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"8th Workshop on Three-Dimensional Modelling of
Seismic Waves Generation, Propagation and their Inversion"

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RAY THEORY
(Overheads)

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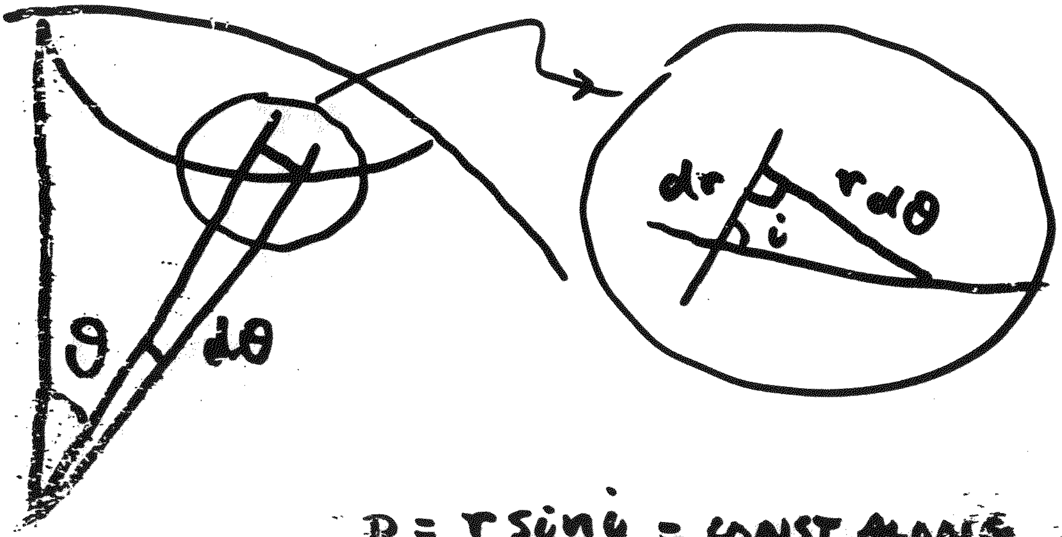
CLASSICAL RAY THEORY



$$p = \frac{r \sin i}{c} = \text{CONST. ALONG RAY} \quad \text{(SNELL'S LAW)}$$

$$= \text{"RAY PARAMETER"}$$

c = WAVE SPEED = $c(r)$

i = ANGLE RAY MAKES LOCALLY WITH VERTICAL

$$r\,d\theta = dr \tan i = dr \frac{pc/r}{\sqrt{1 - \frac{p^2c^2}{r^2}}}$$

$$\Rightarrow \boxed{\theta = \int \frac{pc}{r^2}\left(1 - \frac{p^2c^2}{r^2}\right)^{-1/2} dr}$$

$$dt = \frac{1}{c}\frac{dr}{\cos i} = \frac{dr}{c}\left(1 - \frac{p^2c^2}{r^2}\right)^{-1/2}$$

$$\boxed{t = \int \frac{1}{c}\left(1 - \frac{p^2c^2}{r^2}\right)^{-1/2} dr}$$

EQUATION OF MOTION AND PLANE WAVE SOLUTIONS:

$$t_{ij,j} = \rho \ddot{u}_i$$

u_i = ELASTIC DISPLACEMENT

t_{ij} = STRESS TENSOR

$$t_{ij} = \mu (u_{i,j} + u_{j,i}) + \lambda u_{k,k} \delta_{ij}$$

ISOTROPIC HOOKE'S LAW

PLANE WAVE SOLUTIONS:

CONSIDER WAVE TRAVELLING IN x-DIRECTION

WRITE $\underline{u} = (u, v, w)$

P-WAVE

$$u = U e^{i(\omega t - kx)}$$
$$v = 0$$
$$w = 0$$

k = WAVENUMBER $= \omega/\alpha$

α = P-WAVE SPEED $= \sqrt{\frac{\lambda + 2\mu}{\rho}}$

U = const.

S-WAVE

$$u = 0$$
$$v = V e^{i(\omega t - kx)}$$
$$w = 0$$

$k = \omega/\beta$

β = S-WAVE SPEED $= \sqrt{\mu/\rho}$

V = const.

WHAT IS THE ENERGY FLUX?

WE NEED TO FIND THE RATE OF WORKING OF ONE SIDE OF A PLANE $\perp^r$ x-AXIS ON THE OTHER.

LET $\underline{n}$ be a unit vector in x-direction

$$\underline{n} = (1, 0, 0)$$

Traction $= t_i = t_{ij} n_j \, ds$

where ds = element of area

Rate of working $= t_{ij} n_j \, ds \, \dot{u}_i$ (force × velocity)

P-WAVE

$$\tau_{xx} = (\lambda + 2\mu) ik U e^{i(\omega t - kx)}$$

$$\text{Energy flux} = Re\{(\lambda+2\mu)\, ik\, U e^{i(\omega t - kx)}\} \times Re\{ i\omega U e^{i(\omega t - kx)}\}$$

$\Rightarrow$ ENERGY FLUX AVERAGED OVER A CYCLE

$$= \frac{1}{2} |U|^2 \omega k (\lambda + 2\mu)$$

$$= \frac{1}{2} |U|^2 \omega^2 \rho \alpha$$

UNITS: ENERGY PER UNIT TIME PER UNIT AREA.

S-WAVE SIMILARLY

$$\tau_{xy} = \mu i k V e^{i(\omega t - kx)}$$

$$\text{Energy flux} = Re\{\mu i k V e^{i(\omega t - kx)}\} \times Re\{i \omega V e^{i(\omega t - kx)}\}$$

ENERGY FLUX AVERAGED OVER A CYCLE

$$= \frac{1}{2}|V|^2 \omega k \mu$$

$$= \frac{1}{2}|V|^2 \omega^2 \rho \beta$$

ASYMPTOTIC THEORY

1-DIMENSIONAL CASE

THE BASIC IDEA OF THE ASYMPTOTIC OR RAY THEORIES IS THAT IN MEDIA IN WHICH THE WAVE VELOCITIES AND DENSITY VARY SLOWLY WAVES PROPAGATE IN MUCH THE SAME WAY AS IN HOMOGENEOUS MEDIA.

CONSIDER A P-WAVE PROPAGATING IN THE x-DIRECTION IN A MEDIUM IN WHICH DENSITY AND P-WAVE SPEED ARE ALSO FUNCTIONS OF x.

WAVE EQUATION:

$$\frac{\partial}{\partial x}(\lambda+2\mu)\frac{\partial u}{\partial x} = \rho\frac{\partial^2 u}{\partial t^2}$$

ie.

$$\frac{\partial}{\partial x}\left(\rho\alpha^2\frac{\partial u}{\partial x}\right) = \rho\frac{\partial^2 u}{\partial t^2} \qquad \begin{array}{l}\rho=\rho(x)\\ \alpha=\alpha(x)\end{array}$$

SEEK AN APPROXIMATE SOLUTION OF THE FORM

$$u(x,t) = U(x)\,e^{i\omega(t-\theta(x))}$$

$U=U(x)$, $\theta=\theta(x)$ TO BE DETERMINED.

ω IS CONSIDERED TO BE A LARGE PARAMETER

SUBSTITUTING, WE FIND

$$\frac{\partial u}{\partial x} = \left(-i\omega \frac{\partial \theta}{\partial x} U + \frac{\partial U}{\partial x}\right) e^{i\omega(t-\theta)}$$

$$\frac{\partial}{\partial x}\left(\rho\alpha^2 \frac{\partial u}{\partial x}\right) = \Big\{-\omega^2 \rho\alpha^2 U\left(\frac{\partial \theta}{\partial x}\right)^2$$

$$-i\omega \frac{\partial \theta}{\partial x}\frac{\partial U}{\partial x}\rho\alpha^2$$

$$-i\omega \frac{\partial}{\partial x}\left(\frac{\partial \theta}{\partial x} U\rho\alpha^2\right) + \cdots \Big\} e^{i\omega(t-\theta)}$$

where $\cdots$ indicates terms of lower order in ω

THUS, FROM ω^2 terms :

$$\boxed{\left(\frac{\partial \theta}{\partial x}\right)^2 = \frac{1}{\alpha^2}}$$

EQUATION FOR THE PHASE $\theta(x)$ [EIKONAL EQUATION]

AND FROM ω^1 TERMS

$$\frac{\partial \theta}{\partial x}\frac{\partial U}{\partial x}\rho\alpha^2 + \frac{\partial}{\partial x}\left(\frac{\partial \theta}{\partial x} U\rho\alpha^2\right) = 0$$

ie $$\frac{\partial}{\partial x}\left(\frac{\partial \theta}{\partial x} U^2\rho\alpha^2\right) = 0$$

ie $$\boxed{\frac{\partial}{\partial x}\left(U^2\rho\alpha\right) = 0}$$

$\equiv$ CONSTANT ENERGY FLUX

3-D THEORY — WORKS SIMILARLY

(KARAL & KELLER, J. Acoust. Soc. Am., 31, 694, 1959)

SEEK A SOLUTION OF EQNS. OF MOTION IN FORM

$$u_i = U_i(x, y, z)\, e^{i\omega(t - \theta(x, y, z))}$$

substitute into equation of motion, identify leading powers of ω (ω^2).

DETAILS ARE COMPLICATED.

WE FIND THAT EITHER

$$\boxed{\theta_{,i}\,\theta_{,i} = \frac{1}{\alpha^2} \qquad \text{WITH} \quad U_i \parallel \theta_{,i}}$$

OR

$$\boxed{\theta_{,i}\,\theta_{,i} = \frac{1}{\beta^2} \qquad \text{WITH} \quad U_i \perp^r \theta_{,i}}$$

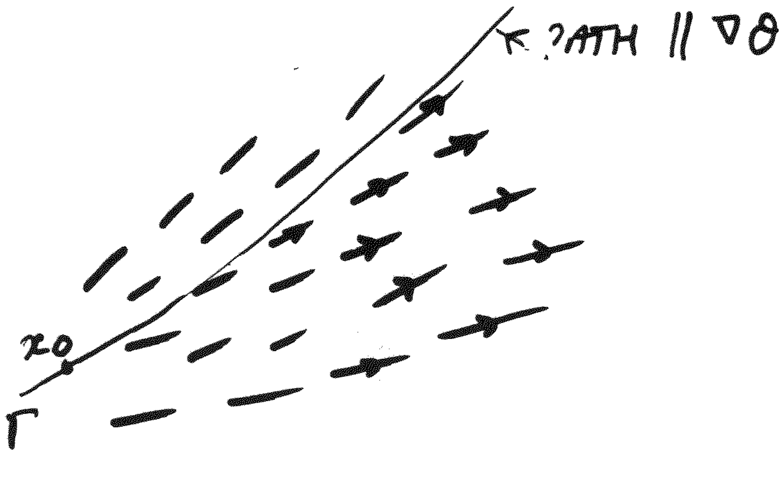
THUS WE GET TWO KINDS OF SOLUTION, CORRESPONDING TO P-WAVES AND TO S-WAVES

THUS, IN BOTH CASES WE OBTAIN FOR THE "TRAVEL TIME" $\vartheta(\underline{x})$ AN EQUATION OF THE FORM

$$\boxed{(\underline{\nabla}\vartheta)^2 = \frac{1}{c^2} \qquad \text{EIKONAL EQUATION}}$$

WHERE $c = \alpha$ FOR P-WAVES, OR $c = \beta$ FOR S-WAVES.

IMAGINE A PATH EVERYWHERE $\parallel$ TO $\underline{\nabla}\vartheta$



WE HAVE $$\vartheta = \vartheta_0 + \int_\Gamma \frac{1}{c}\, ds$$

HOW CAN WE DETERMINE SUCH PATHS?

DIFFERENTIATING THE EIKONAL EQN.

$$2\,\theta_{,i}\,\theta_{,ij} = \frac{\partial}{\partial x_j}\left(\frac{1}{c^2}\right)$$

ie. $$2\,\theta_{,i}\,\theta_{,ji} = \frac{\partial}{\partial x_j}\left(\frac{1}{c^2}\right)$$

but $\theta_{,i}$ is parallel to $\underline{s}$ ie

$$\theta_{,i} = \frac{1}{c}\frac{dx_i}{ds} \qquad \text{(A)}$$

$$\therefore \quad \frac{2}{c}\frac{d\theta_{,j}}{ds} = \frac{\partial}{\partial x_j}\left(\frac{1}{c^2}\right)$$

or $$\frac{d\theta_{,j}}{ds} = \frac{c}{2}\frac{\partial}{\partial x_j}\left(\frac{1}{c^2}\right) = \frac{\partial}{\partial x_j}\left(\frac{1}{c}\right) \qquad \text{(B)}$$

THUS, FROM (A) & (B)

$$\boxed{\begin{aligned} \frac{dx_i}{ds} &= c\,\theta_{,i} \\ \frac{d\theta_{,i}}{ds} &= \frac{\partial}{\partial x_i}\left(\frac{1}{c}\right) \end{aligned}}$$

RAY-TRACING EQUATIONS

ALTERNATIVELY, WRITING

$$\frac{d}{ds} = \frac{1}{c}\frac{d}{d\theta}$$

$$k_i = \omega\,\theta_{,i}$$

WE GET

$$\frac{dx_i}{d\theta} = \frac{c^2}{\omega}k_i = c\frac{k_i}{k}$$

$$\frac{dk_i}{d\theta} = \omega c\frac{\partial}{\partial x_i}\left(\frac{1}{c}\right) = -\frac{\omega}{c}\frac{\partial c}{\partial x^i} = -k\frac{\partial c}{\partial x^i}$$

WHERE $k = (k_i k_i)^{1/2} = \frac{\omega}{c}$

ie

$$\boxed{\begin{aligned} \dot{x}_i &= c\frac{k_i}{k} \\ \dot{k}_i &= -k\frac{\partial c}{\partial x_i} \end{aligned}}$$

THESE REPRESENT THE MOTION OF A "PARTICLE" TRAVELLING AT THE LOCAL WAVE SPEED c, SUFFERING DEFLECTIONS FROM A STRAIGHT-LINE TRAJECTORY DUE TO VELOCITY GRADIENTS THAT ARE NOT $\parallel$ TO THE PATH

A GENERAL WAY OF UNDERSTANDING THE RAY EQUATIONS IS THROUGH THE CONCEPT OF THE LOCAL DISPERSION RELATION BY WHICH WE SHALL MEAN THE RELATION BETWEEN FREQUENCY ω $(= 2\pi/\text{PERIOD})$ AND WAVE-VECTOR $\underline{k}$ $(|k| = 2\pi/\text{WAVELENGTH})$.

THE WAVE VECTOR FOR A WAVE OF THE FORM

$$U e^{i(\omega t - \psi(\underline{x}))}$$

CAN BE DEFINED AS

$$k_i = \frac{\partial \psi}{\partial x_i}$$

THE LOCAL DISPERSION RELATION IS THEN GIVEN BY A FUNCTION $\omega(k_i, x_i)$, SO THE PHASE $\psi(\underline{x})$ SATISFIES AN EQUATION OF THE FORM

$$\omega = \omega\left(\frac{\partial \psi}{\partial x_i}, x_i\right)$$

THE METHOD OF CHARACTERISTICS (ESSENTIALLY THE METHOD GIVEN ABOVE) THEN LEADS TO HAMILTON'S EQUATION

HAMILTON'S EQUATIONS

GIVEN A LOCAL DISPERSION RELATION

$$\omega = \omega(k_i, x_i)$$

THE RAY EQUATIONS ARE

$$\boxed{\begin{aligned} \dot{x}_i &= \frac{\partial \omega}{\partial k_i} \\ \dot{k}_i &= -\frac{\partial \omega}{\partial x_i} \end{aligned}}$$

cf. HAMILTON'S EQNS. FOR A MECHANICAL SYSTEM:
GIVEN THE HAMILTONIAN

$$H(p_i, q_i)$$

THE EVOLUTION OF THE SYSTEM IS GOVERNED BY

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$H(p, q) \Longleftrightarrow \omega(k, x)$

q_i = "GENERALISED COORDINATES"

p_i = "GENERALISED MOMENTA"

LET US USE THIS IDEA TO RE-DERIVE THE RAY EQUATIONS.
THE LOCAL DISPERSION RELATION IS OF THE SIMPLE FORM

$$\omega = c(\underline{x})|\underline{k}|$$

FOR BODY WAVES IN AN ISOTROPIC MEDIUM ($c = \alpha$ or $c = \beta$)

ie $$\omega = c(\underline{x})(k_i k_i)^{1/2}$$

$\therefore$ HAMILTON'S EQUATIONS GIVE

$$\left.\begin{aligned} \dot{x}_i &= c\frac{k_i}{k} \\ \dot{k}_i &= -k\frac{\partial c}{\partial x_i} \end{aligned}\right\} \text{THE SAME AS DERIVED EARLIER}$$

WITH $k \equiv (k_i k_i)^{1/2} = |\underline{k}|$

LET US WRITE DOWN <u>RAY EQUATIONS</u>
<u>FOR AN ANISOTROPIC MEDIUM.</u>

WE HAVE

$$(c_{ijkl}\, u_{k,l})_{,j} + \omega^2 u_i = 0$$

$$\Rightarrow \quad -ik_j\, c_{ijkl}(-ik_l)\, u_k + \omega^2 u_i = 0$$

$$\text{ie} \quad \left(c_{ijkl}\, k_l k_j - \omega^2 \delta_{ik}\right) u_k = 0$$

THUS THE LOCAL DISPERSION RELATION IS

$$\operatorname{Det}\left(c_{ijkl}\, k_l k_j - \omega^2 \delta_{ik}\right) = 0$$

THE DERIVATIVES $\frac{\partial \omega}{\partial x_i}$, $\frac{\partial \omega}{\partial k_i}$ CAN BE FOUND FROM STANDARD PERTURBATION THEORY (RAYLEIGH'S PRINCIPLE)

WE FIND

$$\dot{k}_m = -\frac{\partial \omega}{\partial x_m} = -\frac{1}{2\omega} \frac{\partial c_{ijkl}}{\partial x_m} v_i v_k k_l k_j$$

$$\dot{x}_m = \frac{\partial \omega}{\partial k_m} = \frac{1}{2\omega}\left(c_{ijkm} k_j + c_{imkl} k_l \right) v_i v_k$$

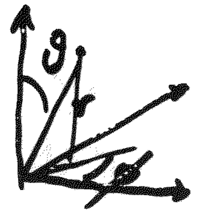
where v_i is a (local) unit eigenvector (CORRESPONDING TO THE WAVE OF INTEREST)

...

ANOTHER ELEGANT PROPERTY OF HAMILTON'S EQUATIONS IS THAT THEY CAN BE WRITTEN DOWN IN <u>ANY COORDINATE SYSTEM</u>

SUPPOSE THAT WE WANT TO DO 3-D RAY TRACING IN SPHERICAL CO-ORDINATES

(r, θ, ϕ)



we have $k_r = \dfrac{\partial \psi}{\partial r}$, $k_\theta = \dfrac{\partial \psi}{\partial \theta}$, $k_\phi = \dfrac{\partial \psi}{\partial \phi}$

and $k = \left(k_r^2 + \dfrac{1}{r^2} k_\theta^2 + \dfrac{1}{r^2 \sin^2\theta} k_\phi^2\right)^{1/2}$

with the usual dispersion relation

$$\omega = c(r, \theta, \phi)\, k$$

WE OBTAIN RAY-TRACING EQUATIONS:

$$\dot{r} = \frac{k_r c}{k}$$

$$\dot{\theta} = \frac{1}{r^2} \frac{k_\theta c}{k}$$

$$\dot{\phi} = \frac{1}{r^2 \sin^2\theta} \frac{k_\phi}{k} c$$

$$\dot{k}_r = -\frac{\partial c}{\partial r}k + \frac{1}{kr}\left(\frac{1}{r^2}k_\theta^2 + \frac{1}{r^2\sin^2\theta}k_\phi^2\right)$$

$$\dot{k}_\theta = -\frac{\partial c}{\partial \theta}k + \frac{\cot\theta}{kr^2\sin^2\theta}k_\phi^2$$

$$\dot{k}_\phi = -\frac{\partial c}{\partial \phi}$$

TO MAKE CONTACT WITH CLASSICAL RAY THEORY IN THE SPHERICAL EARTH LET US NOW SIMPLIFY THESE FOR THE CASE $c = c(r)$

TAKE SOURCE AT $\theta = 0$, $k_\phi = 0$

$$\dot{r} = \frac{k_r c}{k} \qquad \dot{k}_r = -\frac{\partial c}{\partial r}k + \frac{1}{kr^3}k_\theta^2$$

$$\dot{\theta} = \frac{1}{r^2}\frac{k_\theta c}{k} \qquad \dot{k}_\theta = 0$$

$$\dot{\phi} = 0 \qquad \dot{k}_\phi = 0$$

$$\omega = c\left(k_r^2 + \frac{1}{r^2}k_\theta^2\right)^{1/2} = \text{const}$$

WRITE $k_r = \omega p_r$ $\quad k_\theta = \omega p_\theta$

$$p_\theta = \text{const} \qquad p_r^2 + \frac{1}{r^2}p_\theta^2 = \frac{1}{c^2}$$

ie $\quad p_r = \left(\frac{1}{c^2} - \frac{p^2}{r^2}\right)^{1/2}$ $\quad$ ($p \equiv p_\theta$ = "RAY PARAMETER

$$\frac{1}{\dot{r}} = \frac{dt}{dr} = \frac{1}{c}\left(1 - \frac{c^2p^2}{r^2}\right)^{-1/2}$$

$$\frac{d\theta}{dr} = \frac{\dot{\theta}}{\dot{r}} = \frac{pc}{r^2}\left(1 - \frac{c^2p^2}{r^2}\right)^{-1/2}$$

THUS WE OBTAIN THE CLASSICAL RAY INTEGRALS

$$\boxed{\begin{aligned} t &= \int \frac{1}{c}\left(1-\frac{c^2p^2}{r^2}\right)^{-\frac{1}{2}} dr \\ \theta \;(=\text{"}\Delta\text{"}) &= \int \frac{pc}{r^2}\left(1-\frac{c^2p^2}{r^2}\right)^{-\frac{1}{2}} dr \end{aligned}}$$

AMPLITUDES AND WAVEFORMS

BECAUSE RAY THEORY (FOR BODY WAVES) IS FREQUENCY-INDEPENDENT, IT PREDICTS THAT WAVES PROPAGATE WITHOUT ANY CHANGE TO THE WAVEFORM

(JUST AS IN A HOMOGENEOUS MEDIUM)

THE ASYMPTOTIC THEORY CAN BE USED TO DERIVE WAVE AMPLITUDES (BY INVESTIGATING THE TERMS $\propto \omega$)
THE DERIVATION WILL NOT BE GIVEN HERE (SEE LITERATURE) THE RESULT IS

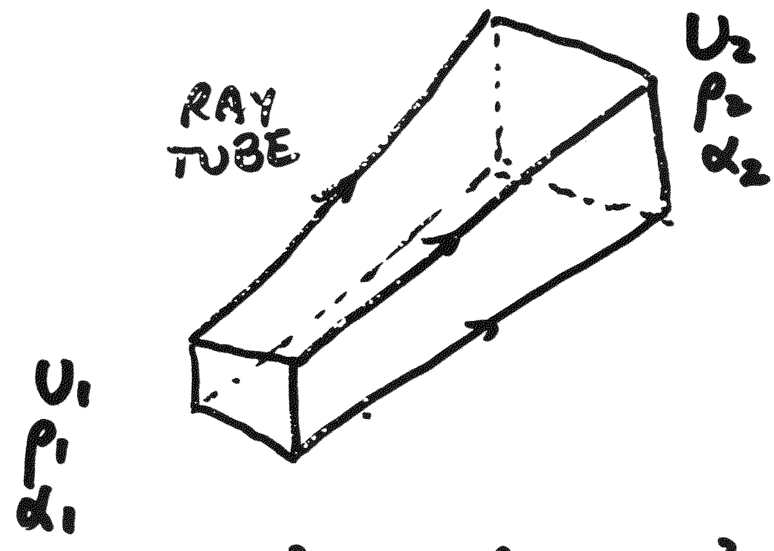
THAT ENERGY FLUX IN A RAY TUBE IS CONSTANT

RECALLING THAT

$$\text{ENERGYFLUX} \propto \rho \alpha U^2 \quad (\text{FOR P-WAVES})$$

THIS MEANS THAT RAY AMPLITUDES VARY INVERSELY AS $\sqrt{\rho\alpha}$ AND ALSO AS $1/\sqrt{A}$

WHERE A is the CROSS-SECTIONAL AREA OF THE RAY TUBE.



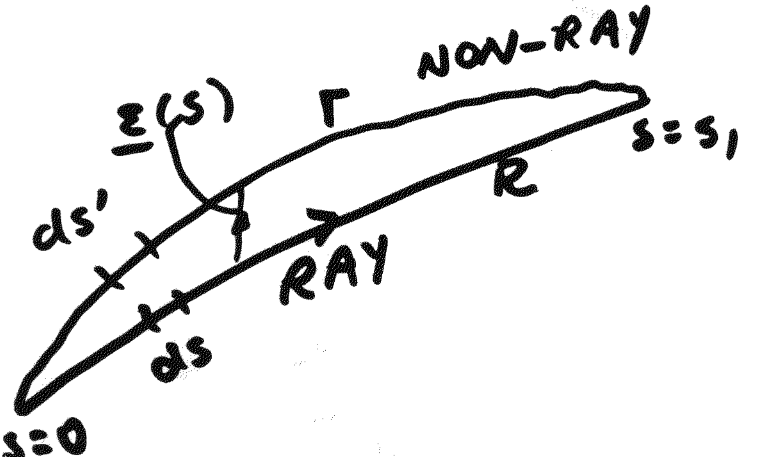
$$U_1^2 \rho_1 \alpha_1 A_1 = U_2^2 \rho_2 \alpha_2 A_2$$

ie $$U_2 = U_1 \sqrt{\frac{\rho_1 \alpha_1 A_1}{\rho_2 \alpha_2 A_2}}$$

(SIMILARLY FOR S-WAVES WITH β INSTEAD OF α)

FERMAT'S PRINCIPLE

TRAVEL TIME IS STATIONARY WITH RESPECT TO PERTURBATIONS OF THE PATH



T' = "TRAVEL TIME CALCULATED ALONG THE NON-RAY Γ"

$$\equiv \int \frac{1}{c(\underline{x}+\underline{\varepsilon})}\, ds'$$

$$= \int \underline{\nabla}\left(\frac{1}{c}\right)\cdot\underline{\varepsilon}\, ds + \int \frac{1}{c}\frac{ds'}{ds}\, ds + O(\varepsilon^2)$$

But $$\frac{ds'}{ds} = \left\{ \frac{d}{ds}(x_i+\varepsilon_i)\frac{d}{ds}(x_i+\varepsilon_i)\right\}^{1/2}$$

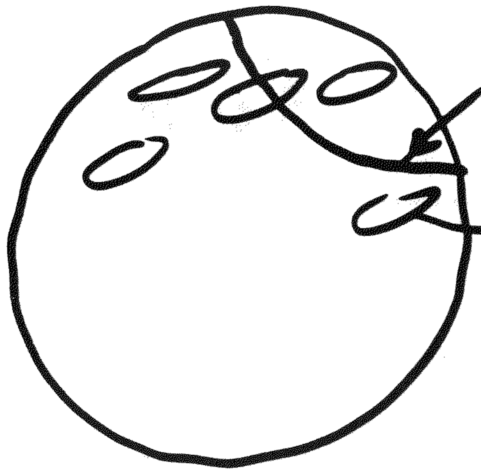
$$\simeq 1 + \frac{d\underline{x}}{ds}\cdot\frac{d\underline{\varepsilon}}{ds}$$

$$\therefore\ \underline{\underline{T'}} = \int \frac{1}{c}\, ds + \int\left(\underline{\nabla}\left(\frac{1}{c}\right)\cdot\underline{\varepsilon} + \frac{1}{c}\frac{d\underline{x}}{ds}\frac{d\underline{\varepsilon}}{ds}\right) ds$$

$$= T + \int\left(\underline{\nabla}\left(\frac{1}{c}\right) + \frac{d}{ds}\frac{1}{c}\frac{d\underline{x}}{ds}\right)\cdot\underline{\varepsilon}\, ds$$

$$+ \left[\frac{1}{c}\frac{d\underline{x}}{ds}\cdot\underline{\varepsilon}\right]_0^{s_1} = \underline{\underline{T}} + O(\varepsilon^2)$$

APPLICATION IN TOMOGRAPHY



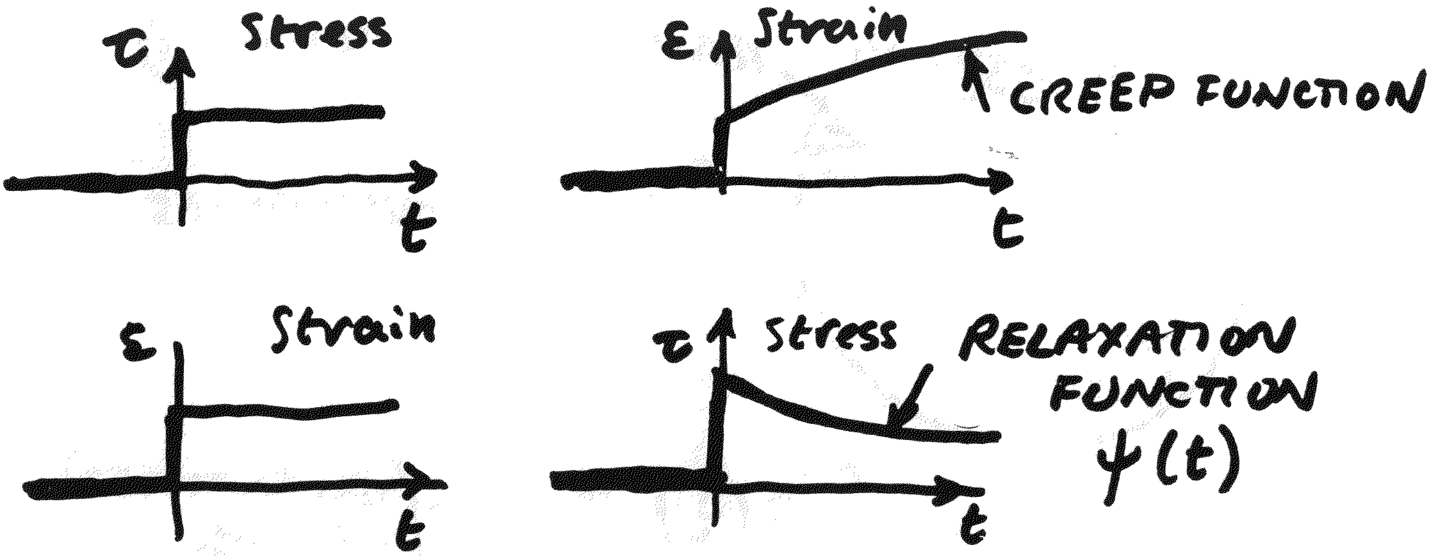
$$\delta T = \int_{\substack{\text{UNPERTURBED}\\ \text{RAY}}} \delta\left(\frac{1}{c}\right) ds + \left(\begin{array}{l}\text{QUANTITIES}\\ \text{OF 2ND}\\ \text{ORDER.}\end{array}\right)$$

NOTE THAT IT IS <u>NOT</u> TRUE

THAT THE PERTURBATION OF THE RAY PATH IS 2ND ORDER.

ATTENUATION AND PHYSICAL DISPERSION OF SEISMIC WAVES

(RECALL DR. YANOVSKAYA'S LECTURES & NOTES)



FOR A SINUSOIDAL SHEAR DISTURBANCE

$$u = U_0 e^{i\omega t}$$

$$\boxed{\tau(t) = \mu(\omega)\varepsilon(t)}$$

WHERE $\mu(\omega)$ IS COMPLEX AND FREQUENCY DEPENDENT

[SIMILARLY FOR COMPRESSION

$$\tau(t) = \kappa(\omega)\varepsilon(t)]$$

WRITING $\bar{\psi}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(t) e^{-i\omega t} dt$

IT IS EASY TO SEE THAT

$$\boxed{\mu(\omega) = i\omega\bar{\psi}(\omega)}$$

IT IS CONVENTIONAL TO DEFINE

$$Q_\mu(\omega) = \frac{Re\,\mu(\omega)}{Im\,\mu(\omega)} \gg 1$$

BUT OFTEN MORE CONVENIENT TO USE

$$q_\mu(\omega) \equiv \frac{1}{Q_\mu(\omega)} \ll 1$$

Writing
$$\frac{1}{v_s} = \sqrt{\frac{\rho}{\mu(\omega)}} = s_1 - i s_2$$

$$\simeq Re\left(\frac{1}{v_s}\right)\left(1 - \tfrac{1}{2} i q_\mu\right)$$

THUS THE EXPRESSION FOR A PLANE WAVE TRAVELLING IN THE x - DIRECTION IS OF THE FORM

$$u \sim U_0\, e^{i\omega(t - x/v_s)}$$

$$= U_0\, e^{-\omega x s_2}\, e^{i\omega(t - x s_1)}$$

with $s_2 = Re\left(\frac{1}{v_s}\right) \cdot \frac{1}{2} q_\mu$

DECAY IN ONE WAVELENGTH

$$\exp\left\{-\omega \frac{2\pi}{\omega s_1} \tfrac{1}{2} q_\mu s_1\right\} = \exp(-\pi q_\mu)$$

AMPLITUDE DECAY FOR S-WAVE

$$= e^{-\pi/Q_\mu} \quad \text{PER CYCLE}$$

Q_μ IS ALSO SOMETIMES DENOTED BY

Q_β (= Q FOR S-WAVES)

CORRESPONINGLY

AMPLITUDE DECAY FOR P-WAVE

$$= e^{-\pi/Q_\alpha} \quad \text{PER CYCLE}$$

where $$Q_\alpha \equiv \frac{-2\,\mathrm{Re}\,(1/v_P)}{\mathrm{Im}\,(1/v_P)}$$

OF COURSE WE HAVE

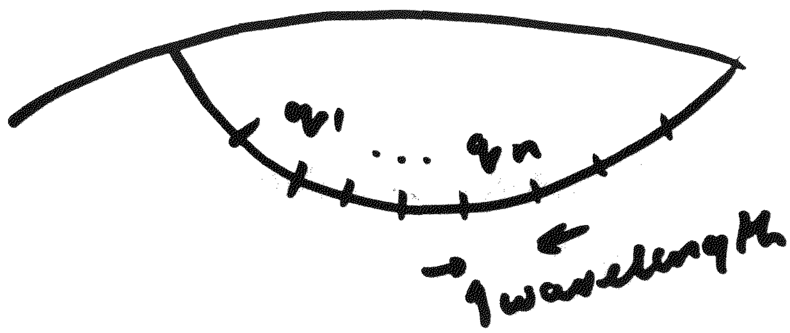
$$v_P = \sqrt{\frac{K + 4/3\,\mu}{\rho}}$$

$$K = (\mathrm{Re}\,K)(1 + i q_K) \quad \text{etc.}$$

SO IT IS EASY TO FIND EXPRESSIONS FOR Q_α IN TERMS OF Q_K, Q_μ. IN PARTICULAR IF $q_K = 0$ (USUALLY A FAIR ASSUMPTION) WE OBTAIN

$$\boxed{q_\alpha = \frac{4}{3}\frac{v_s^2}{v_P^2}\, q_\beta}$$

WITHIN RAY THEORY THIS LEADS TO AN ADDITIONAL AMPLITUDE DECAY



$$e^{-\pi q_1} \times e^{-\pi q_2} \cdots \times e^{-\pi q_N}$$

$$= \exp\left\{ -\tfrac{1}{2}\omega \int q_\mu \, dt \right\}$$

$$= \exp\left\{ -\tfrac{1}{2}\omega t^* \right\}$$

where t^* (t-STAR) IS DEFINED BY

$$\boxed{t^* = \int_{\text{RAY}} q_\mu \, dt}$$

[NOTE STRONG DAMPING OF HIGH FREQUENCY WAVES]

PHYSICAL DISPERSION

WE SAW THAT

$$\mu(\omega) = i\omega \bar{\Psi}(\omega)$$

WHERE $\bar{\Psi}(\omega)$ = F.T. OF RELAXATION FUNCTION $\psi(t)$

WAVE VELOCITY $\left(\sqrt{\frac{\mu}{\rho}}\right)$ IS RELATED TO $Re(\mu)$ AND DAMPING TO $Im(\mu)$.

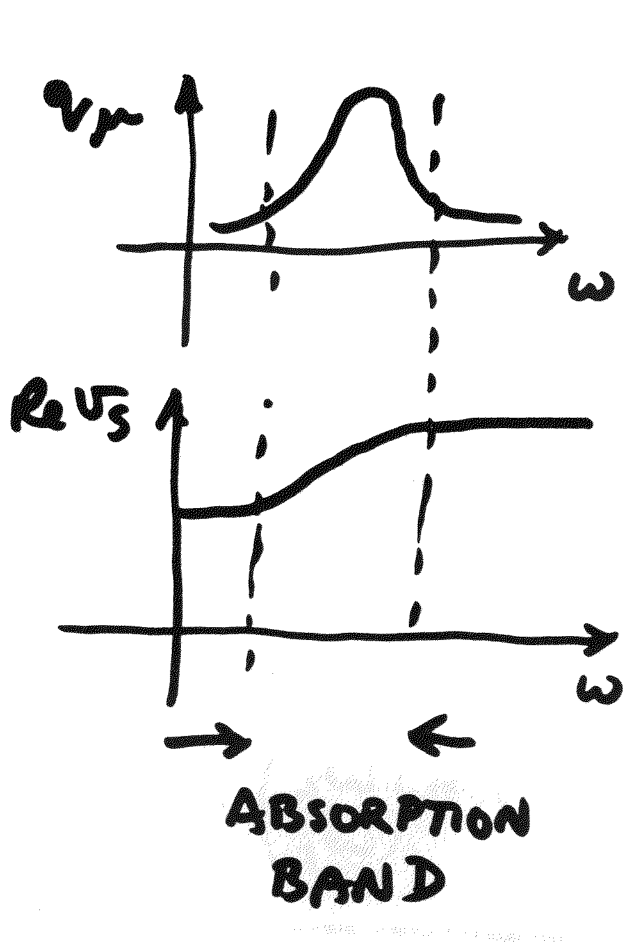
BUT SINCE $\mu(\omega)$ IS THE TRANSFORM OF A SINGLE REAL (CAUSAL) FUNCTION <u>$Re(\mu)$ AND $Im(\mu)$ ARE RELATED.</u>

EG. FOR THE <u>STANDARD LINEAR SOLID</u>
(SEE "WAVE PROPAGATION" NOTES FROM DR. YANOVSKAYA)

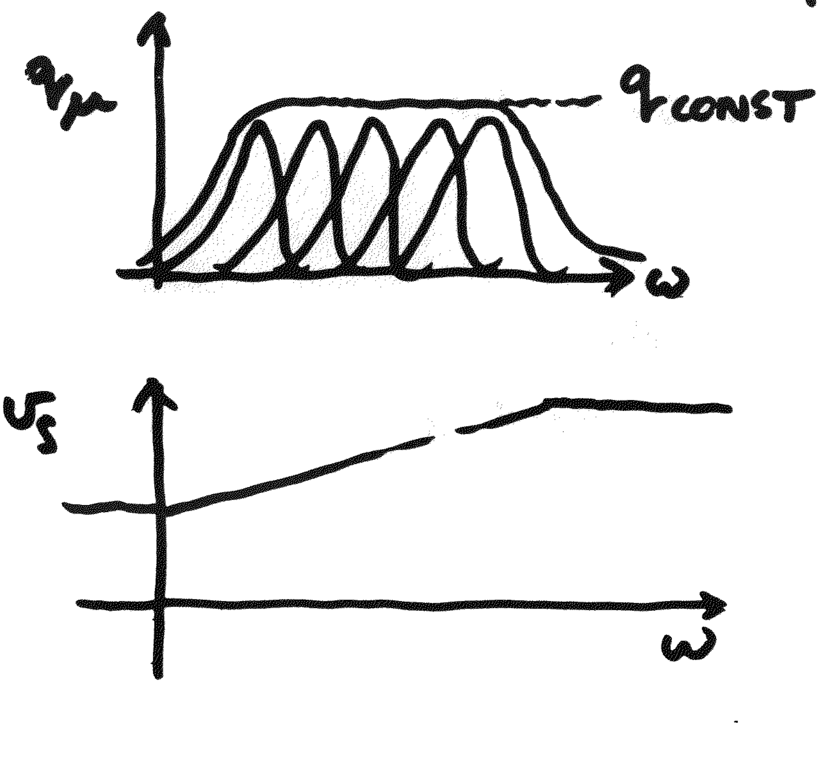
$$\tau + T_\tau \dot{\tau} = \mu_0(\varepsilon + T_\varepsilon \dot{\varepsilon})$$

$$\Rightarrow \quad \mu(\omega) = \frac{\mu_0(1 + i\omega T_\varepsilon)}{1 + i\omega T_\tau}$$

This can be used to find <u>both</u> $q_\mu(\omega)$ AND $v_S(\omega) = Re\sqrt{\frac{\mu(\omega)}{\rho}}$



THUS v_s INCREASES THROUGH THE ABSORPTION BAND. FOR MANY ABSORPTION BANDS v_s INCREASES THROUGHOUT THE RANGE OF CONSTANT q_μ.



QUANTITIVELY IT CAN BE SHOWN THAT APPROXIMATELY, AND WITHIN THE BAND OF CONSTANT Q,

$$\boxed{\frac{d \ln v_s}{d \ln \omega} = \frac{1}{\pi} q_\mu^{\text{CONST}}}$$

OR (INTEGRATING) FOR ω_1, ω_2 WITHIN THE BAND

$$\boxed{\ln \frac{v_s(\omega_2)}{v_s(\omega_1)} = \frac{1}{\pi} q_\mu^{\text{CONST}} \ln\left(\frac{\omega_2}{\omega_1}\right)}$$

THESE LEAD TO A RELATIONSHIP BETWEEN THE <u>DELAY</u> OF A WAVE OF GIVEN FREQUENCY AND IT'S <u>DECAY</u>. –

THE PHENOMENON IS KNOWN AS

<u>PHYSICAL DISPERSION</u>

[SEE LIU, ANDERSON, KANAMORI, GJ. 1976 AND REFERENCES CITED THEREIN]

WE CAN ALSO WRITE FOR THE COMPLEX VELOCITY

$$v(\omega) = v_0\left(1 + \frac{q}{\pi}\ln\frac{\omega}{\omega_0} + \frac{1}{2}iq\right)$$

WHERE v_0 IS THE (REAL) VELOCITY AT REFERENCE FREQUENCY ω_0.

CONSEQUENTLY THE EFFECT ON THE SIGNAL IS REPRESENTED BY

$$\boxed{\exp\left\{-\frac{1}{2}\omega t^*\left(1 - \frac{2i}{\pi}\ln\frac{\omega}{\omega_0}\right)\right\}}$$

THIS REPRESENTS (APPROXIMATELY, AND ASSUMING THAT THE ENTIRE SIGNAL IS WITHIN THE CONSTANT Q_μ BAND) THE TOTAL AFFECT OF ATTENUATION ON THE SIGNAL.