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"8th Workshop on Three-Dimensional Modelling of
Seismic Waves Generation, Propagation and their Inversion"

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Seismology beyond Ray Theory I

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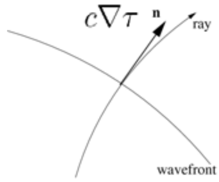
Ray tracing

The eikonal equation



$$c\nabla\tau$$

$$\hat{\boldsymbol{n}} = d\boldsymbol{r}/ds$$

$$\nabla\tau(\boldsymbol{r}) = \frac{1}{c}\frac{d\boldsymbol{r}}{ds}$$

$$|\nabla\tau|^2 = \frac{1}{c^2}$$

The ray tracing equations

$$\nabla\tau(\boldsymbol{r}) = \frac{1}{c}\frac{d\boldsymbol{r}}{ds}$$

can be written as a first order system:

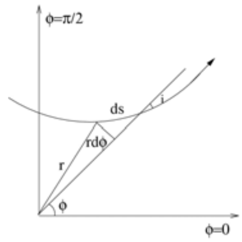
$$\boldsymbol{p} = \frac{1}{c}\frac{d\boldsymbol{r}}{ds}$$

$$\frac{d\boldsymbol{p}}{ds} = \nabla\left(\frac{1}{c}\right)$$

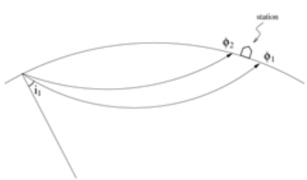
Spherical symmetry

$$\frac{dr}{d\varphi} = r\cot i$$

$$\frac{di}{d\varphi} = \frac{dc}{dr}\frac{r}{c} - 1$$

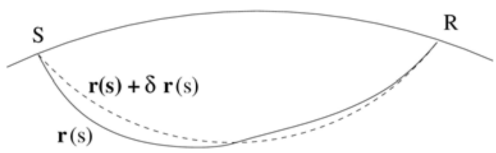


The shooting gallery



$$i_{next} = i_1 + \frac{\varphi - \varphi_1}{\varphi_2 - \varphi_1}(i_2 - i_1)$$

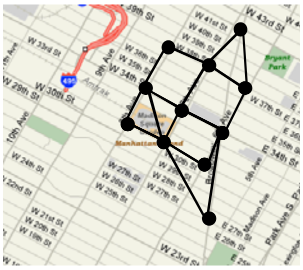
Fermat's principle



$$\delta T = \int_A^B \delta\boldsymbol{r} \cdot \left(\nabla\left(\frac{1}{c}\right) - \frac{d}{ds}\left(\frac{1}{c}\frac{dr}{ds}\right)\right) ds$$

=0

Dijkstra's method



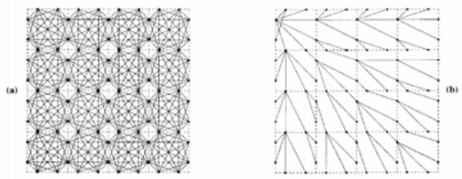
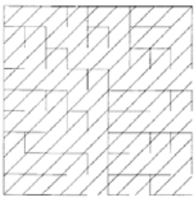
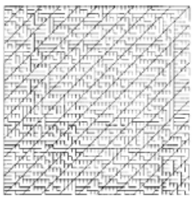


Fig. 1. Cell organization of a network. (a) Dashed lines: cell boundaries. Black circles: nodes. Solid lines: connections. (b) Shortest paths from one node to other nodes in a homogeneous model.

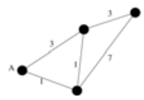
Moser, Geophysics 1991.

"Streets of New York"

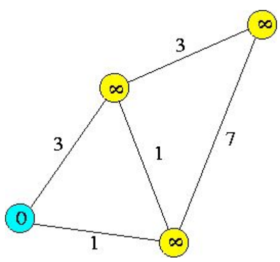




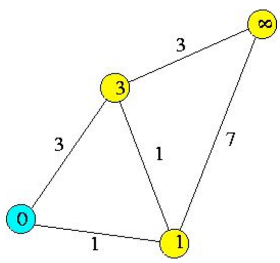
A graph theory exercise



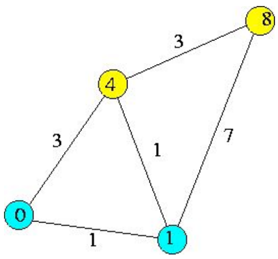
Source node: t=0



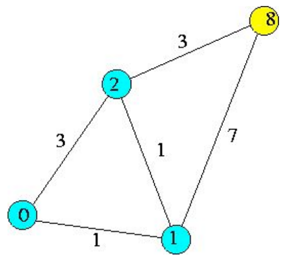
Update time to its neighbours



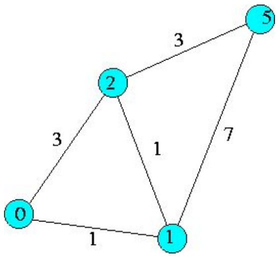
Take smallest time and update neighbours



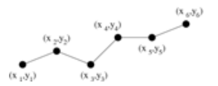
Take again smallest time



and update last node



Bending the ray



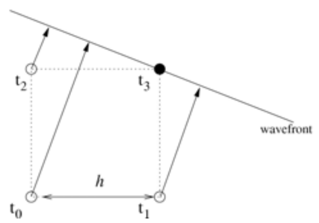
$$T = \sum_{i=2}^{N} \frac{L_i}{\bar{v}_i}$$

$$\frac{\partial L_k}{\partial x_k} = \frac{x_k - x_{k-1}}{L_k}$$

$$\frac{\partial L_{k+1}}{\partial x_k} = \frac{x_k - x_{k+1}}{L_{k+1}}$$

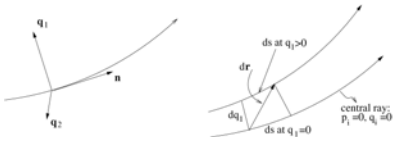
$$\frac{\partial \bar{v}_k}{\partial x_k} = \frac{\partial \bar{v}_{k+1}}{\partial x_k} = \frac{1}{2}\frac{\partial v_k}{\partial x_k}$$

Vidale's finite differencing



$$t_3 = t_0 + \sqrt{2(h/c)^2 - (t_2 - t_1)^2}$$

Paraxial rays



Hamiltonian formulation

$$|\nabla\tau|^2 = \frac{1}{h^2}\left(\frac{\partial\tau}{\partial s}\right)^2 + \left(\frac{\partial\tau}{\partial q_1}\right)^2 + \left(\frac{\partial\tau}{\partial q_2}\right)^2 = \frac{1}{c^2}$$

$$\frac{\partial\tau}{\partial s} = \frac{h}{c}\sqrt{1 - c^2(p_1^2 + p_2^2)} \equiv -\mathcal{H}(q_i, p_i)$$

$$p_i = \partial\tau/\partial q_i$$

Taylor series for the Hamiltonian

$$\mathcal{H}(q_1, q_2, p_1, p_2) =$$

$$\mathcal{H}(0,0,0,0) + \frac{1}{2}c(p_1^2 + p_2^2) + \frac{1}{2}\frac{\partial^2\mathcal{H}}{\partial q_1}q_1^2 + \frac{\partial^2\mathcal{H}}{\partial q_1 \partial q_2}q_1 q_2 + \frac{1}{2}\frac{\partial^2\mathcal{H}}{\partial q_2}q_2^2 =$$

$$-\frac{1}{c} + \frac{1}{2}c(p_1^2 + p_2^2) + \frac{1}{2c^2}\left(\frac{\partial^2 c}{\partial q_1^2}q_1^2 + 2\frac{\partial^2 c}{\partial q_1 \partial q_2}q_1 q_2 + \frac{\partial^2 c}{\partial q_2^2}q_2^2\right)$$

(The p_i and q_i are 0 on the ray itself)

$$\frac{\partial\tau}{\partial s} = \frac{h}{c}\sqrt{1 - c^2(p_1^2 + p_2^2)} \equiv -\mathcal{H}(q_i, p_i)$$

$$\frac{\partial\tau}{\partial s} + \frac{1}{2}c(p_1^2 + p_2^2) + \frac{1}{2c^2}\left(\frac{\partial^2 c}{\partial q_1^2}q_1^2 + 2\frac{\partial^2 c}{\partial q_1 \partial q_2}q_1 q_2 + \frac{\partial^2 c}{\partial q_2^2}q_2^2\right) = \frac{1}{c}$$

Use this with:

$$\tau(s, q_1, q_2) = \tau(s, 0, 0) + \frac{1}{2}\hat{\boldsymbol{q}} \cdot \boldsymbol{H}\hat{\boldsymbol{q}}$$

Ricatti equation for the Hessian

$$\frac{d\boldsymbol{H}}{ds} + c\boldsymbol{H}^2 = -\frac{1}{c^2}\boldsymbol{V}$$

where

$$V_{ij} = \partial^2 c / \partial q_i \partial q_j$$

Paraxial and polar