(X, B)$.

LEMMA 9.2.2. *Let $f: X \rightarrow Y$ be a fibration and let (X, B) be a log pair having Kawamata log terminal singularities over the generic point of Y . Let $\mathbf{B}$ be the induced discriminant $\mathbb{Q}$ -b-divisor of Y . Then*

$$\mathcal{O}_Y(\lceil -\mathbf{B} \rceil) \subseteq f_* \mathcal{O}_X(\lceil \mathbf{A}(X, B) \rceil).$$

PROOF. Clearly this is a local statement on the base. Thus it is enough to show the above inclusion at the level of global sections. Fix a nonzero rational function $a \in k(Y)^\times$ such that $\overline{(a)} + \lceil -\mathbf{B} \rceil \geq 0$. We claim that

$$\overline{(f^*a)} + \lceil \mathbf{A}(X, B) \rceil \geq 0.$$

Let E be a prime b-divisor of X . If $f(\overline{c_X}(E)) = Y$, then $\text{mult}_E(f^*a) = 0$ and $\text{mult}_E(\lceil \mathbf{A}(X, B) \rceil) \geq 0$ by assumption. If $f(\overline{c_X}(E)) \neq Y$, there exists a commutative diagram

$$\begin{array}{ccc} (X, B) & \xleftarrow{\mu_X} & (X', B_{X'}) \\ f \downarrow & & \downarrow f' \\ Y & \xleftarrow{\mu} & Y' \end{array}$$

with the following properties:

- (i) μ and μ_X are birational contractions, $\mu_X: (X', B_{X'}) \rightarrow (X, B)$ is log crepant, and f' is a fiber space.
- (ii) $f'(\overline{c_{X'}}(E))$ is a prime divisor P on Y' .
- (iii) There exists an open set U in Y' such that $P \cap U \neq \emptyset$, $P|_U$ is nonsingular, $f'^{-1}(U)$ is nonsingular and contains a simple normal crossings divisor $\sum_l Q_l$ on $f'^{-1}(U)$ such that $B_{X'|f'^{-1}(U)} = \sum_l b_l Q_l$ and $f'^*(P|_U) = \sum_l m_l Q_l$.
- (iv) There exists l_0 such that $\overline{c_{X'}}(E) = Q_{l_0}$.

The multiplicity $b_P = \text{mult}_P(\mathbf{B})$ is computed as follows

$$1 - b_P = \min_{f'(Q_l)=P} \frac{1 - b_l}{m_l}.$$

By assumption, $\text{mult}_P(a) + \lceil -b_P \rceil \geq 0$, so that $\text{mult}_P(a) + 1 - b_P > 0$. The above formula implies that $v_{Q_{l_0}}(f^*a) + 1 - b_{l_0} > 0$, which means that the b-divisor $\overline{(f^*a)} + \lceil \mathbf{A}(X, B) \rceil$ has non-negative multiplicity at E . $\square$

PROPOSITION 9.2.3. *Let $f: X \rightarrow Y$ be a fiber space and let (X, B) be a log pair having Kawamata log terminal singularities over the generic point of Y . Let $\mathbf{B}$ be the induced discriminant $\mathbb{R}$ -b-divisor of Y . Let $\pi: Y \rightarrow S$ be a proper morphism. Let $(\mathbf{D}_i)_{i \geq 1}$ be a sequence of $\mathbb{R}$ -b-Cartier $\mathbb{R}$ -b-divisors of Y such that the sequence $(f^*\mathbf{D}_i)_{i \geq 1}$ is asymptotically $\mathbf{A}(X, B)$ -saturated, relative to S . Then $(\mathbf{D}_i)_{i \geq 1}$ is asymptotically saturated with respect to $-\mathbf{B}$, relative to S .*

PROOF. We claim that the following inclusion

$$\mathcal{O}_Y(\lceil -\mathbf{B} + \mathbf{D} \rceil) \subseteq f_*\mathcal{O}_X(\lceil \mathbf{A}(X, B) + f^*\mathbf{D} \rceil)$$

holds for every $\mathbb{R}$ -b-Cartier $\mathbb{R}$ -b-divisor $\mathbf{D}$ of Y . Indeed, we may replace $f: (X, B) \rightarrow Y$ birationally, so that $\mathbf{D} = \overline{D}$, where D is an $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisor on Y . Then $(X, B - f^*D)$ is a log pair having Kawamata log terminal singularities over the generic point of Y , with discriminant $\mathbb{R}$ -b-divisor $\mathbf{B} - \mathbf{D}$, and $\mathbf{A}(X, B - f^*D) = \mathbf{A}(X, B) + f^*\mathbf{D}$. The claim follows from Lemma 9.2.2 applied to $f: (X, B - f^*D) \rightarrow Y$.

Let $\nu = \pi \circ f$. By assumption, there exists a positive integer I such that the following inclusion holds for every $I|i, j$:

$$\nu_*\mathcal{O}_X(\lceil \mathbf{A}(X, B) + jf^*\mathbf{D}_i \rceil) \subseteq \nu_*\mathcal{O}_X(jf^*\mathbf{D}_j).$$

We have $\nu_*\mathcal{O}_X(jf^*\mathbf{D}_j) = \pi_*\mathcal{O}_Y(j\mathbf{D}_j)$, and from above we obtain

$$\pi_*\mathcal{O}_Y(\lceil -\mathbf{B} + j\mathbf{D}_i \rceil) \subseteq \nu_*\mathcal{O}_X(\lceil \mathbf{A}(X, B) + jf^*\mathbf{D}_i \rceil).$$

Therefore $\pi_*\mathcal{O}_Y(\lceil -\mathbf{B} + j\mathbf{D}_i \rceil) \subseteq \pi_*\mathcal{O}_Y(j\mathbf{D}_j)$. $\square$

9.3. Non-klt FGA

We present in this section Shokurov's finite generation of FGA algebras in dimension one and two, in the presence of singularities worse than Kawamata log terminal [Sho03, Conjecture 5.26, Example 4.41, Corollary 6.42]. Compared to the original statement, Theorem 9.3.1 contains two simplifications: we no longer assume that the boundary is effective, or that the algebra is ample on the non-klt locus.

THEOREM 9.3.1. *Let (X, B) be a log pair, let $\pi: X \rightarrow S$ be a proper surjective morphism, and let $(\mathbf{D}_i)_{i \geq 1}$ be a sequence of $\mathbb{Q}$ -b-divisors of X such that*

- (i) $i\mathbf{D}_i$ is mobile/ S , for every i .
- (ii) $\mathbf{D}_i \leq \mathbf{D}_j$ for $i|j$.
- (iii) The limit $\lim_{i \rightarrow \infty} \mathbf{D}_i = \mathbf{D}$ is an $\mathbb{R}$ -b-divisor of X .

Assume moreover that the following properties hold:

- (1) $-(K + B)$ is π -nef and π -big.
- (2) $\mathbf{D}_\bullet$ is asymptotically $\mathbf{A}(X, B)$ -saturated over S ; equivalently, there exists a positive integer I such that for every $I|i, j$, the following inclusion holds:

$$\pi_*\mathcal{O}_X(\lceil \mathbf{A}(X, B) + j\mathbf{D}_i \rceil) \subseteq \pi_*\mathcal{O}_X(j\mathbf{D}_j).$$

- (3) There exists an open neighborhood $U \subseteq X$ of $\text{nklt}(X, B)$ such that $\mathbf{D}_{i|U} = \mathbf{D}|_U$ for every $i \geq 1$.

If $\dim(X) \leq 2$, then $\mathbf{D}_i = \mathbf{D}$ for i sufficiently large and divisible.

REMARK 9.3.2. Assumption (1) is redundant if $\dim(X) = 1$.

PROOF. We may assume that S is affine. Since $\mathbf{D}_1$ is mobile/ S , there exists a rational function $a \in k(X)^\times$ such that $\overline{(a)} + \mathbf{D}_1 \geq 0$. By (ii), we obtain $\overline{(a)} + \mathbf{D}_i \geq 0$ for every $i \geq 1$. The two sequences $(\mathbf{D}_i)_i$ and $(\overline{(a)} + \mathbf{D}_i)_i$ satisfy the same properties with respect to (X, B) , and their stabilization is equivalent. Therefore we assume from now on that $\mathbf{D}_i \geq 0$ for every i . After a truncation, we may also assume that $I = 1$ in (2).

(I) Assume $\dim(X) = \dim(S) = 1$. The problem is local, so we may assume that $X = S$ is the germ of a nonsingular curve at a point P . We have $B = b \cdot P$, $\mathbf{D}_i = d_i \cdot P$ and $\mathbf{D} = d \cdot P$. We have $d_i \leq d$ and $\lim_{i \rightarrow \infty} d_i = d$. If $b \geq 1$, then $P \in \text{nktl}(X, B)$, hence $\mathbf{D}_i = \mathbf{D}$ for every i , by (3). Assume now that $b < 1$. Asymptotic saturation is equivalent to

$$\lceil -b + jd_i \rceil \leq jd_j, \forall i, j.$$

Letting i converge to infinity, we obtain

$$\lceil -b + jd \rceil \leq jd_j, \forall j.$$

In particular, $\lceil -b + jd \rceil \leq jd$, which is equivalent to

$$\sup_j \{jd\} \leq b.$$

If $d \notin \mathbb{Q}$, Diophantine Approximation implies $1 \leq b$, contradicting our assumption. Therefore d is rational. Let I' be a positive integer such that $I'd \in \mathbb{Z}$. We infer from the above that $j(d - d_j) \leq \lfloor b \rfloor \leq 0$ for $I'|j$. Therefore $b \geq 0$ and $d_i = d$ for $I'|i$.

(II) Assume $\dim(X) = 1, \dim(S) = 0$. Thus, each $\mathbf{D}_i$ is an effective $\mathbb{Q}$ -divisor D_i of the nonsingular proper curve X . Let $D = \lim_{i \rightarrow \infty} D_i$. If $D = 0$, then $D_i = 0$ for every i . Otherwise, D is an ample $\mathbb{R}$ -divisor. In particular, there exists a positive integer I' such that $\deg(I'D - K - B) > 1$. Asymptotic saturation means that for every i, j , the following inclusion holds

$$H^0(X, \lceil -B + jD_i \rceil) \subseteq H^0(X, jD_j).$$

For fixed j , the divisor $\lceil -B + jD_i \rceil$ coincides with $\lceil -B + jD \rceil$ for some sufficiently large integer i . Therefore for every j we have

$$H^0(X, \lceil -B + jD \rceil) \subseteq H^0(X, jD_j).$$

We have $\lceil -B + jD \rceil = K + \lceil jD - K - B \rceil$ and $\deg[jD - K - B] \geq 2$ for every $I'|j$. Therefore the linear system $|\lceil -B + jD \rceil|$ is base point free for $I'|j$. In particular, asymptotic saturation becomes

$$\lceil -B + jD \rceil \leq jD_j, \forall I'|j.$$

A pointwise argument as in (I) implies that $D_i = D$ for i sufficiently large and divisible.

(III) Assume $\dim(X) = 2$ and $\mathbf{D}_i$ is big/ S for some i ; passing to a truncation, we may assume that bD_i is big/ S for every i .

First of all, if $\dim(S) = 0$, we may also assume that $-(K + B) \cdot (i\mathbf{D}_i)_X > 1$ for every i . Indeed, each $\mathbb{Q}$ -b-divisor $\mathbf{D}_i$ is b-big, and $-(K + B)$ is nef and big. Therefore $-(K + B) \cdot \mathbf{D}_{i,X} > 0$. Since $\mathbf{D}_{i,X} \leq \mathbf{D}_X$, we obtain

$$-(K + B) \cdot \mathbf{D}_X > 0.$$

In particular, $\lim_{i \rightarrow \infty} -(K + B) \cdot (i\mathbf{D}_i)_X = +\infty$, so we may assume after a truncation that $-(K + B) \cdot (i\mathbf{D}_i)_X > 1$ for every i .

The log pair (X, B) has Kawamata log terminal singularities on the open set $V = X \setminus \text{nklt}(X, B)$. Therefore there exist only finitely many geometric valuations E of $k(X)$ such that $c_X(E) \cap V \neq \emptyset$ and $\text{mult}_E \mathbf{A}(X, B) \leq 0$. We consider a log crepant resolution $\mu: (Y, B_Y) \rightarrow (X, B)$ with the following properties:

- (a) Y is nonsingular and B_Y and $\mathbf{D}_{i,Y}$, for every i , are supported by a simple normal crossings divisor on Y .
- (b) $\mathbf{D}_1 = \overline{\mathbf{D}_{1,Y}}$.
- (c) Every valuation E of X , for which $c_X(E) \cap V \neq \emptyset$ and $\text{mult}_E \mathbf{A}(X, B) \leq 0$, has a centre of codimension one on Y .

Let $\mathbf{M} = i\mathbf{D}_i$ for some i . If $\dim(S) = 0$, then

$$-(K_Y + B_Y) \cdot \mathbf{M}_Y = -(K + B) \cdot (i\mathbf{D}_i)_X > 1.$$

Proposition 9.3.3 applies for $\mathbf{M}$ and (Y, B_Y) , hence the linear system $|\mathbf{M}|_Y$ is base point free/ S on $\mu^{-1}(V)$. Equivalently, the restriction of the $\mathbb{Q}$ -b-divisor $\mathbf{E}_Y(\mathbf{D}_i) = \overline{\mathbf{D}_{i,Y}} - \mathbf{D}_i$ to $\mu^{-1}(V)$ is zero. On the other hand, $\mathbf{D}_{i|U} = \mathbf{D}_{1|U}$ by assumption, hence the support of the $\mathbb{Q}$ -divisor $\mathbf{D}_{i,Y} - \mathbf{D}_{1,Y}$ does not intersect $\mu^{-1}(U)$. We infer by (b) that the restriction of the $\mathbb{Q}$ -b-divisor $\mathbf{E}_Y(\mathbf{D}_i)$ to $\mu^{-1}(U)$ is zero. The open sets $\mu^{-1}(U), \mu^{-1}(V)$ cover Y , hence $\mathbf{E}_Y(\mathbf{D}_i) = 0$. This holds for every i , hence we obtain

$$\mathbf{D}_i = \overline{\mathbf{D}_{i,Y}} \text{ for all } i \geq 1.$$

In particular, $\mathbf{D}_Y$ is a π -nef $\mathbb{R}$ -divisor and $\mathbf{D}_Y - (K_Y + B_Y)$ is π -nef and big. Furthermore, $\mu^{-1}(U)$ is an open neighborhood of $\text{nklt}(Y, B_Y)$ and $\mathbf{D}_{i|_{\mu^{-1}(U)}} = \mathbf{D}_{i|_{\mu^{-1}(U)}}$ for every i . We infer by [Amb05b], Theorem 3.3 that $\mathbf{D}_i = \mathbf{D}$ for i sufficiently large and divisible. Note that [Amb05b], Theorem 3.3 is stated only for characteristic sequences of functional algebras, but its proof is valid in our setting.

(IV) Assume $\dim(X) = 2$ and, for all i , $\mathbf{D}_i$ is not big/ S . After a truncation, there exists a rational map with connected fibers $f: X \dashrightarrow Y$, defined over S , such that $\dim(Y) = 1$ and there exists effective $\mathbb{Q}$ -divisors D_i on Y such that $\mathbf{D}_i = f^* \overline{D_i}$ for every $i \geq 1$. It is clear that the sequence $(D_i)_{i \geq 1}$ and its limit $D = \lim_{i \rightarrow \infty} D_i$ satisfy the properties (i)-(iii) in the statement of the theorem.

If $\mu: X' \rightarrow X$ is a resolution of singularities and $\mu^*(K + B) = K_{X'} + B_{X'}$ is the induced crepant log pair structure on X' , then the sequence $(\mathbf{D}_i)_i$ satisfies the same properties (1)-(3) with respect to $(X', B_{X'})$ and $\mu^{-1}(U)$. Therefore we may assume that f is a morphism.

The sequence $(D_i)_i$ is constant in a neighborhood of $f(\text{nklt}(X, B))$. If $f(\text{nklt}(X, B)) = Y$, we are done. Otherwise $f(\text{nklt}(X, B)) \neq Y$, that is (X, B) has Kawamata log terminal singularities over the generic point of Y . In this case, the discriminant B_Y of (X, B) on Y is well defined. Since $(\mathbf{D}_i)_i$ is $\mathbf{A}(X, B)$ -asymptotically saturated, we infer by Proposition 9.2.3 that the sequence $(D_i)_i$ is asymptotically saturated with respect to $\mathbf{A}(Y, B_Y) = -B_Y$.

It is clear that (Y, B_Y) is a log pair structure on Y . Furthermore, the inclusion $\text{nklt}(Y, B_Y) \subset f(\text{nklt}(X, B))$ implies that $D_i = D$ over a neighborhood of $\text{nklt}(Y, B_Y)$. Therefore the sequence $(D_i)_i$ and $(Y, B_Y) \rightarrow S$ satisfy the hypothesis of the theorem, possibly except property (1). We have not used the assumption (1) in the proof of (I) and (II), hence $D_i = D$ for i sufficiently large and divisible. Therefore $\mathbf{D}_i = \mathbf{D}$ for i sufficiently large and divisible. $\square$

PROPOSITION 9.3.3. *Let (X, B) be a 2-dimensional log pair and let $\pi: X \rightarrow S$ be a proper surjective morphism such that $-(K + B)$ is π -nef and π -big. Let $\mathbf{M}$ be a mobile/ S and b -big/ S b -divisor of X such that*

$$\pi_* \mathcal{O}_X(\lceil \mathbf{A}(X, B) + \mathbf{M} \rceil) \subseteq \pi_* \mathcal{O}_X(\mathbf{M}).$$

If $\dim(S) = 0$, assume moreover that $-(K + B) \cdot \mathbf{M}_X > 1$. Then the relative base locus of the linear system $|\mathbf{M}|_X$ is included in the set of points of X where the log pair (X, B) does not have terminal singularities.

PROOF. We may assume that S is affine. Let $\mu: Y \rightarrow X$ be a resolution of singularities such that $\mathbf{M} = \overline{\mathbf{M}}_Y$ and $\text{Supp}(B_Y)$ is a simple normal crossings divisor, where $\mu^*(K + B) = K_Y + B_Y$. The general member $C \in |\mathbf{M}|_Y$ is a nonsingular curve, intersecting $\text{Supp}(B_Y)$ transversely. By saturation, we have

$$H^0(Y, \lceil -B_Y \rceil + C) \subseteq H^0(Y, C).$$

(1) The restriction map $H^0(Y, \lceil -B_Y + C \rceil) \rightarrow H^0(C, \lceil -B_Y + C \rceil|_C)$ is surjective. Indeed, the cokernel is included in

$$H^1(Y, \lceil -B_Y \rceil) = H^1(Y, K_Y + \lceil -\mu^*(K + B) \rceil).$$

Since $-\mu^*(K + B)$ is μ -nef and μ -big, we infer by Kawamata-Viehweg vanishing that $H^1(Y, \lceil -B_Y \rceil) = 0$, hence the claim holds.

(2) The linear system $|\lceil -B_Y + C \rceil|_C|$ is base point free. Indeed, assume that $\dim(S) > 0$. Then C is an affine curve, hence $\mathcal{O}_C(\lceil -B_Y + C \rceil|_C)$ is generated by global sections. If $\dim(S) = 0$, then C is a nonsingular projective curve and the following identity holds by adjunction:

$$\lceil -B_Y + C \rceil|_C = K_C + \lceil -\mu^*(K + B) \rceil|_C.$$

By assumption, we have $\deg(\lceil -\mu^*(K + B) \rceil|_C) \geq 2$. The claim follows now from a standard argument.

(3) B_Y is effective in a neighborhood of C . Indeed, it follows from (1) and (2) that the sheaf $\mathcal{O}_Y(\lceil -B_Y + C \rceil)$ is generated by global sections in a neighborhood of C . Therefore the above saturation implies that $\lceil -B_Y + C \rceil \leq C$ in an open neighborhood of C , which is equivalent to $B_Y \geq 0$ near C .

It is clear that (3) implies that $|\mathbf{M}|_X$ is base point free at the terminal points of (X, B) . $\square$

CHAPTER 10

Glossary

ALESSIO CORTI

If X is a normal variety, I denote by K_X the **canonical class** of X .

If D is a $\mathbb{Q}$ -divisor, then $\lceil D \rceil$, $\lfloor D \rfloor$, $\{D\}$ denote the round up, round down, fractional part of D .

Adjunction: See **inversion of adjunction**.

Ample divisor: A divisor D on a proper normal variety X is ample if the global sections of $\mathcal{O}_X(nD)$ define an embedding of X in projective space. The **Kleiman criterion** states that, if X is $\mathbb{Q}$ -factorial, then D is ample if and only if $D \cdot a > 0$ for all $a \in \overline{NE}X$, the **Mori cone** of X . Because of this, when we say for instance “let D be an ample divisor”, we often have in mind the picture of a divisor which satisfies the condition of the Kleiman criterion, not the usual definition of an ample divisor. Thus, it makes sense to speak of an ample $\mathbb{R}$ -divisor.

The Kleiman criterion implies that, if X is $\mathbb{Q}$ -factorial, then X is projective if and only if the Mori cone $\overline{NE}X \subset N^1(X, \mathbb{R})$ is nondegenerate, that is, it contains no vector subspace of $N^1(X, \mathbb{R})$ other than (0) .

Base locus: If $\mathcal{D}$ is a linear system of Weil divisors on a variety X , the base locus, or fixed locus $\text{Bs } \mathcal{D}$ is the intersection of all the divisors in $\mathcal{D}$; the fixed part is the largest Weil divisor contained in $\text{Bs } \mathcal{D}$.

Base Point Free Theorem: Let (X, B) be a pair with **klt singularities** and L a **nef** Cartier divisor on X . If $L - \varepsilon(K_X + B)$ is nef and **big** on X for some $\varepsilon > 0$, then L is eventually free, that is, $|mL|$ is base point free for some positive integer $m > 0$.

By the **Zariski counterexample**, the Theorem as stated does not hold for a pair (X, B) with **dlt singularities**.

This is a crucial bottleneck of the theory. There are various ways to strengthen the assumptions to make the conclusion valid for dlt pairs. For example one can: require that $L - \varepsilon(K_X + B)$ is **ample**; introduce a special notion of “log big” divisor; use Shokurov’s **LSEPD** trick; etc.

B-divisor: Let X be a normal variety. A (integral) b-divisor (that is, a “birational divisor”) on X is a formal (integral) linear combination

$$\mathbf{D} = \sum m_E E$$

where the sum runs over all the **geometric valuations** with centre on X . If $Y \rightarrow X$ is a **model**, then the trace of $\mathbf{D}$ on Y is the ordinary divisor on Y :

$$\text{tr}_Y \mathbf{D} = \mathbf{D}_Y = \sum_{c_Y E \text{ a divisor}} m_E E.$$

The group of b-divisors on X is denoted by $\mathbf{Div} X$.

The $\mathbb{Q}$ -Cartier closure of a $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor D on X is the b-divisor $\overline{D}$ on X with trace

$$\mathrm{tr}_Y \overline{D} = f^* D$$

on all models $f: Y \rightarrow X$.

A b-divisor $\mathbf{M}$ on X is **mobile** if there is a model $f: Y \rightarrow X$ such that

- $\mathbf{M}_Y$ is a base point free divisor on Y ; in particular, $\mathbf{M}_Y$ is integral and Cartier.
- $\mathbf{M} = \overline{\mathbf{M}_Y}$ (that is, $\mathbf{M}$ “descends” to Y). Strictly speaking, the $\mathbb{Q}$ -Cartier closure $\overline{\mathbf{M}_Y}$ is a b-divisor on Y ; however

$$f_*: \mathbf{Div} Y \xrightarrow{\cong} \mathbf{Div} X$$

is a canonical isomorphism by which we always implicitly identify b-divisors on Y with b-divisors on X .

Big divisor: A divisor D on a normal variety X is big if for some positive integer m , $H^0(X, mD)$ defines a rational map which is birational onto its image. Usually one works with divisors that are **nef** and big. If D is nef and X is proper of dimension n , then D is big if and only if $D^n > 0$.

Boundary: A boundary on a normal variety X is a divisor $B = \sum b_i B_i$ on X where the B_i are distinct prime divisors and

$$0 < b_i \leq 1$$

Usually the coefficients b_i are rational; sometimes they are real. A divisor $B = \sum b_i B_i$ is a **sub-boundary** if $b_i \leq 1$. We never really use sub-boundaries in this book. Sub-boundaries appear naturally in Shokurov’s proof of the **nonvanishing** Theorem. Some features of the **minimal model program** work for sub-boundaries; however, the precise details have never been worked out and one has to be very careful when working with sub-boundaries.

Bounded: A class or set A of algebraic varieties

$$\{X_a \mid a \in A\}$$

is bounded if there is a morphism $f: \mathcal{X} \rightarrow \mathcal{T}$ of algebraic varieties such that every X_a is isomorphic to a fibre of the morphism f . For example, the set of isomorphism classes of Fano manifolds of a given dimension n is bounded in this sense. It is easy to make variant of this notion where A can be a class of various types of object of algebraic geometry.

Canonical class: The canonical divisor class of a normal variety X is the linear equivalence class of a canonical divisor, that is the divisor of a rational differential form, that is, a rational section of the pre-dualizing sheaf ω_X . The pre-dualizing sheaf is the sheaf which Hartshorne denotes ω_X^0 and which is defined there by means of a universal property involving a trace map

$$t: H^n X \rightarrow k$$

and the induced pairing

$$\mathrm{Hom}(\mathcal{F}, \omega_X) \times H^n(X, \mathcal{F}) \rightarrow k$$

assumed to be perfect for all coherent sheaves $\mathcal{F}$ on X .

Cone Theorem: Let X be a proper normal algebraic variety. Denote by $N^1(X, \mathbb{R})$, resp. $N_1(X, \mathbb{R})$, the real vector spaces of Cartier divisors on X , resp. cycles of dimension 1 on X , modulo numerical equivalence. The **Mori cone** $\overline{NE} X$ is the closure, in the real topology, of the convex cone

$$NE(X) = \sum_{C \text{ curve on } X} \mathbb{R}_+[C] \subset N_1(X, \mathbb{R}).$$

$\overline{NE} X$ is a very subtle object; in particular the process of taking the closure of $NE X$ is highly nontrivial and generates a huge number of pages in technical manuals. Mori's cone Theorem is an amazing general result on the structure of the Mori cone: If X is projective and it has **terminal singularities**, $\overline{NE} X$ is locally finitely generated in the half-space $\{K_X < 0\}$. Moreover, if R is a **extremal ray** with $K_X \cdot R < 0$, then $R = \mathbb{R}_+[C]$ for a rational curve $C \subset X$. The cone Theorem is the cornerstone of higher dimensional algebraic geometry. The statement has been generalised in many ways; there is a version where X is replaced with a **pair** (X, B) with **log terminal singularities** and K_X with $K_X + B$; there is a relative version for projective morphisms $f: X \rightarrow Z$; there are even several versions where (X, B) has non-**log canonical singularities**.

Contraction Theorem: If X is projective and has **terminal singularities**, then all **extremal rays** $R \subset \overline{NE} X$ with $K \cdot R < 0$ of the **Mori cone** can be contracted, that is, there is a contraction morphism $f_R: X \rightarrow Y$ characterized by the two properties: (a) f_R contracts a curve $C \subset X$ to a point $y \in Y$ if and only if $[C] \in R$, and (b) $f_{R*} \mathcal{O}_X = \mathcal{O}_Y$. If X is **$\mathbb{Q}$ -factorial**, the contraction is of one of three types:

- f_R is a divisorial contraction if it is birational and the exceptional set contains a divisor. In this case the exceptional set $\text{Exc } f_R \subset X$ is a prime divisor.
- f_R is a small, or flipping, contraction if it is birational and the exceptional set is **small** (i.e., it has codimension ≥ 2).
- f_R is a Mori fibre space if it is not birational, that is, $\dim Y < \dim X$.

Divisorial contractions and flips of small contractions are the steps of the **minimal model program**.

The contraction of an extremal ray $R \subset \overline{NE} X$ with $K_X \cdot R < 0$ is sometimes just called an **extremal contraction**.

As for the **cone Theorem**, the statement can be generalized in various ways; to **pairs** (X, B) with **klt singularities**; to projective morphisms $X \rightarrow Z$; to pairs with non-**log canonical singularities**.

Crepant: A synonym of “nondiscrepant”. A birational morphism $f: X \rightarrow Y$ is crepant if it has zero discrepancy, that is

$$K_X = f^* K_Y.$$

A morphism $f: X \rightarrow Y$ of pairs (X, B_X) and (Y, B_Y) is crepant if $K_X + B_X = f^*(K_Y + B_Y)$.

Different: If X is a variety and $S \subset X$ a codimension 1 subvariety, it is natural to try to compare K_S with $(K_X + S)|_S$ or, more generally, the predualizing sheaf ω_S with $\omega_X(S)|_S$ (however the “restriction” may be defined). In good cases, for a $\mathbb{Q}$ -divisor B one can define a “different”

$\mathbb{Q}$ -divisor $\text{Diff}_S B$ such that the formula

$$K_S + \text{Diff}_S B = (K_X + S + B)|_S$$

holds.

Discrepancy: Let (X, B) be a **pair** of a normal variety X and a divisor B . Let ν be an **geometric valuation** with centre on X ; by definition, this means that there is a uniformization $f: E \subset Y \rightarrow X$ such that $\nu = \text{mult}_E$ where $E \subset Y$ is a divisor. By restricting Y without throwing away E , I may write

$$K_Y = f^*(K_X + B) + aE$$

and then $a = a(\nu, B)$ is the discrepancy of ν (or E). Indeed, $a(\nu, B)$ depends only on ν and the pair (X, B) , not on the uniformization $E \subset Y \rightarrow X$. If $B = \sum b_i B_i$ where the B_i are distinct prime divisors, then the definition implies that $a(B_i, B) = -b_i$.

The discrepancy **b-divisor** of the pair (X, B) is the b-divisor $\mathbf{A} = \mathbf{A}(X, B)$ such that, on any **model** $f: Y \rightarrow X$,

$$K_Y = f^*(K_X + B) + \mathbf{A}(X, B)_Y$$

Dlt model: Divisorial log terminal model. See **model**.

Dlt singularities: Divisorially log terminal singularities. See **log terminal singularities**.

Extremal contraction: See **contraction Theorem**.

Extremal ray: Let N be a finite dimensional real vector space and $C \subset N$ a convex cone. Usually C is a closed cone; often C is nondegenerate, i.e., it contains no vector subspaces other than (0) . A half-line

$$R = \mathbb{R}_+[\mathbf{v}] \subset C$$

is an extremal ray if:

$$\mathbf{v}_1, \mathbf{v}_2 \in C \text{ and } \mathbf{v}_1 + \mathbf{v}_2 \in R \implies \mathbf{v}_1, \mathbf{v}_2 \in R.$$

Fano variety: See **relative weak Fano klt pair**

Finite generation: If a graded algebra R is finitely generated, then $X = \text{Proj } R$ makes sense. Therefore it is natural to ask for natural conditions implying that a graded algebra is finitely generated.

Geometric valuation: Let X be a normal variety. A discrete valuation $\nu: k(X) \rightarrow \mathbb{Z}$ of the function field of X is an geometric valuation with centre on X if

$$\nu = \text{mult}_E, \quad \text{where } E \subset Y \xrightarrow{f} X$$

is a prime divisor on a normal variety Y , and $f: Y \rightarrow X$ is a morphism. The morphism $E \subset Y \rightarrow X$ is sometimes called a uniformization of ν . The centre of E on X is the scheme-theoretic point

$$c_X \nu = f(\text{generic point of } E)$$

In this book we often denote by $\overline{c_X \nu}$ the Zariski closure of $c_X \nu$. It is common practice to abuse notation and identify the valuation ν and the divisor E .

Inversion of Adjunction: Let X be a normal algebraic variety, $S \subset X$ a prime divisor, and $B \subset X$ a **boundary**. Assume that it makes sense to write

$$(K + S + B)_S = K_S + \text{Diff}_S B$$

(see **different**). This is a generalization of the adjunction formula. In this context direct **adjunction** or simply adjunction is the statement: If $(X, S+B)$ has **plt singularities**, resp. **dlt singularities**, then $(S, \text{Diff}_S B)$ has **klt singularities**, resp. **dlt singularities**. The converse to these statements is also true and it is called inversion of adjunction; it is much harder to show.

Kleiman criterion: See **ample divisor**

Klt singularities: Kawamata log terminal singularities. See **log terminal singularities**.

Kodaira trick: Let M be a **big** divisor on a normal variety X ; then for every divisor D on X , $|nM - D| \neq \emptyset$ for all large enough integers n .

LMMP: See **minimal model program**.

Log canonical centre: See **non-klt locus**.

Log canonical model: See **model**.

Log canonical singularities: See **log terminal singularities**.

Log resolution: Let X be a normal variety and let D be a divisor on X .

A log resolution is a proper birational morphism $f: Y \rightarrow X$, such that Y is smooth, the exceptional locus of f is a divisor and the inverse image of D union the exceptional locus has **simple normal crossings**.

Log Terminal Singularities: Let (X, B) be a **pair** of a normal variety X and a **boundary** divisor B .

The pair (X, B) has klt (Kawamata log terminal) singularities if $a(E, B) > -1$ for every **geometric valuation** E with centre on X (see also **discrepancy**). In particular this implies that if $B = \sum b_i B_i$, then all $b_i < 1$.

The pair (X, B) has plt (purely log terminal) singularities if $a(E, B) > -1$ for every geometric valuation E with small centre on X . This allows some of the $b_i = 1$; however the definition implies for example that every connected component of $\lfloor B \rfloor$ is normal.

The pair (X, B) has dlt (divisorially log terminal) singularities if the pair has a **log resolution** $f: Y \rightarrow X$ such that $a(E, B) > -1$ for every geometric valuation E with centre an f -exceptional divisor.

The pair (X, B) has lc (log canonical) singularities if $a(E, B) \geq -1$ for every geometric valuation E with centre on X .

Sometimes one says that the divisor $K_X + B$ is klt, etc., or that (X, B) is a klt pair, etc., to mean that the pair (X, B) has klt singularities, etc.

These definitions are not easy to digest. Note that the syntax in the dlt case is much more subtle than in the other cases; dlt is a very delicate notion and special care must always be exercised when dealing with dlt singularities. For example, dlt is a local property in the Zariski topology but not in the étale (or analytic) topology; by contrast, all other notions are local in the étale and analytic topology.

LSEPD: An acronym for “Locally the Support of an Effective Principal Divisor”. Let X be a normal variety and $f: X \rightarrow Z$ a morphism. An effective divisor D on X is LSEPD over Z if, for all $z \in Z$, there is a regular

function $\varphi \in \mathcal{O}_{Z,z}$ such that $\text{Supp } D = \text{Supp } \text{div } \varphi \circ f$ in a neighbourhood of $f^{-1}\{z\}$. When $Z = X$, we simply say that D is LSEPD.

LSEPD divisors can sometimes be used to show that the conclusion of the **base point free Theorem** holds on a given variety or pair with dlt singularities.

Minimal model: A projective $\mathbb{Q}$ -factorial variety X with terminal singularities and K_X **nef**.

Minimal Model Program: The algorithm which starts with a nonsingular projective variety X and, after finitely many steps, terminates with a **minimal model** or **Mori fibre space**. (One can also start with a projective $\mathbb{Q}$ -factorial variety with terminal singularities.) Each step is either a divisorial contraction or the flip of a small contraction of an **extremal ray** $R \subset \overline{\text{NE}} X$ with $K_X \cdot R < 0$ (see **contraction Theorem**). One sometimes meets the acronym “**MMP**” or the equivalent appellation “Mori program”.

There are variants and generalizations to non- $\mathbb{Q}$ -factorial varieties; varieties with worse kinds of singularities; pairs; and a relative version for varieties over a base.

The minimal model program for pairs is called the log minimal model program (**LMMP**, log Mori program, logarithmic Mori program,...).

MMP: See **minimal model program**.

Mobile b-divisor: See **b-divisor**.

Model: A model of a variety X is a normal variety Y which is birational to X . Models are usually assumed to be proper over a specified or implicitly agreed on base. Often Y is proper over X , that is, it comes with a proper birational morphism $Y \rightarrow X$.

- A **pair** (X, B) is a **dlt model** (divisorially log terminal model) if X is proper, (X, B) has **dlt singularities**, and $K_X + B$ is **nef**.
- (X, B) is a **lc model** (log canonical model) if X is proper, (X, B) has **log canonical singularities**, and $K_X + B$ is **ample**.
- (X, B) is a **wlc model** (weakly log canonical model) if X is proper, (X, B) has **log canonical singularities**, and $K_X + B$ is **nef**.

We also have the following more subtle notion: (X, B) is a **lt** (lc, wlc) model of (Y, D) if (X, B) is a **lt** (lc, wlc) model and there is a birational map $t: Y \dashrightarrow X$ and $a(E, D) \leq a(E, B)$ for all **geometric valuations** E with centre on Y (because X is proper, E always has a centre on X ; see also **discrepancy**).

The point of this definition is that it implies that $H^0(X, n(K_X + B)) = H^0(Y, n(K_Y + D))$ for all positive integers n .

Mori cone: See **cone Theorem**.

Mori fibre space: A contraction $f_R: X \rightarrow Y$ of an extremal ray $R \subset \overline{\text{NE}} X$ with $K_X \cdot R < 0$ which is not birational, i.e., where $\dim Y < \dim X$.

Mori Theory: A collection of results on the **Mori cone** $\overline{\text{NE}}$ and its **extremal rays**.

Morphism: A rational map which is everywhere defined.

Multiplier ideal: Let (X, B) be a pair of a smooth variety X and an effective $\mathbb{Q}$ -divisor $0 \leq B \subset X$. The multiplier ideal sheaf $\mathcal{J}(B) = \mathcal{J}(X, B)$

associated to B is defined to be

$$\mathcal{J}(\mathcal{B}) = \mathcal{O}_X(\lceil \mathbf{A}(X, B) \rceil)$$

where $\mathbf{A}(X, B)$ is the **discrepancy** b-divisor.

Nef divisor: A divisor D on a normal variety X is nef—an acronym for “numerically eventually free”—if $D \cdot C \geq 0$ for every proper curve $C \subset X$.

Negativity Lemma: Let $f: Y \rightarrow X$ be a proper birational morphism with exceptional divisors $E_i \subset Y$. A divisor D on Y is f -effective if D is effective and no E_i appears in D . The negativity Lemma states: If $A + D \equiv \sum a_i E_i$ where A is f -ample and D is f -effective, then all $a_i \leq 0$ (there is also a refined version stating $a_i < 0$ unless something precise happens).

The negativity lemma implies the following statement: If (X, B) is a pair with **klt (dlt, lc, terminal, canonical, etc.) singularities**, X is projective, and $t: X \dashrightarrow Y$ is a divisorial contraction or flip of a small contraction of an extremal ray $R \subset \overline{\text{NE}}(X)$ with $(K_X + B) \cdot R < 0$, then $a(E, B) \leq a(E, t_* B)$ for all **geometric valuations** E with centre on X (see also **discrepancy**). It follows that the pair $(Y, t_* B)$ has klt (dlt, lc, terminal, canonical, etc.) singularities (the dlt case is much harder).

Morally, the higher the discrepancies, the better the singularity; therefore a step of the Mori program “improves singularities”. This is literally true of flips and morally true even of divisorial contractions; indeed, if $f: (E \subset X) \rightarrow Y$ is the divisorial contraction of an extremal ray R and $(K_X + B) \cdot R < 0$, discrepancies of all valuations other than E increase.

Non-klt locus: Let (X, B) be a **pair** of a normal variety X and a $\mathbb{Q}$ -divisor $B \subset X$. The non-klt locus $\text{nklt}(X, B)$ of the pair (X, B) is the complement of the largest Zariski open set $U \subset X$ such that $(U, B|_U)$ has **klt singularities**. In the literature, this is sometimes called the “log canonical set” of the pair (X, B) and denoted $\text{LCS}(X, B)$; we find this terminology and notation misleading and we consistently avoid it in this book,

When the pair (X, B) has **log canonical singularities**, a scheme theoretic point $P \in X$ is a log canonical centre, or *LC* centre, if $P = c_X \nu$ where ν is a *geometric valuation* with centre on X and **discrepancy** $a(X, B) = -1$.

Nonvanishing: Any result where the conclusion is $H^0(X, D) \neq (0)$. Shokurov’s nonvanishing Theorem states: Let X be proper and nonsingular, $B = \sum b_i B_i$ a **sub-boundary** on X with all $b_i < 1$ and such that $\sum B_i$ is **simple normal crossing**; and let L be a **nef** Cartier divisor on X . If $L - \varepsilon(K_X + B)$ is nef and big for some $\varepsilon > 0$, then $H^0(X, mL + \lceil -B \rceil) \neq (0)$ for all sufficiently large integers m .

Pair: A pair (X, B) of a normal variety X and a $\mathbb{Q}$ -divisor B on X such that $K_X + B$ is $\mathbb{Q}$ -Cartier.

In this book B is almost always a **boundary**.

Pl flips: The title of Shokurov’s paper on flips. Here “pl” is an acronym for “pre limiting”.

Plt singularities: Purely log terminal singularities. See **log terminal singularities**.

$\mathbb{Q}$ -factorial: A normal variety X is $\mathbb{Q}$ -factorial if for every Weil divisor D on X , there is a positive integer m such that mD is Cartier.

Being $\mathbb{Q}$ -factorial is a local property in the Zariski topology but not in the étale (or analytic) topology.

It is customary to run the **minimal model program** with $\mathbb{Q}$ -factorial varieties only. This is fine, but there are situations where non- $\mathbb{Q}$ -factorial varieties crop up, even when one's primary interest lies in $\mathbb{Q}$ -factorial varieties. For example if $f: (X, S + B) \rightarrow Z$ is a 4-fold (say) pl flipping contraction, then the 3-folds S is not, in general, $\mathbb{Q}$ -factorial.

There is a version of the minimal model program that runs with non- $\mathbb{Q}$ -factorial varieties; it has unexpected features, for instance it is sometimes necessary to “flip” divisorial contractions.

Relative weak Fano klt pair: A pair (X, B) with **klt singularities**, together with a projective morphism $X \rightarrow Z$ such that $-(K_X + B)$ is relatively **nef** and **big** over Z . The notion generalizes, for example, a Fano manifold and a flipping contraction.

Semiample divisor: Traditionally a divisor D on a proper normal variety X is semiample if $D = f^*A$ is the pull-back of an **ample divisor** A under a morphism $f: X \rightarrow Y$. In the more modern terminology adopted in this book, a semiample divisor is an element of the convex cone of $N^1(X, \mathbb{R})$ generated by semiample divisors (in the traditional sense).

Shokurov algebra: A bounded and asymptotically saturated pbd-algebra. Shokurov's Conjecture states that a Shokurov algebra on a **relative weak Fano klt pair** is finitely generated. This conjecture for varieties of dimension $n - 1$ and the **minimal model program** for varieties of dimension $n - 1$ implies the existence of pl flips in dimension n .

Simple normal crossing: A reduced divisor D on a nonsingular variety X is a simple normal crossing divisor if for each closed point $x \in X$, a local defining equation of D at x can be written as $f = t_1 \cdots t_{j(x)}$ where $t_1, \dots, t_{j(x)}$ is part of a regular system of parameters in the local ring $\mathcal{O}_{X,x}$. You must be aware of the fact that this property is local in the Zariski topology but not in the étale or locally Euclidean topology.

Small: Anything that exists in codimension ≥ 2 . A birational morphism $f: Y \rightarrow X$ is small if the exceptional set is small. A birational map $f: X \dashrightarrow Y$ is small if there are small Zariski closed subsets $E \subset X$ and $F \subset Y$ such that f is an isomorphism of $X \setminus E$ and $Y \setminus F$.

Sub-boundary: See **boundary**.

Sub-klt pair: Let (X, B) be a pair of a normal variety X and a **sub-boundary** divisor B . Then (X, B) has sub-klt singularities if $a(E, K_X + B) > -1$ for all **geometric valuations** E with centre on X .

Terminal and Canonical Singularities: Let X be a normal variety; assume that K_X is $\mathbb{Q}$ -Cartier, that is, there is a positive integer m such that mK_X is a Cartier divisor.

X has terminal, resp. canonical, singularities if $a(E) > 0$, resp. $a(E,) \geq 0$ for every **geometric valuation** E with small centre on X (see also **discrepancy**).

If (X, B) is a pair, then (X, B) has terminal, resp. canonical, singularities if $a(E, B) > 0$, resp. $a(E, B) \geq 0$ for every **geometric valuation** E with small centre on X . (Rather idiosyncratically, this property

is sometimes called “terminal, resp. canonical, in codimension 2” in the literature.)

For example, a surface pair (X, B) is terminal if and only if X is nonsingular and

$$\text{mult}_x B = \sum b_i \text{mult}_x B_i < 1$$

for all $x \in X$.

Termination: An algorithm terminates if it stops in finite time. To show that the **minimal model program** terminates, we are quickly led to show that there is no infinite sequence of flips starting with a given variety X . This statement is called “termination of flips”. **Terminal singularities** are so called because they are the singularities that appear at the termination point of the minimal model program.

Vanishing: The vanishing Theorem of Kawamata and Viehweg is the following generalization of the vanishing Theorem of Kodaira: If X is nonsingular, L is a **nef** and **big** $\mathbb{Q}$ -divisor on X and the fractional part of L has normal crossing support, then $H^i(X, K_X + \lceil L \rceil) = (0)$ for all $i > 0$.

There are very many variants and generalizations of this result; but this is a very long story.

Weak log canonical model: See **model**.

X-method: The method introduced by Kawamata and used by him to prove the foundational results of **Mori theory**. (The terminology was created by the students in Kawamata’s seminar at the University of Tokyo; it is meant to reflect the reputation of the method as a “black art”.) In a nutshell, the method can be described by the following principle: If, for some n , the linear system nM contains a member D which is very singular at x (for example, $K_X + (1/n)D$ is not log canonical), then $x \notin \text{Bs } |K_X + M|$. Shokurov’s saturation property is a sort of contrapositive to the X-method: M is saturated if

$$\text{Mob}[K_X + M] \leq M.$$

When $S \subset X$ is a prime divisor satisfying suitable conditions, Shokurov’s and the statement that M exceptionally saturated implies $M|_S$ canonically saturated is another way to say the X-method.

Zariski counterexample: A nonsingular rational surface S , an elliptic curve $E \sim -K_S$ with $E^2 = -1$, and a **nef** and **big** divisor class L on S such that $L \cdot E = 0$, but $L|_E$ is a nontorsion divisor of degree 0; then $|mL|$ has scheme theoretic base locus E for all integers $m > 0$. Thus the **base point free Theorem** does not hold on the **pair** (S, E) .

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