The Abdus Salam
International Centre for Theoretical Physics

United Nations
Educational, Scientific
and Cultural Organization

International Atomic
Energy Agency

H4.SMR/1775-11

"8th Workshop on Three-Dimensional Modelling of
Seismic Waves Generation, Propagation and their Inversion"

25 September - 7 October 2006

Surface wave inversion for earthquake source study: theory and application for Sumatra megaearthquakes study

B. Bukchin

International Institute of Earthquake Prediction Theory
& Mathematical Geophysics
Moscow, Russia

Surface wave inversion for earthquake source study:

theory and application for Sumatra megaeartquakes study.

B. Bukchin

International Institute of Earthquake Prediction Theory and Mathematical Geophysics, Moscow, Russia

I. Formal description of seismic source

The description of seismic source we will consider is based on the formalism developed by Backus and Mulcahy, 1976.

Statement of the problem.
Motion equation
$$\rho \ddot{u}_i = \sigma_{ij,j} + f_i \qquad (1.1)$$
Hook's law for isotropic medium
$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij} \qquad (1.2)$$
Initial conditions
$$\dot{\mathbf{u}} \equiv \mathbf{u} \equiv 0, t < 0 \qquad (1.3)$$
Boundary conditions
$$\sigma_{ij} n_j |_{S_0} = 0 \qquad (1.4)$$
Here **u** – displacement vector; σ_{ij} – elements of symmetric 3x3 stress tensor; i,j=1,2,3 and the summation convention for repeated subscripts is used; $\sigma_{ij,j} = \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j}$; ε_{ij} – elements of symmetric 3x3 strain tensor and $\varepsilon_{ij} = 0.5(u_{i,j} + u_{j,i})$; ρ - density; f_i – components of external force; n_j – components of the normal to the free surface S_0.

Solution of the problem (1.1)-(1.4) can be given by formula
$$u_i(\mathbf{x},t) = \int_0^T d\tau \int_\Omega G_{ij}(\mathbf{x},\mathbf{y},t-\tau) f_j(\mathbf{y},\tau) dV_y \qquad (1.5)$$
or
$$u_i(\mathbf{x},t) = \int_0^T d\tau \int_\Omega H_{ij}(\mathbf{x},\mathbf{y},t-\tau) \dot{f}_j(\mathbf{y},\tau) dV_y \qquad (1.6)$$
Here G_{ij} is the Green's function,
$$H_{ij}(\mathbf{x},\mathbf{y},t) = \int_0^t G_{ij}(\mathbf{x},\mathbf{y},\tau) d\tau, \qquad (1.7)$$
$\mathbf{x} \in \Omega$ and $0 < t < T$ are the space region and time interval where $\dot{f}$ is not identically zero.

Sources of seismic disturbances

We will consider internal sources only (earthquakes). In this case any external forces are absent. We must then set $\mathbf{f} \equiv 0$ in equation (1.1), so that the only solution that satisfies the homogeneous initial (1.3) and boundary (1.4) conditions, as well as Hook's law (1.2) will be $\mathbf{u} \equiv 0$. Non-zero displacements cannot arise in the medium, unless at least one of the above conditions is not true.

Following Backus and Mulcahy, 1976, we assume seismic motion to be caused by a departure from ideal elasticity (from Hook's law) within some volume of the medium Ω at some time interval $0 < t < T$.

Let $\mathbf{u}(\mathbf{x},t)$ be the actual displacements, $\boldsymbol{\sigma}(\mathbf{x},t)$ - correspondent stresses, if Hook's law is valid, $\mathbf{s}(\mathbf{x},t)$ - actual stresses.

Let the difference
$$\Gamma(\mathbf{x},t) = \boldsymbol{\sigma}(\mathbf{x},t) - \mathbf{s}(\mathbf{x},t), \qquad (1.8)$$

called the *stress glut tensor* or *moment tensor density*, is not identically zero for $0 < t < T$ and $\mathbf{x} \in \Omega$.
T we define as source duration, and Ω - source region. Within this region and time interval (and only there) the tensor $\dot{\boldsymbol{\Gamma}}(\mathbf{x}, t)$ is not identically zero as well.

Replacing $\boldsymbol{\sigma}(\mathbf{x},t)$ by $\mathbf{s}(\mathbf{x},t)$ in equation (1.1), using definition (1.8) and the absence of external forces ($\mathbf{f} \equiv 0$) we can rewrite the motion equation (1.1) in form

$$\rho \ddot{u}_i = s_{ij,j}$$

or

$$\rho \ddot{u}_i = \sigma_{ij,j} + g_i \tag{1.9}$$

where

$$g_i = -\Gamma_{ij,j} \ . \tag{1.10}$$

Equation (1.10) defines the equivalent force $\mathbf{g}$. Using formula (1.6) with f_i replaced by g_i , definition (1.10) and Gauss theorem we have for displacements

$$u_i(\mathbf{x},t) = \int_0^T d\tau \int_\Omega H_{ij,k}(\mathbf{x},\mathbf{y},t-\tau)\dot{\Gamma}_{jk}(\mathbf{y},\tau)dV_y \, , \tag{1.11}$$

where H_{ij} is differentiated with respect to y_k .
If the inelastic motions are concentrated at a surface Σ, then

$$u_i(\mathbf{x},t) = \int_0^T d\tau \int_\Sigma H_{ij,k}(\mathbf{x},\mathbf{y},t-\tau)\dot{\Gamma}_{jk}(\mathbf{y},\tau)d\Sigma_y \, . \tag{1.12}$$

Relation of stress glut (moment tensor density) with classic definition of moment tensor M :

$$\mathbf{M} = \int_0^T dt \int_\Omega \dot{\boldsymbol{\Gamma}}(\mathbf{y},t)dV_y \quad . \tag{1.13}$$

Normalizing moment tensor we define seismic moment M_0 :

$\mathbf{M}=M_0\mathbf{m}$, where tensor $\mathbf{m}$ is normalized by condition $\mathrm{tr}(\mathbf{m}^{\mathrm{T}}\mathbf{m}) = \sum_{i,j=1}^{3} m_{ij}^2 = 2$, $\mathbf{m}^{\mathrm{T}}$ is transposed tensor $\mathbf{m}$.

Stress glut moment for special types of seismic sources

1. Discontinuity of displacement $\Delta\mathbf{u}$ at a surface Σ in isotropic medium (stress is continuous):

$$\Gamma_{ij}(\mathbf{x},t) = \lambda \Delta u_k(\mathbf{x},t) n_k(\mathbf{x}) \delta_{ij} + \mu [n_i(\mathbf{x}) \Delta u_j(\mathbf{x},t) + n_j(\mathbf{x}) \Delta u_i(\mathbf{x},t)]. \tag{1.14}$$

Here $\mathbf{n}(\mathbf{x})$ is the normal to the surface Σ, and seismic disturbances are given by formula (1.12).

2. In the case of tangential (shear) dislocation we have
$\Delta u_k n_k \equiv 0$ and formula (1.14) takes form

$$\Gamma_{ij}(\mathbf{x},t) = \mu [n_i(\mathbf{x}) \Delta u_j(\mathbf{x},t) + n_j(\mathbf{x}) \Delta u_i(\mathbf{x},t)]. \tag{1.15}$$

3. Instant point tangential dislocation occurred in the point $\mathbf{x}=\mathbf{0}$ at time $t=0$:

$$\dot{\Gamma}_{ij}(\mathbf{x},t) = M_0 m_{ij} \delta(t) \delta(\mathbf{x}), \tag{1.16}$$

where $m_{ij} = n_i a_j + n_j a_i$, $\mathbf{a} = \Delta\mathbf{u}/|\Delta\mathbf{u}|$ and $M_0 = \mu \, |\Delta \mathbf{u}|$.

Phenomena of matrix $\mathbf{m}$

Tr**m** = 0. The eigenvalues of matrix **m** are: 1, -1 and 0. The eigenvector correspondent to 1 defines the direction of maximum extension, and the eigenvector correspondent to -1 defines the direction of maximum compression. Such a source is called double couple.

As it follows from formula (1.12) an instant point double couple excites a displacement field of the form

$$u_i(\mathbf{x},t) = M_0 H_{ik,l}(\mathbf{x},\mathbf{0},t) m_{kl} \,. \tag{1.17}$$

We have for Fourier transforms **H**(**x**,**y**,ω) and **G**(**x**,**y**,ω) from equation (1.7):

$$\mathbf{H}(\mathbf{x},\mathbf{y},\omega) = \frac{1}{\mathrm{i}\omega}\mathbf{G}(\mathbf{x},\mathbf{y},\omega)\,, \tag{1.18}$$

where i is the imaginary unit, and ω is angular frequency.
As result the spectrum of displacements is given by formula

$$u_i(\mathbf{x},\omega) = \frac{1}{\mathrm{i}\omega} M_0 m_{kl} G_{ik,l}(\mathbf{x},\mathbf{0},\omega)\,. \tag{1.19}$$

Relation between the displacement field and stress glut moments

We assume that following product can represent the time derivative of stress glut tensor:

$$\dot{\Gamma}(\mathbf{x},t) = f(\mathbf{x},t)\mathbf{m}\,, \tag{1.20}$$

where $f(\mathbf{x},t)$ is non-negative function and **m** is a uniform normalized moment tensor.
The moment $f^{(l,n)}_{k_1\ldots k_l}(\mathbf{q},\tau)$ of spatial degree l and temporal degree n with respect to point **q** and instant of time τ is a tensor of order l and is given by formula

$$f^{(l,n)}_{k_1\ldots k_l}(\mathbf{q},\tau) = \int_V dV \int_0^\infty f(\mathbf{x},t)(x_{k_1}-q_{k_1})\cdots(x_{k_l}-q_{k_l})(t-\tau)^n\,dt\,, \tag{1.21}$$

$k_1,\ldots,k_l$=1,2,3.
Replacing $H_{ij}(\mathbf{x},\mathbf{y},t-\tau)$ in equation (1.11) by its Taylor series in powers of **y** and in powers of τ, we get:

$$u_i(\mathbf{x},t) = \sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!\,n!} m_{jk} f^{(l,n)}_{k_1\ldots k_l}(\mathbf{0},0)\frac{\partial^n}{\partial t^n}\frac{\partial}{\partial y_{k_1}}\cdots\frac{\partial}{\partial y_{k_l}}\frac{\partial}{\partial y_k}H_{ij}(\mathbf{x},\mathbf{y},t)\Big|_{\mathbf{y}=\mathbf{0}}\,. \tag{1.22}$$

Using formulae (1.18) and (1.22) we have following equation for the spectrum of displacements:

$$u_i(\mathbf{x},\omega) = \sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{l!\,n!} m_{jk} f^{(l,n)}_{k_1\ldots k_l}(\mathbf{0},0)(\mathrm{i}\,\omega)^{n-1}\frac{\partial}{\partial y_{k_1}}\cdots\frac{\partial}{\partial y_{k_l}}\frac{\partial}{\partial y_k}G_{ij}(\mathbf{x},\mathbf{y},\omega)\Big|_{\mathbf{y}=\mathbf{0}}\,. \tag{1.23}$$

Here we assume that the point **y**=**0** and the instant t=0 belong to the source region and the time of the source activity respectively.

When the spectra of displacements $u_i(\mathbf{x},\omega)$ and Green's function $G_{ij}(\mathbf{x},\mathbf{y},\omega)$ have been low pass filtered, the terms in equation (1.23) start to decrease with l and n increasing at least as rapidly as $(\omega \mathrm{T})^{l+n}$ (T is the source duration, and ωT<1), and one might then restrict to considering finite sums only.

We will take into account in the following sections only the first terms in formula (1.23) for $l+n \le 2$.

II. Source inversion in moment tensor approximation

The first term in (1.23) corresponding to l=0, n=0, describes the spectra of displacements $u_i(\mathbf{x},\omega)$ excited by an instant point source (compare with formula (1.19) taking into account that seismic moment is equal to zero moment of function $f(\mathbf{x},t)$: $M_0=f^{(0,0)}$). For a source with nonzero size and duration this term approximates $u_i(\mathbf{x},\omega)$ with high accuracy for periods much longer then source duration. Performing the inversion of long period seismic waves we describe the earthquake by an instant point source. As it was mentioned in previous section, an instant point source can be given by moment tensor - a symmetric 3x3 matrix $\mathbf{M}$. Seismic moment M_0 is defined by equation $M_0 = \sqrt{\frac{1}{2}\mathrm{tr}(\mathbf{M}^{\mathrm{T}}\mathbf{M})}$, where $\mathbf{M}^{\mathrm{T}}$ is transposed moment tensor $\mathbf{M}$, and $\mathrm{tr}(\mathbf{M}^{\mathrm{T}}\mathbf{M}) = \sum_{i,j=1}^{3} M_{ij}^{\ 2}$. Moment tensor of any event can be presented in the form $\mathbf{M} = M_0\mathbf{m}$, where matrix $\mathbf{m}$ is normalized by condition $\mathrm{tr}(\mathbf{m}^{\mathrm{T}}\mathbf{m}) = 2$.

We'll consider a double couple instant point source (a pure tangential dislocation) at a depth h. Such a source can be given by 5 parameters: double couple depth, its focal mechanism which is characterizing by three angles: strike, dip and slip or by two unit vectors (direction of principal tension **T** and direction of principal compression **P**) and seismic moment M_0. Four of these parameters we determine by a systematic exploration of the four dimensional parametric space, and the 5-th parameter M_0 - solving the problem of minimization of the misfit between observed and calculated surface wave amplitude spectra for every current combination of all other parameters.

Under assumptions mentioned above the relation between the spectrum of displacements $u_i(\mathbf{x},\omega)$ and moment tensor $\mathbf{M}$ can be expressed by formula (1.19) rewritten below in slightly different form:

$$u_i(\mathbf{x},\omega) = \frac{1}{\mathrm{i}\,\omega}[M_{jl}\frac{\partial}{\partial y_l}G_{ij}(\mathbf{x},\mathbf{y},\omega)] \qquad (2.1)$$

i,j = 1,2,3 and the summation convention for repeated subscripts is used. $G_{ij}(\mathbf{x},\mathbf{y},\omega)$ in equation (2.1) is the spectrum of Green function for the chosen model of medium and wave type (see Levshin, 1985; Bukchin, 1990), $\mathbf{y}$ - source location. We will discuss the inversion of surface wave spectra, so $G_{ij}(\mathbf{x},\mathbf{y},\omega)$ is the spectrum of surface wave Green function. We assume that the paths from the earthquake source to seismic stations are relatively simple and are well approximated by weak laterally inhomogeneous model (Woodhouse, 1974; Babich *et al.*, 1976). The surface wave Green function in this approximation is determined by the near source and near receiver velocity structure, by the mean phase velocity of wave, and by geometrical spreading. We assume that waves propagate from the source to station along great circles. Under these assumptions the amplitude spectrum $|u_i(\mathbf{x},\omega)|$ defined by formula (2.1) does not depend on the average phase velocity of the wave. In such a model the errors in source location do not affect the amplitude spectrum (Bukchin, 1990). The average phase velocities of surface waves are usually not well known. For this reason as a rule we use only amplitude spectra of surface waves for determining source parameters under consideration. We use observed surface wave phase spectra only for very long periods. Correcting the spectra for attenuation we use laterally homogeneous model for quality factor. At the end of this lecture we will consider the effects related to surface wave focusing caused by rays deviation from great circle, and to laterally inhomogeneity of attenuation model.

Surface wave amplitude spectra inversion

If all characteristics of the medium are known, the representation (2.1) gives us a system of equations for parameters defined above. Let us consider now a grid in the space of these 4 parameters. Let the models of the media be given. Using formula (2.1) we can calculate the amplitude spectra of surface waves at the points of observation for every possible combination of values of the varying parameters. Comparison of calculated and observed amplitude spectra give us a residual $\varepsilon^{(i)}$ for every point of observation, every wave and every frequency ω. Let $u^{(i)}(\mathbf{x},\omega)$ be any observed value of the spectrum, i = 1,…,N; $\varepsilon^{(i)}_{\mathrm{amp}}$ - corresponding residual of $|u^{(i)}(\mathbf{x},\omega)|$. We define the normalized amplitude residual by formula

$$\varepsilon_{\mathrm{amp}}(h,\mathbf{T},\mathbf{P}) = \left[\left(\sum_{i=1}^{N} \varepsilon_{\mathrm{amp}}^{(i)\,2}\right) \Big/ \left(\sum_{i=1}^{N} |u^{(i)}(\mathbf{x},\omega|^2\right)\right]^{1/2} . \qquad (2.2)$$

The optimal values of the parameters that minimize $\varepsilon_{\mathrm{amp}}$ we consider as estimates of these parameters. We search them by a systematic exploration of the four-dimensional parameter space. To characterize the degree of resolution of every of these source characteristics we calculate partial residual functions. Fixing the value of one of varying parameters we put in correspondence to it a minimal value of the residual $\varepsilon_{\mathrm{amp}}$ on the set of all possible values of the other parameters. In this way we define one residual function on scalar argument and two residual functions on vector argument corresponding to the scalar and two vector varying parameters: $\varepsilon_h(h)$, $\varepsilon_{\mathbf{T}}(\mathbf{T})$ and $\varepsilon_{\mathbf{P}}(\mathbf{P})$. The value of the parameter for which the corresponding function of the residual attains its minimum we define as estimate of this parameter. At the same time these functions characterize the degree of resolution of the corresponding parameters. From geometrical point of view these functions describe the lower boundaries of projections of the 4-D surface of functional ε on the coordinate planes. A sketch illustrating the definition of partial residual functions is given in figure 1.
Here one of 4 parameters is picked out as ‘parameter 1’, and one of coordinate axis corresponds to this parameter. Another coordinate axis we consider formally as 3-D space of the rest 3 parameters. Plane Σ is orthogonal to the axis ‘parameter 1’ and cross it in a point p_0 . Curve L is the intersection of the plane Σ and the surface of functional ε. As one can see from the figure the point $\varepsilon_1(p_0)$ belong to the boundary of projection of the surface of functional ε, and at the same time it corresponds to a minimal value of the residual ε on the set of all possible values of the other 3 parameters while ‘parameter 1’ is equal to the value p_0. So, as it is accepted in engineering we characterize our surface by its 4 projections on coordinate planes.

It is well known that the focal mechanism cannot be uniquely determined from surface wave amplitude spectra. There are four different focal mechanisms radiating the same surface wave amplitude spectra. These four equivalent solutions represent two pairs of mechanisms symmetric with respect to the vertical axis, and within the pair differ from each other by the opposite direction of slip.

To get a unique solution for the focal mechanism we have to use in the inversion additional observations. For these purpose we use very long period phase spectra of surface waves or polarities of P wave first arrivals.

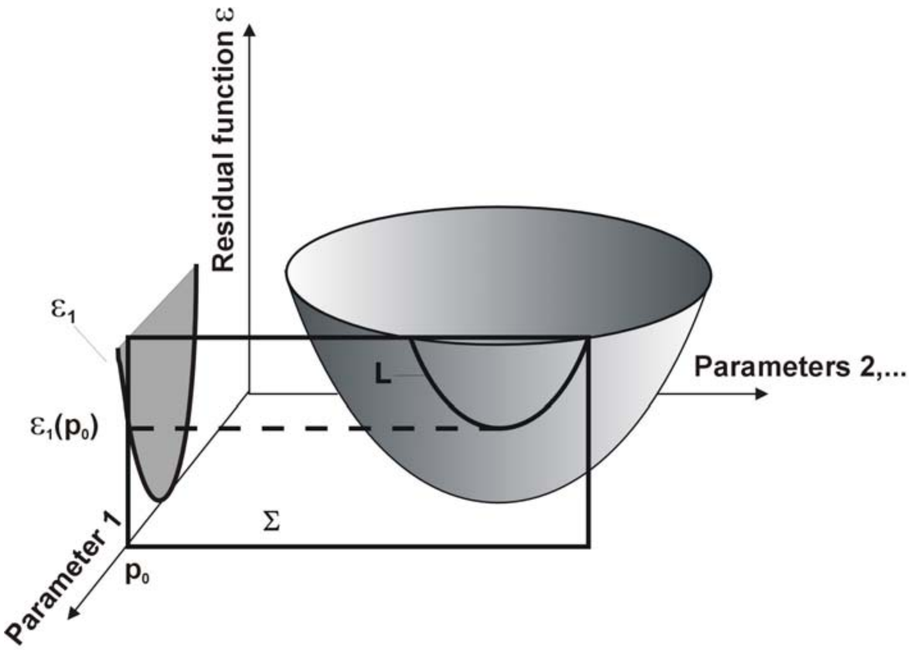

Figure 1. A sketch illustrating the definition of partial residual functions.

Joint inversion of surface wave amplitude and phase spectra

Using formula (2.1) we can calculate for chosen frequency range the phase spectra of surface waves at the points of observation for every possible combination of values of the varying parameters. Comparison of calculated and observed phase spectra give us a residual $\varepsilon_{ph}^{(i)}$ for every point of observation, every wave and every frequency ω. We define the normalized phase residual by formula

$$\varepsilon_{ph}(h, \varphi, \mathbf{T}, \mathbf{P}) = \frac{1}{\pi}\left[\left(\sum_{i=1}^{N} \varepsilon_{ph}^{(i)\,2}\right) / N\right]^{1/2} . \qquad (2.3)$$

We determine the joint residual ε by formula

$$\varepsilon = 1 - (1 - \varepsilon_{ph})(1 - \varepsilon_{amp}) . \qquad (2.4)$$

To characterize the resolution of source characteristics we calculate partial residual functions in the same way as was described above.

Joint inversion of surface wave amplitude spectra and P wave polarities

Calculating radiation pattern of P waves for every current combination of parameters we compare it with observed polarities. The misfit obtained from this comparison we use to calculate a joint residual of surface wave amplitude spectra and polarities of P wave first arrivals. Let ε_{amp} be the residual of surface wave amplitude spectra, ε_p - the residual of P wave first arrival polarities (the number of wrong polarities divided by the full number of observed polarities), then we determine the joint residual ε by formula

$$\varepsilon = 1 - (1 - \varepsilon_p)(1 - \varepsilon_{amp}) . \qquad (2.5)$$

For this type of inversion we calculate partial residual functions to characterize the resolution of parameters under determination in the same way as it was described for two first types.

Before inversion we apply to observed polarities a smoothing procedure (see Lasserre *et al.*, 2001), which we will describe here briefly.

Let us consider a group of observed polarities (+1 for compression and -1 for dilatation) radiated in directions deviating from any medium one by a small angle. This group is presented in the inversion procedure by one polarity prescribing to this medium direction. If the number of one of two types of polarities from this group is significantly larger then the number of opposite polarities, then we prescribe this polarity to this medium direction. If no one of two polarity types can be considered as preferable, then all these polarities will not be used in the inversion. To make a decision for any group of *n* observed polarities we calculate the sum $m = n_+ - n_-$, where n_+ is the number of compressions and $n_- = n - n_+$ is the number of dilatations. We consider one of polarity types as preferable if $|m|$ is larger then its standard deviation in the case when +1 and -1 appear randomly with this same probability 0.5. In this case n_+ is a random value distributed following the binomial low. For its average we have $M(n_+) = 0.5n$, and for dispersion $D(n_+) = 0.25n$. Random value m is a linear function of n_+ such that $m = 2n_+ - n$. So following equations are valid for the average, for the dispersion, and for the standard deviation σ of value m

$M(m) = 2M(n_+) - n = n - n = 0$, $D(m) = 4D(n_+) = n$, and $\sigma(m) = \sqrt{n}$.

As a result, if the inequality $|m| \geq \sqrt{n}$ is valid then we prescribe +1 to the medium direction if $m > 0$, and -1 if $m < 0$.

III. Second moments approximation. Characteristics of source shape and evolution in time.

We present here a technique based on the description of seismic source distribution in space and in time by integral moments (see Bukchin *et al.*, 1994; Bukchin, 1995; Gomez, 1997 a, b). We assume that the time derivative of stress glut tensor $\dot{\mathbf{\Gamma}}$ can be represented in form (1.20). Following Backus and Mulcahy, 1976 we will define the source region by the condition that function $f(\mathbf{x},t)$ is not identically zero and the source duration is the time during which nonelastic motion occurs at various points within the source region, i.e., $f(\mathbf{x},t)$ is different from zero.

Spatial and temporal integral characteristics of the source can be expressed by corresponding moments of the function $f(\mathbf{x},t)$ (Backus, 1977a; Bukchin *et al.*, 1994). These moments can be estimated from the seismic records using the relation between them and the displacements in seismic waves, which we will consider later. In general case stress glut rate moments of spatial degree 2 and higher are not uniquely determined by the displacement field (Pavlov, 1994; Das & Kostrov, 1997). But in the case when equation (1.20) is valid such uniqueness takes place (Backus, 1977b; Bukchin, 1995).

Following equations define the spatio-temporal moments of function $f(\mathbf{x},t)$ of total degree (both in space and time) 0, 1, and 2 with respect to point $\mathbf{q}$ and instant of time τ.

$$f^{(0,0)} = \int_V dV \int_0^\infty f(\mathbf{x},t)dt\,, \quad f_i^{(1,0)}(\mathbf{q}) = \int_V dV \int_0^\infty f(\mathbf{x},t)(x_i - q_i)dt\,,$$

$$f^{(0,1)}(\tau) = \int_V dV \int_0^\infty f(\mathbf{x},t)(t-\tau)dt\,, \quad f^{(0,2)}(\tau) = \int_V dV \int_0^\infty f(\mathbf{x},t)(t-\tau)^2 dt\,,$$

$$f_i^{(1,1)}(\mathbf{q},\tau) = \int_V dV \int_0^\infty f(\mathbf{x},t)(x_i - q_i)(t-\tau)dt\,, \qquad (3.1)$$

$$f_{ij}^{(2,0)}(\mathbf{q}) = \int_V dV \int_0^\infty f(\mathbf{x},t)(x_i - q_i)(x_j - q_j)dt$$

Using these moments we will define integral characteristics of the source. Source location is estimated by the spatial centroid $\mathbf{q}_c$ of the field $f(\mathbf{x},t)$ defined as

$$\mathbf{q}_c = \mathbf{f}^{(1,0)}(\mathbf{0}) / M_0 \ , \qquad (3.2)$$

where $M_0 = f^{(0,0)}$ is the scalar seismic moment.

Similarly, the temporal centroid τ_c is estimated by the formula

$$\tau_c = f^{(0,1)}(0) / M_0 \ . \qquad (3.3)$$

The source duration is Δt estimated by $2\Delta\tau$, where

$$(\Delta\tau)^2 = f^{(0,2)}(\tau_c) / M_0 \ . \qquad (3.4)$$

The spatial extent of the source is described by matrix **W**,

$$\mathbf{W} = \mathbf{f}^{(2,0)}(\mathbf{q}_c) / M_0 \ . \qquad (3.5)$$

The mean source size in the direction of unit vector **r** is estimated by value $2l_r$, defined by formula

$$l_r^2 = \mathbf{r}^{\mathrm{T}}\mathbf{W}\mathbf{r} \, , \qquad (3.6)$$

where $\mathbf{r}^{\mathrm{T}}$ is the transposed vector. From (3.5) and (3.6) we can estimate the principal axes of the source. There directions are given by the eigenvectors of the matrix **W**, and the lengths are defined by correspondent eigenvalues: the length of the minor semi-axis is equal to the least eigenvalue, and the length of the major semi-axis is equal to the greatest eigenvalue.

In the same way, from the coupled space time moment of order (1,1) the mean velocity **v** of the instant spatial centroid (Bukchin, 1989) is estimated as

$$\mathbf{v} = \mathbf{w} / (\Delta\tau)^2 \ , \qquad (3.7)$$

where $\mathbf{w} = \mathbf{f}^{(1,1)}(\mathbf{q}_c, \tau_c) / M_0$.

The relation between integral estimates and real characteristics of source duration and spatial extent depends on the distribution of moment rate density in time and over the fault. Figure 2 illustrates this relation in the case of Gaussian distributions. In this case 99% confidence duration is 2.5 times larger then the integral estimate, and 99% confidence axis length is 3 times larger then correspondent integral estimate.

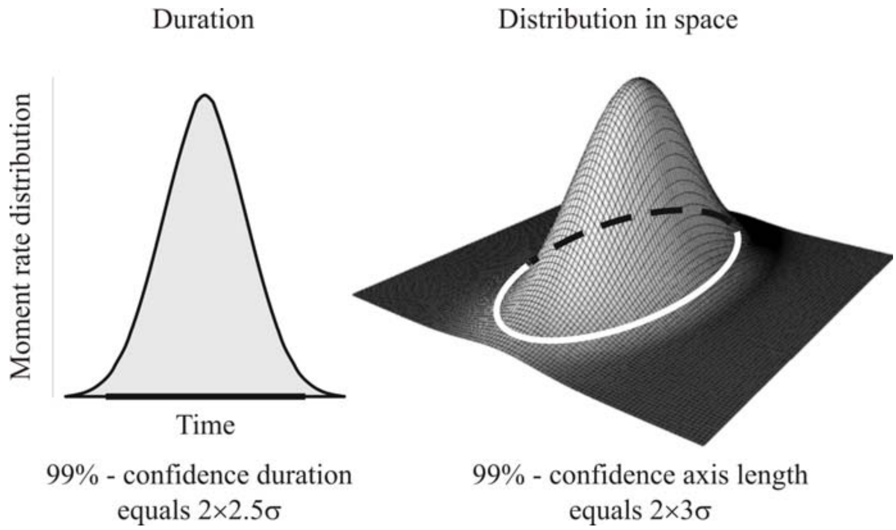

Fig. 2. Relation between integral estimates and real characteristics of source duration and spatial extent.

Now we will consider the low frequency part of the spectra of the i^{th} component of displacements in Love or Rayleigh wave $u_i(\mathbf{x},\omega)$. It is assumed that the frequency ω is small, so that the duration of the source is small in comparison with the period of the wave, and the source size is small as compared with the wavelength. It is assumed that the origin of coordinate system is located in the point of spatial centroid $\mathbf{q}_c$ (i.e. $\mathbf{q}_c = \mathbf{0}$) and that time is measured from the instant of temporal centroid, so that $\tau_c = 0$. With this choice the first degree moments with respect to the spatial origin **x**=**0** and to the temporal origin t=0 are zero, i.e. $\mathbf{f}^{(1,0)}(\mathbf{0}) = \mathbf{0}$ and $f^{(0,1)}(0) = 0$.

Under this assumptions, taking into account in formula (1.23) only the first terms for $l + n \le 2$ we can express the relation between the spectrum of displacements $u_i(\mathbf{x},\omega)$ and the spatio-temporal moments of the function $f(\mathbf{x},t)$ by following formula (Bukchin,1995)

$$u_i(\mathbf{x},\omega) = \frac{1}{\mathrm{i}\omega} M_0 M_{jl} \frac{\partial}{\partial y_l} G_{ij}(\mathbf{x},\mathbf{0},\omega) + \frac{1}{2\,\mathrm{i}\,\omega} f_{mn}^{(2,0)}(\mathbf{0}) M_{jl} \frac{\partial}{\partial y_m}\frac{\partial}{\partial y_n}\frac{\partial}{\partial y_l} G_{ij}(\mathbf{x},\mathbf{0},\omega)$$

$$- f_m^{(1,1)}(\mathbf{0},0) M_{jl} \frac{\partial}{\partial y_m}\frac{\partial}{\partial y_l} G_{ij}(\mathbf{x},\mathbf{0},\omega) + \frac{\mathrm{i}\omega}{2} f^{(0,2)}(0) M_{jl} \frac{\partial}{\partial y_l} G_{ij}(\mathbf{x},\mathbf{0},\omega)\,, \qquad (3.8)$$

i,j,l,m,n = 1,2,3 and the summation convention for repeated subscripts is used. $G_{ij}(\mathbf{x},\mathbf{y},\omega)$ in equation (3.8) is the spectrum of Green function for the chosen model of medium and wave type. We assume that the paths from the earthquake source to seismic stations are well approximated by weak laterally inhomogeneous model. Under this assumption, as it was mentioned above, the amplitude spectrum $|u_i(\mathbf{x},\omega)|$ defined by formula (3.8) does not depend on the average phase velocity of the wave, and the errors in source location do not affect the amplitude spectrum.

If all characteristics of the medium, depth of the best point source and seismic moment tensor are known (determined, for example, using the spectral domain of longer periods) the representation (3.8) gives us a system of linear equations for moments of the function $f(\mathbf{x},t)$ of total degree 2. But as we mentioned considering moment tensor approximation the average phase velocities of surface waves are usually not well known. For this reason, we use only amplitude spectrum of surface waves for determining these moments, in spite of non-linear relation between them.

Let us consider a plane source. All moments of the function $f(\mathbf{x},t)$ of total degree 2 can be expressed in this case by formulas (3.2)-(3.7) in terms of 6 parameters: Δt - estimate of source duration, $l_{\max}$ - estimate of maximal mean size of the source, φ_l - estimate of the angle between the direction of maximal size and strike axis, $l_{\min}$ - estimate of minimal mean size of the source, v - estimate of the absolute value of instant centroid mean velocity **v** and φ_v - the angle between **v** and strike axis.

Using the Bessel inequality for the moments under discussion we can obtain the following constrain for the parameters considered above (Bukchin, 1995):

$$v^2 \Delta t^2 \left(\frac{\cos^2 \varphi}{l_{\max}^2} + \frac{\sin^2 \varphi}{l_{\min}^2} \right) \le 1\,, \qquad (3.9)$$

where φ is the angle between major axis of the source and direction of **v**.

Assuming that the source is a plane fault and representation (1.20) is valid let us consider a rough grid in the space of 6 parameters defined above. These parameters have to follow inequality (3.9). Let models of the media be given and the moment tensor be fixed as well as the depth of the best point source. Let the fault plane (one of two nodal planes) be identified. Using formula (3.8) we can calculate the amplitude spectra of surface waves at the points of observation for every possible combination of values of the varying parameters.

Comparison of calculated and observed amplitude spectra give us a residual $\varepsilon^{(i)}$ for every point of observation, every wave and every frequency ω. Let $u^{(i)}(\mathbf{r},\omega)$ be any observed value of the spectrum, i = 1,…,N; $\varepsilon^{(i)}$ - corresponding residual of $|u^{(i)}(\mathbf{r},\omega)|$. We define the normalized amplitude residual by formula

$$\varepsilon(\Delta t, l_{\max}, l_{\min}, \varphi_l, v, \varphi_v) = \left[\left(\sum_{i=1}^{N} \varepsilon^{(i)2}\right) \Big/ \left(\sum_{i=1}^{N} |u^{(i)}(\mathbf{r},\omega|^2\right)\right]^{1/2} . \quad (3.10)$$

The optimal values of the parameters that minimize ε we consider as estimates of these parameters. We search them by a systematic exploration of the six dimensional parameter space. To characterize the degree of resolution of every of these source characteristics we calculate partial residual functions in the same way as was described in previous section. We define 6 functions of the residual corresponding to the 6 varying parameters: $\varepsilon_{\Delta t}(\Delta t)$, $\varepsilon_{l_{\max}}(l_{\max})$, $\varepsilon_{l_{\min}}(l_{\min})$, $\varepsilon_{\varphi_l}(\varphi_l)$, $\varepsilon_v(v)$ and $\varepsilon_{\varphi_v}(\varphi_v)$. The value of the parameter for which the corresponding function of the residual attains its minimum we define as estimate of this parameter. At the same time these functions characterize the degree of resolution of the corresponding parameters.

IV. Example of application

We illustrate the technique by results of its application for a study of two largest earthquakes in the last four decades: Sumatra-Andaman earthquake occurred on 26 December 2004 and Nias earthquake occurred on 28 March 2003.

(a) Sumatra-Andaman earthquake 26 December 2004.

To estimate the best double couple, duration and geometry of the source we have used amplitude spectra of second and third orbits of fundamental Love and Rayleigh modes in spectral range from 500 to 650 seconds. The records were processed by frequency-time and polarization analysis package. We selected 24 Love wave records and 22 Rayleigh wave records from IRIS and GEOSCOPE stations. Their azimuthal distribution is given in figure 3.

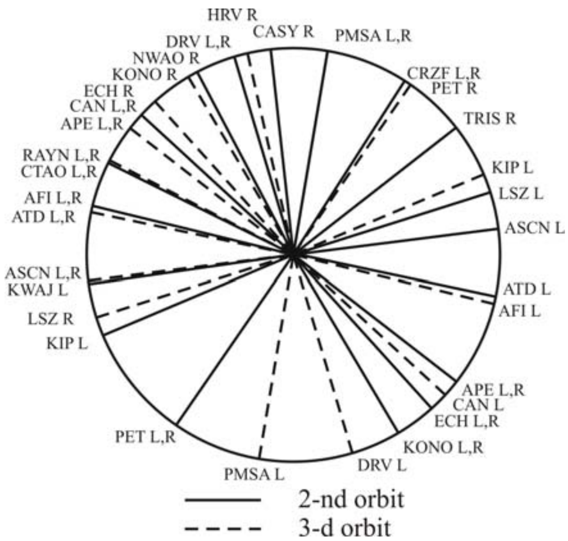

Fig. 3 Azimuthal distribution of radiation of waves used for inversion.
L and R after the name of station denotes Love and Rayleigh wave correspondingly.

In the source region and under the receivers, we used the 3SMAC model (Ricard et al. 1996) for the crust and the PREM model below. We used the quality factor given by the PREM model for attenuation correction. The moment tensor describing the source in instant point source approximation is obtained by joint inversion of surface wave amplitude spectra and first arrival polarities at worldwide stations (Lasserre *et al.* 2001). The solution gives a mechanism described by the following values of strike, dip and slip: 330°, 8°, 105° respectively (see figure 5). The estimate of source depth is equal to 13 km. The estimated value of seismic moment is $0.52 \cdot 10^{23}$ N·m.

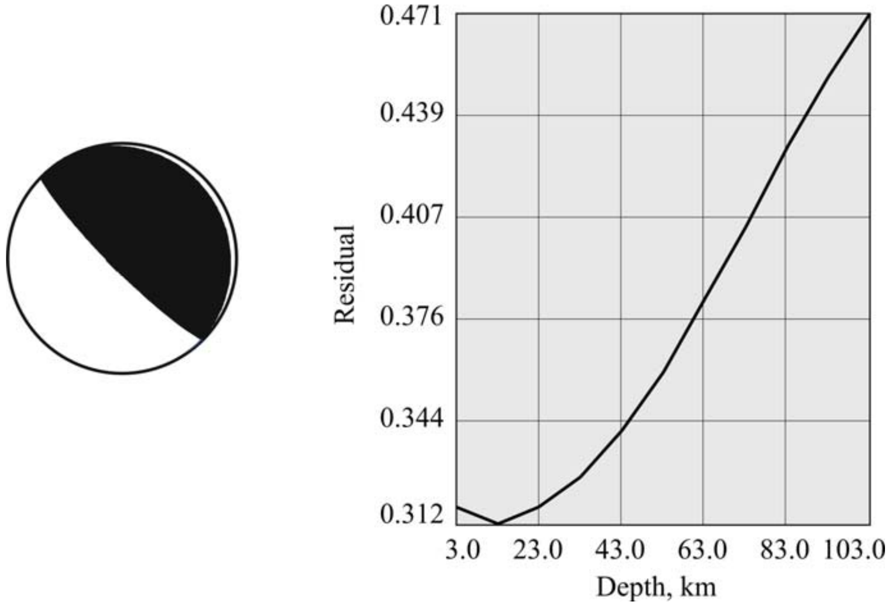

Fig. 4. Double couple solution and source depth resolution curve.
P1: 330°, 8°, 105°; P2: 135°, 82°, 88°. $M_0 = 0.52 \cdot 10^{23}$ N·m

Determining 2-nd moments of moment tensor density we consider the nodal plane dipping to the northeast as a fault plane. We fixed source depth (13km) and focal mechanism obtained in instant point source approximation. Usually when double couple parameters are obtained from periods long enough to consider the source as an instant and point, we fix seismic moment as well. But in this case the periods are not sufficiently long, so we recalculated seismic moment determining source 2nd moments. As it was mentioned above we estimate the duration and the geometry of the source from the same amplitude spectra of fundamental Love and Rayleigh modes in the same spectral band (from 500 to 650 seconds) that was used for inversion in instant point source approximation.

Our final estimate of seismic moment is equal to $0.84 \cdot 10^{23}$ Nm. The residual functions for integral estimates are given in figure 5. The inversion yields the integral estimate of duration being about 160 s, a characteristic source length (major axis length) of 300 - 400 km. The minor axis length is poorly resolved, lying between 0 and 200 km. The average instant centroid velocity estimate is about 2 km/s. The angles giving the major axis and velocity vector orientations are measured clockwise on the footwall starting from the strike axis. They are consistent with each other and correspondent residual functions attain their minimum values at 15°.

The propagation of rupture may be characterized by directivity ratio *d* proposed by McGuire (2002). This parameter is defined as the ratio of the average velocity of the instant centroid over the apparent rupture velocity equal to $l_{max}/\Delta t$. For a unilateral rupture where slip nucleates at one end of a rectangular fault and propagates to the other at a uniform rupture velocity with a uniform slip distribution, $d = 1$. For a symmetric bilateral rupture

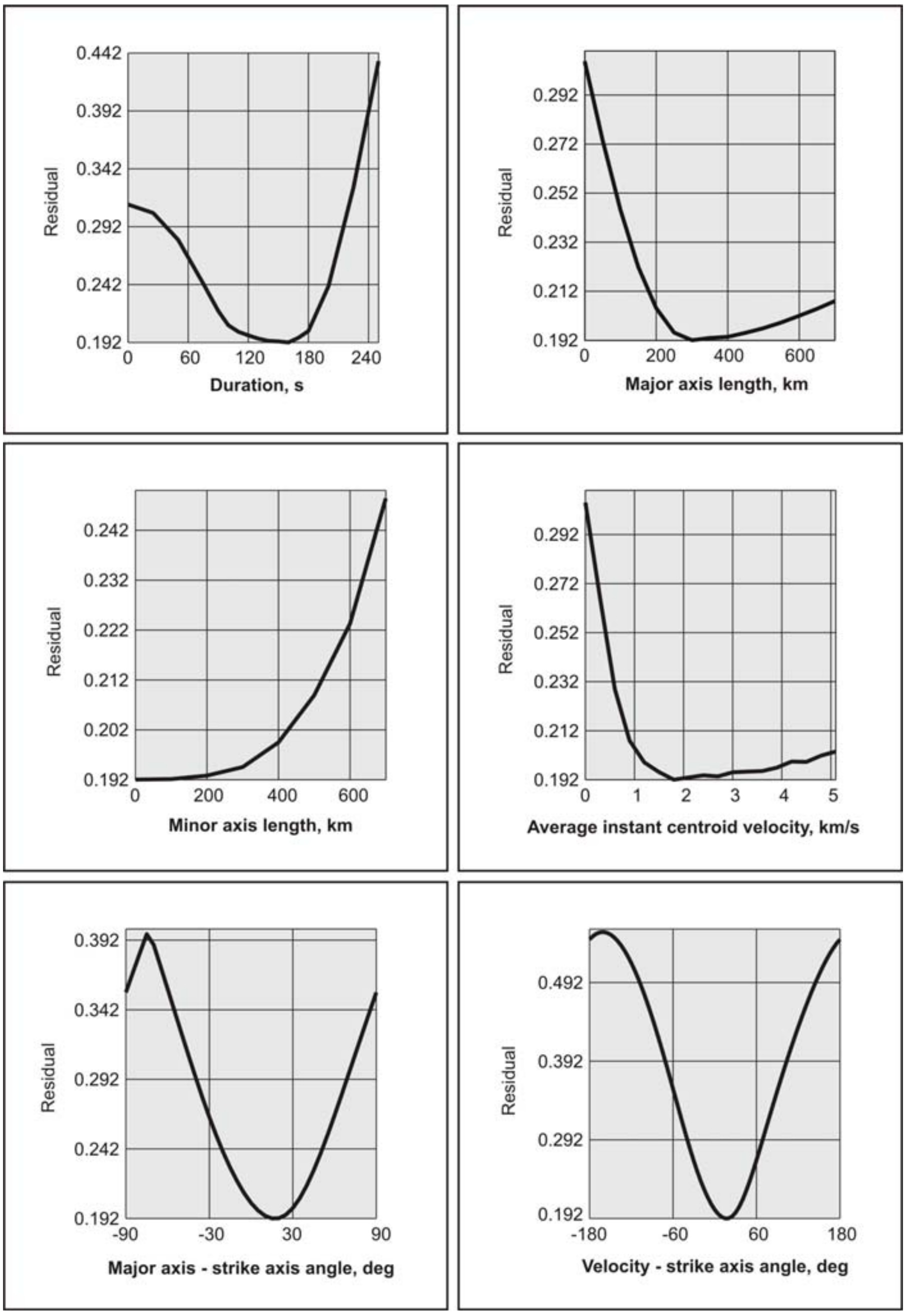

Fig. 5. Residual functions for source integral characteristics.

that initiates in the middle and propagates to both ends of a fault at uniform rupture velocity with uniform slip distribution, d = 0. Predominantly bilateral ruptures correspond to $0 \le d < 0.5$ while predominantly unilateral ruptures correspond to $0.5 < d \le 1$. We find d = 0.9 for our model. This value shows unilateral (northward) rupture propagation.

Multiplying the integral estimate of duration by factor 2.5 (see figure 2) we get for source process duration the value being equal to 400s. Multiplying the integral estimates for principal axes length by factor 3 we get for maximum size 1050 km, and for minor axis 300km.

We compared our estimate of source duration with similar integral estimate calculated directly as second moment of moment-rate function shown in figure 6, reported by (Ammon *et al.* 2005). The value obtained for this estimate is equal to 187 s, which is not so different from our integral estimate 160s.

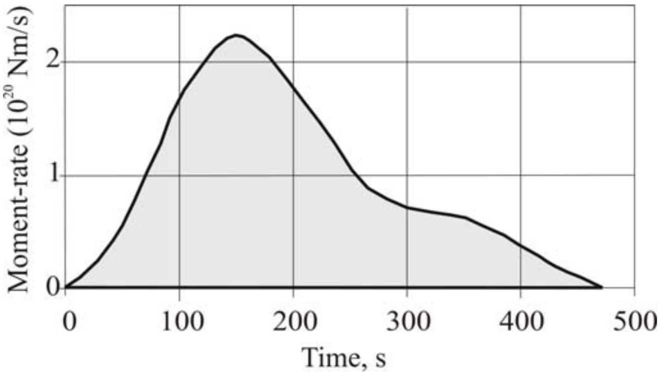

Fig. 6. Moment-rate function

The efficiency of second moments approximation is illustrated by figure 7, showing observed surface wave amplitude spectra and synthetic spectra calculated in two approximations: in instant point source approximation and in 2-nd moments approximation. As on can see from the figure the misfit between observations and synthetics is significantly less when synthetic spectra is calculated in 2-nd moments approximation.

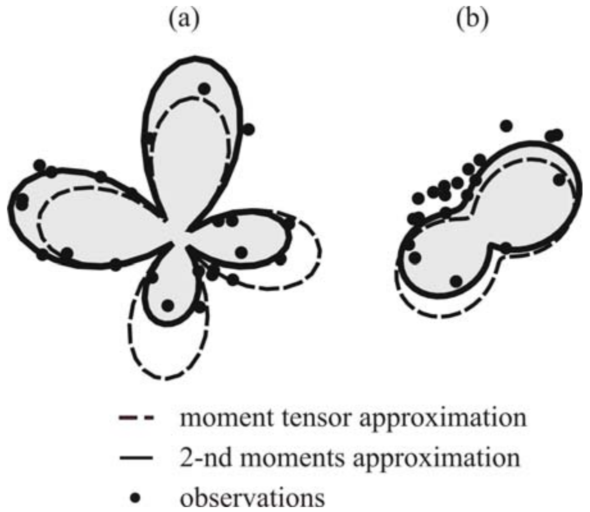

Fig. 7. Radiation patterns of fundamental (a) Love and (b) Rayleigh modes and observed amplitudes at T=550s.

(b) Nias earthquake 28 March 2005.

To obtain the moment tensor describing the source in instant point source approximation we used amplitude spectra of second and third orbits of fundamental Love and Rayleigh modes in spectral range from 250 to 500 seconds. The modes were isolated by frequency-time analysis approach. We selected 26 Love wave records and 29 Rayleigh wave records from IRIS and GEOSCOPE stations. Their azimuthal distribution is given in figure 8.

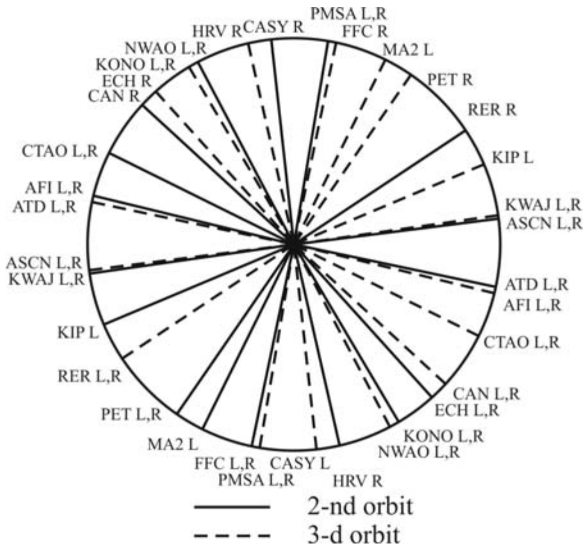

Fig. 8. Azimuthal distribution of radiation of waves used for inversion.
L and R after the name of station denotes Love and Rayleigh wave correspondingly.

In the source region and under the receivers, we used the 3SMAC model (Ricard et al. 1996) for the crust and the PREM model below. We used the quality factor given by the PREM model for attenuation correction. The best double couple for this event and its depth were estimated by joint inversion of surface wave amplitude spectra and first arrival polarities. The solution gives following focal mechanism: 315°, 10°, 90° for values of strike, dip and slip angles correspondingly (see figure 9). The estimate of source depth is equal to 8-10 km. The estimated value of seismic moment is $0.85 \cdot 10^{22}$ N·m.

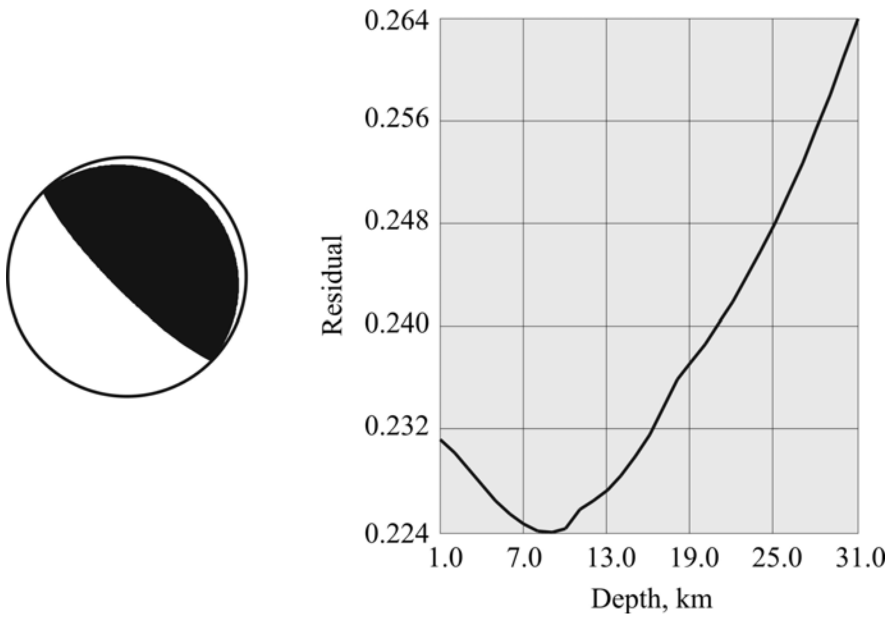

Fig. 9. Double couple solution and source depth resolution curve.
P1: 315°, 10°, 90°; P2: 135°, 80°, 90°. $M_0 = 0.81 \cdot 10^{22}$ N·m

Determining 2-nd moments of moment tensor density we consider the nodal plane dipping to the northeast as a fault plane. We fixed source depth, focal mechanism and seismic moment obtained in instant point source approximation. The 2-nd moment determination was performed in period band from 175 to 300s using 1-st orbits only, and using two higher orbits. The results for these two data samples are very similar. The integral estimates obtained from complete data set using Love and Rayleigh fundamental modes are poor resolved. The inversion of Rayleigh wave spectra only give similar optimum values, but their resolution is much better. The residual functions for integral estimates are given in figure 10. The integral estimate of duration is equal to 40 s, the integral estimate of major axis length is about 150 km. The minor axis length is poorly resolved, lying between 0 and 50 km. The average instant centroid velocity estimate is about 3 km/s. The angles giving the major axis and velocity vector orientations are equal to 30° and 195° correspondingly.

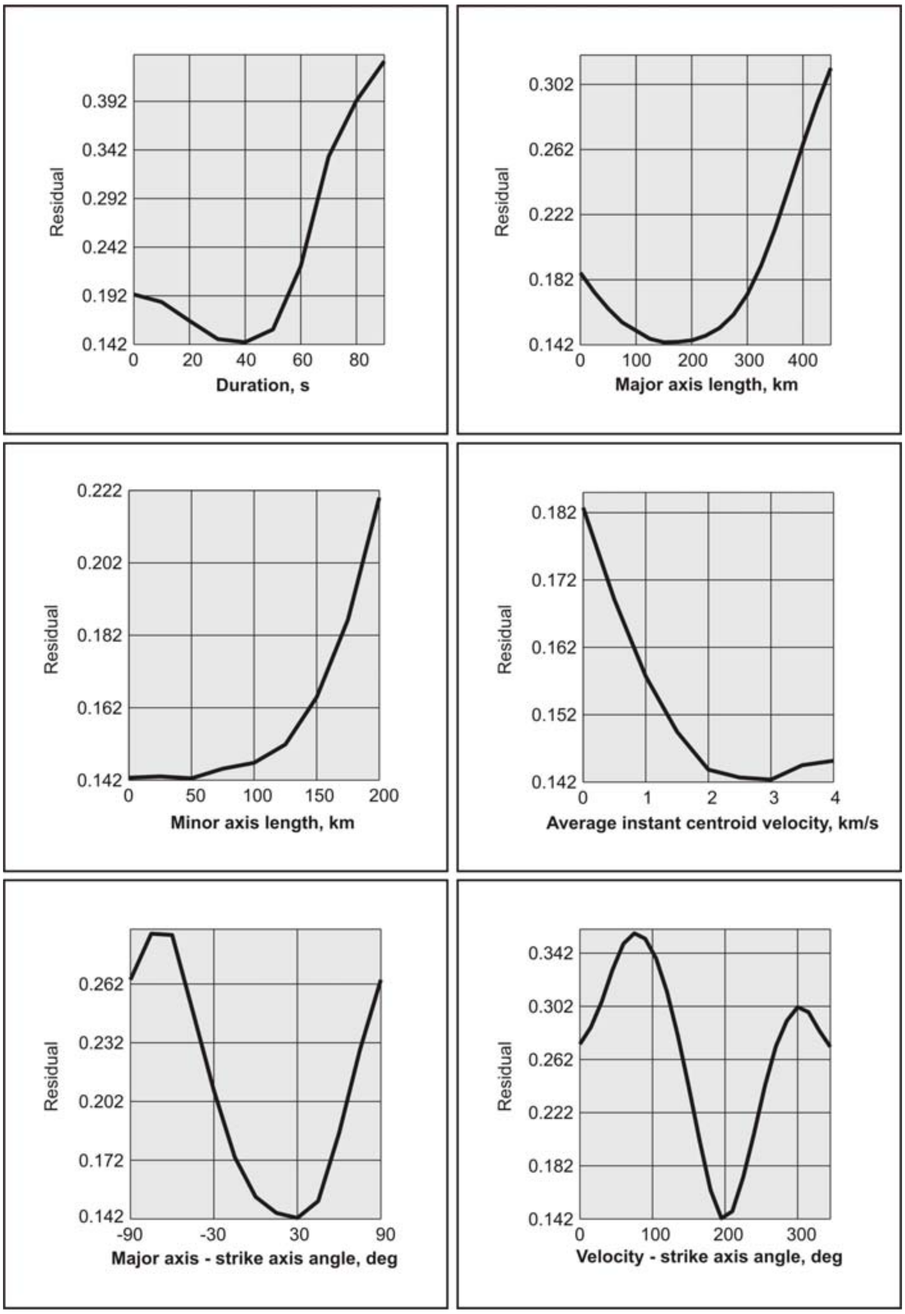

Fig. 10. Residual functions for source integral characteristics.

The directivity ratio *d* for this event is equal to 0.8, which means that rupture propagation is predominantly unilateral to the south-south-east.
Multiplying the integral estimate of duration by factor 2.5 we get for source process duration the value being equal to 100s. Multiplying the integral estimates for principal axes length by factor 3 we get for maximum size 450km, and for minimum size up to150 km.
We compared our estimate of source duration with similar integral estimate calculated as second moment of moment-rate function (see figure 11), reported by Yamanaka (2005). The value obtained for this estimate is equal to 46.5s, which is comparable with our integral estimate equal to 40s.

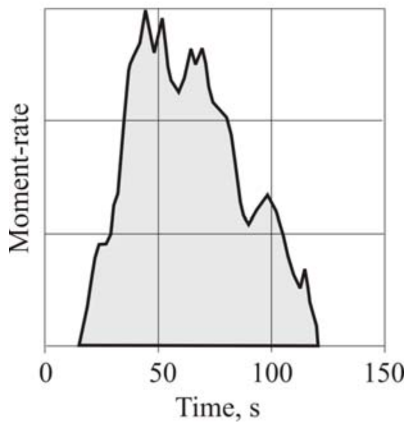

Fig. 11. Moment-rate function

Observed Rayleigh wave amplitude spectra and synthetic spectra calculated in instant point source approximation and in 2-nd moments approximation are presented in figure 13. The misfit between observations and synthetics is less when synthetic spectra is calculated in 2-nd moments approximation.

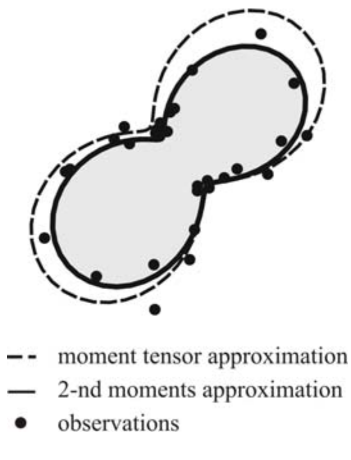

Fig.12. Radiation patterns of fundamental Rayleigh mode and observed amplitudes at T=254s.

Ellipses characterizing the spatial extension of the sources over the aftershocks for both considered earthquakes are shown in the figure 13. The arrows show the directions of rupture propagation. The axes of ellipses are obtained by multiplication of integral estimates by factor 3 corresponding to 2D Gaussian distribution of moment-rate.

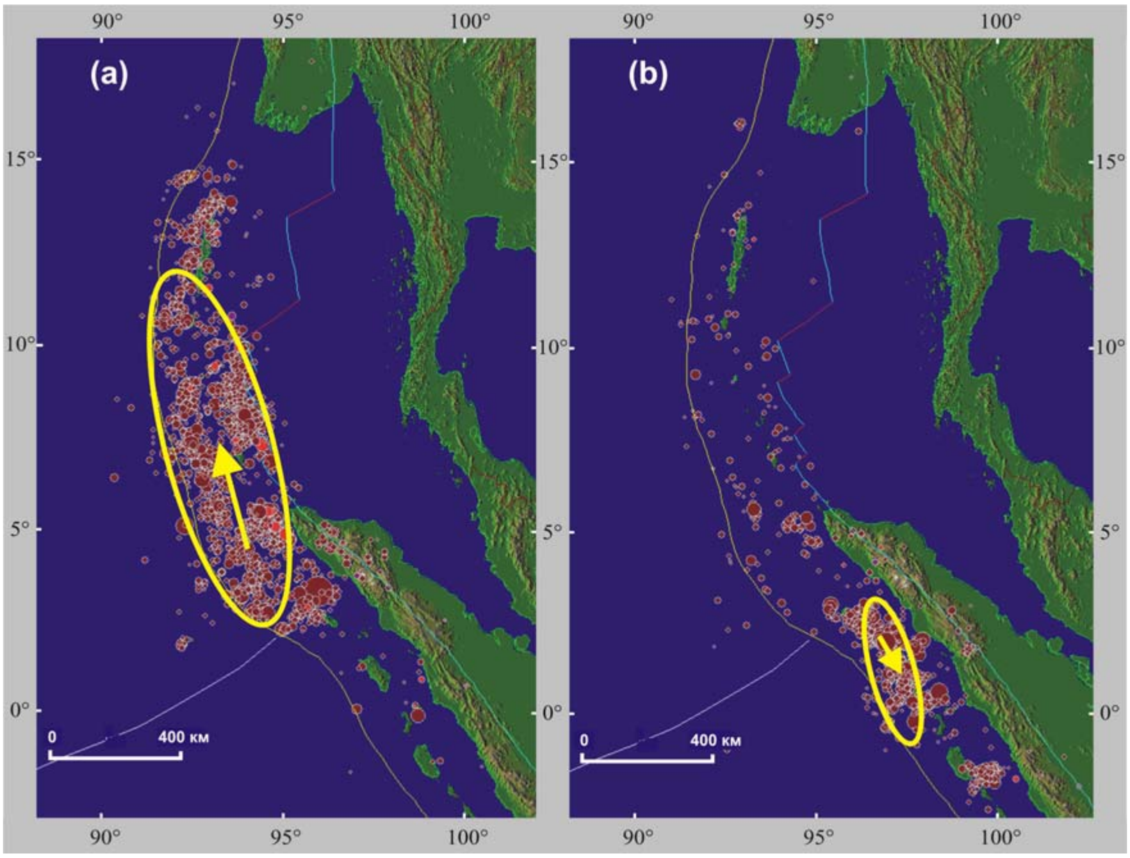

Fig. 13. Ellipses characterizing the spatial extension of the sources over the aftershock clouds for (a) 26.12.2004 earthquake and for (b) 28.03.2005 earthquake. The arrows show the directions of rupture propagation.

As one can see from the figure 13 b the size of the major ellipse axis is in a good agreement with the distribution of aftershocks of the 28.03.2005 event. Similar ellipse characterizing the shape of the source region for Sumatra-Andaman earthquake (figure 13 a) doesn't cover the Northern segment of the aftershock cloud. But GPS observations (Vigny *et al.* 2005) show significant deformations in this region. At the same time tomographic source models based on analysis of different seismological observations (Lay *et al.* 2005, Ammon *et al.* 2005) suggest weak deformations in this region. This contradiction can be explained (Lay *et al.* 2005, Ammon *et al.* 2005) by slow deformations occurred in the Northern segment of the fault, which were not registered by seismic instruments.

References

Charles J. Ammon, Chen Ji, Hong-Kie Thio, David Robinson, Sidao Ni, Vala Hjorleifsdottir, Hiroo Kanamori, Thorne Lay, Shamita Das, Don Helmberger, Gene Ichinose, Jascha Polet, David Wald, 2005. Rupture Process of the 2004 Sumatra-Andaman Earthquake, *Science*, 308, 133-1139.

V.M. Babich, B.A. Chikachev and T.B. Yanovskaya, 1976. Surface waves in a vertically inhomogeneous elastic half-space with weak horizontal inhomogeneity, *Izv. Akad. Nauk SSSR, Fizika Zemli,* 4, 24-31.

G. Backus and M. Mulcahy, 1976. Moment tensors and other phenomenological descriptions of seismic sources. Pt.1. Continuous displacements, *Geophys. J. R. astr. Soc.*, 46, 341-362.

G. Backus, 1977a. Interpreting the seismic glut moments of total degree two or less, *Geophys. J. R. astr. Soc.*, 51, 1-25.

G. Backus, 1977b. Seismic sources with observable glut moments of spatial degree two, *Geophys. J. R. astr. Soc.*, 51, 27-45.
B.G. Bukchin, 1989. Estimation of earthquake source parameters In: V.I. Keilis-Borok (Editor), Seismic surface waves in a laterally inhomogeneous earth. Kluwer Academic Publishers Dordrecht, 229-245.
B.G. Bukchin, 1990. Determination of source parameters from surface waves recordings allowing for uncertainties in the properties of the medium, *Izv. Akad. Nauk SSSR, Fizika Zemli,* 25, 723-728.
B.G. Bukchin, A.L. Levshin, L.I. Ratnikova, B. Dost and G. Nolet, 1994. Estimation of spatio-temporal source parameters for the 1988 Spitak, Armenia Earthquake, *Computational Seismology and Geodynamics*, 25, English Transl. 156-161, Am. Geophys. Union.
B.G. Bukchin, 1995. Determination of stress glut moments of total degree 2 from teleseismic surface waves amplitude spectra, *Tectonophysics*, 248, 185-191.
Das, S. & Kostrov, B.V., 1997. Determination of the polynomial moments of the seismic moment rate density distributions with positivity constraints, *Geophys. J. Int.*,131,115-126.
J.M. Gomez, B. Bukchin, R. Madariaga and E.A. Rogozhin, 1997a. A study of the Barisakho, Georgia earthquake of October 23, 1992 from broad band surface and body waves, *Geophyz. J. Int,* V. 129, No. 3, pp 613--623
J.M. Gomez, B. Bukchin, R. Madariaga, E.A. Rogozhin and B.M. Bogachkin, 1997b. Space-Time study of the 19 August 1992 Susamyr earthquake, Kyrgyzstan, Journal of Seismology, V.1, N 3, pp 219-235.
A.V. Lander, 1989. Frequency-time analysis. In: V.I. Keilis-Borok (Editor), Seismic surface waves in a laterally inhomogeneous earth. Kluwer Academic Publishers Dordrecht, 153-163.
Lasserre, C., Bukchin, B., Bernard, P., Tapponnier, P., Gaudemer, Y., Mostinsky, A., and Dailu, R., Sources parameters and tectonic origin of the 1996 June 1 Tianzhu (Mw = 5:2) and 1995 July 21 Yongden (Mw = 5:6) earthquakes near the Haiyuan fault (Gansu, China), Geophys. J. Int., 144 (1), pp 206-220, 2001.
Lay, T., Kanamori, H., Ammon, C. J., Nettles, M., Ward, S.N., Aster, R.C., Beck, S. L., Bilek, S.L., Brudzinski, M.R., Butler, R., DeShon, H.R., Ekström, G., Satake, K., Sipkin S. The Great Sumatra-Andaman Earthquake of 26 December 2004. *Science*, 2005, 308, 1127-1133.
A.L. Levshin, 1985. Effects of lateral inhomogeneity on surface wave amplitude measurements, *Annles Geophysicae*, 3, 4, 511-518.
McGuire, J. J., Zhao, L., and Jordan, T. H., Teleseismic inversion for the second-degree moments of earthquake space-time distributions, Geophys. J. Int., 145 (3), 661-678, 2001.
Ricard Y., H-C. Nataf & J-P. Montagner, 1996. The three-dimensional seismological model a priori constrained: Confrontation with seismic data. *J. Geophys. Res.*, 101, pp 8457-6472.
Pavlov, V.M., 1994. On non-uniqueness of the inverse problem for a seismic source – II. Treatment in term of polynomial moments, *Geophys. J. Int.*, 119, 497-496.
Vigny, C., Simons, W.J.F., Abu, S., Bamphenyu, R., Satirapod, C., Choosakul, N., Subarya, C., Socquet, A., Omar, K., Abidin, H.Z., Ambrosius, B.A.C. Insight into the 2004 Sumatra-Andaman earthquake from GPS measurements in southeast Asiua. *Nature,* 2005. Vol. 436, doi:10.1038/nature03937, p. 201-206.
J.H. Woodhouse, 1974. Surface waves in the laterally varying structure. *Geophys. J. R. astr. Soc.*, 90, 12, 713-728.