The Abdus Salam International Centre for Theoretical Physics

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COLLEGE ON

PHYSICS OF NANO-DEVICES

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Introduction in Bosonization II

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Effective Hamiltonian: A simple Example

The effective-mass Hamiltonian

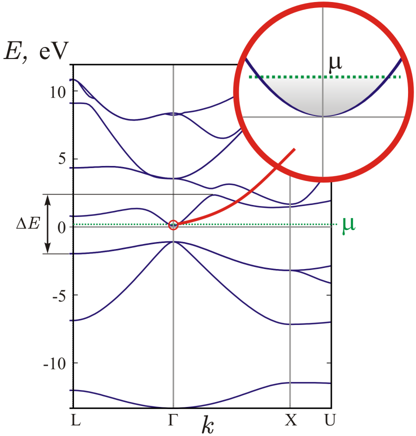
$$H = \int d^3x \psi^\dagger(\vec{r}) \left(-\frac{\nabla^2}{2m^*} - \mu \right) \psi(\vec{r})$$

is an example of an Effective Hamiltonian. Its validity range is determined by conditions

$$\mu \ll \Delta E, \quad T \ll \Delta E, \quad \omega \ll \Delta E$$

and

$$ka \ll 1$$



The Effective Hamiltonian describes the low-energy long-wavelength physics of the system disregarding the high energy detail.

Universality

The effective-mass Hamiltonian

- Reduces all the complexity of the spectrum to a single parameter m^*
- Describes an enormous range of doped materials with completely different chemical composition and lattice structure

Definition

The set of physical Hamiltonians with common low energy effective Hamiltonian is called the universality class.

The model whose Hamiltonian is the effective Hamiltonian is also often called the universality class.

In This Part of the Course...

- We shall try to understand the origins of universality, that is what makes completely different systems alike at low energies. This will lead us to the concept of scaling.
- We shall develop a mathematical formalism, called the Renormalization Group. This will give us a quantitative tool for constructing effective Hamiltonians.
- We shall establish that the Luttinger Model is a low energy effective theory for a universality class of one-dimensional interacting many-particle systems, called Luttinger Liquids .
- We shall briefly discuss some known physical systems belonging to this universality class.

Scaling

Here we introduce some ideas of scaling theory using an example of a 1+1 -dimensional free-fermion hopping model.

Formalization of The Problem

Physical properties of a system are encoded in correlation functions. For example, for a system of non-relativistic spinless fermions in $1+1$ D the particle distribution function and the tunneling density of states can be found from

$$G(x,t) = -i\langle T\psi(x,t)\psi^{\dagger}(0)\rangle, \quad G(k,\omega) = \int dxdte^{-ikx+i\omega t}G(x,t)$$

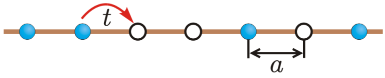
Low energy, or infrared, limit corresponds to small k and ω (large x and t). The effective Hamiltonian should "generate" the long distance asymptotics of correlation functions. For example,

$$G(\lambda x, \lambda t), \qquad \lambda \to \infty$$

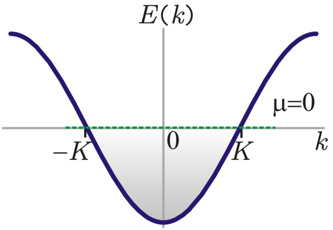
A Simple Example: Spinless Lattice Fermions

Spinless fermions $\{c_i, c_j^\dagger\} = \delta_{i,j}$

$$H = -\sum_{i=-\infty}^{\infty} t\, c_i^\dagger c_{i+1} + \text{h.c.}$$



In Fourier space $c_j = \int \frac{dk}{2\pi\sqrt{a}} e^{ikaj} c(k)$

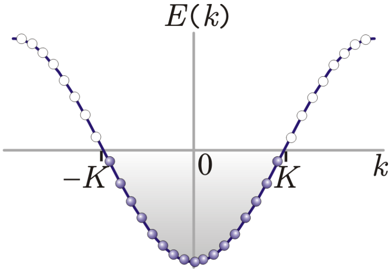


The model Hamiltonian

$$H = \int \frac{dk}{2\pi} c^\dagger(k) E(k) c(k)$$

$$E(k) = -2t\cos(ka)$$

Lattice Fermions. Low Energy Excitations.



All low energy processes only involve fermions near the Fermi points $k = K$ and $k = -K$.

Introduce two types of Fermions:

$$\psi_R(k) = c(K + k)$$
$$\psi_L(k) = c(-K + k)$$

Denote by $\psi(x) = c_j/\sqrt{a}$ where $x = aj$ then

$$\psi(x) = e^{iKx}\psi_R(x) + e^{-iKx}\psi_L(x), \qquad \psi_{R,L} = \int_{-K}^{K} \frac{dk}{2\pi}\psi_{R,L}(k)e^{ikx}$$

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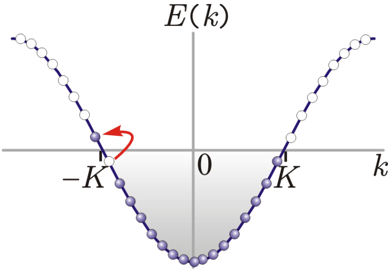

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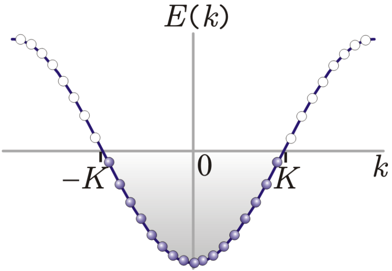

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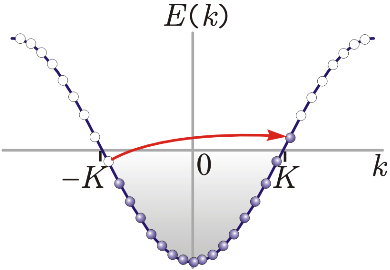

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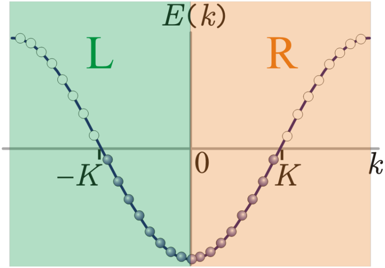

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The Gradient Expansion I

In $\psi_{L,R}(k)$ basis the Hamiltonian becomes

$$H = \int \frac{dk}{2\pi} \left[\psi_L^\dagger(k) E(-K+k) \psi_L(k) + \psi_R^\dagger(k) E(K+k) \psi_R(k) \right]$$

Note: here we dropped the limits of integration since high k energies do not contribute to low energy spectrum.

The gradient expansion:

$$H = \int \frac{dk}{2\pi} (vk + c_3 k^3 + \dots) \left[\psi_L^\dagger(k) \psi_L(k) - \psi_R^\dagger(k) \psi_R(k) \right]$$

where $v = 2at$, $c_3 = ta^3/3$.

The Gradient Expansion II

In space-time domain

$$H = H_0 + \int dx \sum_{n\geq 1} \gamma_n \left[Q_{L,n}(x) - Q_{R,n}(x)\right]$$

where H_0 is the Dirac Hamiltonian

$$H_0 = v \int dx (i\psi_L^\dagger \partial_x \psi_L - i\psi_R^\dagger \partial_x \psi_R)$$

and Q are local operators of the form

$$Q_{\alpha,n}(x) = \psi_\alpha^\dagger(x)(\partial_x)^{2n+1}\psi_\alpha(x), \qquad \alpha = L, R$$

Zooming Out

Scaling Transformation

$$x \to \lambda x, \quad t \to \lambda t, \quad \psi \to \frac{1}{\sqrt{\lambda}}\psi$$

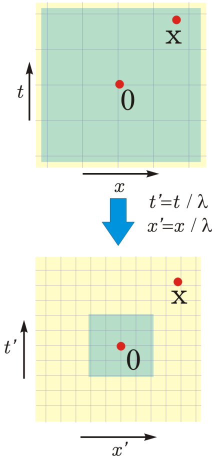
$$H[\psi, \psi^\dagger] \to \lambda H\left[\frac{\psi}{\sqrt{\lambda}}, \frac{\psi^\dagger}{\sqrt{\lambda}}\right]$$

The result of scaling transformation

$$H \to \tilde{H} = H_0 + \int dx \tilde{\gamma}_n \left[Q_{L,n}(x) - Q_{R,n}(x)\right]$$

where

$$\tilde{\gamma}_n = \frac{1}{\lambda^{2n}}\gamma_n$$



Marginal and Irrelevant Operators

We can draw the following conclusions:

- The Dirac-Like part of the Hamiltonian is not affected by scaling. We shall call such operators marginal.
- Other terms in the gradient expansion scale to zero at large distances, that is

$$\tilde{\gamma}_a \to 0 \quad \text{when} \quad \lambda \to \infty.$$

  In we shall call such operators irrelevant.
- The effective low-energy Hamiltonian of the system is the Hamiltonian of a free massless Dirac field

Infrared Asymptotics of Green's Function

Introduce a shorthand $\mathbf{x} = (x, t)$.

$$G_R(\mathbf{x}) = -i\langle T\psi_R(\mathbf{x})\psi_R^\dagger(0)\rangle$$

Apply scaling

$$G_R(\lambda\mathbf{x}) = -\frac{i}{\lambda}\langle T\psi(\mathbf{x})\psi^\dagger(0)\rangle_{\tilde{H}}$$

In the large λ limit $\tilde{H} = H_0$

$$G_{R,L}(\lambda\mathbf{x}) \to \frac{1}{\lambda(x \mp vt)}, \qquad \lambda \to \infty$$

Summary

For a system of free spinless fermions on a lattice at a half-filling

$$H = -\sum_{i=-\infty}^{\infty} t\, c_i^\dagger c_{i+1} + \text{h.c.}$$

we derived the effective low energy Hamiltonian, which is the hamiltonian of a free massless Dirac field. At large distances

$$\langle T c_j(t) c_0^\dagger(0)\rangle = e^{i2Kx} G_R(x,t) + e^{-i2Kx} G_L(x,t),$$

where $K = \pi/2a$, $x = aj$ and $G_{R,L}$ are chiral fermion propagators in free massless Dirac theory

$$G_R(\mathbf{x}) = \frac{1}{x - vt}, \qquad v = 2at_H$$

Renormalization Group

Here we generalize the scaling approach to systems with interactions

Scaling of the Luttinger Hamiltonian

$$H = v \int dx \left[\psi_L^\dagger i\partial_x \psi_L - v\psi_R^\dagger i\partial_x \psi_R + \gamma \rho_L(x)\rho_R(x) \right]$$

Apply the scaling transformation

$$x \to \lambda x, \quad t \to \lambda t, \quad \psi \to \frac{1}{\sqrt{\lambda}}\psi, \quad H[\psi, \psi^\dagger] \to \lambda H \left[\frac{\psi}{\sqrt{\lambda}}, \frac{\psi^\dagger}{\sqrt{\lambda}} \right]$$

$$H \to \tilde{H} = H$$

This would imply that *e.g.*

$$G_R(\lambda \mathbf{x}) = \frac{1}{\lambda} G_R(\mathbf{x})$$ which is wrong!!!

Quantum Corrections to Scaling

In actual fact, from exact solution

$$G_R(\mathbf{x}) = \langle T\psi_R(x,t)\psi_R^\dagger(0)\rangle = \frac{c}{(x - v_c t)^{\Delta}(x + v_c t)^{\bar{\Delta}}}$$

it follows that

$$G_R(\lambda\mathbf{x}) = \frac{1}{\lambda^{\Delta+\bar{\Delta}}} G_R(\mathbf{x}), \qquad \Delta + \bar{\Delta} = \frac{K + K^{-1}}{2}$$

Due to interactions scaling properties of operators change! This is a result divergencies in perturbation theory, which introduce the ultraviolet cutoff scale.

Generalized Scaling Theory

Consider some field theory with a set of local operators $Q_a(x)$ and a Hamiltonian H

$$H = \int dx \gamma_a Q_a(x).$$

Write a generalized scaling relation for a correlation function

$$\langle T Q_{a_1}(\lambda \mathbf{x}_1) \dots Q_{a_N}(\lambda \mathbf{x}_N) \rangle_H = \langle T \tilde{Q}_{a_1}(\mathbf{x}_1) \dots \tilde{Q}_{a_N}(\mathbf{x}_N) \rangle_{\tilde{H}}$$

where

$$\tilde{Q}_a = \Lambda_{ab} Q_b$$

If we calculate the Hamiltonian $\tilde{H}$ and the transformation matrix Λ as a function of λ, we can derive the effective theory.

The Beta Function

There is a relation between $\tilde{H}$ and Λ

$$i\lambda\frac{\partial}{\lambda\partial t}\langle TQ_a(\lambda\mathbf{x})\dots\rangle_H = i\frac{\partial}{\partial t}\langle T\tilde{Q}_a(\mathbf{x})\dots\rangle_{\tilde{H}} \Rightarrow$$

$$\lambda\langle T[Q_a(\lambda\mathbf{x}), H]\dots\rangle_H = \langle T[\tilde{Q}_a(\mathbf{x}), \tilde{H}]\dots\rangle_{\tilde{H}}$$

One immediately finds

$$\tilde{H} = \int dx\tilde{\gamma}_a Q_a(x), \qquad \tilde{\gamma}_a = \lambda^2\Lambda_{ab}\gamma_b$$

In the infinitesimal form the β-function appears

$$\lambda\frac{d}{d\lambda}\gamma_a = \beta_a(\vec{\gamma}), \qquad \beta_a = 2\gamma_a + \lambda\frac{d}{d\lambda}\Lambda_{ab}\gamma_b$$

Back to Non-Interacting Example.

In the non-interacting lattice model we had

$$Q_a \sim \psi^\dagger \partial_x^{2n+1} \psi.$$

Under the scaling transformation $\mathbf{x} \to \lambda \mathbf{x}$

$$Q_a \to \frac{1}{\lambda^{2n+2}} Q_a, \quad \Rightarrow \quad \Lambda_{ab} = \frac{1}{\lambda^{2n+2}} \delta_{ab}$$

The beta-function is given by

$$\beta_a = 2\gamma_a + \lambda \frac{d}{d\lambda} \Lambda_{ab} \gamma_b = -2n\gamma_a$$

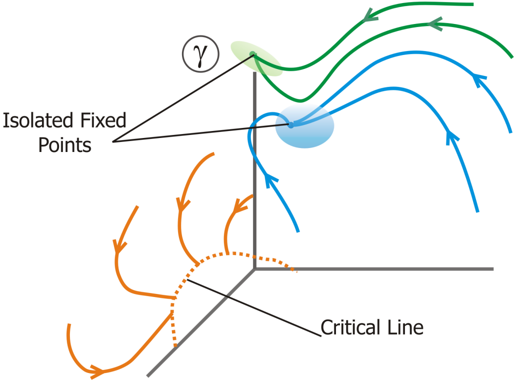
RG Flows and Fixed Points

The trajectories of the equation

$$\lambda \frac{d}{d\lambda} \gamma_a = \beta_a(\vec{\gamma})$$

have fixed points in the space of couplings at points where

$$\forall a \qquad \beta_a(\vec{\gamma}) = 0$$



Lyapunov Analysis of the Fixed Point

Let $\vec{\gamma}^*$ be some fixed point. Then for a small deviation

$$\delta\vec{\gamma} = \vec{\gamma} - \vec{\gamma}^*, \qquad \lambda \frac{d}{d\lambda}\delta\gamma_a = \hat{T}_{ab}\delta\gamma_b$$

The right hand side can be diagonalized

$$\delta\vec{\gamma} = U\vec{g}, \qquad U^{-1}TU = \mathrm{diag}(2 - h_1, 2 - h_2, \dots)$$

In the vicinity of the critical point γ^* there exists a basis of local operators which have definite scaling dimensions h_a (quasi-primary fields)

$$\Phi_a = U^{-1}_{ab} Q_b, \qquad \tilde{\Phi}_a(\mathbf{x}) = \lambda^{-h_a}\Phi_a(\lambda\mathbf{x})$$

Stability of the Fixed Point

Write the effective Hamiltonian near γ^* in terms of quasi-primaries

$$H^* = H_0^* + \sum_a \int dx g_a \Phi_a(x)$$

Note that only quasi-primaries consistent with fundamental symmetries of the system are allowed in this expression!

As we have just seen, for small enough g_a

$$\lambda \frac{d}{d\lambda} g_a = (2 - h_a) g_a$$

The fixed point is only stable if the dimensions of quasi-primaries allowed in the Hamiltonian by symmetries satisfy $h_a > 2$.

Universality Classes

Stable fixed points attract RG flows from some vicinity in the space of couplings γ. All systems, whose parameters are inside this vicinity will have the same infrared description given by the fixed point of the RG flow. Stability of a fixed point is achieved by removing relevant operators (that is operators of dimension $h < 2$) from the theory by either some symmetry or using fine-tuning.

Universality classes are scale-invariant effective field theories, which do not contain dangerous operators, that is relevant quasiprimary fields respecting the fundamental symmetries of the system.

The Luttinger Liquid

The Luttinger Model is a stable fixed point of RG for spinless fermion systems with conserved particle number and momentum. The corresponding universality class is called the Luttinger Liquids.

Using bosonization one can derive the complete spectrum of dimensions of quasiprimaries. The list of dangerous fields is

operator	dimension	physical meaning
$\partial_x \theta$	1	current carrying state
$\partial_x \phi$	1	shift of the chemical potential
$\psi_R^\dagger \psi_L + h.c. = \cos(2\phi)$	$K < 1$	bulk backskattering

Problem

Find missing "dangerous" fields and explain their meaning.

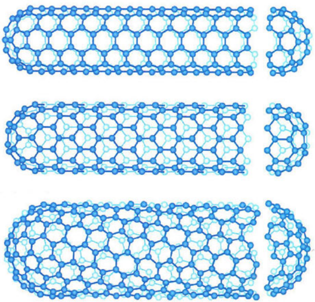
Summary

- We have found that in interacting quantum systems naiive scaling of fields does not lead to the right answers
- We have reviewed some results from a more rigorous Renormalization Group approach
- We have seen that in the RG theory rescaling generates a flow of effective Hamiltonians in the space of coupling constants.
- We analyzed the fixed points of this flow and derived the criterion for the fixed point to be a universality class
- We have shown that the Luttinger model is a universality class of clean one-dimensional interacting systems.

Examples of Physical Systems

Here we shall briefly discuss some known examples of Luttinger Liquids encountered in condensed matter physics.

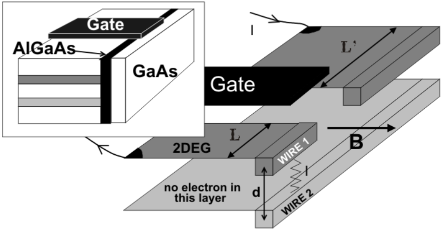
Carbon Nanotubes



R. Egger, A. Bachtold, M. Fuhrer, M. Bockrath, D. Cobden, P. McEuen, cond-mat/0008008

Armchair nanotubes are metallic. Zig-zag and Chiral can be either metallic or semiconducting. There are four conducting channels, labelled by two projections of spin and two projections of isospin. Each conducting channel is a Luttinger Liquid. There is a density-density coupling between the channels. The typical Luttinger parameter is quite small $K \sim 0.2$

Semiconductor Quantum Wires



O.M. Auslaender *et al.* , Science **295**, 825 (2002)

Good one-dimensional wires became available with the cleaved edge overgrouth technique. Luttinger Liquid effects have been investigated on these devices. Previously Luttinger liquid effects were observed in V-groove quantum wires.

Quantum Hall Edge States

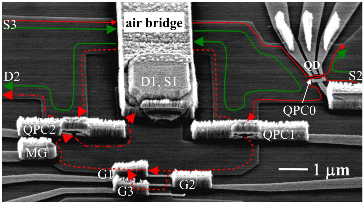


Fig. 1

I. Neder, M. Heiblum, Y. Levinson, D. Mahalu, V. Umansky, cond-mat/0508024

A single quantum Hall edge is described by the chiral Luttinger liquid (which would be a subject of a separate lecture course). However, two edges brought together make a nice Luttinger Liquid.