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WORKSHOP ON THE FUTURE OF IONOSPHERIC RESEARCH FOR SATELLITE NAVIGATION AND POSITIONING: ITS RELEVANCE FOR DEVELOPING COUNTRIES

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Lecture Notes

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Propagation of Radio Waves in the Fluctuating Ionosphere: Ionospheric Scintillation

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Preface

This compendium is a concise version of a course of lectures on wave propagation in random media that was given by the author at the University of Uppsala, Sweden few years ago . The full version of the course was published by the presenting author in co-authorship with Dr. Bengt Lundborg [Zernov and Lundborg 1993]. Dr. Lundborg prepared the initial English draft version of the compendium. The draft was subsequently elaborated by both authors to its final form. Full course of lectures is regularly delivered by Prof. N.N.Zernov at the University of St.Petersburg, Russia to the students specializing in radio wave propagation. It gives main notions, definitions and basic ideas of a series of methods employed in the theory of wave propagation in random media. In the present version all the methods are discussed in their most simple form, i.e., when the background medium is assumed to be homogeneous. This makes description of the methods more transparent and less overloaded by (sometimes boring) transformations and manipulations. At the same time it should be mentioned that in the works of numerous authors (including the presenting author) methods have been extended to the more practical and important case of the inhomogeneous background media. The full version of the course outlines many of these extensions.

The widely known books by Tatarski [1961; 1967], Rytov [1976], Rytov, Kravtsov and Tatarski [1978; 1987; 1988; 1989], Ishimaru [1978], Yeh and Liu [1972], Budden [1985] and Kravtsov and Orlov [1980] have been used for the basic contents of the lectures, without giving specific references to these books.

Along with the basic items, the author additionally included in the list of references a series of papers pertinent to the subject under consideration, which reflect to some extent the up-to-date status of the theory of wave propagation in random media. Among others the papers written by the author in co-authorship with his colleagues are also presented in the List of References, which are devoted to the problems of propagation of the high frequency wave fields in the ionospheric reflection and transionospheric fluctuation channels of propagation (the copies of some of those papers are also applied). In these papers one can additionally find numerous references to the papers of many other authors , who worked into similar problems.

Contents

Chapter 1

Introduction to the theory of random functions

In many cases wave propagation through the ionosphere can be described by the Helmholtz' equation for a component of the electromagnetic field:

$$\nabla^2 E + k^2 \epsilon(\mathbf{r}, \omega, t) E = 0 \tag{1.1}$$

supplemented by appropriate boundary conditions. The isotropic approximation (no geomagnetic field) is quite sufficient to describe the essential effects of the fluctuations of the ionosphere, at least neglecting the effects of ordinary-extraordinary wave coupling. By this we avoid a lot of mathematical complexity, since we can use instead of the permittivity tensor the scalar dielectric permittivity

$$\epsilon(\mathbf{r}, \omega, t) = 1 - \frac{e^2 N(\mathbf{r}, t)}{m \varepsilon_0 \omega^2} \tag{1.2}$$

Here $N(\mathbf{r}, t)$ is the electron density as a function of space and time coordinates. This may be a smooth and slowly varying regular function, in which case we have a deterministic wave propagation problem which can be treated with traditional methods such as ray tracing. In more realistic models of the ionosphere $N(\mathbf{r}, t)$ includes local inhomogeneities and fluctuations. When we describe the influence of fluctuations we can write

$$N(\mathbf{r}, t) = \langle N(\mathbf{r}) \rangle + \Delta N(\mathbf{r}, t) \tag{1.3}$$

The function $\langle N(\mathbf{r}) \rangle$ is a regular function which represents the *background*, the average large-scale density of the ionosphere, and is assumed independent of time. The *fluctuations* are expressed by the quantity $\Delta N(\mathbf{r}, t)$, which is a *zero-mean random function*, i.e. it fulfils

$$\langle \Delta N(\mathbf{r}, t) \rangle = 0 \tag{1.4}$$

With (1.3) the wave equation (1.1) is a stochastic differential equation. This is the topic of the present course and to solve (1.1) we therefore need some tools from this branch of mathematics. Hence we devote this Chapter to an introduction to the theory of random functions.

1.1 Random values

We denote a *random value* by the symbol ξ. If we restrict the treatment to the case of continuous random variables, the probability that ξ takes a value in the interval $(x, x+\mathrm{d}x)$ is expressed by means of the *probability density function* (PDF) $w(x)$ as follows:

$$P(x \le \xi \le x + \mathrm{d}x) = w(x)\ \mathrm{d}x \tag{1.5}$$

The interval of ξ may be finite, $\xi \in [a, b]$, or infinite, $\xi \in (-\infty, +\infty)$.

By means of the probability density function, *moments of* ξ can be constructed:

$$\langle \xi \rangle = \int x\, w(x)\ \mathrm{d}x \tag{1.6a}$$

$$\left\langle \xi^2 \right\rangle = \int x^2\, w(x)\ \mathrm{d}x \tag{1.6b}$$

and so on. In fact, for any deterministic function $f(x)$, its average is given by

$$\langle f(\xi) \rangle = \int f(x)\, w(x)\ \mathrm{d}x \tag{1.6c}$$

An alternative approach to the description using the probability density function is the *characteristic function* for ξ:

$$\varphi_\xi(u) = \left\langle e^{i\xi u} \right\rangle \tag{1.7}$$

where u is a deterministic variable. If the domain of definition for ξ is infinite, the characteristic function obviously forms a Fourier transform pair with the probability density function:

$$\varphi_\xi(u) = \int\limits_{-\infty}^{+\infty} w(x)\ e^{ixu}\ \mathrm{d}x \tag{1.7a'}$$

$$w(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} \varphi_\xi(u)\ e^{-ixu}\ \mathrm{d}u \tag{1.7b'}$$

1.2 The definition of probability

In order to be a probability, P must possess the following three properties.
1. The property of positive-definiteness:

$$P \ge 0 \tag{1.8a}$$

2. Probabilities of mutually excluding events A_k are additive:

$$P\left(\bigcup_k A_k\right) = \sum_k P\left(A_k\right) \tag{1.8b}$$

3. The deterministic event D has probability unity:

$$P(D) = 1 \tag{1.8c}$$

The last property leads to the normalization property of the probability density function:

$$\int w(x)\ \mathrm{d}x = 1 \tag{1.9}$$

1.3 Random functions

$\xi(Q)$ is a *random function* if for every value Q of the independent variable, $\xi(Q)$ is a random value. In the most general case ξ is a function of space and time: $Q = \{\mathbf{r}, t\}$. In special cases we may have $Q = \{\mathbf{r}\}$ or $Q = \{t\}$. The case $\xi(\mathbf{r})$ is called a *random field* and the case $\xi(t)$ is called a *random process.*

1.4 Probability density functions

With random functions the concept of probability density is much more complicated than for random values. Since Q may take a continuum of values it is now necessary to describe how the probabilities for neighbouring Q's are related to each other.

We now have the first-order probability density function

$$w_1(\xi, Q)\ \mathrm{d}\xi = P(\xi \le \xi(Q) \le \xi + \mathrm{d}\xi) \tag{1.10a}$$

which corresponds to the definition (1.5). Furthermore we have a set of higher-order probability density functions:

$$w_2(\xi_1, Q_1, \xi_2, Q_2)\ \mathrm{d}\xi_1\mathrm{d}\xi_2 = P(\xi_1 \le \xi(Q_1) \le \xi_1 + \mathrm{d}\xi_1; \xi_2 \le \xi(Q_2) \le \xi_2 + \mathrm{d}\xi_2) \tag{1.10b}$$

.....

$$w_n(\xi_1, Q_1, ..., \xi_n, Q_n)\ \mathrm{d}\xi_1...\mathrm{d}\xi_n = P(\xi_1 \le \xi(Q_1) \le \xi_1 + \mathrm{d}\xi_1; ...; \xi_n \le \xi(Q_n) \le \xi_n + \mathrm{d}\xi_n) \tag{1.10c}$$

.....

For a rigorous and complete description of the random function, the entire infinite set of multi-dimensional PDF's is reqired.

The following are necessary properties of probability density functions.

1. They are invariant for permutations of each pair of arguments:

$$w_n(\xi_1, Q_1, ...\xi_i, Q_i, ...\xi_j, Q_j, ..., \xi_n, Q_n) = w_n(\xi_1, Q_1, ...\xi_j, Q_j, ...\xi_i, Q_i, ..., \xi_n, Q_n) \tag{1.11a}$$

2. The PDF's of lower order can be obtained from those of higher order:

$$w_k(\xi_1, Q_1, ...\xi_k, Q_k) = \int w_n(\xi_1, Q_1, ...\xi_k, Q_k, \xi_{k+1}, Q_{k+1}, ..., \xi_n, Q_n)\ \mathrm{d}\xi_{k+1}...\mathrm{d}\xi_n \tag{1.11b}$$

3. Normalization property (note that the dependencies on all Q_i vanish in the integration):

$$\int w_n(\xi_1, Q_1, ...\xi_n, Q_n)\ \mathrm{d}\xi_1...\mathrm{d}\xi_n = 1 \tag{1.11c}$$

1.5 Moments of random functions

Corresponding to the set of PDF's we may also construct moments of $\xi(Q)$ of arbitrary orders. To start with we have the first-order moment

$$M_1(Q) = \langle \xi(Q) \rangle = \int \xi\, w_1(\xi, Q)\ \mathrm{d}\xi \tag{1.12a}$$

The second-order moment is defined by

$$B_2(Q_1, Q_2) = \langle \xi(Q_1)\, \xi(Q_2) \rangle = \int \xi_1 \xi_2\, w_2(\xi_1, Q_1, \xi_2, Q_2)\ \mathrm{d}\xi_1 \mathrm{d}\xi_2 \tag{1.12b}$$

In describing energy level fluctuations of ionospheric signals, fourth-order moments are employed.

Because of the property (1.11b) we may also obtain M_1 from the second-order PDF as follows:

$$M_1(Q_1) = \int \xi_1\, w_2(\xi_1, Q_1, \xi_2, Q_2)\ \mathrm{d}\xi_1 \mathrm{d}\xi_2 \tag{1.13}$$

Similarly each moment of a lower order can in general be calculated using a higher-order PDF with integration over the extra variables.

1.6 Correlation functions and their properties

An important second-order moment is the *correlation function*

$$\psi_\xi(Q_1, Q_2) = \Big\langle [\xi(Q_1) - \langle \xi(Q_1) \rangle]\ [\xi(Q_2) - \langle \xi(Q_2) \rangle] \Big\rangle$$

$$= \int [\xi(Q_1) - \langle \xi(Q_1) \rangle]\ [\xi(Q_2) - \langle \xi(Q_2) \rangle]\ w_2(\xi_1, Q_1, \xi_2, Q_2)\ \mathrm{d}\xi_1 \mathrm{d}\xi_2 \tag{1.14}$$

It is easy to show that this definition is equivalent to

$$\psi_\xi(Q_1, Q_2) = B_2(Q_1, Q_2) - \langle \xi(Q_1) \rangle\, \langle \xi(Q_2) \rangle \tag{1.15}$$

In particular, the second term in (1.15) vanishes when the regular part has been subtracted so that $\xi(Q)$ is a zero-mean random function.

When we deal with *complex random functions*

$$\zeta(Q) = \xi(Q) + i\,\eta(Q) \tag{1.16}$$

where $\xi(Q)$ and $\eta(Q)$ both are real random functions, the $2n$-dimensional PDF's must be introduced $w_{2n}(\xi_1, Q_1, ..., \xi_n, Q_n; \eta_1, Q_1, ..., \eta_n, Q_n)$ for each n to get a complete description. All properties introduced above for the random functions apply also for the complex random functions. It is convenient to have a special notation for the zero-mean random part of a random function $\zeta(Q)$. Hence we put a tilde above such quantities:

$$\tilde{\zeta}(Q) = \zeta(Q) - \langle\zeta(Q)\rangle \tag{1.17}$$

In the complex case we are now dealing with two correlation functions, defined as follows:

$$\psi_\zeta(Q_1, Q_2) = \left\langle \tilde{\zeta}(Q_1)\tilde{\zeta}^*(Q_2)\right\rangle \ , \qquad \textit{first correlation function} \tag{1.18a}$$

$$\tilde{\psi}_\zeta(Q_1, Q_2) = \left\langle \tilde{\zeta}(Q_1)\tilde{\zeta}(Q_2)\right\rangle \ , \qquad \textit{second correlation function} \tag{1.18b}$$

The first correlation function has three important properties.

1. Hermitean property:

$$\psi_\zeta(Q_1, Q_2) = \psi^*_\zeta(Q_2, Q_1) \tag{1.19a}$$

2. It fulfils the inequality

$$|\psi_\zeta(Q_1, Q_2)|^2 \leq \sigma^2_\zeta(Q_1)\ \sigma^2_\zeta(Q_2) \tag{1.19b}$$

where the variance or statistical dispersion $\sigma^2_\zeta(Q)$ is a real quantity:

$$\sigma^2_\zeta(Q) = \sigma^2_\xi(Q) + \sigma^2_\eta(Q) = \psi_\zeta(Q, Q) \tag{1.20}$$

Introducing the *correlation coefficient*

$$K_\zeta(Q_1, Q_2) = \frac{\psi_\zeta(Q_1, Q_2)}{\sigma_\zeta(Q_1)\ \sigma_\zeta(Q_2)} \tag{1.21}$$

we may use the property (1.19b) to obtain the following property of the correlation coefficient:

$$|K_\zeta(Q_1, Q_2)| \leq 1 \tag{1.19b'}$$

3. For an arbitrary deterministic function $u(Q)$, the following inequality holds:

$$\int \psi_\zeta(Q_1, Q_2)\ u(Q_1)u^*(Q_2)\ \mathrm{d}Q_1\mathrm{d}Q_2 \geq 0 \tag{1.19c}$$

(property of positive definiteness).

1.7 Statistical homogeneity

We have two forms of statistical homogeneity for a random function $\xi(Q)$, viz. in the strict and in the wide or weak sense. The corresponding terms for random processes are *stationarity* in the strict and in the weak sense.

Statistical homogeneity in the strict sense amounts to the property of translational invariance for arbitrary-order PDF's, i.e. for any ΔQ, any $Q_1, ..., Q_n$ and any w_n one has

$$w_n(\xi_1, Q_1, ..., \xi_n, Q_n) = w_n(\xi_1, Q_1 + \Delta Q, ..., \xi_n, Q_n + \Delta Q) \tag{1.22}$$

When (1.22) pertains it is easily shown that (1.12a,b) simplifies to

$$M_1(Q) = \int \xi\, w_1(\xi, 0)\, \mathrm{d}\xi = M_1(0) = \text{const.} \tag{1.23a}$$

$$B_2(Q_1, Q_2) = \int \xi_1 \xi_2\, w_2(\xi_1, 0, \xi_2, Q_2{-}Q_1)\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 = B_2(0, Q_2{-}Q_1) = B_2(Q_2{-}Q_1) \tag{1.23b}$$

Eqs. (1.23a,b) are, in fact, the conditions for *statistical homogeneity in the wider sense*. These conditions can hold even if (1.22) is not known to hold, and hence the conditions (1.23a,b) are weaker than (1.22).

We can now reformulate the properties (1.19a–c) for the correlation function in the case of statistical homogeneity as follows:

1. The Hermiticity gives in this case

$$\psi_\zeta(Q) = \psi^*_\zeta(-Q) \tag{1.24a}$$

This can also be expressed

$$\psi_\zeta(Q) = a_\zeta(Q) + i\, b_\zeta(Q) \tag{1.24a'}$$

$$\psi^*_\zeta(-Q) = a_\zeta(-Q) - i\, b_\zeta(-Q) \tag{1.24a''}$$

where $a_\zeta(Q)$ is an *even function* and $b_\zeta(Q)$ is an *odd function*.

2. The inequality

$$|\psi_\zeta(Q)| \le \sigma^2_\zeta = \psi_\zeta(0) = \text{const.} \tag{1.24b}$$

3. The positive-definiteness is the same as (1.19c) with obvious changes in arguments.

As **illustrations** of statistical homogeneity, let us consider a few examples of random fields, where the correlation function

$$\psi_\xi(\mathbf{r}_1 - \mathbf{r}_2) = \psi_\xi(\mathbf{r}) \tag{1.25}$$

in general depends on all spatial coordinates, $\psi_\xi(x, y, z)$. This function can be *anisotropic*, e.g.

$$\psi_\xi(\mathbf{r}) = \sigma^2_\xi \exp\left(-\frac{x^2}{2a^2} - \frac{y^2}{2b^2} - \frac{z^2}{2c^2}\right) \tag{1.26}$$

with three spatial scales a, b and c along the arbitrarily oriented x-, y- and z-axes. A more general case of this can be written $\psi_\xi(\mathbf{r}) = \psi_\xi(x/a, y/b, z/c)$ which is typical in the case of ionospheric field-aligned irregularities.

As examples of statistically homogeneous *isotropic* random fields,

$$\psi_\xi(\mathbf{r}) = \psi_\xi(r) \tag{1.27}$$

we may consider, e.g.

$$\psi_\xi(r) = \sigma_\xi^2 \exp\left(-\frac{r^2}{2a^2}\right) \tag{1.28}$$

or

$$\psi_\xi(r) = \sigma_\xi^2 \exp\left(-\frac{r}{2a}\right) \tag{1.29}$$

We may define an *effective scale size* ℓ_{ef} through

$$\sigma_\xi^2\, \ell_{ef} = \int_0^\infty \psi_\xi(r)\, \mathrm{d}r \tag{1.30}$$

if this integral converges (convergence is not always the case).

1.8 Spectra of random functions

We can formally write for the random function the Fourier representation

$$\zeta(\mathbf{r}, t) = \iiiint \overline{\zeta}(\boldsymbol{\kappa}, \omega) \exp[+i(\boldsymbol{\kappa} \cdot \mathbf{r} - \omega t)]\, \mathrm{d}\boldsymbol{\kappa}\, \mathrm{d}\omega \tag{1.31a}$$

and its Fourier transform

$$\overline{\zeta}(\boldsymbol{\kappa}, \omega) = \frac{1}{(2\pi)^4} \iiiint \zeta(\mathbf{r}, t) \exp[-i(\boldsymbol{\kappa} \cdot \mathbf{r} - \omega t)]\, \mathrm{d}\mathbf{r}\, \mathrm{d}t \tag{1.31b}$$

In these expressions $\mathbf{r} = \{x, y, z\} = \{x_1, x_2, x_3\}$ and the spatial frequency is $\boldsymbol{\kappa} = \{\kappa_1, \kappa_2, \kappa_3\}$. Elements of volume integration are written in the usual way, i.e. $\mathrm{d}\mathbf{r}$ denotes $\mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$ and $\mathrm{d}\boldsymbol{\kappa}$ denotes $\mathrm{d}\kappa_1\, \mathrm{d}\kappa_2\, \mathrm{d}\kappa_3$. The overlining is used to denote the Fourier conjugate of the random function. We shall in most of the following simplify the notations by writing many-dimensional integrals with only one integral sign, the dimension of an integral being obvious from the element of integration.

However, the integrals (1.31a,b) are in practice not always convergent. For example if $\xi(\mathbf{r}, t)$ is statistically homogeneous, its spectrum is a delta function. In contrast, correlation function spectra are in general convergent in the mean-square sense. Hence the Fourier transform pair involving the first correlation function is in most cases well-defined:

$$\langle\, \psi_\zeta(\mathbf{r}_1, t_1, \mathbf{r}_2, t_2) =$$

$$\int \left\langle \overline{\tilde{\zeta}}(\boldsymbol{\kappa}_1,\omega_1)\, \overline{\tilde{\zeta}}^*(\boldsymbol{\kappa}_2,\omega_2)\right\rangle$$

$$\exp[i(\boldsymbol{\kappa}_1\cdot\mathbf{r}_1-\omega_1 t_1-\boldsymbol{\kappa}_2\cdot\mathbf{r}_2+\omega_2 t_2)]\ \mathrm{d}\boldsymbol{\kappa}_1\mathrm{d}\boldsymbol{\kappa}_2\mathrm{d}\omega_1\mathrm{d}\omega_2 \qquad (1.32a)$$

$$\left\langle \overline{\tilde{\zeta}}(\boldsymbol{\kappa}_1,\omega_1)\, \overline{\tilde{\zeta}}^*(\boldsymbol{\kappa}_2,\omega_2)\right\rangle = \frac{1}{(2\pi)^8}\int \psi_\zeta(\mathbf{r}_1,t_1,\mathbf{r}_2,t_2)$$

$$\exp[-i(\boldsymbol{\kappa}_1\cdot\mathbf{r}_1-\omega_1 t_1-\boldsymbol{\kappa}_2\cdot\mathbf{r}_2+\omega_2 t_2)]\ \mathrm{d}\mathbf{r}_1\mathrm{d}\mathbf{r}_2\mathrm{d}t_1\mathrm{d}t_2 \qquad (1.32b)$$

When speaking about random function spectra, we shall therefore understand the Fourier transform pairs for correlation functions.

1.9 Spatial spectral expansions for homogeneous random fields

When the correlation function for a random field has the simpler form for statistical homogeneity, the transform corresponding to (1.32b) is

$$\left\langle \overline{\tilde{\zeta}}(\boldsymbol{\kappa}_1)\, \overline{\tilde{\zeta}}^*(\boldsymbol{\kappa}_2)\right\rangle = \frac{1}{(2\pi)^6}\int \psi_\zeta(\mathbf{r}_1-\mathbf{r}_2)\exp[-i(\boldsymbol{\kappa}_1\cdot\mathbf{r}_1-\boldsymbol{\kappa}_2\cdot\mathbf{r}_2)]\ \mathrm{d}\mathbf{r}_1\mathrm{d}\mathbf{r}_2 \quad (1.33)$$

Performing in this integral the change of variables

$$\mathbf{r}_1\to\mathbf{r}_1\ , \qquad \mathbf{r}_1-\mathbf{r}_2\to\boldsymbol{\rho} \qquad (1.34)$$

with the Jacobian having a determinant of absolute value unity,

$$\mathrm{d}\mathbf{r}_1\ \mathrm{d}\mathbf{r}_2 = \begin{vmatrix} \dfrac{\partial\mathbf{r}_1}{\partial\mathbf{r}_1} & \dfrac{\partial\mathbf{r}_2}{\partial\mathbf{r}_1} \\ \dfrac{\partial\mathbf{r}_1}{\partial\boldsymbol{\rho}} & \dfrac{\partial\mathbf{r}_2}{\partial\boldsymbol{\rho}} \end{vmatrix} \mathrm{d}\mathbf{r}_1\ \mathrm{d}\boldsymbol{\rho} = \mathrm{d}\mathbf{r}_1\ \mathrm{d}\boldsymbol{\rho} \qquad (1.35)$$

we can simplify the correlation function spectrum (1.33) as follows

$$\left\langle \overline{\tilde{\zeta}}(\boldsymbol{\kappa}_1)\, \overline{\tilde{\zeta}}^*(\boldsymbol{\kappa}_2)\right\rangle = \frac{1}{(2\pi)^3}\int \psi_\zeta(\boldsymbol{\rho})\exp[-i\boldsymbol{\kappa}_2\cdot\boldsymbol{\rho}]\ \mathrm{d}\boldsymbol{\rho}\$$$

$$\frac{1}{(2\pi)^3}\int \exp[-i(\boldsymbol{\kappa}_1-\boldsymbol{\kappa}_2)\cdot\mathbf{r}_1]\ \mathrm{d}\mathbf{r}_1 = \phi_\zeta(\boldsymbol{\kappa}_2)\ \delta(\boldsymbol{\kappa}_1-\boldsymbol{\kappa}_2) \qquad (1.36)$$

where $\delta(\boldsymbol{\kappa})$ is the Dirac delta function of a vector argument

$$\delta(\boldsymbol{\kappa}) = \frac{1}{(2\pi)^3}\int \exp[-i\boldsymbol{\kappa}\cdot\mathbf{r}]\ \mathrm{d}\mathbf{r} \qquad (1.37)$$

and

$$\phi_\zeta(\boldsymbol{\kappa}) = \frac{1}{(2\pi)^3}\int \psi_\zeta(\boldsymbol{\rho})\exp[-i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}]\ \mathrm{d}\boldsymbol{\rho} \qquad (1.38b)$$

The inverse of (1.38b) is

$$\psi_\zeta(\boldsymbol{\rho}) = \int \phi_\zeta(\boldsymbol{\kappa}) \exp[+i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}]\ \mathrm{d}\boldsymbol{\kappa} \tag{1.38a}$$

Taking the complex conjugate of (1.38b), changing then the sign of the variable, $\boldsymbol{\rho} \to -\boldsymbol{\rho}$, and using finally the property (1.24a), we find

$$\phi_\zeta(\boldsymbol{\kappa}) = \phi_\zeta^*(\boldsymbol{\kappa}) \tag{1.39}$$

i.e. $\phi_\zeta(\boldsymbol{\kappa})$ is a real function.

An interesting *special case* is when also $\psi_\zeta(\mathbf{r})$ is *real.* Then (1.38a,b) can be evaluated with the cosine transform:

$$\psi_\zeta(\mathbf{r}) = \int \phi_\zeta(\boldsymbol{\kappa}) \cos(\boldsymbol{\kappa}\cdot\mathbf{r})\ \mathrm{d}\boldsymbol{\kappa} \tag{1.40a}$$

$$\phi_\zeta(\boldsymbol{\kappa}) = \frac{1}{(2\pi)^3} \int \psi_\zeta(\boldsymbol{\rho}) \cos(\boldsymbol{\kappa}\cdot\boldsymbol{\rho})\ \mathrm{d}\boldsymbol{\rho} \tag{1.40b}$$

Hence $\psi_\zeta(\mathbf{r})$ and $\phi_\zeta(\boldsymbol{\kappa})$ are then even functions of their arguments.

As another *special case* we shall also consider *the isotropic case* $\psi_\zeta(\mathbf{r}) = \psi_\zeta(r)$. In spherical coordinates (r, θ, φ) with $\boldsymbol{\kappa}$ as polar axis we then have $\mathrm{d}\mathbf{r} = r^2 \sin\theta\ \mathrm{d}r\ \mathrm{d}\theta\, \mathrm{d}\varphi$. Then (1.38b) yields

$$\phi_\zeta(\kappa) = \frac{1}{2\pi^2\kappa} \int_0^\infty r\ \psi_\zeta(r) \sin(\kappa r)\ \mathrm{d}r \tag{1.41b}$$

This expression involves the sine transform, and hence, by using the formula for the inverse sine transform, we find

$$\psi_\zeta(r) = \frac{4\pi}{r} \int_0^\infty \kappa\ \phi_\zeta(\kappa) \sin(\kappa r)\ \mathrm{d}\kappa \tag{1.41a}$$

We shall conclude this section by pointing out a consequence of the fact that the arguments of the correlation functions must be dimensionless. If the typical characteristic scale in ordinary space is ℓ_ζ, then the spatial spectrum has the typical scale size $\kappa_\zeta \sim \ell_\zeta^{-1}$, as can be seen from

$$\phi_\zeta(\boldsymbol{\kappa}) = \frac{1}{(2\pi)^3} \int \psi_\zeta\left(\frac{\mathbf{r}}{\ell_\zeta}\right) \exp\left[-i\boldsymbol{\kappa}\,\ell_\zeta \cdot \frac{\mathbf{r}}{\ell_\zeta}\right]\ \mathrm{d}\mathbf{r} =$$
$$\left(\frac{\ell_\zeta}{2\pi}\right)^3 \int \psi_\zeta(\boldsymbol{\alpha}) \exp[-i\boldsymbol{\kappa}\,\ell_\zeta\cdot\boldsymbol{\alpha}]\ \mathrm{d}\boldsymbol{\alpha} \sim \phi_\zeta(\boldsymbol{\kappa}\ell_\zeta) \tag{1.42}$$

so that $\kappa_\zeta\ \ell_\zeta \sim 1$. The quantity $\boldsymbol{\alpha}$ is the dimensionless space variable $\mathbf{r}/\ell_\zeta$.

1.10 Spatial and frequency expansions

When the correlation function is homogeneous in time as well as in position, $\psi_\zeta(\mathbf{r}_1 - \mathbf{r}_2, t_1 - t_2)$, we find as a natural generalization of the spatially homogeneous case the correlation function spectrum

$$\left\langle \bar{\bar{\zeta}}(\boldsymbol{\kappa}_1, \omega_1)\, \bar{\bar{\zeta}}^*(\boldsymbol{\kappa}_2, \omega_2) \right\rangle = \phi_\zeta(\boldsymbol{\kappa}_2, \omega_2)\, \delta(\boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_2)\, \delta(\omega_1 - \omega_2) \tag{1.43}$$

where

$$\phi_\zeta(\boldsymbol{\kappa}, \omega) = \frac{1}{(2\pi)^4} \int \psi_\zeta(\mathbf{r}, t) \exp[-i(\boldsymbol{\kappa} \cdot \mathbf{r} - \omega t)]\; \mathrm{d}\mathbf{r}\; \mathrm{d}t \tag{1.44}$$

Integrating this over frequency and using the δ-function (1.37), we find the *pure spatial spectrum* of the homogeneous correlation function:

$$\phi_\zeta^{sp}(\boldsymbol{\kappa}) = \int \phi_\zeta(\boldsymbol{\kappa}, \omega)\; \mathrm{d}\omega = \frac{1}{(2\pi)^3} \int \psi_\zeta(\mathbf{r}, 0) \exp[-i\boldsymbol{\kappa} \cdot \mathbf{r}]\; \mathrm{d}\mathbf{r} \tag{1.45}$$

Integrating over the spatial variable we find, instead, the *pure frequency spectrum*:

$$\phi_\zeta^{fr}(\omega) = \int \phi_\zeta(\boldsymbol{\kappa}, \omega)\; \mathrm{d}\boldsymbol{\kappa} = \frac{1}{2\pi} \int \psi_\zeta(0, t) \exp[+i\omega t]\; \mathrm{d}t \tag{1.46}$$

1.11 The model of frozen drift

This model is useful when there is a macroscopic velocity $\mathbf{v}$ involved, which dominates over the internal isotropically fluctuating velocities of the medium as for, e.g. the transionospheric signal from a satellite moving with high speed. Then the correlation function can be written

$$\psi_\zeta(\mathbf{r}, t) = \psi_\zeta(\mathbf{r} - \mathbf{v}t) \tag{1.47}$$

The spectrum of this function is expressed by (1.44). Introducing there the variable transformation

$$t \to t\ , \qquad \mathbf{r} - \mathbf{v}\, t \to \boldsymbol{\rho} \tag{1.48}$$

and noting that the determinant modulus of the Jacobian here, as with (1.34), is unity, we obtain the correlation function spectrum

$$\phi_\zeta(\boldsymbol{\kappa}, \omega) = \frac{1}{(2\pi)^4} \int \psi_\zeta(\boldsymbol{\rho}) \exp[-i(\boldsymbol{\kappa} \cdot \boldsymbol{\rho} + \boldsymbol{\kappa} \cdot \mathbf{v}\, t - \omega t)]\; \mathrm{d}\boldsymbol{\rho}\; \mathrm{d}t = \phi_\zeta(\boldsymbol{\kappa})\; \delta(\omega - \boldsymbol{\kappa} \cdot \mathbf{v}) \tag{1.49}$$

with $\phi_\zeta(\boldsymbol{\kappa})$ given by (1.38b).

To express the pure frequency spectrum (1.46), which now is

$$\phi_\zeta^{fr}(\omega) = \int \phi_\zeta(\boldsymbol{\kappa})\; \delta(\omega - \boldsymbol{\kappa} \cdot \mathbf{v})\; \mathrm{d}\boldsymbol{\kappa} \tag{1.50}$$

we split $\boldsymbol{\kappa}$ into its parallel and perpendicular components:

$$\boldsymbol{\kappa} = \{\kappa_{\parallel}, \boldsymbol{\kappa}_{\perp}\} \; ; \qquad \text{where} \quad \kappa_{\parallel} = \boldsymbol{\kappa} \cdot \mathbf{v}/v \; , \quad \boldsymbol{\kappa}_{\perp} = \boldsymbol{\kappa} - \kappa_{\parallel}\mathbf{v}/v \tag{1.51}$$

Hence we obtain the result

$$\phi_{\zeta}^{fr}(\omega) = \int \phi_{\zeta}(\kappa_{\parallel}, \boldsymbol{\kappa}_{\perp}) \; \delta(\omega - \kappa_{\parallel} v) \; \mathrm{d}\kappa_{\parallel} \; \mathrm{d}\boldsymbol{\kappa}_{\perp} = \tfrac{1}{v} \int \phi_{\zeta} \left(\frac{\omega}{v}, \boldsymbol{\kappa}_{\perp} \right) \; \mathrm{d}\boldsymbol{\kappa}_{\perp} \tag{1.52}$$

1.12 Quasi-homogeneous fields

To define the concept of quasi-homogeneity we consider again the general spatial correlation function

$$\psi_{\zeta}(\mathbf{r}_1, \mathbf{r}_2) = \sigma_{\zeta}(\mathbf{r}_1) \; \sigma_{\zeta}(\mathbf{r}_2) \; K_{\zeta}(\mathbf{r}_1, \mathbf{r}_2) \tag{1.53}$$

In the special case of statistical homogeneity (1.53) can be written

$$\psi_{\zeta}(\mathbf{r}_1 - \mathbf{r}_2) = \sigma_{\zeta}^2 \; K_{\zeta}(\mathbf{r}_1 - \mathbf{r}_2) \tag{1.54}$$

If, on the other hand, we introduce the new spatial variables:

$$\mathbf{R} = \tfrac{1}{2} \; (\mathbf{r}_1 + \mathbf{r}_2) \; , \qquad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \tag{1.55}$$

the general correlation function (1.53) can be written

$$\psi_{\zeta}(\mathbf{R}, \mathbf{r}) = \sigma_{\zeta} \left(\mathbf{R} + \tfrac{1}{2}\mathbf{r} \right) \; \sigma_{\zeta} \left(\mathbf{R} - \tfrac{1}{2}\mathbf{r} \right) \; K_{\zeta}(\mathbf{R}, \mathbf{r}) \tag{1.56}$$

Quasi-homogeneity implies that two spatial scales are involved, one scale ℓ_{ef} for the relative variable $\mathbf{r}$ and another L_{ef} for the central variable $\mathbf{R}$, and that

$$\ell_{ef} \ll L_{ef} \tag{1.57}$$

Homogeneity, in particular, is the limiting case when $L_{ef} = \infty$ and then ℓ_{ef} is the only spatial scale.

In the quasi-homogeneous case we may approximate (1.56) as follows:

$$\psi_{\zeta}(\mathbf{R}, \mathbf{r}) = \sigma_{\zeta}^2(\mathbf{R}) \; K_{\zeta}(\mathbf{R}, \mathbf{r}) \tag{1.58}$$

The Fourier transform pair in the fast variable corresponding to (1.38a,b) is now

$$\psi_{\zeta}(\mathbf{R}, \mathbf{r}) = \int \phi_{\zeta}(\mathbf{R}, \boldsymbol{\kappa}) \exp[+i\boldsymbol{\kappa} \cdot \mathbf{r}] \; \mathrm{d}\boldsymbol{\kappa} \tag{1.59a}$$

$$\phi_{\zeta}(\mathbf{R}, \boldsymbol{\kappa}) = \frac{1}{(2\pi)^3} \int \psi_{\zeta}(\mathbf{R}, \mathbf{r}) \exp[-i\boldsymbol{\kappa} \cdot \mathbf{r}] \; \mathrm{d}\mathbf{r} \tag{1.59b}$$

The function $\phi_{\zeta}(\mathbf{R}, \boldsymbol{\kappa})$ signifies a *local spectrum*, which depends slowly on $\mathbf{R}$.

1.13 The concept of ergodicity

The theoretical introduction in this Chapter is based upon averaging over statistical ensembles, expressed by the angular brackets $\langle \ \rangle$. Experimentally we are confined to measurements on a concrete realization of the ensemble. Then the averages of any deterministic function $F[\xi(\mathbf{r},t)]$, e.g. the mean value, must be formed by time averaging

$$\overline{F[\xi(\mathbf{r},t)]} = \tfrac{1}{T} \int\limits_0^T F[\xi(\mathbf{r},t)] \; \mathrm{d}t \tag{1.60}$$

The important *principle of ergodicity* implies that ensemble averages are equal to time averages as $T \to \infty$, i.e.

$$\lim_{T\to\infty} \overline{F[\xi(\mathbf{r},t)]} = \langle F[\xi(\mathbf{r},t)]\rangle \tag{1.61}$$

A necessary condition for a stationary process to be ergodic is that

$$\lim_{T\to\infty} \tfrac{1}{T} \int\limits_0^T \psi_F(t) \; \mathrm{d}t = 0 \tag{1.62}$$

1.14 The structure functions and random fields with stationary increments

The correlation function for the real random field $\xi(\mathbf{r})$ is according to (1.18) given by

$$\psi_\xi(\mathbf{r}_1,\mathbf{r}_2) = \left\langle \tilde{\xi}(\mathbf{r}_1)\; \tilde{\xi}(\mathbf{r}_2)\right\rangle \tag{1.63}$$

Another commonly used second-order moment is the *structure function*, which in the *general case* is defined by

$$D_\xi(\mathbf{r}_1,\mathbf{r}_2) = \left\langle [\tilde{\xi}(\mathbf{r}_1) - \tilde{\xi}(\mathbf{r}_2)]^2\right\rangle = \left\langle \tilde{\xi}^2(\mathbf{r}_1)\right\rangle + \left\langle \tilde{\xi}^2(\mathbf{r}_2)\right\rangle - 2\left\langle \tilde{\xi}(\mathbf{r}_1)\; \tilde{\xi}(\mathbf{r}_2)\right\rangle \tag{1.64}$$

In the situation of *statistical homogeneity* (Section 1.7) the structure function depends only on the difference variable $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. From (1.64) we easily find

$$D_\xi(\mathbf{r}) = 2\;[\sigma_\xi^2 - \psi_\xi(\mathbf{r})] = 2\;[\psi_\xi(0) - \psi_\xi(\mathbf{r})] \tag{1.65}$$

Since the random functions considered in this Section are real, the Fourier transform pair $\psi_\xi(\mathbf{r})$, $\phi_\xi(\boldsymbol{\kappa})$ can be expressed by the cosine transform, (1.40a,b). It is interesting to obtain a transform pair relating $D_\xi(\mathbf{r})$ and $\phi_\xi(\boldsymbol{\kappa})$ to each other. Using (1.40a), we can express (1.65) as follows:

$$D_\xi(\mathbf{r}) = 2 \int \phi_\xi(\boldsymbol{\kappa})\; [1 - \cos(\boldsymbol{\kappa}\cdot\mathbf{r})]\; \mathrm{d}\boldsymbol{\kappa} \tag{1.66a}$$

The gradient of this formula is

$$\nabla D_\xi(\mathbf{r}) = 2\int \boldsymbol{\kappa}_1\, \phi_\xi(\boldsymbol{\kappa}_1)\ \sin(\boldsymbol{\kappa}_1\cdot\mathbf{r})]\ \mathrm{d}\boldsymbol{\kappa}_1 \tag{1.67}$$

Multiplying on both sides by $\exp[-i\boldsymbol{\kappa}\cdot\mathbf{r}]$, expressing the sine with exponential functions and integrating over $\mathbf{r}$, we obtain as the next step

$$\int [\nabla D_\xi(\mathbf{r})]\exp[-i\boldsymbol{\kappa}\cdot\mathbf{r}]\ \mathrm{d}\mathbf{r} =$$

$$-i\int \boldsymbol{\kappa}_1\, \phi_\xi(\boldsymbol{\kappa}_1)\ \{\exp[+i(\boldsymbol{\kappa}_1-\boldsymbol{\kappa})\cdot\mathbf{r}] - \exp[-i(\boldsymbol{\kappa}_1+\boldsymbol{\kappa})\cdot\mathbf{r}]\}\ \mathrm{d}\boldsymbol{\kappa}_1\mathrm{d}\mathbf{r} =$$

$$-i\ (2\pi)^3\int \boldsymbol{\kappa}_1\, \phi_\xi(\boldsymbol{\kappa}_1)\ [\delta(\boldsymbol{\kappa}_1-\boldsymbol{\kappa}) - \delta(\boldsymbol{\kappa}_1+\boldsymbol{\kappa})]\ \mathrm{d}\boldsymbol{\kappa}_1 = -2i\ (2\pi)^3\boldsymbol{\kappa}\, \phi_\xi(\boldsymbol{\kappa}) \tag{1.68}$$

After scalar multiplication with $\boldsymbol{\kappa}$ in the above, we may write the result

$$\phi_\xi(\boldsymbol{\kappa}) = \frac{i}{2(2\pi)^3\kappa^2}\int \boldsymbol{\kappa}\cdot\nabla D_\xi(\mathbf{r})\exp[-i\boldsymbol{\kappa}\cdot\mathbf{r}]\ \mathrm{d}\mathbf{r} \tag{1.69}$$

Since ψ_ξ is an even function, ∇D_ξ must be odd, and hence we arrive at the following form:

$$\phi_\xi(\boldsymbol{\kappa}) = \frac{1}{16\pi^3\kappa^2}\int \boldsymbol{\kappa}\cdot\nabla D_\xi(\mathbf{r})\sin\boldsymbol{\kappa}\cdot\mathbf{r}\ \mathrm{d}\mathbf{r} \tag{1.66b}$$

which we shall adopt as the inverse of (1.66a). We point out that when $\phi_\xi(\boldsymbol{\kappa})$ has a singularity $\sim 1/\kappa^\alpha$ as $\kappa \to 0$, it may happen that $D_\xi(\mathbf{r})$ exists but $\psi_\xi(\mathbf{r})$ cannot be constructed because of the singularity; cf. (1.40a) and (1.66a). For the convergence of (1.40a) the singularity of $\phi_\xi(\boldsymbol{\kappa})$ must fulfil $\alpha < 3$, but for the convergence of (1.66a) it is sufficient that $\alpha < 5$.

Eqs. (1.66a,b) have been obtained formally for the case of statistical homogeneity, when

$$\psi_\xi(\mathbf{r}_1,\mathbf{r}_2) = \psi_\xi(\mathbf{r}_1-\mathbf{r}_2) \tag{1.70}$$

It can be shown that they remain valid for the more general case when the structure function D_ξ obeys the corresponding relation

$$D_\xi(\mathbf{r}_1,\mathbf{r}_2) = D_\xi(\mathbf{r}_1-\mathbf{r}_2) \tag{1.71}$$

but nothing is stated about ψ_ξ. We remark that a necessary condition for (1.71) is that

$$\langle\xi(\mathbf{r}_1)\rangle - \langle\xi(\mathbf{r}_2)\rangle = a\ (\mathbf{r}_1-\mathbf{r}_2) \tag{1.72}$$

where a is a constant. This means that $\xi(\mathbf{r})$ has a stationary increment, so (1.71) is the definition of the *random fields with stationary increments.*

Chapter 2

Statement of the problems in the statistical theory of wave propagation

2.1 Stationary and quasi-stationary forms of Maxwell's equations

We shall denote the arbitrary time-dependent fields by calligraphic letters; $\boldsymbol{\mathcal{E}}(\mathbf{r},t)$, $\boldsymbol{\mathcal{H}}(\mathbf{r},t)$, $\boldsymbol{\mathcal{D}}(\mathbf{r},t)$ and $\boldsymbol{\mathcal{B}}(\mathbf{r},t) = \mu_0\,\boldsymbol{\mathcal{H}}(\mathbf{r},t)$.

Maxwell's equations for the electromagnetic field (without outer sources, currents and magnetic polarization) are

$$\nabla\cdot\boldsymbol{\mathcal{D}} = 0 \tag{2.1a}$$

$$\nabla\cdot\boldsymbol{\mathcal{B}} = 0 \tag{2.1b}$$

$$\nabla\times\boldsymbol{\mathcal{H}} = \frac{\partial\boldsymbol{\mathcal{D}}}{\partial t} \tag{2.1c}$$

$$\nabla\times\boldsymbol{\mathcal{E}} = -\mu_0\,\frac{\partial\boldsymbol{\mathcal{H}}}{\partial t} \tag{2.1d}$$

Eliminating $\boldsymbol{\mathcal{H}}$ between (2.1c) and (2.1d) and using a well-known formula from vector analysis, we easily obtain the wave equation

$$\nabla\,\nabla\cdot\boldsymbol{\mathcal{E}} - \nabla^2\,\boldsymbol{\mathcal{E}} = -\mu_0\,\frac{\partial^2\boldsymbol{\mathcal{D}}}{\partial t^2} \tag{2.2}$$

To develop this equation further, we need the constitutive relation between $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{D}}$. In its most general form for a cold plasma, involving temporal dispersion but being local in space, it is given by the linear functional

$$\boldsymbol{\mathcal{D}}(\mathbf{r},t) = \int\limits_{-\infty}^{t} \varepsilon(\mathbf{r},t,t')\,\boldsymbol{\mathcal{E}}(\mathbf{r},t')\,\mathrm{d}t' \tag{2.3}$$

For a nondispersive medium we have locality also in time, and the permittivity is given by the expression

$$\varepsilon(\mathbf{r},t,t') = \varepsilon[\mathbf{r},\tfrac{1}{2}(t+t')]\;\delta(t-t') = \varepsilon(\mathbf{r},t)\;\delta(t-t') \tag{2.4}$$

so that (2.3) simply becomes

$$\boldsymbol{\mathcal{D}}(\mathbf{r},t) = \varepsilon(\mathbf{r},t)\;\boldsymbol{\mathcal{E}}(\mathbf{r},t) \tag{2.5}$$

This form is, however, not applicable to a plasma.

The fields of a *harmonic process* are characterized by the simple time-dependence

$$\boldsymbol{\mathcal{E}}(\mathbf{r},t) = \mathbf{E}(\mathbf{r})\;e^{-i\omega t} \tag{2.6}$$

Here $\mathbf{E}$ is the complex amplitude of the field. For such fields time derivatives can be obtained by replacement with a simple multiplication:

$$\frac{\partial}{\partial t} \to -i\,\omega \tag{2.7}$$

However, in *quasi-harmonic processes*, where the medium is slowly changing its properties with time, we have time-dependent amplitudes, i.e. instead of (2.6) we have to work with

$$\boldsymbol{\mathcal{E}}(\mathbf{r},t,\alpha t) = \mathbf{E}(\mathbf{r},\alpha t)\;e^{-i\omega t} \tag{2.8}$$

where the slowness of the time-dependence in $\mathbf{E}$ is indicated by the smallness of the formal parameter α. The rule (2.7) is still valid for fields like (2.8) if $\alpha \ll \omega$, i.e. if

$$\omega T \gg 1 \tag{2.9}$$

where $T = \alpha^{-1}$ is the characteristic time-scale of the slow changes of the medium.

For a *stationary plasma* the permittivity depends only on the difference between the time arguments and the constitutive relation (2.3) then takes the form

$$\boldsymbol{\mathcal{D}}(\mathbf{r},t) = \int_{-\infty}^{t} \varepsilon(\mathbf{r},t-t')\;\boldsymbol{\mathcal{E}}(\mathbf{r},t')\;\mathrm{d}t' \tag{2.10}$$

For a harmonic process in this dispersive medium the fields are

$$\boldsymbol{\mathcal{E}}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},\omega)\;e^{-i\omega t} \tag{2.11a}$$

$$\boldsymbol{\mathcal{D}}(\mathbf{r},t) = \mathbf{D}(\mathbf{r},\omega)\;e^{-i\omega t} \tag{2.11b}$$

Using (2.10), one finds then the following relation between the amplitudes

$$\mathbf{D}(\mathbf{r},\omega) = \tilde{\varepsilon}(\mathbf{r},\omega)\;\mathbf{E}(\mathbf{r},\omega) \tag{2.12}$$

where $\tilde{\varepsilon}(\mathbf{r},\omega)$, i.e. the dielectric permittivity of the frequency component ω, is the Fourier transform of the permittivity function $\varepsilon(\mathbf{r},t-t')$. For causality reasons we then put

$$\varepsilon(\mathbf{r},t-t') = \begin{cases} \varepsilon(\mathbf{r},t-t') \ , & t' < t \\ 0 \ , & t' > t \end{cases} \tag{2.13}$$

In order to simplify our notations we shall work with the *relative dielectric permittivity* $\epsilon(\mathbf{r},\omega)$, defined through [cf. (1.2)]

$$\tilde{\varepsilon}(\mathbf{r},\omega) = \varepsilon_0 \ \epsilon(\mathbf{r},\omega) \tag{2.14}$$

We now put (2.12) and (2.14) into (2.2) and use also (2.7) to obtain the wave equation

$$\nabla\,\nabla\cdot\mathbf{E} - \nabla^2\,\mathbf{E} = \mu_0\varepsilon_0\omega^2 \ \epsilon(\mathbf{r},\omega) \ \mathbf{E} \tag{2.15}$$

We rewrite the factor in front of ϵ by means of the vacuum wave number k as follows:

$$\mu_0\varepsilon_0\omega^2 = \frac{\omega^2}{c^2} = k^2 \tag{2.16}$$

The wave equation, valid for a Fourier component of the field in the stationary plasma, may hence be written

$$\nabla\,\nabla\cdot\mathbf{E} - \nabla^2\,\mathbf{E} - k^2 \ \epsilon(\mathbf{r},\omega) \ \mathbf{E} = 0 \tag{2.15$'$}$$

The general case of a non-stationary plasma with constitutive relation of type (2.3) is much more difficult. We shall treat this case in the *quasi-harmonic approximation*, i.e. we assume that the non-stationary time dependence of ε is slow enough so that we can write the relation (2.3) on the form

$$\boldsymbol{\mathcal{D}}(\mathbf{r},t) = \int\limits_{-\infty}^{+\infty} \varepsilon[\mathbf{r},t-t',\tfrac{1}{2}\alpha(t+t')] \ \boldsymbol{\mathcal{E}}(\mathbf{r},t') \ \mathrm{d}t' \tag{2.17}$$

Consequently we shall introduce a slow time dependence also into the fields [cf. (2.11a,b)]

$$\boldsymbol{\mathcal{E}}(\mathbf{r},t,\alpha t) = \mathbf{E}(\mathbf{r},\omega,\alpha t) \ e^{-i\omega t} \tag{2.18a}$$

$$\boldsymbol{\mathcal{D}}(\mathbf{r},t,\alpha t) = \mathbf{D}(\mathbf{r},\omega,\alpha t) \ e^{-i\omega t} \tag{2.18b}$$

With (2.18a) and the introduction of a new integration variable, $t-t' \to \tau$, (2.17) can be written

$$\boldsymbol{\mathcal{D}}(\mathbf{r},t) = \int\limits_{-\infty}^{+\infty} \varepsilon[\mathbf{r},\tau,\alpha(t-\tfrac{1}{2}\tau)] \ \mathbf{E}[\mathbf{r},\omega,\alpha(t-\tau)] \ \exp[+i\omega(\tau-t)] \ \mathrm{d}\tau \tag{2.19}$$

Neglecting τ beside t in the slow argument of $\mathbf{E}$ and ε, we get

$$\boldsymbol{\mathcal{D}}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},\omega,\alpha t) \ e^{-i\omega t} \int\limits_{-\infty}^{+\infty} \varepsilon(\mathbf{r},\tau,\alpha t) \ e^{i\omega\tau} \ \mathrm{d}\tau \tag{2.20}$$

Hence we find in the slowly non-stationary case the same constitutive relation (2.12) as for the strictly stationary case, i.e.

$$\mathbf{D}(\mathbf{r},\omega,\alpha t) = \tilde{\varepsilon}(\mathbf{r},\omega,\alpha t)\ \mathbf{E}(\mathbf{r},\omega,\alpha t) \tag{2.21}$$

where $\tilde{\varepsilon}(\mathbf{r},\omega,\alpha t)$ is the Fourier transform of ε over the fast time variable. This kind of transform is justified if the time-scale T of change of the medium introduced earlier is much larger than some characteristic internal time-scale t_0 of the dispersive plasma, i.e. $T \gg t_0$.

What is this characteristic time t_0 ? We have already introduced the condition (2.9) justifying the rule (2.7) for replacing time derivatives by multiplication. This condition is, however, not sufficient here. Instead we consider the relative permittivity for a cold plasma with collision frequency ν :

$$\epsilon(\omega) = 1 - \frac{e^2\ N(\mathbf{r})}{m\varepsilon_0\omega^2(1+i\nu/\omega)} \tag{2.22}$$

We shall use this expression to get a qualitative estimate of the response of the plasma to an impulse (δ-function, which has spectrum of amplitude unity):

$$\varepsilon(\tau) = \int\limits_{-\infty}^{+\infty} \varepsilon_0\epsilon(\omega)\ e^{-i\omega\tau}\ \mathrm{d}\omega \tag{2.23}$$

We note that the integrand through (2.22) has two poles, one at $\omega = 0$ and the other at $\omega = -i\nu$. To get the physically acceptable solution the integration must pass below the pole at the origin of the complex ω-plane. We may then close the path of integration across $\omega \to -i\infty$, so that it circumvents the other pole at $\omega = -i\nu$. Residue evaluation then shows that $\varepsilon(\tau) \sim e^{-\nu\tau}$, i.e. the plasma response time-scale is $t_0 \sim \nu^{-1}$. Hence we conclude that the condition of validity of the quasi-stationary constitutive relation (2.21) is

$$\nu T \gg 1 \tag{2.24}$$

This is a much stronger condition than (2.9) and it must be valid also at the peak of the ray where ν has its minimum. When (2.24) is fufilled we have, in consequence of (2.21), the same wave equation (2.15) as in the stationary case, with the relative dielectric permittivity now slowly time-dependent, $\epsilon(\mathbf{r},\omega,\alpha t)$.

We have still not made use of the first Maxwell equation (2.1a) in our wave equation. With (2.11a,b), (2.12) and (2.14), eq. (2.1a) takes the form

$$\epsilon\ \nabla\cdot\mathbf{E} + \nabla\epsilon\cdot\mathbf{E} = 0 \tag{2.25}$$

By means of this relation, eq. (2.15′) can be written

$$\nabla^2\,\mathbf{E} + \nabla\ \left(\frac{\nabla\epsilon}{\epsilon}\cdot\mathbf{E}\right) + k^2\ \epsilon\ \mathbf{E} = 0 \tag{2.26}$$

To get a final formula we shall further split the dielectric permittivity into a static regular background part and the superimposed quasi-stationary fluctuations:

$$\epsilon = \epsilon_0(\mathbf{r}, \omega) + \epsilon(\mathbf{r}, \omega, t) \tag{2.27}$$

Hence we arrive at

$$\nabla^2 \mathbf{E} + \nabla \left(\frac{\nabla\epsilon_0 + \nabla\epsilon}{\epsilon_0 + \epsilon} \cdot \mathbf{E} \right) + k^2 \, (\epsilon_0 + \epsilon) \, \mathbf{E} = 0 \tag{2.28}$$

which is our main stochastic wave equation. The middle term in this equation is the depolarization term. It can be neglected in many cases so that we have a simpler form of (2.28):

$$\nabla^2 \mathbf{E} + k^2 \, (\epsilon_0 + \epsilon) \, \mathbf{E} = 0 \tag{2.29}$$

This is essentially (1.1).

2.2 Two approaches in the statistical theory of wave propagation

2.2.1 Construction of the solution for a particular realization

The first, and in this report the major, approach for obtaining a solution of the wave propagation problem is the direct solving of the wave equation for a realization of the fluctuations. Different approaches to this are employed for different regimes in Chapters 3–6. In one particular case only, can we give an exact representation of the field. This is the case treated in Chapter 3, when the wave is propagating in a regular homogeneous half-space, bounded by an infinite random screen where the initial stochastic field is given at the boundary.

When the field has been constructed for a realization, moments of the field can be obtained by averaging.

2.2.2 Direct construction of the equations for the moments

Another approach is to construct equations for the propagation of the moments, i.e. wave equations directly governing quantities such as $\langle \mathbf{E} \rangle$, $\langle \mathbf{E} \cdot \mathbf{E}^* \rangle$ etc. For example, from (2.29) we directly can write down the equation for the mean field:

$$\nabla^2 \, \langle \mathbf{E} \rangle + k^2 \, \epsilon_0 \, \langle \mathbf{E} \rangle + k^2 \, \langle \epsilon \, \mathbf{E} \rangle = 0 \tag{2.30}$$

We see that in this equation there appears a new unknown quantity, the moment $\langle \epsilon \, \mathbf{E} \rangle$, which cannot be split into the component parts of its argument. In order to treat this kind of equation we must therefore find a way to handle such quantities. This is difficult in general, but can be done in some special cases. It is the subject of the later chapters of these notes.

2.3 Plane waves in homogeneous media

The Helmholtz' equation (2.29) for a component of the electric field in the regular homogeneous medium (ϵ_0 = constant) is given by

$$\nabla^2 E + k^2 \, \epsilon_0 \, E = 0 \tag{2.31}$$

Substituting a solution on the form of an exponential function $\exp(i\boldsymbol{\kappa} \cdot \mathbf{r})$ into (2.31), we find that the spatial frequency $\boldsymbol{\kappa}$ must fulfil a dispersion relation of the form $\kappa^2 - k^2 \, \epsilon_0 = 0$. Hence the solutions can be written

$$E = A \ \exp(i\boldsymbol{\kappa} \cdot \mathbf{r}) = A \ \exp(i\kappa \boldsymbol{\ell} \cdot \mathbf{r}) = A \exp[i\kappa(\ell_x x + \ell_y y + \ell_z z)] \tag{2.32}$$

with the wave number

$$\kappa = k\sqrt{\epsilon_0} \tag{2.33a}$$

the wave vector

$$\boldsymbol{\kappa} = \kappa \, \boldsymbol{\ell} \tag{2.33b}$$

and

$$\boldsymbol{\ell} = \{\ell_x, \ell_y, \ell_z\} \tag{2.33c}$$

being the unit vector in the direction of propagation: $|\boldsymbol{\ell}| = 1$, $\quad \ell_x^2 + \ell_y^2 + \ell_z^2 = 1$. We also remark that the wavelength is $\lambda = 2\pi/\kappa$.

2.4 Plane wave expansion of a spherical wave

The Green's function for the wave equation in vacuum is the solution of the wave equation for a point source in an arbitrary point $\mathbf{r}_0$:

$$\nabla^2 G + k^2 \, G = \delta(\mathbf{r} - \mathbf{r}_0) = \delta(x - x_0) \, \delta(y - y_0) \, \delta(z - z_0) \tag{2.34}$$

with the necessary boundary condition at infinity. With the time-dependence on the form $\exp(-i\omega t)$, outgoing spherical waves increase their phase with the distance from the source. Since the Green's function shall be a purely outgoing wave, we then have the well-known solution

$$G(\mathbf{r} - \mathbf{r}_0) = -\frac{1}{4\pi} \, \frac{\exp(ik|\mathbf{r} - \mathbf{r}_0|)}{|\mathbf{r} - \mathbf{r}_0|} \tag{2.35}$$

We want to obtain an expansion of this solution in terms of plane waves (2.32). To this end we attempt a solution on the form

$$G(\mathbf{r} - \mathbf{r}_0) = \iint g(z, \alpha, \beta) \exp[i(\alpha x + \beta y)] \; \mathrm{d}\alpha \, \mathrm{d}\beta \tag{2.36}$$

When we introduce this expansion into (2.34), we also replace the δ-functions of x and y by their Fourier integrals:

$$\delta(x - x_0) = \frac{1}{2\pi} \int \exp[i\alpha(x - x_0)] \; \mathrm{d}\alpha \tag{2.37a}$$

$$\delta(y-y_0) = \frac{1}{2\pi}\int \exp[i\beta(y-y_0)]\ \mathrm{d}\beta \tag{2.37b}$$

In this way we get the following ordinary differential equation for the function g:

$$\frac{\mathrm{d}^2 g}{\mathrm{d}z^2} + \left(k^2-\alpha^2-\beta^2\right)\ g = \frac{1}{4\pi^2}\ \exp[-i(\alpha x_0+\beta y_0)]\ \delta(z-z_0) \tag{2.38}$$

The solution of this equation which is continuous at z_0 and the derivative of which has a step equal to the coefficient of the δ-function at z_0, is given by

$$g(z,z_0) = \frac{\exp[-i(\alpha x_0+\beta y_0)]}{4\pi^2\ 2i\ \sqrt{k^2-\alpha^2-\beta^2}}\ \exp\left[+i\ \sqrt{k^2-\alpha^2-\beta^2}\ (z-z_0)\right]\ , \qquad z \geq z_0 \tag{2.39a}$$

$$g(z,z_0) = \frac{\exp[-i(\alpha x_0+\beta y_0)]}{4\pi^2\ 2i\ \sqrt{k^2-\alpha^2-\beta^2}}\ \exp\left[-i\ \sqrt{k^2-\alpha^2-\beta^2}\ (z-z_0)\right]\ , \qquad z \leq z_0 \tag{2.39b}$$

In conclusion, the Green's function (2.35) with (2.36) and (2.39a,b) can be written

$$G(\mathbf{r}-\mathbf{r}_0) = -\frac{1}{4\pi}\ \frac{\exp(ik|\mathbf{r}-\mathbf{r}_0|)}{|\mathbf{r}-\mathbf{r}_0|} =$$

$$= \frac{1}{8\pi^2 i}\iint \frac{\exp\left\{+i\left[\alpha(x-x_0)+\beta(y-y_0) \pm \sqrt{k^2-\alpha^2-\beta^2}\,(z-z_0)\right]\right\}}{\sqrt{k^2-\alpha^2-\beta^2}}\ \mathrm{d}\alpha\ \mathrm{d}\beta \tag{2.40}$$

Hence the expansion of the spherical wave in plane waves is

$$\frac{\exp(ik|\mathbf{r}-\mathbf{r}_0|)}{|\mathbf{r}-\mathbf{r}_0|} =$$

$$= -\frac{1}{2\pi\, i}\iint \frac{\exp\left\{+i\left[\alpha(x-x_0)+\beta(y-y_0) \pm \sqrt{k^2-\alpha^2-\beta^2}\,(z-z_0)\right]\right\}}{\sqrt{k^2-\alpha^2-\beta^2}}\ \mathrm{d}\alpha\ \mathrm{d}\beta \tag{2.41}$$

where the plus and minus signs pertain to $z \geq z_0$ and $z \leq z_0$, respectively. Later we will also need the derivative

$$\frac{\partial}{\partial z}\ \frac{\exp(ik|\mathbf{r}-\mathbf{r}_0|)}{|\mathbf{r}-\mathbf{r}_0|} =$$

$$= \mp\frac{1}{2\pi}\iint \exp\left\{+i\left[\alpha(x-x_0)+\beta(y-y_0) \pm \sqrt{k^2-\alpha^2-\beta^2}\,(z-z_0)\right]\right\}\ \mathrm{d}\alpha\ \mathrm{d}\beta \tag{2.42}$$

We introduce in these expansions some simplifying notations:

$$\mathbf{r} = \{\boldsymbol{\rho}, z\} \ , \qquad \text{where} \qquad \boldsymbol{\rho} = \{x, y\} \tag{2.43}$$

$$\mathbf{r}_0 = \{\boldsymbol{\rho}_0, z_0\} \ , \qquad \text{where} \qquad \boldsymbol{\rho}_0 = \{x_0, y_0\} \tag{2.44}$$

$$\boldsymbol{\kappa} = \{\alpha, \beta\} \tag{2.45}$$

Then we obtain from (2.41), (2.42)

$$\frac{\exp(ik|\mathbf{r}-\mathbf{r}_0|)}{|\mathbf{r}-\mathbf{r}_0|} = -\frac{1}{2\pi\, i} \iint \frac{\exp\left\{+i\left[\boldsymbol{\kappa}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}_0) \pm \sqrt{k^2-\kappa^2}\,(z-z_0)\right]\right\}}{\sqrt{k^2-\kappa^2}}\, \mathrm{d}\boldsymbol{\kappa} \tag{2.41$'$}$$

$$\frac{\partial}{\partial z}\,\frac{\exp(ik|\mathbf{r}-\mathbf{r}_0|)}{|\mathbf{r}-\mathbf{r}_0|} = \mp\frac{1}{2\pi} \iint \exp\left\{+i\left[\boldsymbol{\kappa}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}_0) \pm \sqrt{k^2-\kappa^2}\,(z-z_0)\right]\right\} \mathrm{d}\boldsymbol{\kappa} \tag{2.42$'$}$$

The integration is to be performed over the entire $\boldsymbol{\kappa}$-surface, but the partial waves are oscillatory in z only when $\kappa < k$. Note that the integrand of (2.41′) has a weak singularity at $\kappa = \pm k$; these two points are also branch points of the integrand. To do a correct evaluation of the integral with the integrand as an analytic function, it is therefore necessary to define branch cuts from these points and to keep to those definitions. In order to get the correct solution it is necessary to fix these branches so that $\operatorname{Im}\sqrt{k^2-\kappa^2} \geq 0$.

The partial waves with $\kappa > k$ are evanescent and sometimes do not influence the solution at a distance from z_0.

Chapter 3

Boundary given as an infinite stochastic screen

Consider a satellite emitting a VHF signal which passes through the ionosphere down to earth. Due to the irregularities of the ionosphere this signal has stochastic features when it reaches the bottom of the ionized layers. If we treat the wave propagation problem below the ionosphere as a free-space problem, we may consider the bottom ionosphere as an infinite screen where the stochastic field is given as a boundary condition.

Given this boundary field we are, in fact, able to obtain an exact representation for the field in the half-space below the ionosphere. However, the problem how the ionospheric fluctuations act to set up the field at the boundary then still remains.

3.1 Field representation in the half-space

We let the boundary be at $z = 0$ and assume that the complex amplitude of the monochromatic (possibly slowly time-varying) field is given there on the form

$$E(\boldsymbol{\rho}, 0) = E_0(\boldsymbol{\rho}) \tag{3.1}$$

With this boundary condition we shall construct the field in the half-space $z > 0$ as the solution to the wave equation

$$\nabla^2 E + k^2 E = 0 \tag{3.2}$$

We shall proceed similarly as in Section 2.5 and write the solution as an expansion of plane waves:

$$E(\boldsymbol{\rho}, z) = \int f(z, \alpha, \beta)\ \exp[+i(\alpha x + \beta y)]\ \mathrm{d}\alpha\ \mathrm{d}\beta = \int f(z, \boldsymbol{\kappa})\ \exp[+i\boldsymbol{\kappa} \cdot \boldsymbol{\rho}]\ \mathrm{d}\boldsymbol{\kappa} \tag{3.3}$$

Then (3.2) is transformed into an ordinary differential equation for f :

$$\frac{\mathrm{d}^2 f}{\mathrm{d}z^2} + \left(k^2 - \kappa^2\right)\ f = 0 \tag{3.4}$$

Our boundary condition at $z = +\infty$ is that the waves there be purely outgoing, and hence the acceptable solutions of (3.4) have the form

$$f(z, \boldsymbol{\kappa}) = A(\boldsymbol{\kappa})\ \exp\left(+i\sqrt{k^2 - \kappa^2}\ z\right) \tag{3.5}$$

The boundary condition (3.1) on the screen at $z = 0$ gives the coefficient function as the Fourier transform of the boundary field:

$$A(\boldsymbol{\kappa}) = f(0, \boldsymbol{\kappa}) = \tilde{E}_0(\boldsymbol{\kappa}) \tag{3.6}$$

$$\tilde{E}_0(\boldsymbol{\kappa}) = \frac{1}{(2\pi)^2} \int E_0(\boldsymbol{\rho})\ \exp(-i\boldsymbol{\kappa}\cdot\boldsymbol{\rho})\ \mathrm{d}\boldsymbol{\rho} \tag{3.7}$$

When we use (3.6) and (3.5) in (3.3), we get the result

$$E(\boldsymbol{\rho}, z) = \int \tilde{E}_0(\boldsymbol{\kappa})\ \exp\left[+i\left(\boldsymbol{\kappa}\cdot\boldsymbol{\rho} + \sqrt{k^2 - \kappa^2}\ z\right)\right]\ \mathrm{d}\boldsymbol{\kappa} \tag{3.8}$$

Thus we have obtained an exact representation of the field for $z > 0$ in terms of an expansion in plane waves over spatial frequencies $\boldsymbol{\kappa}$. In order to obtain an alternative representation, in terms of an integral over the boundary, we introduce the transform (3.7) into (3.8), which yields

$$E(\boldsymbol{\rho}, z) = \frac{1}{(2\pi)^2} \iint E_0(\boldsymbol{\rho}_0)\ \exp\left\{+i\left[\boldsymbol{\kappa}\cdot(\boldsymbol{\rho} - \boldsymbol{\rho}_0) + \sqrt{k^2 - \kappa^2}\ z\right]\right\}\ \mathrm{d}\boldsymbol{\kappa}\ \mathrm{d}\boldsymbol{\rho}_0 \tag{3.9}$$

Replacing the integral of the exponential function over $\boldsymbol{\kappa}$ by the left-hand side of (2.42′), we arrive at the formula

$$E(\boldsymbol{\rho}, z) = -\frac{1}{2\pi} \int E_0(\boldsymbol{\rho}_0)\ \frac{\partial}{\partial z}\ \left[\frac{\exp(+ik|\mathbf{r} - \boldsymbol{\rho}_0|)}{|\mathbf{r} - \boldsymbol{\rho}_0|}\right]\ \mathrm{d}\boldsymbol{\rho}_0 \tag{3.10}$$

This is also an exact representation of the field for $z > 0$, but now in terms of spherical waves emanating from the source points $\boldsymbol{\rho}_0$.

In Sections 3.2–3.4 we will obtain various approximations of this representation.

3.2 Far-field (wave) zone

To obtain an approximation of the integral representation, we first calculate the derivative in (3.10):

$$\frac{\partial}{\partial z}\ \left[\frac{\exp(+ik|\mathbf{r} - \boldsymbol{\rho}_0|)}{|\mathbf{r} - \boldsymbol{\rho}_0|}\right] = ikz\ \left(1 - \frac{1}{ik\,|\mathbf{r} - \boldsymbol{\rho}_0|}\right)\ \frac{\exp(+ik|\mathbf{r} - \boldsymbol{\rho}_0|)}{|\mathbf{r} - \boldsymbol{\rho}_0|^2} \tag{3.11}$$

The second term within the large brackets can be neglected if $k|\mathbf{r}-\boldsymbol{\rho}_0| \gg 1$ or, rather,

$$kz \gg 1 \tag{3.12}$$

i.e. for field points many wavelengths away from the screen. We adopt (3.12) as the definition of the *far-field* or *wave zone* and find in this approximation from (3.10) the integral representation

$$E(\boldsymbol{\rho}, z) = \frac{kz}{2\pi i} \int E_0(\boldsymbol{\rho}_0) \, \frac{\exp(+ik|\mathbf{r}-\boldsymbol{\rho}_0|)}{|\mathbf{r}-\boldsymbol{\rho}_0|^2} \, \mathrm{d}\boldsymbol{\rho}_0 \tag{3.13}$$

3.3 The Fresnel approximation

The next step of approximation of our integral representation involves the Taylor expansion of the distance term

$$|\mathbf{r}-\boldsymbol{\rho}_0| = \sqrt{(\boldsymbol{\rho}-\boldsymbol{\rho}_0)^2 + z^2} \tag{3.14}$$

for large z, i.e.

$$|\mathbf{r}-\boldsymbol{\rho}_0| = z \left\{ 1 + \frac{(\boldsymbol{\rho}-\boldsymbol{\rho}_0)^2}{2z^2} + O\left[\frac{(\boldsymbol{\rho}-\boldsymbol{\rho}_0)^4}{z^4} \right] \right\} \tag{3.15}$$

The *Fresnel approximation* amounts to neglecting terms beyond the quadratic in this expansion when using it in the exponential function in (3.13). This is justified if the neglected terms introduce phase errors much less than unity in the integrand. Hence the condition of validity of the Fresnel approximation is that

$$kL_0^4 \ll z^3 \tag{3.16}$$

Here we have introduced a characteristic scale $L_0 = |\boldsymbol{\rho}-\boldsymbol{\rho}_0|$ of the initial field distribution, which is related to the size of the illuminated area of the screen. Later in the random case we will find that it is the characteristic scale of the correlation function.

It is easily seen that the first term of the Taylor expansion is sufficient in the denominator in (3.13), i.e. $|\mathbf{r}-\boldsymbol{\rho}_0| \approx z$. Thus we arrive at the following expansion for the field in the Fresnel approximation:

$$E(\boldsymbol{\rho}, z) = \frac{k\, e^{ikz}}{2\pi i\, z} \int E_0(\boldsymbol{\rho}_0) \, \exp\left[\frac{+ik(\boldsymbol{\rho}-\boldsymbol{\rho}_0)^2}{2z} \right] \mathrm{d}\boldsymbol{\rho}_0 \tag{3.17}$$

3.4 The Fraunhofer approximation

Finally we expand the square in the argument of the exponential function in (3.17):

$$(\boldsymbol{\rho}-\boldsymbol{\rho}_0)^2 = \rho^2 - 2\boldsymbol{\rho}\cdot\boldsymbol{\rho}_0 + \rho_0^2 \tag{3.18}$$

We obtain the *Fraunhofer approximation* by neglecting ρ_0^2 in this expression. This is permissible if the error introduced in the argument is much less than unity, i.e. if $k\rho_0^2/(2z) \ll 1$ or, if we use the characteristic length L_0 introduced in the previous section,

$$k\,L_0^2 \ll z \tag{3.19}$$

Another way to write this condition is

$$L_0 \ll \ell_{fr} \tag{3.19$'$}$$

where

$$\ell_{fr} = \sqrt{\lambda\, z} \tag{3.20}$$

is the size of the main Fresnel zone. The Fraunhofer approximation then turns out to be

$$E(\boldsymbol{\rho}, z) = \frac{k}{2\pi i\, z}\ \exp\left[+ik\left(z + \frac{\rho^2}{2z}\right)\right]\ \int E_0(\boldsymbol{\rho}_0)\ \exp\left[-ik\frac{\boldsymbol{\rho}\cdot\boldsymbol{\rho}_0}{z}\right]\ \mathrm{d}\boldsymbol{\rho}_0 \tag{3.21}$$

We see that this representation is a kind of plane wave expansion. If the integration is carried out over the entire $\boldsymbol{\rho}_0$-plane, then another way to write the field is

$$E(\boldsymbol{\rho}, z) = \frac{2\pi k}{i\, z}\ \exp\left[+ik\left(z + \frac{\rho^2}{2z}\right)\right]\ \tilde{E}_0\left(\frac{k\boldsymbol{\rho}}{z}\right) \tag{3.21$'$}$$

3.5 Mean field and correlation function for the field in the half-space

When we talk about spectra we usually mean spectra of correlation functions. As we have already mentioned, the spectrum of the random function itself is not always defined. To obtain the mean field we shall therefore use the integral representation (3.9), which involves the initial stochastic field as a function of the position on the screen. Thus we have

$$\langle E(\boldsymbol{\rho}, z)\rangle = \frac{1}{(2\pi)^2}\ \iint \langle E_0(\boldsymbol{\rho}')\rangle\ \exp\left\{+i\left[\boldsymbol{\kappa}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}') + \sqrt{k^2-\kappa^2}\ z\right]\right\}\ \mathrm{d}\boldsymbol{\kappa}\ \mathrm{d}\boldsymbol{\rho}' \tag{3.22}$$

We shall assume a statistically homogeneous initial field, which means that

$$\langle E_0(\boldsymbol{\rho}')\rangle = E_{00} = \mathrm{const} \tag{3.23}$$

With this condition the integration over $\boldsymbol{\rho}'$ in (3.22) gives a delta function; $\delta(\boldsymbol{\kappa})$. Hence the result of (3.22) is simply a plane wave of constant amplitude in the z-direction:

$$\langle E(\boldsymbol{\rho}', z)\rangle = E_{00}\ e^{ikz} \tag{3.24}$$

The correlation function for the fluctuating part $\tilde{E}$ of the field is from (3.8)

$$\{\psi_E(\boldsymbol{\rho}_1, z_1, \boldsymbol{\rho}_2, z_2) = \left\langle \tilde{E}(\boldsymbol{\rho}_1, z_1)\ \tilde{E}^*(\boldsymbol{\rho}_2, z_2)\right\rangle\ = \int \left\langle \tilde{E}_0(\boldsymbol{\kappa}_1)\ \tilde{E}_0^*(\boldsymbol{\kappa}_2)\right\rangle$$

$$\cdot \exp\left[+i\left(\boldsymbol{\kappa}_1\cdot\boldsymbol{\rho}_1-\boldsymbol{\kappa}_2\cdot\boldsymbol{\rho}_2\right)+i\sqrt{k^2-\kappa_1^2}\,z_1+\left(i\sqrt{k^2-\kappa_2^2}\right)^*\,z_2\right]\,\mathrm{d}\boldsymbol{\kappa}_1\,\mathrm{d}\boldsymbol{\kappa}_2 \tag{3.25}$$

For the statistically homogeneous initial field considered here we have according to (1.36)

$$\left\langle \tilde{E}_0(\boldsymbol{\kappa}_1)\,\tilde{E}_0^*(\boldsymbol{\kappa}_2)\right\rangle = \phi_{E_0}(\boldsymbol{\kappa}_1)\,\delta(\boldsymbol{\kappa}_1-\boldsymbol{\kappa}_2) \tag{3.26}$$

Hence, from (3.25) we get

$$\psi_E(\boldsymbol{\rho}_1,z_1,\boldsymbol{\rho}_2,z_2)=\int\phi_{E_0}(\boldsymbol{\kappa})$$

$$\exp\left[+i\boldsymbol{\kappa}\cdot(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)+i\sqrt{k^2-\kappa^2}\,z_1+\left(i\sqrt{k^2-\kappa^2}\right)^*\,z_2\right]\,\mathrm{d}\boldsymbol{\kappa} \tag{3.27}$$

The physical solution corresponds to the following branch of the square root:

$$\sqrt{k^2-\kappa^2}=\begin{cases}\sqrt{k^2-\kappa^2}\;, & \kappa<k\\ i\,\sqrt{|k^2-\kappa^2|}\;, & \kappa>k\end{cases} \tag{3.28}$$

Hence part of the exponent in (3.27) can be written

$$\exp\left[i\sqrt{k^2-\kappa^2}\,z_1+\left(i\sqrt{k^2-\kappa^2}\right)^*\,z_2\right]=\begin{cases}\exp\left[i\sqrt{k^2-\kappa^2}\,(z_1-z_2)\right]\;, & \kappa<k\\ \exp\left[-\sqrt{|k^2-\kappa^2|}\,(z_1+z_2)\right]\;, & \kappa>k\end{cases} \tag{3.28$'$}$$

We see from the lower case of this that the integrand vanishes if $\kappa>k$ and $z_1,\ z_2$ lie several wavelengths away from the screen. Thus we may confine the integration to a circle of radius k in the $\boldsymbol{\kappa}$-surface:

$$\psi_E(\boldsymbol{\rho}_1,z_1,\boldsymbol{\rho}_2,z_2)\approx\int\limits_{\kappa\le k}\phi_{E_0}(\boldsymbol{\kappa})\,\exp\left[+i\left(\boldsymbol{\kappa}\cdot\boldsymbol{\rho}+\sqrt{k^2-\kappa^2}\,z\right)\right]\,\mathrm{d}\boldsymbol{\kappa} \tag{3.29}$$

where we have introduced the relative coordinates

$$\boldsymbol{\rho}=\boldsymbol{\rho}_1-\boldsymbol{\rho}_2\;,\qquad z=z_1-z_2 \tag{3.30}$$

This result shows that with $E_0(\boldsymbol{\rho})$ statistically homogeneous, also the field in the half-space is statistically homogeneous in this approximation. Our result is exact in the transverse $\boldsymbol{\rho}$-variable, but an approximation in the longitudinal variable z. Below we will study special cases of the result (3.29) in more detail.

3.6 Longitudinal and transverse correlation functions in the limiting case of small-scale field fluctuations on the screen

We shall begin by considering *small-scale fluctuations*, i.e. cases when the spatial scale ℓ_{E_0} of the fluctuations is small compared to the wavelength. Formally we express the condition for this in the following inequality

$$k\,\ell_{E_0} \ll 1 \tag{3.31}$$

where k is the wave number of the radiation. We have already seen [eq. (1.42)] that the spatial spectrum of the initial field has the scale $\kappa_{E_0} \sim \ell_{E_0}^{-1}$. Hence the condition (3.31) can also be written

$$k \ll \kappa_{E_0} \tag{3.31$'$}$$

When we now integrate (3.29) over the area $\kappa \leq k$, eq. (3.31) shows us that ϕ_{E_0} is non-zero over a much larger area than that. Hence $\phi_{E_0}(\boldsymbol{\kappa})$ does not vary so much over the area of integration and can be approximated by a constant value :

$$\psi_E(\boldsymbol{\rho}, z) \approx \phi_{E_0}(0) \int\limits_{\kappa\leq k} \exp\left[+i\left(\boldsymbol{\kappa}\cdot\boldsymbol{\rho} + \sqrt{k^2-\kappa^2}\,z\right)\right]\,\mathrm{d}\boldsymbol{\kappa} \tag{3.32}$$

We shall now study this expression in two special cases. First the case of *purely transverse correlation function*, when we put $z = 0$ (i.e. $z_1 = z_2$). Then we have

$$\psi_E(\boldsymbol{\rho}, 0) \approx \phi_{E_0}(0) \int\limits_{\kappa\leq k} \exp\left[+i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}\right]\,\mathrm{d}\boldsymbol{\kappa} \tag{3.33}$$

With polar coordinates $\boldsymbol{\kappa} \to \kappa,\ \varphi$, this gives

$$\psi_E(\boldsymbol{\rho}, 0) \approx \phi_{E_0}(0) \int\limits_0^{2\pi} \mathrm{d}\varphi \int\limits_0^{k} \exp\left[+i\kappa\rho\cos\varphi\right]\,\kappa\,\mathrm{d}\kappa =$$

$$= 2\pi\,\phi_{E_0}(0)\,k\,J_1(k\rho)/\rho \tag{3.34}$$

This result involves the Bessel function of the first order, $J_1(x)$, and denoting the first zero of the Bessel function by $x = d_1$, we may find an estimate of the characteristic transverse scale-size ℓ_E of the field in the half-space through

$$k\,\ell_{E\perp} = d_1 \approx 3.8 \tag{3.35}$$

In the other case, the *purely longitudinal correlation function*, we put $\boldsymbol{\rho} = 0$ in (3.32) to obtain

$$\psi_E(0, z) \approx \phi_{E_0}(0) \int\limits_{\kappa\leq k} \exp\left[+i\sqrt{k^2-\kappa^2}\,z\right]\,\mathrm{d}\boldsymbol{\kappa} \tag{3.36}$$

When we once again introduce polar coordinates, we can immediately carry out the φ-integration so that

$$\psi_E(0, z) \approx 2\pi\,\phi_{E_0}(0) \int\limits_0^{k} \exp\left[+i\sqrt{k^2-\kappa^2}\,z\right]\,\kappa\,\mathrm{d}\kappa$$

$$= 2\pi\ \phi_{E_0}(0)\ \frac{\mathrm{d}}{\mathrm{d}z} \int\limits_0^k \frac{\kappa}{i\sqrt{k^2-\kappa^2}} \exp\left[+i\sqrt{k^2-\kappa^2}\,z\right]\ \mathrm{d}\kappa$$

$$= 2\pi\ \phi_{E_0}(0)\ \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{1}{z}\ \left(e^{ikz}-1\right)\right] \tag{3.37}$$

Hence the characteristic longitudinal scale-size $\ell_E^{(z)}$ is of the order k^{-1}, i.e of the order λ. From these estimates we may conclude that the correlation volume is several units larger in the transverse directions than in the longitudinal direction.

Finally we shall consider the *variance*, i.e. the statistical dispersion, for the case of small-scale fluctuations in the half-space. From (3.32), we get

$$\sigma_E^2 = \psi_E(0,0) = \phi_{E_0}(0) \int\limits_{\kappa\le k} \mathrm{d}\boldsymbol{\kappa} = \pi\, k^2 \phi_{E_0}(0) \tag{3.38}$$

Expressing $\phi_{E_0}(0)$ with its Fourier transform, this yields the estimate

$$\sigma_E^2 = \frac{\pi\, k^2}{(2\pi)^2} \int \psi_{E_0}(\boldsymbol{\rho})\ \mathrm{d}\boldsymbol{\rho} \approx \frac{k^2}{4\pi}\ \pi \ell_{E_0}^2 \psi_{E_0}(0) = \tfrac{1}{4}\ (k\ell_{E_0})^2\ \sigma_{E_0}^2 \tag{3.39}$$

In view of the condition for small-scale fluctuations of the initial field, eq. (3.31), we hence find that the field fluctuations in the half-space fulfil

$$\sigma_E^2 \ll \sigma_{E_0}^2 \tag{3.40}$$

3.7 Longitudinal and transverse correlation functions in the limiting case of large-scale field fluctuations on the screen

In the case of *large-scale fluctuations* the situation is the opposite of (3.31), i.e. the spatial scale ℓ_{E_0} of the fluctuations fulfils

$$k\ \ell_{E_0} \gg 1 \tag{3.41}$$

Because of $\kappa_{E_0} \sim \ell_{E_0}^{-1}$ this condition can also be written

$$k \gg \kappa_{E_0} \tag{3.41$'$}$$

In the integral (3.29), $\phi_{E_0}(\boldsymbol{\kappa})$ is now non-zero only over a small part of the area $\kappa \le k$ of integration. Under these circumstances we may expand the square root in the exponent as follows:

$$\sqrt{k^2-\kappa^2} \approx k - \frac{\kappa^2}{2k} \tag{3.42}$$

In this limiting case we may then write the correlation function (3.29), also extending the integration to infinity,

$$\psi_E(\boldsymbol{\rho}, z) \approx e^{ikz} \int\limits_{-\infty}^{+\infty} \phi_{E_0}(\boldsymbol{\kappa}) \; \exp\left[+i\left(\boldsymbol{\kappa}\cdot\boldsymbol{\rho} - \frac{\kappa^2 z}{2k}\right)\right] \; \mathrm{d}\boldsymbol{\kappa} \tag{3.43}$$

This expression may now be used to determine the transverse and longitudinal correlation functions, just as in the previous section. For the *transverse correlation function* we find simply

$$\psi_E(\boldsymbol{\rho}, 0) \approx \int\limits_{-\infty}^{+\infty} \phi_{E_0}(\boldsymbol{\kappa}) \; \exp\left[+i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}\right] \; \mathrm{d}\boldsymbol{\kappa} = \psi_{E_0}(\boldsymbol{\rho}) \tag{3.44}$$

i.e. it is the same as for the initial field and hence $\ell_{E\perp} \approx \ell_{E_0}$. Of course we then also have

$$\sigma_E^2 \approx \sigma_{E_0}^2 \tag{3.45}$$

The *longitudinal correlation function*, on the other hand, is given by

$$\psi_E(0, z) \approx e^{ikz} \int\limits_{-\infty}^{+\infty} \phi_{E_0}(\boldsymbol{\kappa}) \; \exp\left[-i\frac{\kappa^2 z}{2k}\right] \; \mathrm{d}\boldsymbol{\kappa} \tag{3.46}$$

This integral cannot be solved in the general case, but we shall obtain an estimate of the order of magnitude of the longitudinal scale-size of the field. We note that the exponent in (3.46) has a stationary point at $\kappa = 0$. Hence the main contributions to the integral come from the main Fresnel zone defined by

$$\frac{\kappa_{E_0}^2 \ell_E^{(z)}}{2k} = 1 \tag{3.47}$$

i.e. the correlation function in practice varies only for $|z| < \ell_E^{(z)}$, where, according to (3.41),

$$\ell_E^{(z)} \sim k\,\ell_{E_0}^2 \gg \ell_{E_0} \tag{3.48}$$

To summarize our results (3.44) and (3.48), we can say that it is typical for the correlation function for fields radiated by large-scale inhomogeneities that it is non-zero within an elongated volume with transverse dimension equal to the scale-size of the initial field and longitudinal dimension much larger than that. This is in contrast to the previous case of small-scale inhomogeneities.

We shall also say a few words about the *coherence function* for the case of large-scale inhomogeneities on the screen. It is related to the correlation function, but not centred on the mean value:

$$\begin{aligned} B_E(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z_1, z_2) &= \langle E(\boldsymbol{\rho}_1, z_1)\; E^*(\boldsymbol{\rho}_2, z_2)\rangle \\ &= \psi_E(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z_1, z_2) + \langle E(\boldsymbol{\rho}_1, z_1)\rangle \; \langle E^*(\boldsymbol{\rho}_2, z_2)\rangle \end{aligned} \tag{3.49}$$

For statistically homogeneous fields the mean field has constant amplitude equal to its initial value [cf. (3.23,24)]; hence

$$B_E(\boldsymbol{\rho},0) = \psi_E(\boldsymbol{\rho},0) + |E_{00}|^2 \tag{3.50}$$

The mean energy of the field is directly obtained from this and it is seen to be independent of position in space:

$$W = B_E(0,0) = \psi_E(0,0) + |E_{00}|^2 = \sigma_{E_0}^2 + |E_{00}|^2 \tag{3.51}$$

3.8 The case of a pure phase screen

In some important applications the initial field has constant amplitude, only the phase is stochastic, i.e.

$$E_0 = A\ \exp[iS(\boldsymbol{\rho})] \tag{3.52}$$

In the following we shall put the amplitude equal to unity, $A = 1$. The phase S is a new zero-mean random field, $\langle S(\boldsymbol{\rho})\rangle = 0$, which we assume has a Gaussian probability density function:

$$w(S) = \frac{1}{\sqrt{2\pi}\,\sigma_S}\ \exp\left(-\frac{S^2}{2\sigma_S^2}\right) \tag{3.53}$$

Using the integral representations (3.8) and (3.9), we obtain the mean field

$$\begin{aligned}\langle E(\boldsymbol{\rho},z)\rangle &= \int \left\langle \tilde{E}_0(\boldsymbol{\kappa})\right\rangle\ \exp\left[+i\left(\boldsymbol{\kappa}\cdot\boldsymbol{\rho} + \sqrt{k^2-\kappa^2}\ z\right)\right]\ \mathrm{d}\boldsymbol{\kappa} \\ &= \frac{1}{(2\pi)^2}\ \int \langle E_0(\boldsymbol{\rho}')\rangle\ \exp\left\{+i\left[\boldsymbol{\kappa}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}') + \sqrt{k^2-\kappa^2}\ z\right]\right\}\ \mathrm{d}\boldsymbol{\kappa}\,\mathrm{d}\boldsymbol{\rho}'\end{aligned} \tag{3.54}$$

Since we know that the average initial field is a constant we can take it outside the integral. We then obtain a δ-function from the integration over $\boldsymbol{\rho}'$ so that (3.54) gives

$$\langle E(\boldsymbol{\rho},z)\rangle = \langle \exp[iS(\boldsymbol{\rho})]\rangle\ e^{ikz} \tag{3.55}$$

With (3.53), the average of the exponential function is

$$\langle \exp[iS(\boldsymbol{\rho})]\rangle = \frac{1}{\sqrt{2\pi}\sigma_S}\int_{-\infty}^{+\infty} \exp\left(+iS - \frac{S^2}{2\sigma_S^2}\right)\ \mathrm{d}S \tag{3.56}$$

This is transformed into a standard integral by rewriting the exponent as a square expression:

$$-\frac{1}{2\sigma_S^2}\ \left(S^2 - 2iS\sigma_S^2 - \sigma_S^4 + \sigma_S^4\right) = -\frac{\sigma_S^2}{2} - \frac{1}{2\sigma_S^2}\ \left(S - i\sigma_S^2\right)^2 \tag{3.57}$$

The result of the evaluation is

$$\langle \exp[iS(\boldsymbol{\rho})] \rangle = \exp\left(-\tfrac{1}{2}\sigma_S^2\right) \tag{3.58}$$

which gives the mean field

$$\langle E(\boldsymbol{\rho}, z) \rangle = \exp\left(-\tfrac{1}{2}\sigma_S^2 + ikz\right) \tag{3.59}$$

We remark that (3.58) is a universal result for normally distributed random functions S.

Next we consider the second-order moment. Using the integral representation (3.9), we find

$$\begin{aligned} B_E(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z_1, z_2) &= \langle E(\boldsymbol{\rho}_1, z_1)\; E^*(\boldsymbol{\rho}_2, z_2) \rangle \\ &= \frac{1}{(2\pi)^4} \int \langle \exp\{+i[S(\boldsymbol{\rho}') - S(\boldsymbol{\rho}'')]\} \rangle \; \mathrm{d}\boldsymbol{\kappa}_1 \, \mathrm{d}\boldsymbol{\kappa}_2 \, \mathrm{d}\boldsymbol{\rho}' \, \mathrm{d}\boldsymbol{\rho}'' \cdot \\ \exp\left\{+i\left[\boldsymbol{\kappa}_1 \cdot (\boldsymbol{\rho}_1 - \boldsymbol{\rho}') + \sqrt{k^2 - \kappa_1^2}\, z_1 - \boldsymbol{\kappa}_2 \cdot (\boldsymbol{\rho}_2 - \boldsymbol{\rho}'') - \sqrt{k^2 - \kappa_2^2}\, z_2\right]\right\}& \end{aligned} \tag{3.60}$$

The integrand contains $\langle \exp\{+i[S(\boldsymbol{\rho}') - S(\boldsymbol{\rho}'')]\} \rangle$. Since the exponent is normally distributed, we may again use the property (3.58):

$$\begin{aligned} \langle \exp\{+i[S(\boldsymbol{\rho}') - S(\boldsymbol{\rho}'')]\} \rangle &= \exp\left\{-\tfrac{1}{2}\left\langle [S(\boldsymbol{\rho}') - S(\boldsymbol{\rho}'')]^2 \right\rangle\right\} \\ &= \exp\left[-\tfrac{1}{2} D_S(\boldsymbol{\rho}' - \boldsymbol{\rho}'')\right] = f_S(\boldsymbol{\rho}' - \boldsymbol{\rho}'') \end{aligned} \tag{3.61}$$

where D_S is the structure function. Introducing this result into (3.60) togehter with the change of variables

$$\boldsymbol{\rho}'' \to \boldsymbol{\rho}'' \;, \qquad \boldsymbol{\rho}' - \boldsymbol{\rho}'' \to \boldsymbol{\rho} \tag{3.62}$$

and specializing to the *transverse moment* by putting $z_1 = z_2 = z$ we obtain

$$\begin{aligned} B_E(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) &= \frac{1}{(2\pi)^4} \int f_S(\boldsymbol{\rho}) \; \mathrm{d}\boldsymbol{\kappa}_1 \, \mathrm{d}\boldsymbol{\kappa}_2 \, \mathrm{d}\boldsymbol{\rho} \, \mathrm{d}\boldsymbol{\rho}'' \\ \cdot\; \exp\left\{+i\left[\boldsymbol{\kappa}_1 \cdot (\boldsymbol{\rho}_1 - \boldsymbol{\rho}'' - \boldsymbol{\rho}) + \sqrt{k^2 - \kappa_1^2}\, z - \boldsymbol{\kappa}_2 \cdot (\boldsymbol{\rho}_2 - \boldsymbol{\rho}'') - \sqrt{k^2 - \kappa_2^2}\, z\right]\right\}& \end{aligned} \tag{3.63}$$

Here the integration over $\boldsymbol{\rho}''$ may be carried out directly; it gives the function $\delta(\boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_2)$. With $\boldsymbol{\kappa}_1 = \boldsymbol{\kappa}_2 = \boldsymbol{\kappa}$, (3.63) can hence be written

$$B_E(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) = \frac{1}{(2\pi)^2} \int f_S(\boldsymbol{\rho}) \; \exp\left\{+i\left[\boldsymbol{\kappa} \cdot (\boldsymbol{\rho}_1 - \boldsymbol{\rho}) - \boldsymbol{\kappa} \cdot \boldsymbol{\rho}_2\right]\right\} \; \mathrm{d}\boldsymbol{\kappa} \, \mathrm{d}\boldsymbol{\rho} \tag{3.64}$$

We see that also the $\boldsymbol{\kappa}$-integration gives a delta function, $\delta(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2 - \boldsymbol{\rho})$, so the final result is

$$B_E(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) = f_S(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2) = \exp\left[-\tfrac{1}{2} D_S(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)\right] \tag{3.65}$$

Clearly this transverse coherence function is independent of z. According to (1.65) we have

$$D_S(\boldsymbol{\rho}) = 2[\sigma_S^2 - \psi_S(\boldsymbol{\rho})] \tag{3.66}$$

with $\boldsymbol{\rho} = \boldsymbol{\rho}_1 - \boldsymbol{\rho}_2$. Hence we may write the result (3.65) as follows:

$$B_E(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) = \exp\left\{-[\sigma_S^2 - \psi_S(\boldsymbol{\rho})]\right\} \tag{3.67}$$

At the same time we have, according to the definitions of the second-order moments,

$$B_E(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) = \psi_E(\boldsymbol{\rho}, z) + E_{00}^2 \tag{3.68}$$

where E_{00} is the initial mean field, which is real according to (3.59). Hence the following relation holds for the correlation functions of the field and the phase of the field:

$$\psi_E(\boldsymbol{\rho}, z) = B_E(\boldsymbol{\rho}, z) - E_{00}^2 = \exp\left[\psi_S(\boldsymbol{\rho}) - \sigma_S^2\right] - \exp\left[-\sigma_S^2\right] \tag{3.69}$$

We shall consider this expression in two limiting cases, thereby making use of the correlation coefficient (1.21):

$$\psi_S = \sigma_S^2 \, K_S(\boldsymbol{\rho}) \tag{3.70}$$

In the limiting case of *weak fluctuations*, $\sigma_S^2 \ll 1$, Taylor expansions of the exponential functions in (3.69) give

$$\psi_E^{(w)} \approx \sigma_S^2 \, K_S(\boldsymbol{\rho}) = \psi_S(\boldsymbol{\rho}) \tag{3.71}$$

The condition for *strong fluctuations* is $\sigma_S^2 \gg 1$. Then the second exponential function in (3.69) vanishes and in the first one we may use the Taylor expansion of the correlation coefficient:

$$K_S(\boldsymbol{\rho}) = 1 - \tfrac{1}{2}\,|K_S''(0)|\,\rho^2 \tag{3.72}$$

Hence the result is

$$\psi_E^{(s)} \approx \exp\left[-\tfrac{1}{2}\,|K_S''(0)|\,\sigma_S^2\,\rho^2\right] = \exp\left[-\tfrac{1}{2}\,|\psi_S''(0)|\,\rho^2\right] \tag{3.73}$$

According to (3.67), (3.68) the average energy is

$$\left\langle |E|^2 \right\rangle = B_E(0) = \sigma_E^2 + E_{00}^2 = 1 \tag{3.74}$$

3.9 Fluctuations of the amplitude and phase of the field, generated by a phase screen

Sometimes it is of interest to construct the moments for amplitude and phase separately. To this end we shall use the Fresnel representation (3.17), where we separate the fluctuating part of a particular realization of the initial field from the mean field:

$$E(\boldsymbol{\rho}, z) = \frac{k\, e^{ikz}}{2\pi i\, z} \int [E_0(\boldsymbol{\rho}_0) - E_{00} + E_{00}] \; \exp\left[\frac{+ik(\boldsymbol{\rho} - \boldsymbol{\rho}')^2}{2z}\right] \; \mathrm{d}\boldsymbol{\rho}' \tag{3.75}$$

The resulting field is also separated into deterministic and zero-mean fluctuating parts:

$$E(\boldsymbol{\rho}, z) = [E_{00} + a(\boldsymbol{\rho}, z)] \; e^{ikz} \tag{3.76}$$

where

$$a(\boldsymbol{\rho}, z) = \frac{k}{2\pi i\, z} \int \tilde{E}_0(\boldsymbol{\rho}') \exp\left[\frac{+ik(\boldsymbol{\rho} - \boldsymbol{\rho}')^2}{2z}\right] \; \mathrm{d}\boldsymbol{\rho}' \tag{3.77}$$

with the definition

$$\tilde{E}_0(\boldsymbol{\rho}_0) = E_0(\boldsymbol{\rho}_0) - E_{00} \tag{3.78}$$

If we further separate the fluctuating field into its real and imaginary parts,

$$a(\boldsymbol{\rho}, z) = a_1(\boldsymbol{\rho}, z) + i\; a_2(\boldsymbol{\rho}, z) \tag{3.79}$$

we may write

$$E(\boldsymbol{\rho}, z) = A(\boldsymbol{\rho}, z) \; \exp[i\Sigma(\boldsymbol{\rho}, z)] \; e^{ikz} \tag{3.80}$$

with the amplitude and phase given by

$$A = \sqrt{(E_{00} + a_1)^2 + a_2^2} \tag{3.81a}$$

$$\Sigma = \arctan\left(\frac{a_2}{E_{00} + a_1}\right) \tag{3.81b}$$

Far away from the screen, where the condition (3.19) for the Fraunhofer approximation applies, many inhomogeneities contribute to the field in any particular point. Then, according to the central limit theorem, a_1 and a_2 are normally distributed and this distribution can be used to construct the moments. In more general cases the representation (3.77) has to be dealt with, usually in the approximation of *weak fluctuations*. Since $\sigma_E^2 = \sigma_1^2 + \sigma_2^2$, we have from (3.74)

$$\sigma_1^2 + \sigma_2^2 + E_{00}^2 = 1 \tag{3.82}$$

When the σ_1 and σ_2 are small we thus can express (3.81a,b) by means of some terms in their Taylor expansions, thereby making use of (3.82). Hence,

$$A \approx 1 + a_1 - \tfrac{1}{2}\, a_1^2 \tag{3.83a}$$

$$\Sigma \approx a_1 - a_1 a_2 \tag{3.83b}$$

We may now express moments by means of these expressions, e.g. the mean values

$$\langle A \rangle = 1 - \tfrac{1}{2}\, \langle a_1^2 \rangle \tag{3.84a}$$

$$\langle \Sigma \rangle = -\, \langle a_1 a_2 \rangle = \psi_{a_1 a_2}(0) \tag{3.84b}$$

and the correlation functions

$$\psi_A = \langle a_1(\boldsymbol{\rho}_1, z_1)\ a_1(\boldsymbol{\rho}_2, z_2)\rangle = \psi_{a_1} \tag{3.85a}$$

$$\psi_\Sigma = \langle a_2(\boldsymbol{\rho}_1, z_1)\ a_2(\boldsymbol{\rho}_2, z_2)\rangle = \psi_{a_2} \tag{3.85b}$$

as well as the cross-correlation function (mutual correlation function)

$$\psi_{A\Sigma} = \psi_{a_1 a_2}(\boldsymbol{\rho}, z) \tag{3.85c}$$

which also appears in (3.84b).

The correlation functions (3.85a–c) can be expressed by means of the two correlation functions (1.18a,b) introduced earlier, i.e.

$$\psi_a = \langle a(\boldsymbol{\rho}_1, z_1)\ a^*(\boldsymbol{\rho}_2, z_2)\rangle \tag{3.86a}$$

$$\tilde{\psi}_a = \langle a(\boldsymbol{\rho}_1, z_1)\ a(\boldsymbol{\rho}_2, z_2)\rangle \tag{3.86b}$$

With these we get the relations

$$\psi_{a_1} = \tfrac{1}{2}\mathrm{Re}\left[\psi_a + \tilde{\psi}_a\right] \tag{3.87a}$$

$$\psi_{a_2} = \tfrac{1}{2}\mathrm{Re}\left[\psi_a - \tilde{\psi}_a\right] \tag{3.87b}$$

$$\psi_{a_1 a_2} = \tfrac{1}{2}\mathrm{Im}\left[\psi_a - \tilde{\psi}_a\right] \tag{3.87c}$$

To construct (3.86a,b) we use the integral representation (3.77). In this way we get

$$\psi_a(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) = \frac{k^2}{4\pi^2\, z^2} \int \psi_{E_0}(\boldsymbol{\rho}'-\boldsymbol{\rho}'') \exp\left[+\frac{ik}{2z}\ (\boldsymbol{\rho}_1 - \boldsymbol{\rho}')^2 - \frac{ik}{2z}\ (\boldsymbol{\rho}_2 - \boldsymbol{\rho}'')^2\right]\ \mathrm{d}\boldsymbol{\rho}'\ \mathrm{d}\boldsymbol{\rho}'' \tag{3.88a}$$

$$\tilde{\psi}_a(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) = \frac{-k^2}{4\pi^2\, z^2} \int \tilde{\psi}_{E_0}(\boldsymbol{\rho}'-\boldsymbol{\rho}'') \exp\left[+\frac{ik}{2z}\ (\boldsymbol{\rho}_1 - \boldsymbol{\rho}')^2 + \frac{ik}{2z}\ (\boldsymbol{\rho}_2 - \boldsymbol{\rho}'')^2\right]\ \mathrm{d}\boldsymbol{\rho}'\ \mathrm{d}\boldsymbol{\rho}'' \tag{3.88b}$$

Performing the integrations possible and putting $\boldsymbol{\rho} = \boldsymbol{\rho}_1 - \boldsymbol{\rho}_2$, we obtain

$$\psi_a(\boldsymbol{\rho}, z) = \psi_{E_0}(\boldsymbol{\rho}, z) \tag{3.89a}$$

$$\tilde{\psi}_a(\boldsymbol{\rho}, z) = \frac{k}{4\pi\, i\, z} \int \tilde{\psi}_{E_0}(\boldsymbol{\rho}) \exp\left[+\frac{ik}{4z}\ (\boldsymbol{\rho} - \boldsymbol{\rho}')^2\right]\ \mathrm{d}\boldsymbol{\rho}' \tag{3.89b}$$

With these expressions and (3.87a–c), we finally find the following expressions for the correlation functions (3.85a,b):

$$\psi_{A,\Sigma}(\boldsymbol{\rho}, z) = \frac{1}{2}\left\{\psi_S \mp \frac{k}{4\pi\, z} \int\limits_{-\infty}^{+\infty} \psi_S(\boldsymbol{\rho}')\ \sin\left[\frac{k(\boldsymbol{\rho} - \boldsymbol{\rho}')^2}{4z}\right]\ \mathrm{d}\boldsymbol{\rho}'\right\} \tag{3.90a, b}$$

where $(-)$ pertains to ψ_A and $(+)$ to ψ_Σ . Similarly we find for the cross-correlation function (3.85c):

$$\psi_{A\Sigma}(\boldsymbol{\rho}, z) = \frac{k}{8\pi\, z} \int\limits_{-\infty}^{+\infty} \psi_S(\boldsymbol{\rho}')\ \cos\left[\frac{k(\boldsymbol{\rho} - \boldsymbol{\rho}')^2}{4z}\right]\ \mathrm{d}\boldsymbol{\rho}' \tag{3.90c}$$

Chapter 4

Geometrical-optics approximation for media with large-scale inhomogeneities

It is known that large-scale inhomogeneities predominantly scatter radiation in the forward direction. Appropriate consideration of the differential cross section of linearly, or elliptically polarized electromagnetic field scattered by the large scale inhomogeneities shows that the depolarization effects are very small in these cases. The latter allows, to the zero approximation, confine consideration of the geometrical optics approximation for electromagnetic field in the scalar approximation. According to this, one can expect that the dominant terms of the solutions of the scalar and vector problems differ only insignificantly for large-scale inhomogeneities, unless, the effects of rotation of the plane of polarization are of interest. This is the case in the geometrical-optics approximation to be dealt with now. It is one of the methods providing solutions for wave propagation through media with large-scale inhomogeneities. The solutions are expressed in terms of rays which are formally similar to particle trajectories in classical mechanics. We shall confine our consideration by the scalar case for the Helmholtz' equation.

4.1 Asymptotic representation of the solution of Helmholtz' equation as a series of inverse powers of the wave number

We consider the scalar form of the wave equation (2.29), i.e.

$$\nabla^2 E + k^2\ \epsilon(\mathbf{r}, t)\ E = 0 \tag{4.1}$$

The permittivity ϵ so far contains the overall inhomogeneous background as well as the fluctuations and may be slowly time-dependent in the quasi-stationary approximation.

In a homogenous medium, $\epsilon =$ const., the solutions of (4.1) are plane waves, $E = A\ \exp(i\boldsymbol{\kappa} \cdot \mathbf{r})$, as we saw in Section 2.4. If ϵ varies slowly with position in the medium we can expect the solutions to bear some resemblance to plane waves, and therefore we shall attempt a solution on a form with amplitude and phase separated as follows:

$$E = A(\mathbf{r})\ \exp[ik\varphi(\mathbf{r})] \tag{4.2}$$

From this expression we immediately find the derivatives

$$\nabla E = \nabla A\, e^{ik\varphi} + ik\, \nabla\varphi\ A\ e^{ik\varphi} \tag{4.3}$$

$$\nabla^2 E = \nabla^2 A\ e^{ik\varphi} + 2ik\, \nabla A \cdot \nabla\varphi\ e^{ik\varphi} + ik\, \nabla^2\varphi\ A\ e^{ik\varphi} - k^2\ (\nabla\varphi)^2\ A\ e^{ik\varphi} \tag{4.4}$$

Substituting (4.2) into (4.1) we obtain an equation which is equivalent to Helmholtz' equation:

$$k^2\ \left[\epsilon(\mathbf{r}) - (\nabla\varphi)^2\right]\ A + ik\ \left[2\nabla A \cdot \nabla\varphi + A\ \nabla^2\varphi\right] + \nabla^2 A = 0 \tag{4.5}$$

This is an unseparable equation in the two unknown functions A and φ. In order to solve it we expand A in an infinite (asymptotic) series in negative powers of the wave number:

$$A(\mathbf{r}) = \sum_{m=0}^{\infty} \frac{A_m(\mathbf{r})}{(ik)^m} \tag{6.6}$$

while we assume φ to be independent of k. Treating k as a variable, which formally tends to infinity, we may equate each power of k in (4.5) to zero. In this way we replace (4.5) by an equation for φ and an infinite sequence of equations for A_m:

$$(\nabla\varphi)^2 = \epsilon(\mathbf{r}) \tag{4.7a}$$

$$2\,\nabla A_0 \cdot \nabla\varphi + A_0\ \nabla^2\varphi = 0 \tag{4.7b}$$

$$2\,\nabla A_1 \cdot \nabla\varphi + A_1\ \nabla^2\varphi = -\nabla^2 A_0 \tag{4.7c}$$

$$....$$

$$2\,\nabla A_m \cdot \nabla\varphi + A_m\ \nabla^2\varphi = -\nabla^2 A_{m-1} \tag{4.7d}$$

$$....$$

4.2 Eikonal and transport equations

The function φ determining the phase in (4.2) is also called eikonal and (4.7a) is known as the *eikonal equation*. The second equation (4.7b) is called the *main transport equation*. Formally the whole series of amplitude terms A_m can be constructed, but here we shall restrict ourselves to solving the eikonal equation and the main transport equation for A_0, neglecting all A_m with $m \geq 1$. Since the series (4.6) is asymptotic it is in general not convergent. With our truncation it will, however, accurately represent the solution if k is large enough.

4.2.1 Ray equations as the characteristic equations for the eikonal equation

First we shall discuss the eikonal equation (4.7a), which we may write in rectangular coordinates $\mathbf{r} = \{x_1, x_2, x_3\}$ as

$$\left(\frac{\partial\varphi}{\partial x_1}\right)^2 + \left(\frac{\partial\varphi}{\partial x_2}\right)^2 + \left(\frac{\partial\varphi}{\partial x_3}\right)^2 = \epsilon(\mathbf{r}) \tag{4.8}$$

In the language of the theory of partial differential equations, we can say that for every first order partial differential equation there can be associated a set of ordinary differential equations describing trajectories being characteristic for the solutions of the original equation. These trajectories, the rays, are orthogonal to the surfaces of constant φ. At the same time the eikonal equation (8.8) is the Hamilton-Jacobi equation for the motion of a classical particle . In accordance with the methods of classical mechanics (see GOLDSTEIN [1969], Chapter 9; ORLOV, KRAVTSOV [1980]) the Hamilton equations can hence be written for the same trajectories.

For the coordinates $\mathbf{r} = \{x_1, x_2, x_3\}$ we introduce formally the conjugate momenta

$$p_i = \frac{\partial\varphi}{\partial x_i}\,, \qquad i = 1, 2, 3 \tag{4.9}$$

or

$$\mathbf{p} = \frac{\partial\varphi}{\partial\mathbf{r}} = \nabla\varphi \tag{4.9$'$}$$

With the Hamiltonian

$$\mathcal{H}(\mathbf{r}, \mathbf{p}) = \tfrac{1}{2}\left[p^2 - \epsilon(\mathbf{r})\right] \tag{4.10}$$

the eikonal equation then takes the form of the *Hamilton-Jacobi equation* of a classical particle at location $\mathbf{r}$ with momentum $\mathbf{p}$ moving in the potential field $\epsilon(\mathbf{r})$:

$$\mathcal{H}(\mathbf{r}, \mathbf{p}) = 0 \tag{4.11}$$

or explicitly

$$p_1^2 + p_2^2 + p_3^2 - \epsilon(\mathbf{r}) = 0 \tag{4.11$'$}$$

In the techniques of analytical mechanics surfaces of constant φ are obtained and the particle trajectories are always orthogonal to these surfaces. The trajectories are the solutions of a set of ordinary differential equations

$$\frac{\mathrm{d}x_i}{\frac{\partial \mathcal{H}}{\partial p_i}} = -\frac{\mathrm{d}p_i}{\frac{\partial \mathcal{H}}{\partial x_i}} = \frac{\mathrm{d}\varphi}{\sum_{j=1}^{3} p_j \frac{\partial \mathcal{H}}{\partial p_j}} = \mathrm{d}\tau \ , \qquad i = 1,2,3 \tag{4.12}$$

where τ is the parameter of location along the trajectories. Eqs. (4.12) are the six Hamilton equations for the classical particle

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\tau} = \frac{\partial \mathcal{H}}{\partial \mathbf{p}} \tag{4.13a}$$

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau} = -\frac{\partial \mathcal{H}}{\partial \mathbf{r}} \tag{4.13b}$$

and the equation for Hamilton's characteristic function

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \sum_{i=1}^{3} p_i \frac{\partial \mathcal{H}}{\partial p_i} \tag{4.13c}$$

with solution

$$\varphi = \varphi_0 + \int_{\tau_0}^{\tau} \sum_{i=1}^{3} p_i \frac{\partial \mathcal{H}}{\partial p_i} \ \mathrm{d}\tau \tag{4.14}$$

Introducing the refractive index n through

$$n^2(\mathbf{r}) = \epsilon(\mathbf{r}) \tag{4.15}$$

and using our particular Hamiltonian according to (4.10), eqs. (4.13a,b) take the form

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\tau} = \mathbf{p} \tag{4.16a}$$

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau} = n \ \nabla n = \tfrac{1}{2}\nabla \left(n^2\right) \tag{4.16b}$$

These are our six ray equations, which have to be supplemented by six initial conditions. We shall assume that these are given through the initial phase distribution $\varphi_0(\xi,\eta)$ over a boundary surface $\mathbf{r}_0(\xi,\eta)$, described by the arbitrary parameters (ξ,η) , together with the initial momenta $\partial\varphi_0/\partial\mathbf{r}_0$ defined from

$$\frac{\partial \varphi_0}{\partial \mathbf{r}_0}\cdot\frac{\partial \mathbf{r}_0}{\partial \xi} = \frac{\partial \varphi_0}{\partial \xi} \ , \qquad \frac{\partial \varphi_0}{\partial \mathbf{r}_0}\cdot\frac{\partial \mathbf{r}_0}{\partial \eta} = \frac{\partial \varphi_0}{\partial \eta} \tag{4.17}$$

the third momentum being determined from these through the fulfilment of the eikonal equation on the boundary surface. With (5.16a,b), we may now, in principle, construct rays emanating from each point on the boundary. These rays are in general not perpendicular to the boundary, unless this boundary is a surface of constant phase. The parameters (ξ,η,τ) can be used as new curvlinear coordinates, (ξ, η) selecting the ray and τ denoting the position along the ray. These are orthogonal if the phase is constant over the initial surface.

Instead of the parameter τ it is convenient to use the length s along the ray. From (4.16a) we have $(\mathrm{d}s)^2 = (\mathrm{d}\mathbf{r})^2 = p^2\,(\mathrm{d}\tau)^2$. Using also (4.11′) and (4.15) then obtain

$$\mathrm{d}s = n\,\mathrm{d}\tau \tag{4.18}$$

Hence we may write the ray equations:

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \frac{\mathbf{p}}{n} \tag{4.19a}$$

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}s} = \nabla n = \frac{1}{2\,n}\,\nabla\,(n^2) \tag{4.19b}$$

Introducing instead of $\mathbf{p}$ the unit vector along the ray

$$\boldsymbol{\ell} = \frac{\mathbf{p}}{n} \tag{4.20}$$

the ray equations finally take the form

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \boldsymbol{\ell} \tag{4.21a}$$

$$\frac{\mathrm{d}(n\boldsymbol{\ell})}{\mathrm{d}s} = \nabla n \tag{4.21b}$$

Sometimes one meets, instead of the two first-order (vector) equations (4.21a,b), a second-order ray equation. This equation is easily derived from (5.21a,b) as follows:

$$\frac{\mathrm{d}(n\boldsymbol{\ell})}{\mathrm{d}s} = n\,\frac{\mathrm{d}\boldsymbol{\ell}}{\mathrm{d}s} + \boldsymbol{\ell}\,\left(\nabla n \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\right) = n\,\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}s^2} + \boldsymbol{\ell}\,\left(\boldsymbol{\ell}\cdot\nabla n\right) = \nabla n \tag{4.22}$$

Rearranging the terms we find

$$n\,\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}s^2} = \nabla n - \boldsymbol{\ell}\,\left(\boldsymbol{\ell}\cdot\nabla n\right) = \nabla_\perp\,n \tag{4.23}$$

where $\nabla_\perp$ is the part of the gradient transverse to the ray direction. Alternatively we may also write this equation as follows:

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}s^2} = \nabla_\perp\,\ln n \tag{4.23′}$$

The solving of the ray equations will be discussed in later Sections.

4.2.2 The main transport equation and the ray tube divergence

Let us now return to the main transport equation (4.7b). Obviously we can also write this equation as follows:

$$\nabla \cdot \left(A_0^2 \, \nabla\varphi\right) = 0 \tag{4.24}$$

Integrating this expression over an arbitrary volume $\mathcal{V}$ not containing the source of the waves, we obtain with Gauss' theorem

$$\int_{\mathcal{V}} \nabla \cdot \left(A_0^2 \, \nabla\varphi\right) \, \mathrm{d}\mathcal{V} = \oint_{\mathcal{S}} A_0^2 \, \nabla\varphi \cdot \mathrm{d}\boldsymbol{\mathcal{S}} = 0 \tag{4.25}$$

We choose this volume as a *ray tube* or a *ray pencil*. Then $\nabla\varphi$ is perpendicular to $\mathrm{d}\boldsymbol{\mathcal{S}}$ except for the end surfaces. Furthermore $|\nabla\varphi| = n$ according to the eikonal equation so that (counting the scalar surface element $\mathrm{d}\mathcal{S}$ positive in the direction of the ray) $\nabla\varphi \cdot \mathrm{d}\boldsymbol{\mathcal{S}}_1 = -n \, \mathrm{d}\mathcal{S}$ on the end surface towards the source and $\nabla\varphi \cdot \mathrm{d}\boldsymbol{\mathcal{S}}_2 = +n \, \mathrm{d}\mathcal{S}$ on the other end surface. Then (4.25) gives

$$\int_{\mathcal{S}_1} A_0^2 \, n \, \mathrm{d}\mathcal{S} = \int_{\mathcal{S}_2} A_0^2 \, n \, \mathrm{d}\mathcal{S} \tag{4.26}$$

If we choose an infinitesimally thin ray tube, we can thus draw the conclusion that

$$A_0^2 \, n \, \mathrm{d}\mathcal{S} = \text{const.} \tag{4.27}$$

along the tube. Hence we can express the amplitude for any point along the ray tube if its initial value is given

$$A_0^2(\mathbf{r}_0) \, n(\mathbf{r}_0) \, \mathrm{d}\mathcal{S}_0 = A_0^2(\mathbf{r}) \, n(\mathbf{r}) \, \mathrm{d}\mathcal{S} \tag{4.27$'$}$$

i.e. we have

$$A_0(\mathbf{r}) = A_0(\mathbf{r}_0) \sqrt{\frac{n(\mathbf{r}_0) \, \mathrm{d}\mathcal{S}_0}{n(\mathbf{r}) \, \mathrm{d}\mathcal{S}}} \tag{4.27$''$}$$

If we also account for the phase through (4.14) and the ray equations, we can construct the field in the geometrical-optics approximation:

$$E(\mathbf{r}) = A_0(\mathbf{r}_0) \sqrt{\frac{n(\mathbf{r}_0) \, \mathrm{d}\mathcal{S}_0}{n(\mathbf{r}) \, \mathrm{d}\mathcal{S}}} \, \exp\left\{ ik \left[\varphi_0(\mathbf{r}_0) + \int_{\mathbf{r}_0}^{\mathbf{r}} n(\mathbf{r}) \, \mathrm{d}s \right] \right\} \tag{4.28}$$

In order to use this expression we first have to know the phase and amplitude distributions $\varphi_0(\mathbf{r}_0)$ and $A_0(\mathbf{r}_0)$ on the initial surface. Then the rays from each point, $\mathbf{r}_0 \to \mathbf{r}$, must be constructed and from these rays the phase function $\varphi(\mathbf{r})$. We also have to determine the ray pencil divergence $\mathrm{d}\mathcal{S}/\mathrm{d}\mathcal{S}_0$. In the case of multi-path, rays emanating from several points on the initial surface coincide in the same point; then also the fields of these have to be added to get the total field $E(\mathbf{r})$.

In the general case of three-dimensional ray pencils we have the cross section

$$\mathrm{d}\mathcal{S} = \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}s} \tag{4.29}$$

where $\mathrm{d}\mathcal{V}$ is the elementary volume in the ray variables (ξ, η, s) , ξ and η being the variables along the coordinate curves on the reference surface. If the rays are described through the known functions

$$\begin{aligned} x &= x(\xi, \eta, s) \\ y &= y(\xi, \eta, s) \\ z &= z(\xi, \eta, s) \end{aligned} \tag{4.30}$$

then we have

$$\mathrm{d}\mathcal{V} = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \mathcal{D}(s)\ \mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}s \tag{4.31}$$

where $\mathcal{D}(s)$ is the determinant of the Jacobian:

$$\mathcal{D}(s) = \begin{vmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial s} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} & \dfrac{\partial y}{\partial s} \\ \dfrac{\partial z}{\partial \xi} & \dfrac{\partial z}{\partial \eta} & \dfrac{\partial z}{\partial s} \end{vmatrix} \tag{4.32}$$

Hence the cross section of the ray pencil is given by

$$\mathrm{d}\mathcal{S} = \mathcal{D}(s)\ \mathrm{d}\xi\,\mathrm{d}\eta \tag{4.33}$$

As a result the single-ray field representation in the three-dimensional geometry is just (4.28) with the following expression for the ratio of the cross sections:

$$\frac{\mathrm{d}\mathcal{S}_0}{\mathrm{d}\mathcal{S}} = \frac{\mathcal{D}(s_0)}{\mathcal{D}(s)} \tag{4.34}$$

If we look more closely at the amplitude expression (4.27″), we see that it has singularities where:
(i) $\mathrm{n}(\mathbf{r}) = 0$ (plasma resonance),
(ii) $\mathrm{dS} = 0$ (near caustics).
These conditions represent breakdowns of geometrical optics. Later we shall see that there is a third restriction for the validity of the geometrical-optics approximation which requires the scale-size of the inhomogeneities to be large in comparison with the main Fresnel zone size. This condition cannot be obtained within geometrical-optics theory. To derive it more general considerations, involving full-wave type solutions of the Helmholtz' equation, are necessary.

4.3 Methods for constructing the solutions of the geometrical-optics equations

Numerical methods for solving the ray equations are widespread but will not be dealt with here. One formulation of the ray equations in earth-centered spherical coordinates is known under the name of Haselgrove's equations [Haselgrove, 1954]. A computer program based on these equations has been produced by Jones and Stephenson [Jones and Stephenson, 1975; Jones, 1968]. A modern computer system for ray tracing, *RaTS*, which is based on the same equations has been developed at the Uppsala division of IRF.

4.3.1 Additive separation of variables for Cartesian geometry

Here we shall study a few cases where an analytical solution can be constructed. For simplicity we shall assume a two-dimensional geometry, i.e. our point source at the origin x_0, z_0 is in reality a line source. Consider the 2-dimensional eikonal equation

$$\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2 = \epsilon(x,z) \tag{4.35}$$

If we have an ionospheric structure of the particular form

$$\epsilon(x,z) = \epsilon_1(x) + \epsilon_2(z) \tag{4.36}$$

we may attempt a solution with additive separation of variables:

$$\varphi(x,z) = \varphi_1(x) + \varphi_2(z) \tag{4.37}$$

Substitution of this into (4.35) gives

$$\left(\frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\right)^2 - \epsilon_1(x) = \epsilon_2(z) - \left(\frac{\mathrm{d}\varphi_2}{\mathrm{d}z}\right)^2 = \alpha^2 \tag{4.38}$$

where α^2 is the separation constant. The separated equations are consequently

$$\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} = \sqrt{\epsilon_1(x) + \alpha^2} \tag{4.39a}$$

$$\frac{\mathrm{d}\varphi_2}{\mathrm{d}z} = \sqrt{\epsilon_2(z) - \alpha^2} \tag{4.39b}$$

resulting in the following expression for the eikonal:

$$\varphi = \int_{x_0}^{x} \sqrt{\epsilon_1(x) + \alpha^2}\ \mathrm{d}x + \int_{z_0}^{z} \sqrt{\epsilon_2(z) - \alpha^2}\ \mathrm{d}z + \varphi_0 \tag{4.40}$$

with φ_0 being the phase given by the initial condition at the boundary point (x_0, z_0) .

If we put $\varphi_0 = 0$ and further specialize to a plane stratified ionosphere, $\epsilon(x,z) = \epsilon_2(z)$, $\epsilon_1(x) = 0$, we find from (4.40)

$$\varphi = \alpha\,(x - x_0) + \int\limits_{z_0}^{z} \sqrt{\epsilon(z) - \alpha^2}\;\mathrm{d}z \tag{4.41}$$

Earlier we saw that the ray direction is along the gradient of the eikonal, $\mathrm{d}\mathbf{r} = \nabla\varphi\;\mathrm{d}s/n$; see e.g. (4.9′) and (4.19a). With this in mind we can easily express the local slope of the rays corresponding to the case (4.41)

$$\tan\theta(z) = \frac{\mathrm{d}x}{\mathrm{d}z} = \frac{\partial\varphi}{\partial x}\bigg/\frac{\partial\varphi}{\partial z} = \frac{\alpha}{\sqrt{\epsilon(z) - \alpha^2}} \tag{4.42}$$

Integrating this differential equation we get the ray curve expressed in the form of x as a function of z :

$$x - x_0 = \int\limits_{z_0}^{z} \frac{\alpha\;\mathrm{d}z}{\sqrt{\epsilon(z) - \alpha^2}} \tag{4.43}$$

If we introduce the notation

$$\phi(z,\alpha) = \int\limits_{z_0}^{z} \sqrt{\epsilon(z) - \alpha^2}\;\mathrm{d}z \tag{4.44}$$

we may write the eikonal (4.41) as follows:

$$\varphi = \alpha\,(x - x_0) + \phi(z,\alpha) \tag{4.41′}$$

Because of this, and since we have

$$\frac{\partial\phi}{\partial\alpha} = \phi'_\alpha = -\int\limits_{z_0}^{z} \frac{\alpha\;\mathrm{d}z}{\sqrt{\epsilon(z) - \alpha^2}} \tag{4.45}$$

it is also possible to write the ray expression (4.43) as follows:

$$\frac{\partial\varphi}{\partial\alpha} = x - x_0 + \phi'_\alpha = 0 \tag{4.43′}$$

We shall illustrate the practical use of these ray expressions in a few cases. We choose the starting points of the rays on the ground level $z_0 = 0$ where we have free-space propagation, $\epsilon(0) = 1$. Then we see from (4.42) that the parameter α is determined by θ_0 , the initial angle of the ray with the vertical:

$$\tan\theta_0 = \frac{\alpha}{\sqrt{1 - \alpha^2}} \qquad \Rightarrow \qquad \alpha = \sin\theta_0 \tag{4.46}$$

The situation when α is fixed and the starting point x_0 of the rays is varied corresponds to an initial wave which is truly plane if the phase is arranged to be constant over a surface perpendicular to the incident rays. To the left we show a case with a rather low frequency where the rays are forming a horizontal *caustic* at the height of reflection $\tilde{z}$ for which the local plasma frequency fulfils $\omega_p(\tilde{z}) = \omega \cos\theta_0$. The caustic represents a situation where several rays are merging and interferring with each other. To the right we show penetrating rays of a higher frequency; $\omega \cos\theta_0 > \omega_{crit}$, where ω_{crit} is the critical frequency of the ionosphere.

Now we consider a fixed-frequency point source at the origin, $x_0 = 0$, $z_0 = 0$, while the launching direction α is varied. This is the case of a physical transmitter located to the earth surface. In the upper case we show low-frequency rays where even the vertical ray is reflected; $\omega < \omega_{crit}$. There we find a caustic which is no longer horizontal; hence the formation of caustics depends on the initial conditions. In the lower case the frequency is higher, $\omega > \omega_{crit}$, so that some rays are penetrating. Besides the *exterior caustic* seen also in the upper case we now also have an *interior caustic* formed by the returning high-elevation rays. The crossing-point of this caustic with the ground is called the skip distance; no real rays are reaching the ground at closer distance to the source. The points defining these caustics are defined by the ray expression (4.43′) together with the condition

$$\phi''_{\alpha\alpha} = 0 \tag{4.47}$$

as we shall see later on in this Section.

We shall now take a closer look at the amplitude expressions (4.27). At distances within the free-space area from this source the phase is then constant over cylindrical surfaces. We choose a reference cylinder of radius r_0 , where we put the amplitude

$$A_{00} = \frac{1}{\sqrt{r_0}} \tag{4.48}$$

To determine the amplitude at arbitrary points we have to calculate the ratio $\mathrm{d}\mathcal{S}_0/\mathrm{d}\mathcal{S}$, which is rather simple for the two-dimensional case and central ray field. We then choose an infinitesimal surface element (or infinitesimal element of length in our two-dimensional problem) on this cylinder

$$\mathrm{d}\mathcal{S}_0 = r_0 \ \mathrm{d}\theta_0 \tag{4.49}$$

The rays bounding this element form a ray pencil through space. Using the ray expression (4.43′), we may easily express the horizontal distance between these rays at a fixed height z :

$$\mathrm{d}x = |\phi''_{\alpha\alpha}| \ \frac{\mathrm{d}\alpha}{\mathrm{d}\theta_0} \ \mathrm{d}\theta_0 = |\phi''_{\alpha\alpha}| \ \cos\theta_0 \ \mathrm{d}\theta_0 \tag{4.50}$$

We then find the following cross-section of the ray pencil at this height:

$$\mathrm{d}\mathcal{S} = \mathrm{d}x \ \cos\theta(z) = |\phi''_{\alpha\alpha}| \ \cos\theta_0 \ \cos\theta \ \mathrm{d}\theta_0 \tag{4.51}$$

Hence the surface ratio involved in the amplitude expression (4.27″) is given by

$$\frac{\mathrm{d}\mathcal{S}_0}{\mathrm{d}\mathcal{S}} = \frac{r_0}{|\phi''_{\alpha\alpha}|\ \cos\theta_0\ \cos\theta} \tag{4.52}$$

We shall rewrite the cosines in this expression by means of (4.46):

$$\cos\theta_0 = \sqrt{1-\alpha^2} \tag{5.53a}$$

and by means of (4.42):

$$\cos\theta(z) = \frac{\sqrt{\epsilon(z)-\alpha^2}}{\sqrt{\epsilon(z)}} \tag{4.53b}$$

obtaining

$$\frac{\mathrm{d}\mathcal{S}_0}{\mathrm{d}\mathcal{S}} = \frac{r_0\sqrt{\epsilon}}{|\phi''_{\alpha\alpha}|\ \sqrt{1-\alpha^2}\ \sqrt{\epsilon-\alpha^2}} \tag{4.54}$$

Putting this, together with (4.48) and (4.15), into (4.27″), the amplitude expression becomes

$$A_0(x,z) = \frac{1}{|\phi''_{\alpha\alpha}(z,\alpha)|^{\frac{1}{2}}\ [1-\alpha^2]^{\frac{1}{4}}\ [\epsilon(z)-\alpha^2]^{\frac{1}{4}}} \tag{4.55}$$

According to (4.28), (4.41′) we now have the *single-ray field representation*

$$E(x,z) = \text{const.}\ \frac{\exp\{ik[\alpha x + \phi(z,\alpha)]\}}{|\phi''_{\alpha\alpha}(z,\alpha)|^{\frac{1}{2}}\ [1-\alpha^2]^{\frac{1}{4}}\ [\epsilon(z)-\alpha^2]^{\frac{1}{4}}} \tag{4.56}$$

with the x-coordinate expressed as a function of z through the ray expression (4.43′). In connection with this representation we wish to emphasize some important points:

(i) In the case of *multiple rays* crossing the same point in space, several expressions like (4.56) have to be superposed (vector addition in the case of electromagnetic waves).

(ii) For *rays before reflection* or penetrating rays the phase function ϕ is given by (4.44). For *rays reflected* at height $\tilde{z}$, where $\epsilon(\tilde{z})-\alpha^2=0$, this phase function has to be generalized to

$$\phi(\alpha,z) = \int_0^{\tilde{z}} \sqrt{\epsilon(z)-\alpha^2}\ \mathrm{d}z + \int_z^{\tilde{z}} \sqrt{\epsilon(z)-\alpha^2}\ \mathrm{d}z - \frac{\pi}{2\,k} \tag{4.44′}$$

The term $\pi/2k$ added here is a phase shift $-\pi/2$ at reflection from the caustic which cannot be derived within geometrical optics.

(iii) The ray expression (4.43), (4.43′) similarly after reflection has to be generalized to

$$x - x_0 = \int_0^{\tilde{z}} \frac{\alpha\ \mathrm{d}z}{\sqrt{\epsilon(z)-\alpha^2}} + \int_z^{\tilde{z}} \frac{\alpha\ \mathrm{d}z}{\sqrt{\epsilon(z)-\alpha^2}} \tag{4.43″}$$

(iv) The quartic root of $\epsilon(z) - \alpha^2$ in the denominatior of (4.55) has a zero as $z \to \tilde{z}$. Simultaneously the square root of $|\phi''_{\alpha\alpha}|$ tends to ∞, however, so that the amplitude is finite through the reflection.
(v) When the observation point is on the ground, we find from (4.43″) the covered distance of the ray

$$D(\alpha) = 2 \int_0^{\tilde{z}} \frac{\alpha \, \mathrm{d}z}{\sqrt{\epsilon(z) - \alpha^2}} \tag{4.57}$$

From this expression it is possible to draw a $D(\alpha)$-curve, pertaining to the actual electron density height profile $N_e(z)$. There are two particular example for two frequencies; one with $\omega > \omega_{crit}$, where we can see the skip distance, and another with $\omega < \omega_{crit}$, where all distances $0 < D < \infty$ are obtained. In the physical ionosphere the 1-hop distance is limited by the earth curvature, the height of the reflecting layer and the absorption of low-elevation rays.

4.3.2 Perturbation theory in the geometrical-optics approximation

We now decompose the permittivity into a background term $\epsilon_0(\mathbf{r})$, for which we assume the geometrical-optics field representation according to the preceding Sections is known, and a small perturbation term $\epsilon(\mathbf{r})$, describing local inhomogeneities which may be deterministic or random. This is similar to the decomposition we introduced in the Chapter on single scattering. Then we may write the total eikonal and main transport equations (4.7a,b) and the corresponding ray equations (4.16a,b) as follows:

$$(\nabla\varphi)^2 = \epsilon_0(\mathbf{r}) + \epsilon(\mathbf{r}) \tag{4.58}$$

$$2\,\nabla A_0 \cdot \nabla\varphi + A_0\, \nabla^2\varphi = 0 \tag{4.59}$$

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}\tau^2} = \tfrac{1}{2}\, \nabla\, [\epsilon_0(\mathbf{r}) + \epsilon(\mathbf{r})] \tag{4.60}$$

Utilizing the solution of the background problem, we shall in the following introduce perturbation expansions of the unknowns in the above equations and eventually obtain the first-order corrections due to the perturbations.

The perturbation approach presented in this Section is good for short paths, e.g. for 1-hop propagation. For long paths such as propagation in ionospheric ducts more complicated approaches, involving two-scale expansions, are necessary.

Perturbation theory for the eikonal equation

We introduce for the eikonal the following expansion

$$\varphi = \varphi_0 + \varphi_1 + ... \tag{4.61}$$

so that the eikonal equation (4.58) takes the form

$$(\nabla\varphi_0 + \nabla\varphi_1 + ...)^2 = \epsilon_0(\mathbf{r}) + \epsilon(\mathbf{r}) \tag{4.62}$$

Identifying successive orders of smallness in this equation, we find the background equation as the 0:th order equation

$$(\nabla\varphi_0)^2 = \epsilon_0(\mathbf{r}) \tag{4.63a}$$

where hence φ_0 is assumed known, the first-order equation

$$2\,\nabla\varphi_1 \cdot \nabla\varphi_0 = \epsilon(\mathbf{r}) \tag{4.63b}$$

the second-order equation

$$2\,\nabla\varphi_2 \cdot \nabla\varphi_0 = -(\nabla\varphi_1)^2 \tag{4.63c}$$

and so on.

With the variable s denoting distance along the ray we know from (4.9) that

$$|\nabla\varphi_0| = \frac{\mathrm{d}\varphi_0}{\mathrm{d}s} \tag{4.64}$$

Using the unit vector $\boldsymbol{\ell}$ along the ray we can then write (4.63a) as follows:

$$\nabla\varphi_0 = \sqrt{\epsilon_0[\mathbf{r}(s)]}\;\boldsymbol{\ell} \tag{4.65}$$

Inserting this expression into the first-order equation (4.63b), we get

$$2\,\sqrt{\epsilon_0[\mathbf{r}(s)]}\;\frac{\mathrm{d}\varphi_1}{\mathrm{d}s} = \epsilon[\mathbf{r}(s)] \tag{4.66}$$

and may hence obtain the first-order correction to the phase along the undisturbed ray:

$$\varphi_1(\mathbf{r}) = \int\limits_0^s \frac{\epsilon[\mathbf{r}(s)]}{2\,\sqrt{\epsilon_0[\mathbf{r}(s)]}}\;\mathrm{d}s \tag{4.67}$$

The parallel component of $\nabla\varphi_1$ is according to (4.66) given by

$$\nabla_{\parallel}\,\varphi_1 = \frac{\epsilon}{2\sqrt{\epsilon_0}} \tag{4.68a}$$

If we describe the fluctuations by their variance σ_ϵ, this quantity is $\sim \frac{1}{2}\sigma_\epsilon/\sqrt{\epsilon_0}$. The perpendicular component of $\nabla\varphi_1$, on the other hand, can to good approximation be obtained from (4.67) as follows:

$$\nabla_{\perp}\,\varphi_1 = \int\limits_0^s \frac{\nabla_{\perp}\,\epsilon}{2\,\sqrt{\epsilon_0}}\;\mathrm{d}s \tag{4.68b}$$

Assuming that the rays are passing through an area of length L containing many irregularities of characteristic length ℓ_ϵ, we may obtain the following estimate for this component $\nabla_\perp \varphi_1 \sim \frac{1}{2}\sigma_\epsilon L/(\ell_\epsilon \sqrt{\epsilon_0})$. Putting the above two estimates together we have

$$\frac{\nabla_\parallel \varphi_1}{\nabla_\perp \varphi_1} \sim \frac{\ell_\epsilon}{L} \ll 1 \tag{4.69}$$

i.e. after passage through many irregularities the parallel part of the gradient of the correction is much smaller than the transverse part.

We mention here briefly one application of the result (4.67), viz. as a way to account for the effect of low absorption, $\nu/\omega \ll 1$. Let us separate the permittivity into real and imaginary parts:

$$\epsilon = \mathrm{Re}\ \epsilon + i\ \mathrm{Im}\ \epsilon \tag{4.70}$$

and identify the real part of this with the background. The dissipated power is then expressed by (4.67) through the imaginary part of the phase:

$$\varphi_1 = \frac{i}{2} \int_0^s \frac{\mathrm{Im}\ \epsilon(s,0)}{\sqrt{\mathrm{Re}\ \epsilon(s,0)}}\ \mathrm{d}s \tag{4.71}$$

We shall also use the result (4.67) in determining the change of angle of propagation due to a perturbation. According to (5.20) the unit vector along the direction of propagation is

$$\boldsymbol{\ell} = \frac{\nabla\varphi}{\sqrt{\epsilon_0 + \epsilon}} \approx \frac{\nabla\varphi_0 + \nabla\varphi_1}{\sqrt{\epsilon_0}[1 + \epsilon/(2\epsilon_0)]} \approx \frac{\nabla\varphi_0}{\sqrt{\epsilon_0}} + \frac{\nabla\varphi_1}{\sqrt{\epsilon_0}} - \frac{\nabla\varphi_0}{\sqrt{\epsilon_0}}\ \frac{\epsilon}{2\epsilon_0} \tag{4.72}$$

With the direction $\boldsymbol{\ell}_0 = \nabla\varphi_0/\sqrt{\epsilon_0}$ of the undisturbed ray and with the use of the first-order equation (4.63b), we hence we find the first-order correction to the ray direction

$$\boldsymbol{\ell} - \boldsymbol{\ell}_0 = \frac{1}{\sqrt{\epsilon_0}}\ [\nabla\varphi_1 - \boldsymbol{\ell}_0\,(\boldsymbol{\ell}_0 \cdot \nabla\varphi_1)] = \frac{\nabla_\perp \varphi_1}{\sqrt{\epsilon_0}} \tag{4.73}$$

The results of this subsection can be used when ϵ is a local deterministic inhomogeneity. Later we shall also use them in applications when ϵ is a random function.

Perturbation theory for the main transport equation

Instead of the amplitude A_0 of the wave we shall here introduce the level

$$\chi = \ln A_0 \tag{4.74}$$

With this notation the main transport equation (4.59) can be written

$$2\ \nabla\chi \cdot \nabla\varphi + \nabla^2\varphi = 0 \tag{4.75}$$

Together with the expansion (4.61) for φ we now introduce a corresponding expansion for the level:

$$\chi = \chi_0 + \chi_1 + ... \tag{4.76}$$

In this way we obtain from (4.75) the zero:th- and first-order equations

$$2\, \nabla\chi_0 \cdot \nabla\varphi_0 + \nabla^2\varphi_0 = 0 \tag{4.77a}$$

$$2\, \nabla\chi_0 \cdot \nabla\varphi_1 + 2\, \nabla\chi_1 \cdot \nabla\varphi_0 + \nabla^2\varphi_1 = 0 \tag{4.77b}$$

The solution of (4.77a) is already known according to Section 4.2.2. By virtue of (4.69) we neglect in (4.77b) the parallel component of $\nabla\varphi_1$. Since $\nabla\chi_0$ is mainly parallel to the ray direction we shall, furthermore, neglect the first term in (4.77b) obtaining the approximate first-order equation

$$2\, \nabla\chi_1 \cdot \nabla\varphi_0 = -\nabla_\perp^2\, \varphi_1 \tag{4.78}$$

Using once again (4.65) we find that this equation can be written

$$2\, \sqrt{\epsilon_0}\, \frac{\mathrm{d}\chi_1}{\mathrm{d}s} = -\nabla_\perp^2\, \varphi_1 \tag{4.79}$$

which leads to the following first-order correction to the level:

$$\chi_1(s) = -\int_0^s \frac{\nabla_\perp^2\, \varphi_1}{2\, \sqrt{\epsilon_0}}\, \mathrm{d}s \tag{4.80}$$

Perturbation theory for the ray equations

In treating the ray equation (4.60) we now introduce the ray path expansion

$$\mathbf{r}(\tau) = \mathbf{r}_0(\tau) + \mathbf{r}_1(\tau) + ... \tag{4.81}$$

Thereby we must remember that the permittivity terms depend on the perturbed positions, $\epsilon_0[\mathbf{r}_0(\tau) + \mathbf{r}_1(\tau) + ...]$ and $\epsilon[\mathbf{r}_0(\tau) + \mathbf{r}_1(\tau) + ...]$, so that (4.60) to the first order can be written

$$\frac{\mathrm{d}^2(\mathbf{r}_0 + \mathbf{r}_1)}{\mathrm{d}\tau^2} = \tfrac{1}{2}\, [\nabla\, \epsilon_0(\mathbf{r}_0) + (\mathbf{r}_1 \cdot \nabla)\, \nabla\epsilon_0(\mathbf{r}_0) + \nabla\, \epsilon(\mathbf{r}_0)] \tag{4.82}$$

Separating the zero:th and first orders we find the equation for the undisturbed ray

$$\frac{\mathrm{d}^2\mathbf{r}_0}{\mathrm{d}\tau^2} = \tfrac{1}{2}\, \nabla\, \epsilon_0(\mathbf{r}_0) \tag{4.83a}$$

and the equation for the first-order correction

$$\frac{\mathrm{d}^2\mathbf{r}_1}{\mathrm{d}\tau^2} - \tfrac{1}{2}\, (\mathbf{r}_1 \cdot \nabla)\, \nabla\epsilon_0(\mathbf{r}_0) = \tfrac{1}{2}\, \nabla\, \epsilon(\mathbf{r}_0) = \mathbf{F}_1 \tag{4.83b}$$

For each $\mathbf{r}_0$ (4.83b) is a system of linear second-order differential equations in the unknown components of $\mathbf{r}_1$. The quantity $\mathbf{F}_1$ is introduced here only to have a short-hand notation for the right-hand side.

Consider now the known undisturbed rays

$$\mathbf{r}_0 = \mathbf{r}_0(\tau, \beta_m) \tag{4.84}$$

where the parameters β_m are the six initial conditions of the solutions of (4.83a). If we formally differentiate (4.83a) with respect to these parameters we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left(\frac{\partial \mathbf{r}_0}{\partial \beta_m}\right) = \tfrac{1}{2}\left(\frac{\partial \mathbf{r}_0}{\partial \beta_m} \cdot \nabla\right) \nabla\, \epsilon_0(\mathbf{r}_0) \tag{4.85}$$

Hence, if we introduce the six new vectors

$$\boldsymbol{\rho}_m = \frac{\partial \mathbf{r}_0}{\partial \beta_m}\ , \qquad m = 1, ..., 6 \tag{4.86}$$

we find that they satisfy the differential equation

$$\frac{\mathrm{d}^2 \boldsymbol{\rho}_m}{\mathrm{d}\tau^2} - \tfrac{1}{2}\left(\boldsymbol{\rho}_m \cdot \nabla\right) \nabla \epsilon_0(\mathbf{r}_0) = 0 \tag{4.87}$$

i.e. they are solutions of the homogeneous system of equations corresponding to (4.83b).

According to a standard procedure called *the method with variation of the parameters* we may use these vectors to construct the solutions of the inhomogeneous equation by assuming a trial solution on the form

$$\mathbf{r}_1 = \sum_{m=1}^{6} C_m(\tau)\, \boldsymbol{\rho}_m \tag{4.88a}$$

If we let the functions C_m be subject to the constraint

$$\sum_{m=1}^{6} \frac{\mathrm{d}C_m}{\mathrm{d}\tau}\, \boldsymbol{\rho}_m = 0 \tag{4.88b}$$

we may write the derivative of (4.88a) in the form as if C_m were constants:

$$\frac{\mathrm{d}\mathbf{r}_1}{\mathrm{d}\tau} = \sum_{m=1}^{6} C_m(\tau)\, \frac{\mathrm{d}\boldsymbol{\rho}_m}{\mathrm{d}\tau} \tag{4.88c}$$

Hence the second derivative of (4.88a) is

$$\frac{\mathrm{d}^2\mathbf{r}_1}{\mathrm{d}\tau^2} = \sum_{m=1}^{6} \frac{\mathrm{d}C_m}{\mathrm{d}\tau}\, \frac{\mathrm{d}\boldsymbol{\rho}_m}{\mathrm{d}\tau} + \sum_{m=1}^{6} C_m(\tau)\, \frac{\mathrm{d}^2\boldsymbol{\rho}_m}{\mathrm{d}\tau^2} \tag{4.88d}$$

Substituting (4.88a,d) into (4.83b), and using thereby also (4.87), we find the following equation:

$$\sum_{m=1}^{6} \frac{\mathrm{d}C_m}{\mathrm{d}\tau} \frac{\mathrm{d}\boldsymbol{\rho}_m}{\mathrm{d}\tau} = \mathbf{F}_1 \tag{4.89}$$

The result of the procedure just carried out is that we have replaced the second-order differential equation (4.83b) in $\mathbf{r}_1$ by a system of six linear first-order equations, (4.88b) and (4.89), in the functions $C_m(\tau)$. Provided that the system determinant is non-zero, this system can be inverted to take the form

$$\frac{\mathrm{d}C_m}{\mathrm{d}\tau} = \sum_{j=1}^{3} Q_{mj}\, F_{1j} \ , \qquad m = 1, ..., 6 \tag{4.90}$$

The summation here is only taken up to $j = 3$ since three components are zero in the right-hand side column vector of (4.89), (4.88b). The elements Q_{mj} of the 6×6 system matrix are, in principle, known expressions of the matrix elements $\boldsymbol{\rho}_m$ and $\mathrm{d}\boldsymbol{\rho}_m/\mathrm{d}\tau$ of the system (4.89), (4.88b). Hence (4.90) can be integrated to yield

$$C_m(\tau) = C_m(0) + \int_0^{\tau} \sum_{j=1}^{3} Q_{mj}(\tau)\, F_{1j}(\tau)\, \mathrm{d}\tau \ , \qquad m = 1, ..., 6 \tag{4.91}$$

When this solution has been obtained the first-order correction to $\mathbf{r}$ is expressed by means of (4.88a). In the same way the first-order correction to the conjugate momentum, $\mathbf{p}_1 = \mathrm{d}\mathbf{r}_1/\mathrm{d}\tau$, is given by (4.88c).

4.4 Geometrical optics for random fluctuations and homogeneous background

Next we shall use the geometrical-optics technique for describing the statistical properties of wave propagation through a medium with random fluctuations of the dielectric permittivity. We shall decompose the medium as in (4.58), where we assume that the field in the background medium $\epsilon_0(\mathbf{r})$ is known in the geometrical-optics approximation and that the fluctuations are zero-mean, $\langle \epsilon(\mathbf{r}) \rangle = 0$. Our major tool in describing these fluctuations will be the perturbation theory expounded in Section 4.3.2. Hence we shall use the first-order phase correction (4.67), ray-direction correction (4.73) and level correction (4.80) to investigate the fluctuations of the corresponding quantities in the random medium.

Throughout this Section we shall adopt the simplifying assumption of *homogeneous background* $\epsilon_0 =$ const. and *propagation along the z-axis.*

4.4.1 Fluctuations of the phase of the field

Since $\langle \epsilon \rangle = 0$, we can immediately see from (4.67) that the mean of the first-order phase fluctuations is zero:

$$\langle \varphi_1 \rangle = 0 \tag{4.92}$$

Assuming the fluctuations to be statistically homogeneous the correlation function of the phase fluctuations is from (4.77) as follows:

$$\psi_\varphi(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z_1, z_2) = \frac{1}{4\epsilon_0} \int_0^{z_1} \int_0^{z_2} \psi_\epsilon(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2, z' - z'')\ \mathrm{d}z'\, \mathrm{d}z'' \tag{4.93}$$

The integration here is performed over a rectangle in the $z'z''$-plane. Changing the variables into

$$2\,\eta = z' + z''\ , \qquad \zeta = z' - z'' \tag{4.94}$$

and introducing the transverse the difference variable

$$\boldsymbol{\rho} = \boldsymbol{\rho}_1 - \boldsymbol{\rho}_2 \tag{4.95}$$

we can write this integral

$$\psi_\varphi(\boldsymbol{\rho}, z_1, z_2) = \frac{1}{4\epsilon_0} \int_\diamond \psi_\epsilon(\boldsymbol{\rho}, \zeta)\ \mathrm{d}\zeta\, \mathrm{d}\eta \tag{4.96}$$

where the area of integration in the $\zeta\eta$-plane is a parallelogram.

Suppose now that we consider the correlation function after propagation through many irregularities so that $\ell_\epsilon \ll z_1,\, z_2$. Then the integrand is non-zero only close to the η-axis , and we may without introducing significant error extend the ζ-integration to $\pm\infty$ and at the same time perform the integration over η from zero to

$$z_< = \min(z_1, z_2) \tag{4.97}$$

Hence we obtain the result

$$\psi_\varphi(\boldsymbol{\rho}, z_1, z_2) = \frac{z_<}{4\epsilon_0} \int_{-\infty}^{+\infty} \psi_\epsilon(\boldsymbol{\rho}, \zeta)\ \mathrm{d}\zeta \tag{4.98}$$

This result is statistically homogeneous in the transverse direction, but not in the z-direction. Indeed, if we vary z_2 while keeping z_1 fixed, ψ_φ first increases linearly until $z_1 = z_2$ and then remains constant. In particular, we have from (4.98):

$$\sigma_\varphi^2(z) = \psi_\varphi(0, z, z) = \frac{z}{4\epsilon_0} \int_{-\infty}^{+\infty} \psi_\epsilon(0, \zeta)\ \mathrm{d}\zeta \tag{4.99}$$

This integral, as we have mentioned before [eq. (1.30)], is equal to $2\,\ell_\epsilon\,\sigma_\epsilon^2$ if it converges. Then

$$\sigma_\varphi^2(z) = \frac{z\,\ell_\epsilon}{2\epsilon_0}\,\sigma_\epsilon^2 \tag{4.100}$$

We shall use this quantity to express the *longitudinal correlation coefficient*

$$K_\varphi(0,z_1,z_2) = \frac{\psi_\varphi(0,z_1,z_2)}{\sigma_\varphi(z_1)\sigma_\varphi(z_2)} = \begin{cases} \sqrt{z_1/z_2} & z_1 < z_2 \\ \sqrt{z_2/z_1} & z_1 > z_2 \end{cases} \tag{4.101}$$

Finally we recall that the physical phase of the wave is φ multiplied by the wave number k and hence that the correlation function of the phase is

$$\psi_S = k^2\ \psi_\varphi \tag{4.102}$$

4.4.2 Fluctuations of the angle of arrival

We have already considered the deviation $\boldsymbol{\ell} - \boldsymbol{\ell}_0$ of the ray direction due to an inhomogeneity in (4.73). When this deviation is very small the angle ϑ of the deviation is approximately equal to the length of the difference vector

$$\vartheta \approx |\boldsymbol{\ell} - \boldsymbol{\ell}_0| \tag{4.103}$$

In this approximation and with the z-axis as the major direction of propagation we find from (4.73):

$$\vartheta_x = \frac{1}{\sqrt{\epsilon_0}}\,\frac{\partial\varphi_1}{\partial x} \tag{4.104a}$$

$$\vartheta_y = \frac{1}{\sqrt{\epsilon_0}}\,\frac{\partial\varphi_1}{\partial y} \tag{4.104b}$$

Because of (4.92) we directly find that the angular fluctuations are zero-mean:

$$\langle\vartheta_x\rangle = 0\ , \qquad \langle\vartheta_y\rangle = 0 \tag{4.105}$$

To construct the correlation functions of the angular fluctuations from (4.104a,b) we first regard

$$\vartheta_x(\mathbf{r}_1)\,\vartheta_x(\mathbf{r}_2) = \frac{1}{\epsilon_0}\,\frac{\partial^2}{\partial x_1\,\partial x_2}\ \varphi_1(\mathbf{r}_1)\,\varphi_1(\mathbf{r}_2) \tag{4.106}$$

We found in the previous Section that if the permittivity fluctuations are statistically homogeneous, then the phase fluctuations are statistically homogeneous in the transverse direction; cf. (4.98). With $\boldsymbol{\rho} = \{x_1 - x_2, y_1 - y_2\}$ we hence obtain from (4.106):

$$\langle\vartheta_x(\mathbf{r}_1)\,\vartheta_x(\mathbf{r}_2)\rangle = -\frac{1}{\epsilon_0}\,\frac{\partial^2}{\partial x^2}\ \langle\psi_\varphi(\boldsymbol{\rho},z_1,z_2)\rangle \tag{4.106a}$$

and analogously for the remaining two correlation functions:

$$\langle\vartheta_y(\mathbf{r}_1)\,\vartheta_y(\mathbf{r}_2)\rangle = -\frac{1}{\epsilon_0}\,\frac{\partial^2}{\partial y^2}\ \langle\psi_\varphi(\boldsymbol{\rho},z_1,z_2)\rangle \tag{4.106b}$$

$$\langle\vartheta_x(\mathbf{r}_1)\,\vartheta_y(\mathbf{r}_2)\rangle = -\frac{1}{\epsilon_0}\,\frac{\partial^2}{\partial x\,\partial y}\ \langle\psi_\varphi(\boldsymbol{\rho},z_1,z_2)\rangle \tag{4.106c}$$

4.4.3 Level fluctuations

For statistically homogeneous permittivity fluctuations also the first-order level fluctuations turn out to be zero-mean from (4.80):

$$\langle \chi_1 \rangle = 0 \tag{4.107}$$

The *transverse* correlation function of the level fluctuations is from (4.80) and (4.98)

$$\psi_\chi(\boldsymbol{\rho}, z, z) = \frac{1}{4\epsilon_0}\ \nabla_\perp^4 \int_0^z \int_0^z \psi_\varphi(\boldsymbol{\rho}, z', z'')\ \mathrm{d}z'\ \mathrm{d}z''$$

$$= \frac{1}{4\epsilon_0}\ \nabla_\perp^4 \int_0^z \int_0^z \mathrm{d}z'\ \mathrm{d}z''\ \frac{z_<}{4\epsilon_0} \int_{-\infty}^{+\infty} \psi_\epsilon(\boldsymbol{\rho}, \zeta)\ \mathrm{d}\zeta = \frac{\nabla_\perp^4}{16\epsilon_0^2} \int_{-\infty}^{+\infty} \psi_\epsilon(\boldsymbol{\rho}, \zeta)\ \mathrm{d}\zeta \int_0^z \int_0^z z_<\ \mathrm{d}z'\ \mathrm{d}z'' \tag{4.108}$$

The integral over the $z'z''$-surface can easily be evaluated to yield $z^3/3$ so the final result for the transverse correlation function is

$$\psi_\chi(\boldsymbol{\rho}, z, z) = \frac{z^3\ \nabla_\perp^4}{48\epsilon_0^2} \int_{-\infty}^{+\infty} \psi_\epsilon(\boldsymbol{\rho}, \zeta)\ \mathrm{d}\zeta \tag{4.109}$$

In particular the variance of this is given by

$$\sigma_\chi^2(z) = \psi_\chi(0, z, z) = \frac{z^3\ \nabla_\perp^4}{48\epsilon_0^2} \int_{-\infty}^{+\infty} \psi_\epsilon(0, \zeta)\ \mathrm{d}\zeta \tag{4.110}$$

Sometimes it is convenient to use, instead of (4.98), (4.109), the spectral representation of the permittivity fluctuations

$$\psi_\epsilon(\boldsymbol{\rho}, z) = \int_{-\infty}^{+\infty} \phi_\epsilon(\boldsymbol{\kappa}, \kappa_z)\ \exp[+i\,(\boldsymbol{\kappa} \cdot \boldsymbol{\rho} + \kappa_z z)]\ \mathrm{d}\boldsymbol{\kappa}\ \mathrm{d}\kappa_z \tag{4.111}$$

With this representation we find from (4.99), (4.102), thereby using also the δ-function, the variance of the phase fluctuations

$$\sigma_S^2(z) = \frac{k^2 z}{4\epsilon_0} \int_{-\infty}^{+\infty} \phi_\epsilon(\boldsymbol{\kappa}, \kappa_z)\ e^{i\kappa_z \zeta}\ \mathrm{d}\boldsymbol{\kappa}\ \mathrm{d}\zeta = \frac{\pi k^2 z}{2\epsilon_0} \int_{-\infty}^{+\infty} \phi_\epsilon(\boldsymbol{\kappa}, 0)\ \mathrm{d}\boldsymbol{\kappa} \tag{4.112}$$

In the same way we get from (4.110) the variance of the level fluctuations

$$\sigma_\chi^2(z) = \frac{\pi z^3}{24\epsilon_0^2} \int_{-\infty}^{+\infty} \kappa^4\ \phi_\epsilon(\boldsymbol{\kappa}, 0)\ \mathrm{d}\boldsymbol{\kappa} \tag{4.113}$$

4.4.4 Relative contributions of phase and level fluctuations

We shall now make a qualitative comparison of the orders of magnitude of the phase fluctuations (4.102) with (4.99) and the level fluctuations (4.110). To obtain estimates of the orders of magnitude of these quantities we use the effective scale-size of the irregularities ℓ_ϵ introduced in (4.100), thereby noting that the spatial spectrum of the fluctuations has the scale-size ℓ_ϵ^{-1}. Hence we get

$$\sigma_S^2 \sim \frac{k^2 z \ell_\epsilon \sigma_\epsilon^2}{2\epsilon_0} \tag{4.114a}$$

$$\sigma_\chi^2 \sim \frac{z^3 \ell_\epsilon \sigma_\epsilon^2}{48\epsilon_0^2 \ell_\epsilon^4} \tag{4.144b}$$

We now introduce the *wave parameter*

$$D = \frac{\sqrt{\lambda z}}{\ell_\epsilon} \tag{4.115}$$

to describe the relative magnitudes of these fluctuations to obtain

$$\frac{\sigma_S^2}{\sigma_\chi^2} \sim \frac{k^2 \ell_\epsilon^4}{z^2} = \left(\frac{\ell\epsilon}{\sqrt{\lambda z}}\right)^4 = D^{-4} \tag{4.116}$$

In the next Chapter on Rytov's method we shall obtain the geometrical-optics approximation as the limiting case

$$D \to 0 \tag{4.117}$$

of that more general method. Hence the practical applicability of the geometrical-optics method is for small D and there, according to (4.116), the fluctuations are mainly in the phase and not in the level.

4.4.5 Mean field in the geometrical optics approximation

With propagation along the z-axis through the homogeneous background the field expression (4.2) in the first approximation takes the form

$$E(\boldsymbol{\rho}, z) = \exp\left[\chi_1 + i\ (k\sqrt{\epsilon_0}\, z + S_1)\right] \approx \exp\left[+i\ (k\sqrt{\epsilon_0}\, z + S_1)\right] \tag{4.118}$$

where we neglect the level fluctuations by virtue of (4.116).

With propagation through many irregularities the phase correction

$$S_1 = \frac{k}{2\sqrt{\epsilon_0}} \int_0^z \epsilon(\boldsymbol{\rho}, z)\ \mathrm{d}z \tag{4.119}$$

is a normally distributed random value. Hence the mean field may be calculated in the same way as we used in Chapter 3, eqs. (3.55–59):

$$\langle E \rangle = \exp\left(i\, k\sqrt{\epsilon_0}\, z\right)\ \langle \exp\left(iS_1\right)\rangle = \exp\left(+ik\sqrt{\epsilon_0}\, z - \tfrac{1}{2}\sigma_S^2\right) \tag{4.120}$$

Substituting also the expression (4.100) for the variance of the phase fluctuations, we get the final result

$$\langle E \rangle = \exp\left(i\,k\sqrt{\epsilon_0}\,z - \frac{k^2\sigma_\epsilon^2\ell_\epsilon}{4\epsilon_0}\;z \right) \tag{4.121}$$

As we see from this result the mean field decreases exponentially with propagation distance in the medium. This is called extinction and is due to redistribution of the energy into the random component.

That energy is not lost can be seen from the mean energy, which according to (4.118) is independent of distance:

$$\langle E\,E^* \rangle = 1 \tag{4.122}$$

4.4.6 Pulse propagation through the fluctuating ionosphere

Now also some considerations about pulse propagation through the fluctuating ionosphere. What we have done so far in the geometrical-optics method applies to a single frequency. Regarding the spectral decomposition of a pulse we thus have for a Fourier component the representation

$$E(\mathbf{r},\omega) = E_0^g(\mathbf{r},\omega)\;\exp[+ik\varphi_1(\mathbf{r},\omega)] \tag{4.123}$$

where $E_0^g(\mathbf{r},\omega)$ is the geometrical-optics representation of the undisturbed field with its own phase and $k\varphi_1(\mathbf{r},\omega)$ is the phase fluctuation. If we denote the emitted spectrum of the pulse by $p(\omega)$, we have for a particular realization of the field received in the point $\mathbf{r}$:

$$E(\mathbf{r},t) = \int\limits_{-\infty}^{+\infty} p(\omega)\;E_0^g(\mathbf{r},\omega)\;\exp\{+ik[\varphi_1(\mathbf{r},\omega) - \omega t]\}\;\mathrm{d}\omega \tag{4.124}$$

From this we may construct the mean energy of the received pulse

$$\begin{aligned}\langle E(\mathbf{r},t)\,E^*(\mathbf{r},t)\rangle = &\iint \mathrm{d}\omega_1\,\mathrm{d}\omega_2\;p(\omega_1)\,p^*(\omega_2)\;E_0^g(\mathbf{r},\omega_1)\,E_0^{g*}(\mathbf{r},\omega_2)\\ &\cdot\langle\exp\{+ik[\varphi_1(\mathbf{r},\omega_1) - \varphi_1(\mathbf{r},\omega_2)]\}\rangle\;\exp[-i(\omega_1-\omega_2)\,t]\end{aligned} \tag{4.125}$$

The phase term in the exponent of the average is a zero-mean normally distributed random function, so we may once again make use of the property (3.58), i.e.

$$\langle\exp\{+ik[\varphi_1(\mathbf{r},\omega_1) - \varphi_1(\mathbf{r},\omega_2)]\}\rangle = \exp\left\{-\frac{k^2}{2}\;D_{\varphi_1}(\omega_1,\omega_2)\right\} \tag{4.126}$$

where $D_{\varphi_1}(\omega_1,\omega_2)$ is the structure function (1.64):

$$D_{\varphi_1}(\omega_1,\omega_2) = \left\langle[\varphi_1(\mathbf{r},\omega_1) - \varphi_1(\mathbf{r},\omega_2)]^2\right\rangle \tag{4.127}$$

This representation is appropriate when $\mathbf{r}$ is far from the caustics and strong interference between rays can be disregarded. In the next Chapter we shall consider pulse propagation in the more general case of Rytov's approximation. The results to be obtained there will contain those derivable from (4.125) as a particular case.

Chapter 5

Rytov's method (method of smooth perturbations)

5.1 Equation for the complex phase

As before we consider the scalar Helmholtz' equation for a point source at the origin:

$$\nabla^2 E + k^2 \; [\epsilon_0(\mathbf{r},\omega) + \epsilon(\mathbf{r},\omega,t)] \; E = \delta(\mathbf{r}) \tag{5.1}$$

where the fluctuations ϵ may be slowly time-dependent in the sense of eq. (2.24). As before we assume that the solution E_0 and the corresponding Green's function G of the undisturbed problem are known. We shall now use these to construct an approximate full-wave solution of (5.1) in the so-called Rytov's method. Hence we put

$$E = E_0(\mathbf{r},\omega) \; \exp[\Psi(\mathbf{r},\omega,t)] \tag{5.2}$$

where we have introduced the complex phase Ψ to account for the corrections to the undisturbed field due to the local inhomogeneities. In the forward scattering approximation, which is appropriate in the case of large-scale inhomogeneitites, the phase is subject to the boundary condition $\Psi \to 0$ when $\mathbf{r} \to 0$. When we introduce (5.2) into (5.1) we find that outside the source Ψ must fulfil

$$\nabla^2\Psi \; E_0 \; e^{\Psi} + 2\,\nabla\Psi\cdot\nabla E_0 \; e^{\Psi} + (\nabla\Psi)^2 \; E_0 \; e^{\Psi} + \nabla^2 E_0 \; e^{\Psi} + k^2\,\epsilon_0\, E_0 \; e^{\Psi} + k^2\,\epsilon\, E_0 \; e^{\Psi} = 0 \tag{5.3}$$

The undisturbed field E_0 fulfils the undisturbed wave equation [(5.1) with $\epsilon = 0$] and ensures the correct behaviour of the solution as $\mathbf{r} \to 0$ since $\Psi \to 0$ there. Hence

$$\nabla^2\Psi + (\nabla\Psi)^2 + 2\,\nabla \ln E_0 \cdot \nabla\Psi = -k^2\,\epsilon \tag{5.4}$$

is an exact equation for the new unknown function Ψ outside the source.

5.2 Perturbation series for the complex phase

Next we shall assume that ϵ is a small perturbation, which may be deterministic or random, and expand the phase in a perturbation series

$$\Psi = \Psi_1 + \Psi_2 + ... \tag{5.5}$$

The first-order equation from (5.4) is then given by

$$\nabla^2 \Psi_1 + 2\,\nabla \ln E_0 \cdot \nabla \Psi_1 = -k^2\,\epsilon \tag{5.6a}$$

The second-order equation is subsequently obtained as

$$\nabla^2 \Psi_2 + 2\,\nabla \ln E_0 \cdot \nabla \Psi_2 = -(\nabla \Psi_1)^2 \tag{5.6b}$$

In the following we shall also make use of the average of Ψ_2 which, according to (5.6b) obeys the equation

$$\nabla^2 \langle \Psi_2 \rangle + 2\,\nabla \ln E_0 \cdot \nabla \langle \Psi_2 \rangle = -\left\langle (\nabla \Psi_1)^2 \right\rangle \tag{5.6c}$$

These three equations are all of the same form:

$$\nabla^2 \Psi_m + 2\,\nabla \ln E_0 \cdot \nabla \Psi_m = f_m \tag{5.7}$$

where the right-hand sides are

$$f_1 = -k^2\,\epsilon\,, \qquad f_2 = -(\nabla \Psi_1)^2\,, \qquad f_3 = -\left\langle (\nabla \Psi_1)^2 \right\rangle \tag{5.8}$$

and where, as a mere *notation* not to be confused with the third term in the perturbation expansion, we have introduced

$$\Psi_3 = \langle \Psi_2 \rangle \tag{5.9}$$

To solve (5.7) we now introduce new unknowns w_m through

$$\Psi_m = E_0^{-1}\, w_m \tag{5.10}$$

Since E_0 satisfies the undisturbed wave equation, our equation (5.7) is equivalent to

$$\nabla^2 w_m + k^2\,\epsilon_0\, w_m = f_m\, E_0 \tag{5.11}$$

i.e. w_m satisfy the “undisturbed equation” with new known right-hand sides. Hence they can be written

$$w_m = \int G(\mathbf{r}, \mathbf{r}')\, f_m(\mathbf{r}')\, E_0(\mathbf{r}')\, \mathrm{d}\mathbf{r}' \tag{5.12}$$

and consequently

$$\Psi_m = E_0^{-1}(\mathbf{r}) \int G(\mathbf{r}, \mathbf{r}')\, f_m(\mathbf{r}')\, E_0(\mathbf{r}')\, \mathrm{d}\mathbf{r}' \tag{5.13}$$

The explicit expressions corresponding to (5.8) are then the following:

$$\Psi_1 = -k^2 \, E_0^{-1}(\mathbf{r}) \int G(\mathbf{r},\mathbf{r}') \, \epsilon(\mathbf{r}') \, E_0(\mathbf{r}') \, \mathrm{d}\mathbf{r}' \tag{5.14a}$$

$$\Psi_2 = -E_0^{-1}(\mathbf{r}) \int G(\mathbf{r},\mathbf{r}') \, \left[\nabla\Psi_1(\mathbf{r}')\right]^2 \, E_0(\mathbf{r}') \, \mathrm{d}\mathbf{r}' \tag{5.14b}$$

$$\Psi_3 = \langle\Psi_2\rangle = -E_0^{-1}(\mathbf{r}) \int G(\mathbf{r},\mathbf{r}') \, \left\langle\left[\nabla\Psi_1(\mathbf{r}')\right]^2\right\rangle \, E_0(\mathbf{r}') \, \mathrm{d}\mathbf{r}' \tag{5.14c}$$

The full-wave solution (5.2) may now be expressed (omitting obvious spatial arguments)

$$E = E_0 \, \exp\left[-k^2 \, E_0^{-1} \int G \, \epsilon \, E_0 \, \mathrm{d}\mathbf{r}' - E_0^{-1} \int G \, \left(\nabla\Psi_1\right)^2 \, E_0 \, \mathrm{d}\mathbf{r}' + ...\right] \tag{5.15}$$

If the corrections are very small we may expand the exponential function as follows:

$$E \approx E_0 \left[1 - k^2 \, E_0^{-1} \int G \, \epsilon \, E_0 \, \mathrm{d}\mathbf{r}' + \tfrac{1}{2} k^4 \, E_0^{-2} \iint G \, \epsilon \, E_0 \; G \, \epsilon \, E_0 \, \mathrm{d}\mathbf{r}' \, \mathrm{d}\mathbf{r}'' \right.$$
$$\left. - E_0^{-1} \int G \, \left(\nabla\Psi_1\right)^2 \, E_0 \, \mathrm{d}\mathbf{r}' + ...\right] \tag{5.16}$$

We see that this result is the same as the field in the approximation of single-scattering, plus additional terms corresponding to higher-order scattering. Hence we may conclude that the solution (5.15) obtained in Rytov's method contains a kind of partial (or approximate) summation of the multiple-scattering series.

For the following treatment we now separate Ψ into real and imaginary parts, i.e. into level and phase fluctuations of the field, as follows:

$$\Psi_1 = \chi_1 + i \, S_1 \, , \qquad \Psi_2 = \chi_2 + i \, S_2 \tag{5.17}$$

We may then use the statistical properties of χ_i and S_i to express the statistical properties of the total field. For the correlation functions

$$\psi_\chi = \langle \chi_1 \chi_1 \rangle \tag{5.18a}$$

and

$$\psi_S = \langle S_1 \, S_1 \rangle \tag{5.18b}$$

it is sufficient to know Ψ_1, but for expressing some moments of the entire field at least the terms including Ψ_2 are necessary.

Indeed, for the mean energy we have

$$\langle E(\mathbf{r}) E^*(\mathbf{r}) \rangle = |E_0|^2 \, \langle \exp\left[\Psi_1 + \Psi_2 + \Psi_1^* + \Psi_2^*\right] \rangle = |E_0|^2 \, \langle \exp\left[2\,\chi_1 + 2\,\chi_2\right] \rangle \tag{5.19}$$

Rewriting here $\chi_2 = \langle \chi_2 \rangle + \chi_2 - \langle \chi_2 \rangle$ and noting that $\chi_2 - \langle \chi_2 \rangle$ is a higher-order correction which may be neglected, we can write this average

$$\langle E(\mathbf{r}) E^*(\mathbf{r}) \rangle \approx |E_0|^2 \ \exp\left[2 \langle \chi_2 \rangle\right] \ \langle \exp\left[2 \chi_1\right] \rangle = |E_0|^2 \ \exp\left[2 \langle \chi_2 \rangle + 2 \left\langle \chi_1^2 \right\rangle\right] \tag{5.20}$$

where in the last member we have used the property (3.58) for the zero-mean normally distributed random function χ_1 . Since χ_2 and χ_1^2 are both of order ϵ^2 , it is obvious that we need χ_2 to express this average. Note that our result (5.20) differs from the result $\left\langle |E|^2 \right\rangle = |E_0|^2$ of geometrical optics; cf. (4.122). These two results coincide in the case when $\langle \chi_2 \rangle = -\left\langle \chi_1^2 \right\rangle$. If they differ, (5.20) corresponds to redistribution of energy which is not described by the dominant term in geometrical optics.

The situation is the same for the mean field

$$\begin{aligned} \langle E(\mathbf{r}) \rangle &\approx E_0 \ \langle \exp\left[\langle \chi_2 \rangle + i \ \langle S_2 \rangle + \chi_1 + i\, S_1\right] \rangle \\ &= \exp\left[\langle \chi_2 \rangle + i \ \langle S_2 \rangle\right] \ \langle \exp\left[\chi_1 + i\, S_1\right] \rangle \\ &= E_0 \ \exp\left[\langle \chi_2 \rangle + i \ \langle S_2 \rangle + \tfrac{1}{2} \left\langle \chi_1^2 \right\rangle - \tfrac{1}{2} \left\langle S_1^2 \right\rangle + i \, \langle \chi_1 \, S_1 \rangle\right] \end{aligned} \tag{5.21}$$

Since all terms in the exponent are of order ϵ^2 we need Ψ_2 also here.

Let us consider also the coherence function which is necessary for describing pulse propagation:

$$\Gamma(\mathbf{r}, \omega_1, \omega_2) = E_0(\mathbf{r}, \omega_1)\, E_0^*(\mathbf{r}, \omega_2) \ \langle \exp\left[\Psi(\mathbf{r}, \omega_1) + \Psi^*(\mathbf{r}, \omega_2)\right] \rangle \approx E_0(\omega_1)\, E_0^*(\omega_2)$$

$\langle \exp\left[\chi_1(\omega_1) + i\, S_1(\omega_1) + \chi_2(\omega_1) + i\, S_2(\omega_1)\right.$
$\left. + \chi_1(\omega_2) - i\, S_1(\omega_2) + \chi_2(\omega_2) - i\, S_2(\omega_2)\right] \rangle$ (5.22)

where we have included terms up to Ψ_2 as in the previous cases. Neglecting also $\Psi_2 - \langle \Psi_2 \rangle$ we obtain

$$\begin{aligned} \Gamma_2(\mathbf{r}, \omega_1, \omega_2) = E_0(\omega_1)\, E_0^*(\omega_2) \ &\exp\left\{\langle \chi_2(\omega_1) \rangle + \langle \chi_2(\omega_2) \rangle + i \ \left[\langle S_2(\omega_1) \rangle - \langle S_2(\omega_2) \rangle\right]\right\} \\ &\cdot \exp\left[+\tfrac{1}{2} \ F(\omega_1, \omega_2)\right] \end{aligned} \tag{5.23}$$

with

$$F(\omega_1, \omega_2) = \left\langle \left\{\chi_1(\omega_1) + \chi_1(\omega_2) + i \ \left[S_1(\omega_1) - S_1(\omega_2)\right]\right\}^2 \right\rangle \tag{5.24}$$

In general $\Psi_1 + \Psi_2$ is necessary for constructing this coherence function.

5.3 Forward scattering Fresnel approximation for the complex phase

We shall consider here the most simple case of a homogeneous background with fluctuations in the half-plane $z > 0$ and an incident field in the form of a plane wave in the z-direction. Hence the incident field is

$$E_0 = e^{ikz} \tag{5.25}$$

and the Green's function is given by (2.35), i.e.

$$G(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi} \, \frac{\exp[ik|\mathbf{r} - \mathbf{r}'|]}{|\mathbf{r} - \mathbf{r}'|} \tag{5.26}$$

The first-order complex phase is then according to (5.14a):

$$\Psi_1(x, y, z) = \frac{k^2}{4\pi} \, e^{-ikz} \int\limits_{-\infty}^{+\infty} \mathrm{d}x' \, \mathrm{d}y' \int\limits_{0}^{+\infty} \frac{\exp\left[ik|\mathbf{r} - \mathbf{r}'| + ikz'\right]}{|\mathbf{r} - \mathbf{r}'|} \, \epsilon(x', y', z') \, \mathrm{d}z' \tag{5.27}$$

For our following discussion we shall denote the entire phase of this expression by P:

$$P(x', y', z') = k \, \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} - k \, (z - z') \tag{5.28}$$

We note that it may sometimes be possible to evaluate the $x'y'$-integrals by the method of steepest descents, and then the stationary points of the exponent are given by

$$\frac{\partial P}{\partial x'} = \frac{k \, (x - x')}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} = 0 \tag{5.29a}$$

$$\frac{\partial P}{\partial y'} = \frac{k \, (y - y')}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} = 0 \tag{5.29b}$$

i.e. for a fixed point (x, y, z) of observation these points form a straight line parallel to the z-axis and ending at (x, y, z).

Since rapid spatial phase variations tend to cancel the contributions to integrals of the type (5.27), it is of importance to investigate the regions where P is constant. We shall consider the surfaces

$$P = k \, \sqrt{(\boldsymbol{\rho} - \boldsymbol{\rho}')^2 + (z - z')^2} - k \, (z - z') = \ell\pi \; , \qquad \ell = 0, \; 1, \; 2, \; ... \tag{5.30}$$

where as before we denote the transverse coordinates by $\boldsymbol{\rho} = \{x, y\}$. Introducing the wavelength $\lambda = 2\pi/k$, (5.30) is the same as

$$(\boldsymbol{\rho} - \boldsymbol{\rho}')^2 = \ell\lambda \, (z - z') + \tfrac{1}{4}\ell^2\lambda^2 \tag{5.31}$$

Hence we see that the surfaces of constant phase are rotational paraboloides around the line through the point of observation and parallel to the z-axis. These surfaces cross the line $z' = z$ at the positions $z' = z + \ell\lambda/4$. In particular, the *main Fresnel zone* is the volume within the surface with $\ell = 1$, i.e. the region of space where $0 < P < \pi$.

We shall obtain the Fresnel approximation of (5.27) by expanding the square root in the phase P for $|\boldsymbol{\rho} - \boldsymbol{\rho}'| \ll |z - z'|$, i.e. we shall make use of

$$\sqrt{(\boldsymbol{\rho} - \boldsymbol{\rho}')^2 + (z - z')^2} \approx |z - z'| + \frac{(\boldsymbol{\rho} - \boldsymbol{\rho}')^2}{2 \, |z - z'|} \tag{5.32}$$

to obtain the phase

$$P = \begin{cases} \frac{1}{2}\,k\,(\boldsymbol{\rho}-\boldsymbol{\rho}')^2/(z-z')\ , & z' < z \\ 2k\,(z'-z) + \frac{1}{2}\,k\,(\boldsymbol{\rho}-\boldsymbol{\rho}')^2/(z'-z)\ , & z' > z \end{cases} \tag{5.33}$$

In the denominator of (5.27) the approximation $|z - z'|$ of the square root is sufficient. We then use the result (5.33) to write (5.27) as a sum of two items:

$$\Psi_1(x,y,z) = \frac{k^2}{4\pi} \int\limits_{-\infty}^{+\infty} \mathrm{d}x'\,\mathrm{d}y' \int\limits_{0}^{z} \frac{\exp\left[+ik\,\frac{(\boldsymbol{\rho}-\boldsymbol{\rho}')^2}{2(z-z')}\right]}{z-z'}\,\epsilon(\mathbf{r}')\,\mathrm{d}z'$$

$$+\frac{k^2}{4\pi} \int\limits_{-\infty}^{+\infty} \mathrm{d}x'\,\mathrm{d}y' \int\limits_{z}^{+\infty} \frac{\exp\left[2ik(z'-z) + ik\,\frac{(\boldsymbol{\rho}-\boldsymbol{\rho}')^2}{2(z'-z)}\right]}{z'-z}\,\epsilon(\mathbf{r}')\,\mathrm{d}z' \tag{5.34}$$

It is easily seen that the phase of the second term oscillates much more rapidly with z' than the first term. The volume of integration to the left (for $z' < z$) contains the main Fresnel zone, whereas to the right (for $z' > z$) it crosses into a new zone for every $\lambda/4$. Hence it its obvious that if the fluctuations are of sufficiently large scale so that $k\ell_\epsilon \gg 1$, the oscillations will almost cancel the second integral in (5.34). Neglecting this integral we then obtain

$$\Psi_1(x,y,z) = \frac{k^2}{4\pi} \int\limits_{-\infty}^{+\infty} \mathrm{d}x'\,\mathrm{d}y' \int\limits_{0}^{z} \frac{\exp\left[+ik\,\frac{(\boldsymbol{\rho}-\boldsymbol{\rho}')^2}{2(z-z')}\right]}{z-z'}\,\epsilon(\mathbf{r}')\,\mathrm{d}z' \tag{5.35}$$

This is our desired Rytov's representation of the first-order complex phase. Physically this result corresponds to forward scattering where only the points previously passed by the initial wave contribute to the fluctuating field. It is an approximate solution of the first-approximation equation (5.6a), which with the initial field (6.25) has the form

$$\nabla^2\Psi_1 + 2\,ik\,\frac{\partial\Psi_1}{\partial z} = -k^2\,\epsilon \tag{5.36}$$

It can be easily shown that (5.35) is an exact solution of the similar parabolic equation

$$\nabla_\perp^2\Psi_1 + 2\,ik\,\frac{\partial\Psi_1}{\partial z} = -k^2\,\epsilon \tag{5.37}$$

which solution does not contain waves propagating in the direction opposite to the incident field. The Fresnel propagator

$$\exp\left[+ik\,\frac{(\boldsymbol{\rho}-\boldsymbol{\rho}')^2}{2(z-z')}\right] \tag{5.38}$$

appearing in (5.35) is typical for solutions of parabolic equations.

The second-order complex phase Ψ_2 is given by (5.14b), which can be treated to give the same sort of integral as (5.35). However, we shall not here go into details about this.

5.4 Geometrical optics as a limiting case of Rytov's method

We saw in (5.29a,b) that $x' = x\,,\ y' = y$ are stationary points of the exponent in (5.27). This is also true for (5.35) and we shall now investigate under what conditions the $x'y'$-integrations can be performed by the steepest-descent method.

As we have already discussed, the major contributions to the x'-integral come from the main Fresnel zone, i.e. from x' which fulfil $|x - x'| < \ell_x$, where

$$\frac{k\,\ell_x^2}{2\,(z-z')} = \pi \tag{5.39}$$

or

$$\ell_x = \sqrt{\lambda\,(z-z')} \tag{5.39$'$}$$

For a fixed distance of observation z , this has its maximum for $z' = 0$ and we hence take

$$R_F = \ell_x = \sqrt{\lambda\,z} \tag{5.40}$$

as the scale of the main Fresnel zone. It is now clear that the steepest-descent method can be used in the x'- and y'-directions when the scale of the irregularities is much larger than this parameter, i.e.

$$\ell_\epsilon \gg R_F \tag{5.41}$$

We shall show below that this is, in fact, the third condition of validity for the geometrical-optics approximation. It may also be expressed in terms of the *wave parameter* D introduced in (4.115) as follows:

$$D = \frac{R_F}{\ell_\epsilon} = \frac{\sqrt{\lambda\,z}}{\ell_\epsilon} \ll 1 \tag{5.42}$$

When (5.42) is violated the general integral (5.35) has to be calculated. Next we shall, however, demonstrate the steepest-descent evaluation of the transverse integrations and show the limit of geometrical optics. To this end we expand the relative permittivity of the fluctuations

$$\epsilon(x') = \epsilon(x) + \frac{\partial\epsilon(x)}{\partial x}\,(x'-x) + \tfrac{1}{2}\,\frac{\partial^2\epsilon(x)}{\partial x^2}\,(x'-x)^2 + ... \tag{5.43}$$

Then we have in this approximation, since the first derivative gives an odd integrand,

$$\begin{aligned}\int\limits_{-\infty}^{+\infty} \epsilon(x')\,\frac{\exp\left[\frac{ik\,(x-x')^2}{2\,(z-z')}\right]}{z-z'}\,\mathrm{d}x' &= \epsilon(x)\int\limits_{-\infty}^{+\infty}\frac{\exp\left[\frac{ik\,(x-x')^2}{2\,(z-z')}\right]}{z-z'}\,\mathrm{d}x' \\ &\quad+\tfrac{1}{2}\,\frac{\partial^2\epsilon(x)}{\partial x^2}\int\limits_{-\infty}^{+\infty}\frac{(x'-x)^2}{z-z'}\,\exp\left[\frac{ik\,(x-x')^2}{2\,(z-z')}\right]\,\mathrm{d}x'\end{aligned} \tag{5.44}$$

The first integral here may be evaluated by deforming the path of integration into a line through x tilted an angle $\pi/4$ in the complex x'-plane, i.e. $x' - x = \alpha\ \exp(i\pi/4)$. In this way we obtain

$$e^{i\pi/4} \int\limits_{-\infty}^{+\infty} \frac{\exp\left[-\frac{k\,\alpha^2}{2\,(z-z')}\right]}{z-z'}\ \mathrm{d}\alpha = \frac{e^{i\pi/4}}{z-z'}\ \sqrt{\frac{2\pi\,(z-z')}{k}} \tag{5.45}$$

The integration over y' can be done in the same way, and hence (5.35) with the first term in the Taylor expansion of $\epsilon(x', y', z')$ yields a purely imaginary result:

$$i\,S_1 = \frac{ik}{2} \int\limits_0^z \epsilon(x, y, z')\ \mathrm{d}z' \tag{5.46}$$

This is exactly the first-order correction to the phase obtained in geometrical optics aproximation, eq. (4.67). The second integral in (5.44) can easily be evaluated if we make use of (5.45) as follows:

$$\int\limits_{-\infty}^{+\infty} \xi^2 \exp[i\beta\xi^2]\ \mathrm{d}\xi = -i\ \frac{\partial}{\partial\beta} \int\limits_{-\infty}^{+\infty} \exp[i\beta\xi^2]\ \mathrm{d}\xi = \tfrac{1}{2}\ i\,e^{i\pi/4}\sqrt{\pi/\beta^3} \tag{5.47}$$

Together with the corresponding contribution in the y'-direction this gives the real-valued result

$$\chi_1 = -\tfrac{1}{4} \int\limits_0^z (z-z')\ \nabla_\perp^2 \epsilon(x, y, z')\ \mathrm{d}z' \tag{5.48}$$

The first-order level-correction in the geometrical-optics approximation is given by (4.80). If we substitute φ_1 according to (4.67)(with the unity background dielectric permittivity) into this and perform an integration by parts, we obtain exactly (5.48).

Hence we have verified that the first-order Rytov's representation (5.35) really gives the result of geometrical-optics in the limit (5.42) when the steepest-descent method applies. To summarize, the steepest-descent evaluation of (6.35) can be written

$$\Psi_1 = \frac{ik}{2} \int\limits_0^z \epsilon(x, y, z')\ \mathrm{d}z' - \tfrac{1}{4} \int\limits_0^z (z-z')\ \nabla_\perp^2 \epsilon(x, y, z')\ \mathrm{d}z' \tag{5.49}$$

When (5.42) is violated diffraction effects are essential and then the general integral (5.35) has to be calculated.

5.5 Phase and level fluctuations, their mean values and correlation functions

When we separate the first-order Rytov's representation (6.35) into real and imaginary parts we get

$$\chi_1(\mathbf{r}) = \frac{k^2}{4\pi} \int\limits_{-\infty}^{+\infty} \mathrm{d}x'\, \mathrm{d}y' \int\limits_{0}^{z} \frac{\epsilon(\mathbf{r}')}{z-z'} \, \cos\left[k\, \frac{(\boldsymbol{\rho}-\boldsymbol{\rho}')^2}{2\,(z-z')}\right] \mathrm{d}z' \tag{5.50a}$$

$$S_1(\mathbf{r}) = \frac{k^2}{4\pi} \int\limits_{-\infty}^{+\infty} \mathrm{d}x'\, \mathrm{d}y' \int\limits_{0}^{z} \frac{\epsilon(\mathbf{r}')}{z-z'} \, \sin\left[k\, \frac{(\boldsymbol{\rho}-\boldsymbol{\rho}')^2}{2\,(z-z')}\right] \mathrm{d}z' \tag{5.50b}$$

Since these are linear in the zero-mean quantity ϵ, we have $\langle \chi_1 \rangle = \langle S_1 \rangle = 0$.

In calculating the correlation functions we shall employ the transverse spatial spectrum of the fluctuations; $\tilde{\epsilon}(\boldsymbol{\kappa}, z')$ with $\boldsymbol{\kappa} = \{\kappa_x, \kappa_y\}$. The transverse spatial Fourier transform of (6.35) takes the form of a convolution integral in the x- and y-variables which may be considerably simplified by using the convolution theorem to yield

$$\tilde{\Psi}_1(\boldsymbol{\kappa}, z) = \frac{ik}{2} \int\limits_{0}^{z} \tilde{\epsilon}(\boldsymbol{\kappa}, z') \exp\left[-i\, \frac{(\kappa_x^2+\kappa_y^2)(z-z')}{2\,k}\right] \mathrm{d}z' \tag{5.51}$$

The level fluctuations may be expressed

$$\chi_1(\mathbf{r}) = \tfrac{1}{2}\, [\Psi_1(\mathbf{r}) + \Psi_1^*(\mathbf{r})] \tag{5.52a}$$

where we now shall use

$$\Psi_1(\mathbf{r}) = \int\limits_{-\infty}^{+\infty} \tilde{\Psi}_1(\boldsymbol{\kappa}, z)\, \exp[i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}]\, \mathrm{d}\boldsymbol{\kappa} \tag{5.53}$$

When we substitute (5.51) and (5.53) into (5.52a), we may also utilize the fact that for real-valued fluctuations $\epsilon(\mathbf{r})$ we have

$$\tilde{\epsilon}^*(-\boldsymbol{\kappa}, z) = \tilde{\epsilon}(\boldsymbol{\kappa}, z) \tag{5.54}$$

We then arrive at the following expression:

$$\chi_1(\mathbf{r}) = \frac{k}{2} \int\limits_{-\infty}^{+\infty} \exp[i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}]\, \mathrm{d}\boldsymbol{\kappa} \int\limits_{0}^{z} \tilde{\epsilon}(\boldsymbol{\kappa}, z') \sin\left[\frac{\kappa^2(z-z')}{2\,k}\right] \mathrm{d}z' \tag{5.55}$$

where the simplifying notation $\kappa^2 = \kappa_x^2 + \kappa_y^2$ has been used. In this result we can immediately identify the transverse spectrum of the level for a separate realization of the fluctuations:

$$\tilde{\chi}_1(\boldsymbol{\kappa}, z) = \frac{k}{2} \int\limits_{0}^{z} \tilde{\epsilon}(\boldsymbol{\kappa}, z') \sin\left[\frac{\kappa^2(z-z')}{2\,k}\right] \mathrm{d}z' \tag{5.56a}$$

Analogously we may use the formula

$$S_1(\mathbf{r}) = \tfrac{1}{2i}\ [\Psi_1(\mathbf{r}) - \Psi_1^*(\mathbf{r})] \tag{5.52b}$$

to obtain the result

$$\tilde{S}_1(\boldsymbol{\kappa}, z) = \frac{k}{2}\int\limits_0^z \tilde{\epsilon}(\boldsymbol{\kappa}, z')\cos\left[\frac{\kappa^2(z-z')}{2\,k}\right]\ \mathrm{d}z' \tag{5.56b}$$

We shall next use the result (5.56a) together with

$$\chi_1(\mathbf{r}) = \int\limits_{-\infty}^{+\infty} \tilde{\chi}_1(\boldsymbol{\kappa}, z)\ \exp[i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}]\ \mathrm{d}\boldsymbol{\kappa} \tag{5.57}$$

to construct the transverse correlation function for the level. Assuming statistical homogeneity in the longitudinal direction and noting that χ_1 is real we have by definition

$$\psi_\chi(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) = \int\limits_{-\infty}^{+\infty} \mathrm{d}\boldsymbol{\kappa}_1\ \mathrm{d}\boldsymbol{\kappa}_2\ \langle\tilde{\chi}_1(\boldsymbol{\kappa}_1, z)\ \tilde{\chi}_1(\boldsymbol{\kappa}_2, z)\rangle\ \exp[i\,(\boldsymbol{\kappa}_1\cdot\boldsymbol{\rho}_1 + \boldsymbol{\kappa}_2\cdot\boldsymbol{\rho}_2)] \tag{5.58}$$

Substituting (5.56a) into this, we get

$$\psi_\chi(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) = \frac{k^2}{4}\int\limits_{-\infty}^{+\infty} \mathrm{d}\boldsymbol{\kappa}_1\ \mathrm{d}\boldsymbol{\kappa}_2\ \exp[i\,(\boldsymbol{\kappa}_1\cdot\boldsymbol{\rho}_1 + \boldsymbol{\kappa}_2\cdot\boldsymbol{\rho}_2)]$$

$$\cdot\int\limits_0^z\int\limits_0^z \langle\tilde{\epsilon}(\boldsymbol{\kappa}_1, z')\ \tilde{\epsilon}(\boldsymbol{\kappa}_2, z'')\rangle\ \sin\left[\frac{\kappa_1^2(z-z')}{2\,k}\right]\ \sin\left[\frac{\kappa_2^2(z-z'')}{2\,k}\right]\ \mathrm{d}z'\ \mathrm{d}z'' \tag{5.59}$$

The average appearing in the integrand is the transverse spectrum of the correlation function of the permittivity fluctuations, i.e.

$$\psi_\epsilon(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2, z', z'') = \langle\tilde{\epsilon}(\boldsymbol{\kappa}_1, z')\ \tilde{\epsilon}(\boldsymbol{\kappa}_2, z'')\rangle \tag{5.60}$$

When we have statistical homogeneity this quantity has the form

$$\psi_\epsilon(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2, z', z'') = F_\epsilon(\boldsymbol{\kappa}_1, z'-z'')\ \delta(\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2) \tag{5.61}$$

Under these conditions (5.59) can be written

$$\psi_\chi(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) = \frac{k^2}{4}\int\limits_{-\infty}^{+\infty} \mathrm{d}\boldsymbol{\kappa}\ \exp[i\,\boldsymbol{\kappa}\cdot(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)]$$

$$\cdot\int\limits_0^z\int\limits_0^z F_\epsilon(\boldsymbol{\kappa}, z'-z'')\ \sin\left[\frac{\kappa^2(z-z')}{2\,k}\right]\ \sin\left[\frac{\kappa^2(z-z'')}{2\,k}\right]\ \mathrm{d}z'\ \mathrm{d}z'' \tag{5.62}$$

With $\boldsymbol{\rho} = \boldsymbol{\rho}_1 - \boldsymbol{\rho}_2$ we finally get the following Fourier representation of the correlation function

$$\psi_\chi(\boldsymbol{\rho}, z) = \int\limits_{-\infty}^{+\infty} F_\chi(\boldsymbol{\kappa}, z)\ \exp[i\,\boldsymbol{\kappa}\cdot\boldsymbol{\rho}]\ \mathrm{d}\boldsymbol{\kappa} \tag{5.63}$$

with

$$F_\chi(\boldsymbol{\kappa}, z) = \frac{k^2}{4}\int\limits_0^z\int\limits_0^z F_\epsilon(\boldsymbol{\kappa}, z'-z'')\ \sin\left[\frac{\kappa^2(z-z')}{2\,k}\right]\ \sin\left[\frac{\kappa^2(z-z'')}{2\,k}\right]\ \mathrm{d}z'\ \mathrm{d}z'' \tag{5.64}$$

Now we shall treat (5.64) in a similar way as we did with the integral in Section 4.4.1. Hence we introduce new variables through

$$2\,\eta = z' + z''\ , \qquad \zeta = z' - z'' \tag{5.65}$$

so that (5.64) can be written

$$F_\chi(\boldsymbol{\kappa}, z) = \frac{k^2}{4}\int_\diamondsuit F_\epsilon(\boldsymbol{\kappa}, \zeta)\ \sin\left[\frac{\kappa^2}{2\,k}\left(z-\eta-\tfrac{1}{2}\zeta\right)\right]\ \sin\left[\frac{\kappa^2}{2\,k}\left(z-\eta+\tfrac{1}{2}\zeta\right)\right]\ \mathrm{d}\zeta\ \mathrm{d}\eta \tag{5.66}$$

The symbol $\diamondsuit$ denotes the area of integration in the $\zeta\eta$-plane. With the spatial scale ℓ_ϵ of the fluctuations the scale of the spatial spectrum will be ℓ_ϵ^{-1}, as we have already stressed many times. Since F_ϵ is practically zero for $|\zeta| > \ell_\epsilon$ it is then sufficient to integrate over $|\zeta|$ smaller than this value if $z \gg \ell_\epsilon$,. Since also in practice $\kappa < \ell_\epsilon^{-1}$, the term $\kappa^2\zeta/(4k)$ has a maximum value which is of the order $1/(k\ell_\epsilon) \ll 1$ for the large-scale inhomogeneities considered here. Hence we can omit this term in both sines obtaining

$$F_\chi(\boldsymbol{\kappa}, z) = \frac{k^2}{4}\int_\diamondsuit F_\epsilon(\boldsymbol{\kappa}, \zeta)\ \sin^2\left[\frac{\kappa^2(z-\eta)}{2\,k}\right]\ \mathrm{d}\zeta\ \mathrm{d}\eta \tag{5.67}$$

Because of the restricted area of non-zero F_ϵ, the integration over ζ can without introducing large error be extended to $\pm\infty$ and then we may also take the integration over η right up to the point z:

$$F_\chi(\boldsymbol{\kappa}, z) = \frac{k^2}{4}\int\limits_0^z \mathrm{d}\eta\ \sin^2\left[\frac{\kappa^2(z-\eta)}{2\,k}\right]\ \int\limits_{-\infty}^{+\infty} F_\epsilon(\boldsymbol{\kappa}, \zeta)\ \mathrm{d}\zeta \tag{5.68}$$

The three-dimensional spatial spectrum corresponding to F_ϵ is

$$\phi_\epsilon(\boldsymbol{\kappa}, \kappa_z) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} F_\epsilon(\boldsymbol{\kappa}, \zeta)\ \exp[-i\kappa_z\zeta]\ \mathrm{d}\zeta \tag{5.69}$$

and in particular we have

$$\phi_\epsilon(\boldsymbol{\kappa},0) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} F_\epsilon(\boldsymbol{\kappa},\zeta)\ \mathrm{d}\zeta \tag{5.69$'$}$$

Hence

$$F_\chi(\boldsymbol{\kappa},z) = \frac{\pi k^2}{2}\ \phi_\epsilon(\boldsymbol{\kappa},0) \int\limits_0^z \sin^2\left[\frac{\kappa^2(z-\eta)}{2\,k}\right]\ \mathrm{d}\eta \tag{5.70}$$

Carrying out the straight-forward integration over η, we obtain from this

$$F_\chi(\boldsymbol{\kappa},z) = \frac{\pi k^2\,z}{4}\ \left[1 - \frac{k}{\kappa^2\,z}\sin\left(\frac{\kappa^2\,z}{k}\right)\right]\ \phi_\epsilon(\boldsymbol{\kappa},0) \tag{5.71a}$$

As our final result, the transverse spatial correlation function for the level is then given by

$$\psi_\chi(\boldsymbol{\rho},z) = \frac{\pi k^2\,z}{4} \int\limits_{-\infty}^{+\infty} \left[1 - \frac{k}{\kappa^2\,z}\sin\left(\frac{\kappa^2\,z}{k}\right)\right]\ \phi_\epsilon(\boldsymbol{\kappa},0)\ \exp[i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}]\ \mathrm{d}\boldsymbol{\kappa} \tag{5.72a}$$

The calculation of the transverse spatial correlation function for the phase is completely analogous with the above. Hence we obtain the corresponding result

$$F_S(\boldsymbol{\kappa},z) = \frac{\pi k^2\,z}{4}\ \left[1 + \frac{k}{\kappa^2\,z}\sin\left(\frac{\kappa^2\,z}{k}\right)\right]\ \phi_\epsilon(\boldsymbol{\kappa},0) \tag{5.71b}$$

$$\psi_S(\boldsymbol{\rho},z) = \frac{\pi k^2\,z}{4} \int\limits_{-\infty}^{+\infty} \left[1 + \frac{k}{\kappa^2\,z}\sin\left(\frac{\kappa^2\,z}{k}\right)\right]\ \phi_\epsilon(\boldsymbol{\kappa},0)\ \exp[i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}]\ \mathrm{d}\boldsymbol{\kappa} \tag{5.72b}$$

The results (5.71a,b) and (5.72a,b) are valid generally; not only for very large-scale inhomogeneities when geometrical optics pertains, but also when ℓ_ϵ is less than the Fresnel zone size so that diffraction effects are essential.

It is now interesting to investigate how the limiting case of geometrical optics can be obtained from these results. We know that the largest value of κ for which ϕ_ϵ is non-zero and which is then necessary in the evaluation of the integrals is of the order ℓ_ϵ^{-1}. The argument of the sine function for this κ is $z/(k\,\ell_\epsilon^2) \sim \lambda\,z/\ell_\epsilon^2 = D^2$. If the wave parameter fulfils $D \ll 1$ we may then use the expansion

$$\frac{k}{\kappa^2\,z}\sin\left(\frac{\kappa^2\,z}{k}\right) = \frac{\sin\xi}{\xi} \approx 1 - \frac{\xi^2}{6} + ... \tag{5.73}$$

In the case of level fluctuations the first term of this expansion is cancelled in (5.71a). Hence we have to include the second term which gives

$$F_\chi(\boldsymbol{\kappa},z) = \frac{\pi\,\kappa^4\,z^3}{24}\ \phi_\epsilon(\boldsymbol{\kappa},0) \tag{5.74a}$$

For the phase fluctuations the first term is sufficient which yields

$$F_S(\boldsymbol{\kappa}, z) = \frac{\pi k^2\, z}{2}\; \phi_\epsilon(\boldsymbol{\kappa}, 0) \tag{5.74b}$$

The resulting transverse spatial correlation functions (5.72a,b) are then given by

$$\psi_\chi^G(\boldsymbol{\rho}, z) = \int\limits_{-\infty}^{+\infty} \frac{\pi\, \kappa^4\, z^3}{24}\; \phi_\epsilon(\boldsymbol{\kappa}, 0)\; \exp[i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}]\; \mathrm{d}\boldsymbol{\kappa} \tag{5.75a}$$

$$\psi_S^G(\boldsymbol{\rho}, z) = \int\limits_{-\infty}^{+\infty} \frac{\pi\, k^2\, z}{2}\; \phi_\epsilon(\boldsymbol{\kappa}, 0)\; \exp[i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}]\; \mathrm{d}\boldsymbol{\kappa} \tag{5.75b}$$

where the superscript G indicates that we have really obtained the geometrical-optics limit from Chapter 4. Indeed, (5.75a,b) specialized to $\boldsymbol{\rho} = 0$ are just eqs. (4.112,113).

The opposite limit, when $D \gg 1$, is the Fraunhofer limit of plane wave diffraction. Then the sine can be omitted in the formulas (5.71a,b) for the main part of the domain of integration, giving

$$F_{\chi,S}^F(\boldsymbol{\rho}, z) = \frac{\pi\, k^2\, z}{4}\; \phi_\epsilon(\boldsymbol{\kappa}, 0) \tag{5.76}$$

As a result we get the same expression for the level and phase fluctuations:

$$\psi_{\chi,S}^F(\boldsymbol{\rho}, z) = \int\limits_{-\infty}^{+\infty} \frac{\pi\, k^2\, z}{4}\; \phi_\epsilon(\boldsymbol{\kappa}, 0)\; \exp[i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}]\; \mathrm{d}\boldsymbol{\kappa} \tag{5.77}$$

For intermediate values of D the general formulas (5.72a,b) must be used. We restate them here again in a unified form:

$$\psi_{\chi,S}(\boldsymbol{\rho}, z) = \frac{\pi k^2\, z}{4} \int\limits_{-\infty}^{+\infty} \left[1 \mp \frac{k}{\kappa^2\, z} \sin\left(\frac{\kappa^2\, z}{k}\right)\right]\; \phi_\epsilon(\boldsymbol{\kappa}, 0)\; \exp[i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}]\; \mathrm{d}\boldsymbol{\kappa} \tag{5.72$'$}$$

We remark that all these considerations pertain to the case of an incident plane wave in the z-direction and homogeneous background with fluctuations in the half-space $z > 0$. For ionospheric propagation the mean properties of the fluctuations depend on the height and in the quasi-homogeneous approximation the fluctuation spectrum will then also be a function of η, i.e. $\phi_\epsilon(\boldsymbol{\kappa}, \kappa_z, \eta)$. In this case we cannot take this quantity out of the integration in (6.70) and hence we obtain instead of (5.72a) the expression

$$\psi_\chi(\boldsymbol{\rho}, z) = \frac{\pi k^2}{2} \int\limits_{-\infty}^{+\infty} \exp[i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}]\; \mathrm{d}\boldsymbol{\kappa} \int\limits_0^z \sin^2\left(\frac{\kappa^2\, z}{2\, k}\right)\; \phi_\epsilon(\boldsymbol{\kappa}, 0, \eta)\; \mathrm{d}\eta \tag{5.78a}$$

and instead of (5.72b)

$$\psi_S(\boldsymbol{\rho}, z) = \frac{\pi k^2}{2} \int\limits_{-\infty}^{+\infty} \exp[i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}] \; \mathrm{d}\boldsymbol{\kappa} \int\limits_0^z \cos^2\left(\frac{\kappa^2 \, z}{2\,k}\right) \; \phi_\epsilon(\boldsymbol{\kappa}, 0, \eta) \; \mathrm{d}\eta \qquad (5.78b)$$

Alternatively we may have a slow temporal change of the fluctuations, $\phi_\epsilon(\boldsymbol{\kappa}, \kappa_z, t)$, e.g. in the model of frozen drift. Then we have instead of (5.72'), if we specialize to the variance with $\boldsymbol{\rho} = 0$, the expressions

$$\sigma_{\chi,S}(\boldsymbol{\rho}, z, t) = \frac{\pi k^2 \, z}{4} \int\limits_{-\infty}^{+\infty} \left[1 \mp \frac{k}{\kappa^2 \, z} \sin\left(\frac{\kappa^2 \, z}{k}\right)\right] \; \phi_\epsilon(\boldsymbol{\kappa}, 0, t) \; \mathrm{d}\boldsymbol{\kappa} \qquad (5.79)$$

The frequency spectra of these slow temporal changes are

$$\tilde{\sigma}_{\chi,S}(\boldsymbol{\rho}, z, \Omega) = \frac{k^2 \, z}{8} \int\limits_{-\infty}^{+\infty} \left[1 \mp \frac{k}{\kappa^2 \, z} \sin\left(\frac{\kappa^2 \, z}{k}\right)\right] \; \phi_\epsilon(\boldsymbol{\kappa}, 0, t) \; \exp(i\Omega t) \; \mathrm{d}\boldsymbol{\kappa} \; \mathrm{d}t \qquad (5.80)$$

5.6 The moments of the total field

We have already derived the expression (5.20) for the mean energy in Rytov's method:

$$\langle E \, E^* \rangle = |E_0|^2 \; \exp\left[2\,\langle\chi_2\rangle + 2\,\langle\chi_1^2\rangle\right] \qquad (5.81)$$

as well as (5.21) for the average total field

$$\langle E(\mathbf{r})\rangle = E_0 \; \exp\left[\langle\chi_2\rangle + i\;\langle S_2\rangle + \tfrac{1}{2}\,\langle\chi_1^2\rangle - \tfrac{1}{2}\,\langle S_1^2\rangle + i\,\langle\chi_1\,S_1\rangle\right] \qquad (5.82)$$

It can be shown that the particular relations

$$\langle\chi_2\rangle = -\,\langle\chi_1^2\rangle \qquad (5.83a)$$

$$\langle S_2\rangle = -\,\langle\chi_1 S_1\rangle \qquad (5.83b)$$

are valid when the incident field is a plane wave and the background is homogeneous. Then (5.81,82) specialize to

$$\langle E\,E^*\rangle = |E_0|^2 \qquad (5.84)$$

$$\langle E(\mathbf{r})\rangle = E_0 \; \exp\left[-\tfrac{1}{2}\left(\langle\chi_1^2\rangle + \langle S_1^2\rangle\right)\right] \qquad (5.85)$$

We then also easily find from (5.72') that

$$\langle\chi_1^2\rangle + \langle S_1^2\rangle = \frac{\pi k^2 \, z}{2} \int\limits_{-\infty}^{+\infty} \phi_\epsilon(\boldsymbol{\kappa}, 0) \; \mathrm{d}\boldsymbol{\kappa} \qquad (5.86)$$

5.7 Range of applicability of Rytov's method

It is difficult to write down explicit and rigorous conditions for the validity of the Rytov's approximation. One condition concerns the validity of the Fresnel approximation, i.e. our omission of the next higher term in (6.32). This is justified provided that the phase error in the integrand is much less than unity;

$$\frac{k\,(\boldsymbol{\rho}-\boldsymbol{\rho}')^4}{(z-z')^3} \ll 1 \tag{5.87}$$

However, the main criteria of the validity of the results predicted in the scope of Rytov's approximation is a good fitting the appropriate results of the experimental observations. In particular, while comparing experimentally observed values of the variance of the log-amplitude (level) fluctuations with those predicted by Rytov's approximation, fairly good coincidence is observed up to values of the variance fulfilling the inequality

$$\left\langle \chi_1^2 \right\rangle \lesssim 1 \tag{5.88a}$$

At this, comparison of the experimentally observed values of the scintillation indes S_4 with the same forecasted by Rytov's approximation results in the range of the validity of the latter given by the value of the order

$$\left\langle \chi_1^2 \right\rangle \lesssim 0.1 \tag{5.88b}$$

5.8 Simplest relationships for the dielectric permittivity of plasma with the electron density fluctuations

We already briefly faced this issue in the introductory part of Chapter 1. We still confine ourselves here by the consideration of a cold collisionless plasma, which relative dielectric permittivity in full 3-D case is defined by equation

$$\epsilon = 1 - \frac{e^2\,[N_0(\mathbf{r}) + N(\mathbf{r},t)]}{m\varepsilon_0\omega^2} \tag{5.89}$$

where the full electron density is divided into two parts. Density $N_0(\mathbf{r})$ stands for the slowly varying in space distribution in the background medium of propagation (ionosphere), and $N(\mathbf{r},t)$ represents fluctuations of the electron density in space and time (time dependence should be considered in the sense of the slow time). The fluctuational item $N(\mathbf{r},t)$ in (5.89) cannot be considered as the statistically homogeneous as the absolute values of fluctuations are significantly dependant on the electron density of the background plasma. Alternatively we may write

$$\epsilon = \epsilon_0(z) + \epsilon(\mathbf{r},t) = 1 - \frac{e^2\,N_0(z)}{m\varepsilon_0\omega^2} - \frac{e^2\,N_0(z)}{m\varepsilon_0\omega^2}\,N_f(\mathbf{r},t) \tag{5.90}$$

where it is realistic and reasonable to assume the *relative* density fluctuations

$$N_f(\mathbf{r},t) = \frac{N(\mathbf{r},t)}{N_0(z)} \tag{5.91}$$

to be statistically homogeneous. It is possible to choose this function such that the relative permittivity fluctuations are zero-mean, $\langle\epsilon\rangle = 0$, but the fluctuational part of the relative dielectric permittivity, at best, may only be considered as quasi-homogeneous so that their correlation function has the form

$$\psi_\epsilon(\mathbf{r}_1, \mathbf{r}_2, t_1 - t_2) = \psi_\epsilon(\mathbf{r}_1 - \mathbf{r}_2, \alpha(\mathbf{r}_1 + \mathbf{r}_2), t_1 - t_2) \tag{5.92}$$

Here α is a small parameter.

According to the experience a significant part of random ionospheric inhomogeneities are of the turbulent type. In the inertial interval of wave numbers, localized between $k = 2\pi/L$ and $K = 2\pi/\ell$, the spatial spectrum of the ionospheric turbulence is characterized by the inverse power law as follows:

$$\phi_\epsilon(\boldsymbol{\kappa}, \kappa_z, z) = \frac{C_\epsilon(z)}{\left[1 + \frac{\kappa_x^2}{K_x^2} + \frac{\kappa_y^2}{K_y^2} + \frac{\kappa_z^2}{K_z^2}\right]^{p/2}} \tag{5.93}$$

Quantities ℓ and L are the *inner* and *outer scales* of a turbulence.

Sometimes an exponential decrease is introduced for fluctuations smaller than some *inner scale* or minimum size ℓ of the ionospheric blobs, i.e. for $\kappa > 2\pi/\ell$. The spectral index p in (5.93) is generally of the order 3 – 4.

The spatial spectrum of the relative fluctuations of the ionospheric electron density written like in (5.93) implies that random inhomogeneities are, generally speaking, 3-D inhomogeneous (anysotropic) bodies, i.e. they may have different values of ℓ anf L along different axes of an ortogonal co-ordinate system. Anysotropy of the ionospheric random inhomogeneities is caused by the Earth's magnetic field, so that the inhomogeneities are field-aligned. In mid-latitudes they have sigar-like shapes with the cylindrical symmetry in the plane orthogonal to the magnetic field lines. However, physically more complicated processes occuring in high-latitude and low-latitude ionosphere may result in full 3-D shapes of the ionospheric random inhomogeneities. It should be additionally pointed out that in some cases the inverse power law spatial spectrum may be of a more complicated shape with two different values of the spectral index (two slopes).

5.9 Extension of Rytov's approximation to the case of nhomogeneous background media

We have been dealing with the most simple case when the background medium considered to be homogeneous. However, in the vast majority of the problems of HF, VHF, UHF wave propagation in the ionospheric with the electron density fluctuations one has to treat the case, when the background medium is essentially inhomogeneous. Meeting these needs Rytov's approximation was gradually extended, finally, to the case of full 3-D inhomogeneous background medium. The first extension was performed in [Zernov, 1980] in 2-D problem for the case of plane-layered inhomogeneous background medium. The ray-centres variables were used to construct the complex phases, where the bent paths of propagation defined by the inhomogeneous layered background medium were utilized as reference rays to the appropriate ray-centred co-ordinate system. Further extension was carried out by [Zernov, 1990], [Zernov and Lundborg, 1996].

They suggested the integral representation in terms of diffracting component waves, where each diffracting component wave was constructed employing the smooth perturbation technique. This allowed investigation of the effects of the ionospheric electron density fluctuations on the fields near caustics. Further generalization of the method of smooth perturbation to the layered medium for the 3-D case made it possible solving some realistic problems of HF propagation in the ionosphere with the electron density fluctuations [Gherm and Zernov, 1995, 1998], including HF pulse propagation [Gherm et al., 1997a,b]. Finnally, one will find in [Gherm et al., 2005a] the most general case of Rytov's method, which is valid for an arbitrary 3-D inhomogeneous background medium.

5.10 Pulse propagation

In treating ionospheric propagation of pulses we have to deal with a frequency spectrum of waves, where each component may be desribed, e.g., by the Rytov's solution, or mentioned above method of the geometrical optics. Here we shall follow the formalizm of Rytov's method. Later on, however, we shall see that the case of strong scintillation requires other adequate treatment of the frequency component waves. In particular, the two-freqency coherence function must be constructed by one of the methods capable of describing the case of strong amplitude fluctuations. Now, according to the Rytov's approximation

$$E(\mathbf{r},\omega) = E_0^G(\mathbf{r},\omega)\ \exp[\Psi(\mathbf{r},\omega)] = f_0(z,\omega)\ \exp\{i\,\varphi_0[\mathbf{r},\omega,\alpha(\omega)] + \Psi(\mathbf{r},\omega)\} \tag{5.94}$$

with the geometrical-optics representation (4.56) for a plane-stratified ionosphere as the background field. Hence the background amplitude f_0 is proportional to (4.55) and the background phase is

$$\varphi_0(\mathbf{r},\alpha) = k\,\alpha(\omega)\,x + k\ \phi[z,\omega,\alpha(\omega)] \tag{5.95}$$

It is important here to note the frequency-dependence in $\alpha(\omega)$ which is there to ensure that the undisturbed component ray has an initial direction allowing it to reach the point of observation.

The transionospheric case is the simplest to consider; the treatment of the ionospheric reflection channel requires some extra precautions.

If we assume the spectrum $p(\omega)$ of the emitted pulse, we can express the field at the point of observation as follows:

$$E(\mathbf{r},t) = \int_{-\infty}^{+\infty} p(\omega)\; f_0(z,\omega)\; \exp\left\{i\,\varphi_0[\mathbf{r},\omega,\alpha(\omega)] + \Psi(\mathbf{r},\omega) - i\omega t\right\}\; \mathrm{d}\omega \tag{5.96}$$

The mean energy of the field is expressed by

$$W(\mathbf{r},t) = \int_{-\infty}^{+\infty} \mathrm{d}\omega_1\; \mathrm{d}\omega_2\; p(\omega_1)\, p^*(\omega_2)\; f_0(z,\omega_1)\; f_0^*(z,\omega_2)\Gamma(\mathbf{r},\omega_1,\omega_2)$$

$$\exp\Big\{i\,\varphi_0[\mathbf{r},\omega_1,\alpha(\omega_1)] - i\,\varphi_0[\mathbf{r},\omega_2,\alpha(\omega_2)] - i(\omega_1-\omega_2)t\Big\} \tag{5.97}$$

involving the two-frequency coherence function

$$\Gamma(\mathbf{r},\omega_1,\omega_2) = \Big\langle \exp[\Psi(\mathbf{r},\omega_1) + \Psi^*(\mathbf{r},\omega_2)]\Big\rangle \tag{5.98}$$

We must account for at least the first and second approximations of Ψ in constructing this coherence function, i.e.

$$\Gamma \simeq \Gamma_2(\mathbf{r},\omega_1,\omega_2) = \exp\left[A(\mathbf{r},\omega_1,\omega_2) + i\,B(\mathbf{r},\omega_1,\omega_2)\right] \tag{5.99}$$

with

$$A(\omega_1,\omega_2) = \langle\chi_2(\omega_1)\rangle + \langle\chi_2(\omega_2)\rangle + \tfrac{1}{2}\Big\langle[\chi_1(\omega_1)+\chi_1(\omega_2)]^2\Big\rangle - \tfrac{1}{2}\Big\langle[S_1(\omega_1)-S_1(\omega_2)]^2\Big\rangle \tag{5.100a}$$

$$B(\omega_1,\omega_2) = \langle S_2(\omega_1)\rangle - \langle S_2(\omega_2)\rangle + \Big\langle[\chi_1(\omega_1)+\chi_1(\omega_2)]\;[S_1(\omega_1)-S_1(\omega_2)]\Big\rangle \tag{5.100b}$$

These general expressions can be simplified in some limiting cases. In most cases the problem has a dominant frequency ω_d, e.g. the carrier frequency for narrowband pulses or stationary points, giving the dominant contribution to the integral in (5.97) for wideband pulses. Expanding the moments in a double Taylor series around ω_d and omitting terms higher than the quadratic, we may write the coherence function as follows:

$$\Gamma_2(\mathbf{r},\omega_1,\omega_2) = \exp\left[\beta_0(\omega_d) + \;\beta_1(\omega_d)(\omega_1-\omega_2) - \beta_2(\omega_d)(\omega_1-\omega_2)^2\right] \tag{5.101}$$

In the most general case the expansion coefficients are given by

$$\beta_0(\omega_d) = 2\;\langle\chi_2(\omega_d)\rangle + 2\;\langle\chi_1^2(\omega_d)\rangle \tag{5.102$_0$}$$

$$\beta_1(\omega_d) = \left\langle \frac{\partial S_2(\omega_d)}{\partial \omega} \right\rangle + 2\ \left\langle \chi_1(\omega_d)\ \frac{\partial S_1(\omega_d)}{\partial \omega} \right\rangle \tag{5.102$_1$}$$

$$\beta_2(\omega_d) = \tfrac{1}{2}\ \left\langle \left(\frac{\partial S_1(\omega_d)}{\partial \omega} \right)^2 \right\rangle - \tfrac{1}{4}\ \left\langle \frac{\partial^2 \chi_2(\omega_d)}{\partial \omega^2} \right\rangle - \tfrac{1}{2}\ \left\langle \chi_1(\omega_d)\ \frac{\partial^2 \chi_1(\omega_d)}{\partial \omega^2} \right\rangle \tag{5.102$_2$}$$

By our way of writing the derivatives we understand that ω is to be put $= \omega_d$ after the differentiation is performed.

In the absence of caustics, as e.g. for transionospheric propagation, the relations (5.83a,b) hold. The expressions 6.109_{0-3} are then simplified to

$$\beta_0^{NS}(\omega_d) = 0 \tag{5.103$_0$}$$

$$\beta_1^{NS}(\omega_d) = \left\langle \chi_1(\omega_d)\ \frac{\partial S_1(\omega_d)}{\partial \omega} \right\rangle - \left\langle \frac{\partial \chi_1(\omega_d)}{\partial \omega}\ S_1(\omega_d) \right\rangle \tag{5.103$_1$}$$

$$\beta_2^{NS}(\omega_d) = \tfrac{1}{2}\ \left[\left\langle \left(\frac{\partial \chi_1(\omega_d)}{\partial \omega} \right)^2 \right\rangle + \left\langle \left(\frac{\partial S_1(\omega_d)}{\partial \omega} \right)^2 \right\rangle \right] \tag{5.103$_2$}$$

where the superscript NS stands for "non-singular". In this case we still have diffraction effects and phase fluctuations.

In the geometrical optics limit we have $\chi_1 = 0$ and because of this the further simplification

$$\beta_0^{GO}(\omega_d) = 0 \tag{5.104$_0$}$$

$$\beta_1^{GO}(\omega_d) = 0 \tag{5.104$_1$}$$

$$\beta_2^{GO}(\omega_d) = \tfrac{1}{2}\ \left\langle \left(\frac{\partial S_1(\omega_d)}{\partial \omega} \right)^2 \right\rangle \tag{5.104$_2$}$$

As we know, we have in this case only phase fluctuations and no diffraction effects.

We shall use these representations of the coherence function in the mean-energy integral (5.97). First we note that in the absence of fluctuations, when $\Gamma = 1$ in (6.104), we still have the regular ionospheric dispersion with the mean energy

$$W_0(\mathbf{r}, t) = \int_{-\infty}^{+\infty} \mathrm{d}\omega_1\ \mathrm{d}\omega_2\ p(\omega_1)\, p^*(\omega_2)\ f_0(z, \omega_1)\ f_0^*(z, \omega_2)$$

$$\exp\Big\{ i\, \varphi_0[\mathbf{r}, \omega_1, \alpha(\omega_1)] - i\, \varphi_0[\mathbf{r}, \omega_2, \alpha(\omega_2)] - i(\omega_1 - \omega_2)t \Big\} \tag{5.105}$$

When we introduce the form (6.108) of the coherence function into (5.97) it is possible to carry out the Fourier transformation of the exponential function there with respect to the difference frequency $(\omega_1 - \omega_2)$. Thus we may use the convolution theorem to obtain the following expression for the mean-energy in the general case:

$$W(\mathbf{r},t) = \frac{1}{2\sqrt{\pi\beta_2(\omega_d)}} \int_{-\infty}^{+\infty} \exp\left\{\beta_0(\omega_d) - \frac{[\tau - t + \beta_1(\omega_d)]^2}{4\,\beta_2(\omega_d)}\right\} W_0(\mathbf{r},\tau)\,\mathrm{d}\tau \tag{5.106}$$

This result contains all the effects caused by the inhomogeneities as described by the coefficients β_i. The formula can be specialized to various cases by using the appropriate one of the sets (5.102–104) of these coefficients. It contains also the limiting case of geometrical optics when $\beta_0 = \beta_1 = 0$ and only β_2 is non-zero.

When $4\beta_2 \ll T^2$, where T is the length of the pulse after passage through the regular ionosphere, the exponent in (5.106) varies rapidly compared to the regular pulse W_0 and then the mean energy W is the same as the mean energy W_0 of the regular ionosphere, only delayed somewhat due to the fluctuations and with an extra amplitude factor:

$$W(\mathbf{r},t) = \exp[\beta_0(\omega_d)]\; W_0[\mathbf{r}, t - \beta_1(\omega_d)] \tag{5.107}$$

If, in particular, $\beta_0 = \beta_1 = \beta_2 = 0$ then the regular result (5.105) is recovered.

The condition that β_2 be small implies weak fluctuations. We shall consider some ways of directly calculating the double integral (5.97) for cases when β_2 is considerable compared to T^2.

The simplest case is for *wideband pulses* when the spectrum $p(\omega)$ is a slowly varying function of frequency. Then it is possible to obtain the result by means of the steepest descent method on the following form:

$$W(\mathbf{r},t) = \frac{\pi\sqrt{2}\,|p(\omega_0)|^2\, f_0^2(\omega_0)}{\frac{\mathrm{d}\beta_1(\omega_0)}{\mathrm{d}\omega} + \frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\left[k_0\varphi_0(\omega_0)\right]}\;\exp[\beta_0(\omega_0)] \tag{5.108a}$$

Time enters implicitly into this expression through ω_0 which is given by the equation for the saddle point

$$\beta_1(\omega_0) + \frac{\mathrm{d}}{\mathrm{d}\omega}\left[k_0\varphi_0(\omega_0)\right] = t \tag{5.108b}$$

We see that when (6.115a) is valid the mean energy is independent of β_2. This means that diffraction alone is responsible for the pulse distortion. In more complicated cases of calculating (6.104) the final result depends also on β_2; see [Zernov and Lundborg, 1993].

In the case of *narrowband signals* the spectrum $p(\omega)$ is no longer a slowly varying function. There is no general way to calculate the integral in this case, instead we choose a particular model of the initial pulse, viz. a carrier of center frequency ω_c and a Gaussian amplitude envelope of width T_0 :

$$E_{00}(t) = \exp\left[-\frac{t^2}{2T_0^2} - i\omega_c t\right] \tag{5.109a}$$

with the spectrum

$$p(\omega) = \frac{T_0}{\sqrt{2\pi}} \exp\left[-\frac{T_0^2}{2}\Big(\omega - \omega_c\Big)^2\right] \tag{5.109b}$$

Expanding the exponent in the integrand of (5.97) in a double series around ω_c it is possible to obtain the following result

$$\begin{aligned} W(\mathbf{r}, t) = {} & \frac{T_0^2 f_0^2(\omega_c)\ \exp\left[\beta_0(\omega_c)\right]}{\sqrt{T_0^4 + 4T_0^2\beta_2(\omega_c) + \left(\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\left[k_c\varphi_0(\omega_c)\right]\right)^2}} \\ & \cdot \exp\left\{-\frac{\tau^2(\omega_c)T_0^2}{T_0^4 + 4T_0^2\beta_2(\omega_c) + \left(\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\left[k_c\varphi_0(\omega_c)\right]\right)^2}\right\} \end{aligned} \tag{5.110a}$$

where τ is a translated time due to the propagation:

$$\tau(\omega_c) = t - t_g \tag{5.110b}$$

with the group delay time

$$t_g = \frac{\mathrm{d}}{\mathrm{d}\omega}\Big[k_c\phi_0(\omega_c)\Big] + \beta_1(\omega_c) \tag{5.110c}$$

In this the regular dispersion is taken into account by the function $[k_c\varphi_0(\omega_c)]''$, which is related to the dispersive bandwidth [Lin *et al.*, 1989] of the regular ionosphere; essentially the same quantity is denoted by P_1' in [Lundborg, 1990]. The additional dispersive properties, caused by the fluctuations, are represented by the functions $\beta_0(\omega_c)$, $\beta_1(\omega_c)$ and $\beta_2(\omega_c)$. By analogy with the regular dispersion, the quantity $\beta_2(\omega_c)$ can be used to define a fluctuational bandwidth. When $\beta_0(\omega_c) \neq 0$ and $\beta_1(\omega_c) \neq 0$, diffraction affects also the amplitude of the pulse and the group delay time, as is easily seen from (5.110a,c). It is also easy to see that when all dispersion terms are absent, we recover the energy of the undisturbed pulse

$$W_p(\mathbf{r}, t) = f_0^2(\omega_c)\ \exp\left\{-\frac{t^2}{T_0^2}\right\} \tag{5.111}$$

which has the same shape as the initial pulse.

Chapter 6

Diffusive Markov approximation for the parabolic equation

All the methods considered previously (except the stochastic screen method) are based on the assumption that the local inhomogeneities of the dielectric permittivity are weak. In contrast, the subject of the present Chapter is a method which does not have this limitation. The small parameter of the problem will now be the ratio of the correlation radius to the characteristic scale of the mean field. The equations for the first two moments of the full field will be derived directly using the smallness of the just-mentioned parameter.

In the subsequent treatment the technique of variational or functional derivatives will be used. Therefore we define first the functional derivative and discuss some of its applications.

6.1 The notion of variational derivative

We use for functionals, which give a mapping of the space of functions $\{u(x)\}$ onto the space of numbers, the notation $\varphi[u(x)]$. Then we may define the variational derivative of $\varphi[u(x)]$ through

$$\frac{\delta\varphi[u(x)]}{\delta u(x_0)} = \lim_{\substack{\Delta x \to 0 \\ \max|\Delta u(x)| \to 0}} \frac{\varphi[u(x)+\Delta u(x)] - \varphi[u(x)]}{\int \Delta u(x)\,\mathrm{d}x} \tag{6.1}$$

where $\Delta u(x)$ is a local arbitray change of the functional argument which is non-zero only in a small interval around the point $x = x_0$, for which the derivative is being calculated, and which has an infinitesimal absolute value.

6.1.1 Variational derivative for linear functionals

Using the definition given, we shall first find the variational derivative for a linear functional, which in the most general case may be represented as

$$L[u(x)] = \int A(x)\, u(x)\, \mathrm{d}x \tag{6.2}$$

where $A(x)$ is the kernel of the functional. Then, according to the definition (6.1), we have for the derivative

$$\frac{\delta L[u(x)]}{\delta u(x_0)} = \lim_{\substack{\Delta x \to 0 \\ \max|\Delta u(x)| \to 0}} \frac{\int A\,[u + \Delta u]\,\mathrm{d}x - \int A\,u\,\mathrm{d}x}{\int_{\{x_0\}} \Delta u(x)\,\mathrm{d}x}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \max|\Delta u(x)| \to 0}} \frac{\int A(x)\,\Delta u(x)\,\mathrm{d}x}{\int_{\{x_0\}} \Delta u(x)\,\mathrm{d}x} = \lim_{\overline{x} \to x_0} A(\overline{x}) = A(x_0) \tag{6.3}$$

The subscript $\{x_0\}$ on the integral signs in the denominators indicate that the integration is to be performed over the interval around x_0 where $\Delta u(x)$ is nonzero. The integrals in the numerators, on the other hand, are performed over the interval stated in (6.2).

6.1.2 Derivative of an arbitrary functional

The last member of (6.3) leads to a rule of great practical importance for calculating the variational derivative of a function $u(x)$ as a particular case of a linear functional depending on the parameter x:

$$u(x) = \int \delta(x - y)\, u(y)\, \mathrm{d}y \tag{6.4}$$

Using the rule (6.3) for the variational derivative, we easily find

$$\frac{\delta u(x)}{\delta u(x_0)} = \delta(x - x_0) \tag{6.5}$$

In calculating the variational derivative of an arbitrary functional we may then use the ordinary rules of derivation with the rule (6.5) in the last step. For instance:
(i) For the linear functional $L[u]$ we obtain once again

$$\frac{\delta L[u(x)]}{\delta u(x_0)} = \int A(x)\, \delta(x - x_0)\, \mathrm{d}x = A(x_0) \tag{6.6}$$

(ii) For a quadratic functional $Q[u(x)] = \int B(x_1, x_2)\, u(x_1)\, u(x_2)\, \mathrm{d}x_1\, \mathrm{d}x_2$ we may obtain

$$\frac{\delta Q[u]}{\delta u(x_0)} = \int B(x_1, x_2)\, \delta(x_1 - x_0)\, u(x_2)\, \mathrm{d}x_1\, \mathrm{d}x_2$$

$$+\int B(x_1,x_2)\; u(x_1)\; \delta(x_1-x_0)\; \mathrm{d}x_1\; \mathrm{d}x_2$$

$$= \int B(x_0,x_2)\; u(x_2)\; \mathrm{d}x_2 + \int B(x_1,x_0)\; u(x_1)\; \mathrm{d}x_1 \tag{6.7}$$

This gives

$$\frac{\delta Q[u]}{\delta u(x_0)} = 2 \int B(x_0,x)\; u(x)\; \mathrm{d}x \tag{6.8}$$

for a symmetric kernel $B(x_1,x_2)$.

6.2 Characteristic functional for a random function

The technique of variational derivatives can be employed in developing an alternative description of random functions. To this end we shall now introduce the characteristic functional for a random function. For simplicity we shall consider the case of a one-dimensional random function $\epsilon(x)$. Then we define the characteristic functional $Q_\epsilon[u]$ of this random function through

$$Q_\epsilon[u(x)] = \left\langle \exp\left[i \int \epsilon(x)\; u(x)\; \mathrm{d}x \right] \right\rangle \tag{6.9}$$

where $u(x)$ is an arbitrary determinitstic function. Keeping (6.6) in mind it is now easy to see that

$$\langle \epsilon(x_0) \rangle = \frac{1}{i} \;\; \frac{\delta Q_\epsilon[u]}{\delta u(x_0)}\bigg|_{u=0} \tag{6.10a}$$

$$\langle \epsilon(x_1)\, \epsilon(x_2) \rangle = \frac{1}{(i)^2} \;\; \frac{\delta^2 Q_\epsilon[u]}{\delta u(x_1)\; \delta u(x_2)}\bigg|_{u=0} \tag{6.10b}$$

and in the general case

$$\langle \epsilon(x_1) \ldots \epsilon(x_2) \rangle = \frac{1}{(i)^n} \;\; \frac{\delta^n Q_\epsilon[u]}{\delta u(x_1) \ldots \delta u(x_n)}\bigg|_{u=0} \tag{6.10c}$$

Hence any moment of the random function can be described by means of the characteristic functional (6.9).

6.2.1 The connection between the characteristic function and the probability density function

In the particular case when $u(x)$ has the special form

$$u_n(x) = \sum_{j=1}^{n} u_j\; \delta(x - x_j) \tag{6.11}$$

we find the following expression for the characteristic functional:

$$Q_\epsilon[u_n(x)] = \left\langle \exp\left[i \sum_{j=1}^{n} u_j \ \epsilon_j \right] \right\rangle \tag{6.12}$$

with $\epsilon_j = \epsilon(x_j)$. Assuming that the random function $\epsilon(x)$ is continuous and defined over an infinite interval, (6.12) can be rewritten using the PDF's of ϵ as follows:

$$Q_\epsilon[u_n(x)] = \int_{-\infty}^{+\infty} w_n(x_1, \epsilon_1, ..., x_n, \epsilon_n) \ \exp\left[i \sum_{j=1}^{n} u_j \ \epsilon_j \right] \ \mathrm{d}\epsilon_1 ... \mathrm{d}\epsilon_n \tag{6.13}$$

From this last equation we understand that for continuous random functions with infinite domain of definition, the characteristic functionals (6.12) and the multidimensional PDF's form Fourier transform pairs.

6.2.2 Characteristic functional for a Gaussian zero-mean random field

As an example we shall construct the characteristic functional for the zero-mean normally distributed random function.

Let the one-dimensional random function $\epsilon(x)$ be a normally distributed random field with $\langle \epsilon(x) \rangle = 0$. Then the quantity

$$q = \int \epsilon(x) \ u(x) \ \mathrm{d}x \tag{6.14}$$

which appears in (7.9) is a normally distributed zero-mean random value and, as in (3.58), $\langle e^{iq} \rangle = \exp\left[-\frac{1}{2} \langle q^2 \rangle \right]$. Therefore we only have to construct the variance $\langle q^2 \rangle$ for q from (6.14), which is as follows

$$\langle q^2 \rangle = \int \psi_\epsilon(x_1, x_2) \ u(x_1) \ u(x_2) \ \mathrm{d}x_1 \ \mathrm{d}x_2 \tag{6.15}$$

As a result the final expression for the characteristic functional can be written on the form

$$Q_\epsilon[u] = \exp\left[-\frac{1}{2} \int \psi_\epsilon(x_1, x_2) \ u(x_1) \ u(x_2) \ \mathrm{d}x_1 \ \mathrm{d}x_2 \right] \tag{6.16}$$

We shall now check if (7.16) really gives the correct description of the zero-mean Gaussian random field. Indeed, let us calculate

$$\langle \epsilon(x_0) \rangle = \frac{1}{i} \ \left. \frac{\delta Q_\epsilon[u]}{\delta u(x_0)} \right|_{u=0} = \exp\left[-\frac{1}{2} \int \psi_\epsilon(x_1, x_2) \ u(x_1) \ u(x_2) \ \mathrm{d}x_1 \ \mathrm{d}x_2 \right] \cdot$$

$\left[-\frac{1}{2}\int\psi_\epsilon(x_0,x_2)\,u(x_2)\,\mathrm{d}x_2-\frac{1}{2}\int\psi_\epsilon(x_1,x_0)\,u(x_1)\,\mathrm{d}x_1\right]|_{u=0}$(6.17)

While calculating the second derivative, one needs to differentiate only the second factor in (6.17), since the derivative of the exponent yields zero due to $u=0$. Then we finally get for the second moment the expected result

$$\langle\epsilon(x_1)\,\epsilon(x_2)\rangle=\psi_\epsilon(x_1,x_2) \tag{6.18}$$

6.3 Parabolic approximation of the Helmholtz' equation

It appears to be possible to construct closed equations for the moments of the stochastic differential equation

$$\nabla^2E+k^2\;[\epsilon_0(\mathbf{r},\omega)+\epsilon(\mathbf{r},\omega,t)]\;E=0 \tag{6.19}$$

only in the case when the parabolic approximation can be introduced. We now consider the simplest case with $\epsilon_0(\mathbf{r},\omega)=1$. Then, if we again investigate the large-scale inhomogeneities $k\ell_\epsilon\gg1$, we can attempt a solution of (6.19) on the form

$$E(\mathbf{r},\omega,t)=E_0(\mathbf{r},\omega)\;v(\mathbf{r},\omega,t) \tag{6.20}$$

with $E_0(\mathbf{r},\omega)$ being the solution of the reduced equation (6.19) for $\epsilon=0$. We choose E_0 as a plane wave, propagating in the diirection of the z-axis:

$$E_0(\mathbf{r},\omega)=e^{ikz} \tag{6.21}$$

Substituting E in the form of (6.20) with E_0 given by (6.21) into eq. (6.19), we can easily obtain the following exact equation for the new unknown function v:

$$\frac{\partial^2v}{\partial z^2}+2ik\,\frac{\partial v}{\partial z}+\nabla_\perp^2v+k^2\epsilon\,v=0 \tag{6.22}$$

Here $v=v(\boldsymbol{\rho},z)$, $\boldsymbol{\rho}=\{x,y\}$ and $\nabla_\perp^2=\partial^2/\partial x^2+\partial^2/\partial y^2$. The order of magnitude of the first two terms in (6.22) can be estimated as follows:

$$2ik\,\frac{\partial v}{\partial z}\sim2ik\,\frac{v}{\ell_\epsilon} \tag{6.23a}$$

$$\frac{\partial^2v}{\partial z^2}\sim\frac{v}{\ell_\epsilon^2} \tag{6.23b}$$

and hence for the sum of these two the estimate

$$2ik\,\frac{\partial v}{\partial z}+\frac{\partial^2v}{\partial z^2}\sim2ik\,\frac{\partial v}{\partial z}\left(1+\frac{1}{2ik\ell_\epsilon}\right) \tag{6.24}$$

This shows that the contribution of the second derivative of v is small for the large-scale inhomogeneities. Then we finally obtain the approximate parabolic equation

$$2ik\,\frac{\partial v}{\partial z}+\nabla_\perp^2v+k^2\epsilon\,v=0 \tag{6.25}$$

for the complex amplitude function $v(\boldsymbol{\rho}, z)$ of the plane wave (6.21) in the inhomogeneous medium. We shall also use the integral form of the last equation

$$v(\boldsymbol{\rho}, z) = v(\boldsymbol{\rho}, 0) + \frac{i}{2k} \int_0^z \left[\nabla_\perp^2 v(\boldsymbol{\rho}, \zeta) + k^2 \epsilon(\boldsymbol{\rho}, \zeta)\, v(\boldsymbol{\rho}, \zeta)\right] \,\mathrm{d}\zeta \tag{6.26}$$

where $v(\boldsymbol{\rho}, 0)$ is the given "initial" value of the field $v(\boldsymbol{\rho}, z)$ in the plane $z = 0$. [We again consider the situation when the inhomogeneities $\epsilon(\boldsymbol{\rho}, z)$ occupy the half-space $z \geq 0$].

Eq. (6.26) shows that the field $v(\boldsymbol{\rho}, z)$ obeys the property of "dynamical causality", i.e., the field at the point of observation $(\boldsymbol{\rho}, z)$ depends on the properties of the medium $\epsilon(\boldsymbol{\rho}, z')$ at the points which lie "before" the point of observation $(z' < z)$. This yields the important relation for the functional derivative of the solution of (6.25) [or (6.26)] with respect to the function ϵ:

$$\frac{\delta v(\boldsymbol{\rho}, z)}{\delta \epsilon(\boldsymbol{\rho}', z')} = 0 \;, \qquad z' > z \tag{6.27}$$

which will be used later on.

6.4 Averaging of the parabolic equation

On the basis of eqs. (6.25,26) we shall now derive the equation for the mean field $\langle v \rangle$. Averaging (6.25), we obtain the equation

$$2ik \,\frac{\partial \langle v \rangle}{\partial z} + \nabla_\perp^2 \langle v \rangle + k^2 \langle \epsilon\, v \rangle = 0 \tag{7.28}$$

which is unclosed in the sense that it contains two unknown functions, $\langle v \rangle$ and $\langle \epsilon v \rangle$. The last average is the mutual correlation of the random field ϵ and the solution v of eqs. (6.25,26), which is itself a functional on this random field. For this sort of correlation the Furutsu-Novikov formula can be used, which has the form

$$\langle \epsilon(\boldsymbol{\rho}, z)\, v(\boldsymbol{\rho}, z) \rangle = \int\limits_{-\infty}^{+\infty} \mathrm{d}z' \int\limits_{-\infty}^{+\infty} \mathrm{d}\boldsymbol{\rho}' \;\psi_\epsilon(\boldsymbol{\rho}, z, \boldsymbol{\rho}', z') \,\left\langle \frac{\delta v(\boldsymbol{\rho}, z)}{\delta \epsilon(\boldsymbol{\rho}', z')} \right\rangle \tag{6.29}$$

if $\epsilon(\boldsymbol{\rho}, z)$ is a Gaussian random function with the correlation function $\psi_\epsilon(\boldsymbol{\rho}, z, \boldsymbol{\rho}', z')$. In our case with fluctuations in the half-space $z > 0$ with the property (6.27), eq. (6.29) yields

$$\langle \epsilon(\boldsymbol{\rho}, z)\, v(\boldsymbol{\rho}, z) \rangle = \int\limits_{0}^{z} \mathrm{d}z' \int\limits_{-\infty}^{+\infty} \mathrm{d}\boldsymbol{\rho}' \;\psi_\epsilon(\boldsymbol{\rho}, z, \boldsymbol{\rho}', z') \,\left\langle \frac{\delta v(\boldsymbol{\rho}, z)}{\delta \epsilon(\boldsymbol{\rho}', z')} \right\rangle \tag{6.30}$$

Using the last expression another "unclosed" equation can be written instead of (6.28) on the form

$$2ik \,\frac{\partial \langle v \rangle}{\partial z} + \nabla_\perp^2 \langle v \rangle + k^2 \int\limits_{0}^{z} \mathrm{d}z' \int\limits_{-\infty}^{+\infty} \mathrm{d}\boldsymbol{\rho}' \;\psi_\epsilon(\boldsymbol{\rho}, z, \boldsymbol{\rho}', z') \,\left\langle \frac{\delta v(\boldsymbol{\rho}, z)}{\delta \epsilon(\boldsymbol{\rho}', z')} \right\rangle = 0 \tag{6.31}$$

which contains except $\langle v \rangle$ the unknown function

$$\left\langle \frac{\delta v(\boldsymbol{\rho}, z)}{\delta \epsilon(\boldsymbol{\rho}', z')} \right\rangle = T_1(\boldsymbol{\rho}, z, \boldsymbol{\rho}', z') \tag{6.32}$$

In the general case another equation has to be derived for the new unknown function $T_1(\boldsymbol{\rho}, z, \boldsymbol{\rho}', z')$ and this equation also will be unclosed. In this way one can construct an infinite chain of equations, which can then be terminated after an arbitrary number of steps. We can do this just on the first step to obtain the diffusive Markov approximation, but we shall then need at least one more step to assess the range of validity for the diffusive Markov approximation. First, however, we derive the diffusive Markov approximation itself.

6.4.1 Approximation of δ-correlated random field

To close the equation (6.31) directly we assume as a model of the correlation function ψ_ϵ of the dielectric permittivity fluctuations the expression

$$\psi_\epsilon(\boldsymbol{\rho}, \boldsymbol{\rho}', z - z') = A_\epsilon(\boldsymbol{\rho}, \boldsymbol{\rho}')\, \delta(z - z') \tag{6.33}$$

with

$$\int_{-\infty}^{+\infty} \psi_\epsilon(\boldsymbol{\rho}, \boldsymbol{\rho}', \zeta)\, \mathrm{d}\zeta = A_\epsilon(\boldsymbol{\rho}, \boldsymbol{\rho}') \tag{6.34}$$

This last relation gives the rule for calculating the function $A_\epsilon(\boldsymbol{\rho}, \boldsymbol{\rho}')$ in the transversal variables.

The representation (6.33) provides the property of "statistical causality" for the mean field $\langle v \rangle$. With this formula taken into account one finds from (6.31) the following equation which, as we can see, is local in the longitudinal variable:

$$2ik\, \frac{\partial \langle v \rangle}{\partial z} + \nabla_\perp^2 \langle v \rangle + \tfrac{1}{2}\, k^2 \int_{-\infty}^{+\infty} \mathrm{d}\boldsymbol{\rho}'\; A_\epsilon(\boldsymbol{\rho}, \boldsymbol{\rho}')\; \left\langle \frac{\delta v(\boldsymbol{\rho}, z)}{\delta \epsilon(\boldsymbol{\rho}', z)} \right\rangle = 0 \tag{6.35}$$

The factor $\frac{1}{2}$ in front of the integral has appeared due to the even property of the delta function.

To find the quantity $\langle \delta v(\boldsymbol{\rho}, z)/\delta\epsilon(\boldsymbol{\rho}', z) \rangle$, we differentiate (6.26) with respect to the function $\epsilon(\boldsymbol{\rho}', z')$. Then we find

$$\begin{aligned} \frac{\delta v(\boldsymbol{\rho}, z)}{\delta \epsilon(\boldsymbol{\rho}', z')} &= \frac{i}{2k} \int_0^z \left\{ \left[\nabla_\perp^2 + k^2\, \epsilon(\boldsymbol{\rho}, \zeta)\right]\; \frac{\delta v(\boldsymbol{\rho}, \zeta)}{\delta \epsilon(\boldsymbol{\rho}', z')} + k^2\, v(\boldsymbol{\rho}, \zeta)\, \delta(\boldsymbol{\rho} - \boldsymbol{\rho}')\, \delta(\zeta - z') \right\} \mathrm{d}\zeta \\ &= \frac{i}{2k} \int_0^z \left[\nabla_\perp^2 + k^2\, \epsilon(\boldsymbol{\rho}, \zeta)\right]\; \frac{\delta v(\boldsymbol{\rho}, \zeta)}{\delta \epsilon(\boldsymbol{\rho}', z')}\, \mathrm{d}\zeta + \frac{i\,k}{2}\, v(\boldsymbol{\rho}, z')\, \delta(\boldsymbol{\rho} - \boldsymbol{\rho}') \end{aligned} \tag{6.36}$$

Taking into account the property (6.27) of dynamical causality, we may finally write

$$\frac{\delta v(\boldsymbol{\rho}, z)}{\delta \epsilon(\boldsymbol{\rho}', z')} = \frac{i}{2k} \int\limits_{z'}^{z} \left[\nabla_\perp^2 + k^2\, \epsilon(\boldsymbol{\rho}, \zeta)\right] \frac{\delta v(\boldsymbol{\rho}, \zeta)}{\delta \epsilon(\boldsymbol{\rho}', z')}\, \mathrm{d}\zeta + \frac{i\,k}{2}\, v(\boldsymbol{\rho}, z')\, \delta(\boldsymbol{\rho} - \boldsymbol{\rho}') \tag{6.37}$$

We now put $z = z'$ in (6.37) and average to find the quantity

$$\left\langle \frac{\delta v(\boldsymbol{\rho}, z)}{\delta \epsilon(\boldsymbol{\rho}', z)} \right\rangle = +\frac{i\,k}{2}\, \langle v(\boldsymbol{\rho}, z)\rangle\ \delta(\boldsymbol{\rho} - \boldsymbol{\rho}') \tag{6.38}$$

When we substitute this into (6.35), we may there perform the integration over $\boldsymbol{\rho}'$ and obtain as our ultimate result the following closed equation for the mean field

$$2ik\, \frac{\partial \langle v\rangle}{\partial z} + \nabla_\perp^2 \langle v\rangle + \frac{i\,k^3}{4}\, A_\epsilon(\boldsymbol{\rho}, \boldsymbol{\rho})\ \langle v\rangle = 0 \tag{6.39}$$

6.4.2 The solution of the mean field equation

Let us consider a totally statistically homogeneous random field $\epsilon(\boldsymbol{\rho}, z)$, i.e. we have $A_\epsilon(\boldsymbol{\rho}, \boldsymbol{\rho}') = A_\epsilon(\boldsymbol{\rho} - \boldsymbol{\rho}')$ and $A_\epsilon(\boldsymbol{\rho}, \boldsymbol{\rho}) = A_\epsilon(0)$. In this case we can rewrite (6.39) as follows:

$$\frac{\partial \langle v\rangle}{\partial z} - \frac{i}{2k}\, \nabla_\perp^2 \langle v\rangle + \frac{k^2}{8}\, A_\epsilon(0)\ \langle v\rangle = 0 \tag{6.40}$$

We shall look for solutions of this equation on the form

$$\langle v(\boldsymbol{\rho}, z)\rangle = e^{f(z)}\ w(\boldsymbol{\rho}, z) \tag{6.41}$$

with an unknown function $f(z)$ and with $w(\boldsymbol{\rho}, z)$ satisfying the equation for the field in the medium without fluctuations:

$$\frac{\partial w}{\partial z} - \frac{i}{2k}\, \nabla_\perp^2 w = 0 \tag{6.42}$$

Substituting (6.41) into (6.40) we find the equation for $f(z)$:

$$\frac{\mathrm{d}f}{\mathrm{d}z} = -\frac{k^2}{8}\, A_\epsilon(0) \tag{6.43}$$

which can be easily solved to give

$$f(z) = -\frac{k^2}{8}\, A_\epsilon(0)\, (z - z_0) \tag{6.44}$$

Obviously we can here put the constant z_0 to zero corresponding to the beginning of the half-space with fluctuations. Hence

$$\langle v(\boldsymbol{\rho}, z)\rangle = \exp\left[-\frac{k^2}{8}\, A_\epsilon(0)\, z\right]\ w(\boldsymbol{\rho}, z) \tag{6.45}$$

With the incident field given by (7.21), we see that the expression

$$w(\boldsymbol{\rho}, z) = 1 \tag{6.46}$$

the differential equation (6.42).

Putting together (6.20), (6.21), (6.41), (6.45) and (6.46), we then find the mean field

$$\langle E(\boldsymbol{\rho}, z)\rangle = \exp\left[ikz - \frac{k^2}{8}\, A_\epsilon(0)\, z\right] \tag{6.47}$$

Recalling also the definition (7.34) of $A_\epsilon(\boldsymbol{\rho}, \boldsymbol{\rho}')$, which now gives us

$$A_\epsilon(0) = \int\limits_{-\infty}^{+\infty} \psi_\epsilon(0, \zeta)\ \mathrm{d}\zeta = 2\,\sigma_\epsilon^2\,\ell_\epsilon \tag{6.48}$$

we finally obtain

$$\langle E(\boldsymbol{\rho}, z)\rangle = \exp\left[ikz - \frac{k^2\sigma_\epsilon^2\,\ell_\epsilon}{8}\ z\right] \tag{6.49}$$

This coincides with the representation of the mean field in geometrical optics, eq. (4.121), for the case of $\epsilon_0 = 1$, but now the range of validity for (6.49) is wider than what happened to be the case for the geometrical-optics representation. Later we shall investigate the range of applicability of (7.47,49) in more detail.

It is also of interest to point out that in calculating the total scattering cross-section for large-scale isotropic inhomogeneities, $k\ell_\epsilon \gg 1$, one finds

$$\sigma_0 = \frac{k^2\,A_\epsilon(0)}{4} \tag{6.50}$$

which is twice the extinction coefficient in (6.47) or the extinction coefficient for the “intensity of the mean field”. At the same time it can be shown that the conservation law for the mean energy of the field $\exp[ikz]\,v(\boldsymbol{\rho}, z)$:

$$\langle vv^*\rangle = 1 \tag{6.51}$$

follows from the parabolic equation approximation (6.25). Therefore the energy of the fluctuational part of the field $\tilde{v}$ can be expressed as

$$\langle \tilde{v}\tilde{v}^*\rangle = \langle vv^*\rangle - |\,\langle v\rangle\,|^2 = 1 - \exp[-\sigma_0 z] \tag{6.52}$$

We see from this expression that for distances $z > \sigma_0^{-1}$ the main part of the wave intensity is connected with the random component.

6.5 Equation for the mean field in the case of finite correlation radius

To discuss the range of validity for eqs. (6.39,40) and the solution (6.47), we must derive a more general equation for the mean field, which takes the finite longitudinal correlation radius of the dielectric permittivity into account. This can be done if we could find the function $T_1(\boldsymbol{\rho}, z, \boldsymbol{\rho}', z')$ from (6.32) in some approximation, i.e. if we could solve at least the second equation in the chain of unclosed equations for the mean field $\langle v \rangle$. We need then the function T_1 in the area $z \geq z'$.

First we derive the equation for $\delta v(\boldsymbol{\rho}, z)/\delta \epsilon(\boldsymbol{\rho}', z')$ in the case $z > z'$. We achieve this by taking the variational derivative of (6.25) with respect to $\epsilon(\boldsymbol{\rho}', z')$. Then we get

$$2\,ik\, \frac{\partial}{\partial z}\, \frac{\delta v(\boldsymbol{\rho}, z)}{\delta \epsilon(\boldsymbol{\rho}', z')} + \nabla_\perp^2 \frac{\delta v(\boldsymbol{\rho}, z)}{\delta \epsilon(\boldsymbol{\rho}', z')} + k^2\, \epsilon(\boldsymbol{\rho}, z)\, \frac{\delta v(\boldsymbol{\rho}, z)}{\delta \epsilon(\boldsymbol{\rho}', z')} = 0 \tag{6.53}$$

While averaging the last equation we use once again the Furutsu-Novikov formula for the correlation $\langle \epsilon(\boldsymbol{\rho}, z)\, \delta v(\boldsymbol{\rho}, z)/\delta \epsilon(\boldsymbol{\rho}', z') \rangle$, which gives

$$\langle \epsilon(\boldsymbol{\rho}, z)\, \delta v(\boldsymbol{\rho}, z)/\delta \epsilon(\boldsymbol{\rho}', z') \rangle = \int\limits_0^z \mathrm{d}z'' \int\limits_{-\infty}^{+\infty} \mathrm{d}\boldsymbol{\rho}''\, \psi_\epsilon(\boldsymbol{\rho}, z, \boldsymbol{\rho}'', z'')\, \left\langle \frac{\delta^2 v(\boldsymbol{\rho}, z)}{\delta \epsilon(\boldsymbol{\rho}', z')\, \delta \epsilon(\boldsymbol{\rho}'', z'')} \right\rangle \tag{6.54}$$

If we now assume the δ-function approximation (7.33,34) in deriving the equation for $T_1(\boldsymbol{\rho}, z, \boldsymbol{\rho}', z')$, we then obtain from (6.53,54) the equation

$$2ik\, \frac{\partial}{\partial z}\, T_1 + \nabla_\perp^2 T_1 + \tfrac{1}{2}\, k^2 \int\limits_{-\infty}^{+\infty} \mathrm{d}\boldsymbol{\rho}''\, A_\epsilon(\boldsymbol{\rho}, \boldsymbol{\rho}'')\, \left\langle \frac{\delta^2 v(\boldsymbol{\rho}, z)}{\delta \epsilon(\boldsymbol{\rho}', z')\, \delta \epsilon(\boldsymbol{\rho}'', z)} \right\rangle = 0 \tag{6.55}$$

The second derivative in the integrand can be determined from (6.37) where the change of variables $z' = z''$, $\boldsymbol{\rho}' = \boldsymbol{\rho}''$ has been made and where we have put $z'' = z$. Then (6.37) gives after averaging

$$\left\langle \frac{\delta^2 v(\boldsymbol{\rho}, z)}{\delta \epsilon(\boldsymbol{\rho}', z')\, \delta \epsilon(\boldsymbol{\rho}'', z)} \right\rangle = \frac{ik}{2}\, \left\langle \frac{\delta v(\boldsymbol{\rho}, z)}{\delta \epsilon(\boldsymbol{\rho}', z')} \right\rangle\, \delta(\boldsymbol{\rho} - \boldsymbol{\rho}') \tag{6.56}$$

Finally we then find for the function $T_1(\boldsymbol{\rho}, z, \boldsymbol{\rho}', z')$ from (6.55), with (6.56) taken into account, the equation

$$2ik\, \frac{\partial}{\partial z}\, T_1 + \nabla_\perp^2 T_1 + \frac{i\, k^2}{4}\, A_\epsilon(\boldsymbol{\rho}, \boldsymbol{\rho})\, T_1 = 0 \tag{6.57}$$

which has the same form as (7.39), or for the totally homogeneous fluctuations

$$2ik\, \frac{\partial}{\partial z}\, T_1 + \nabla_\perp^2 T_1 + \frac{i\, k^2}{4}\, A_\epsilon(0)\, T_1 = 0 \tag{7.58}$$

The last of these equations must be supplemented by the initial condition at the point $z' = z$. This is given through (6.38), which we rewrite as

$$T_1(\boldsymbol{\rho}, z, \boldsymbol{\rho}', z) = +\frac{i\, k}{2}\, \langle v(\boldsymbol{\rho}, z) \rangle\, \delta(\boldsymbol{\rho} - \boldsymbol{\rho}') \tag{6.59}$$

Through (6.58,59) we now have a complete formulation of the problem for the function $T_1(\boldsymbol{\rho}, z, \boldsymbol{\rho}', z')$ which appears in the equation (6.31) for the mean field $\langle v \rangle$. Its solution can be written on the form

$$T_1(\boldsymbol{\rho}, z, \boldsymbol{\rho}', z') = \frac{k^2}{4\pi(z-z')} \exp\left[\frac{ik(\boldsymbol{\rho}-\boldsymbol{\rho}')^2}{2(z-z')} - \frac{k^2\, A_\epsilon(0)}{8}(z-z')\right] \langle v(\boldsymbol{\rho}', z')\rangle \tag{6.60}$$

Using (6.60) we finally find from (6.31) the following integro-differential equation for the mean field:

$$\frac{\partial \langle v \rangle}{\partial z} = \frac{i}{2k}\nabla_\perp^2 \langle v \rangle + \frac{ik^3}{8}\int_0^z \frac{\mathrm{d}z'}{z-z'}\int_{-\infty}^{+\infty}\mathrm{d}\boldsymbol{\rho}'$$

$$\cdot \exp\left[\frac{ik(\boldsymbol{\rho}-\boldsymbol{\rho}')^2}{2(z-z')} - \frac{k^2\, A_\epsilon(0)}{8}(z-z')\right] \cdot \psi_\epsilon(\boldsymbol{\rho}-\boldsymbol{\rho}', z-z')\ \langle v(\boldsymbol{\rho}', z')\rangle \tag{6.61}$$

6.5.1 Range of validity for the diffusive Markov approximation

In (.61) we have obtained an equation for the mean field $\langle v \rangle$ which is of a more general form than the diffusive Markov approximation (6.40). If we now investigate under what conditions the form (6.61) may be simplified to the form (6.40), we can therefore describe the range of validity for the diffusive Markov approximation.

First we have to understand under what conditions all functions under the integration sign in (6.61) can be considered as slowly varying in the longitudinal variable, in comparison with the longitudinal variation of the correlation function.

Let the longitudinal correlation radius of ψ_ϵ be $\ell_\parallel$. The longitudinal scale of the other functions is expressed by the quantity $[k^2\, A_\epsilon(0)]^{-1}$. Then, if

$$[k^2\, A_\epsilon(0)]^{-1} > \ell_\parallel \qquad \text{or} \qquad k^2\, A_\epsilon(0)\, \ell_\parallel = k^2\, \sigma_\epsilon^2\, \ell_\epsilon^2 < 1 \tag{6.62}$$

one can write

$$\frac{\partial \langle v \rangle}{\partial z} = \frac{i}{2k}\nabla_\perp^2 \langle v \rangle + \frac{ik^3}{8}\int_0^z \frac{\mathrm{d}z'}{z-z'}\int_{-\infty}^{+\infty}\mathrm{d}\boldsymbol{\rho}' \exp\left[\frac{ik(\boldsymbol{\rho}-\boldsymbol{\rho}')^2}{2(z-z')}\right]\ \psi_\epsilon(\boldsymbol{\rho}-\boldsymbol{\rho}', z-z')\ \langle v(\boldsymbol{\rho}', z)\rangle \tag{6.63}$$

If the exponential function in (7.63) changes fast in $\boldsymbol{\rho}'$ compared to the $\boldsymbol{\rho}'$-dependence in ψ_ϵ, we can calculate the integral in $\boldsymbol{\rho}'$ by the steepest-descent method. The condition for this is found to be

$$k\,\ell_\perp^2 \gg \ell_\parallel \tag{6.64}$$

for the case of an incident plane wave, and the same condition together with the condition

$$k\,a^2 \gg \ell_\parallel \tag{6.65}$$

for an incident beam wave with beam width a. Condition (6.64) is always fulfilled for isotropic inhomogeneities $\ell_{\parallel} = \ell_{\perp}$, for which (6.64) becomes the condition of validity $k\ell_{\epsilon} \gg 1$ for the initial parabolic equation. With the inequalities (6.64,65) the steepest descent method in $\boldsymbol{\rho}'$ gives from (6.63)

$$\frac{\partial \langle v \rangle}{\partial z} = \frac{i}{2k} \nabla_{\perp}^2 \langle v \rangle - \frac{k^2}{4} \langle v \rangle \int_0^z \psi_{\epsilon}(0, z - z')\, \mathrm{d}z' \tag{6.66}$$

If we, finally, also require the following inequality to be fulfilled:

$$z \gg \ell_{\parallel} \tag{6.67}$$

we recover from (6.67) exactly the diffusive Markov approximation. Hence we conclude that the validity of this approximation is described jointly by the inequalities (6.62,64,67). In particular (6.67) shows that the small parameter in the diffusive Markov approximation is the ratio of the longitudinal correlation radius and the characteristic longitudinal scale of the mean field.

6.6 Diffusive Markov approximation for the coherence function

We consider now the transversal coherence function

$$\Gamma(\boldsymbol{\rho}', \boldsymbol{\rho}'', z) = \langle v(\boldsymbol{\rho}', z)\, v^*(\boldsymbol{\rho}'', z) \rangle \tag{6.68}$$

To derive the equation for Γ we recall first that the complex amplitude $v(\boldsymbol{\rho}', z)$ satisfies the parabolic equation

$$2ik \frac{\partial v}{\partial z} + \Delta' v + k^2 \epsilon\, v = 0 \tag{6.69}$$

where $\Delta' = \nabla'^2_{\perp}$ is the Laplacian in the transversal variables $\boldsymbol{\rho}'$. Then the complex conjugate $v^*(\boldsymbol{\rho}'', z)$ satisfies the equation

$$-2ik \frac{\partial v^*}{\partial z} + \Delta'' v^* + k^2 \epsilon\, v^* = 0 \tag{6.70}$$

Multiplying (6.69) by $v^*(\boldsymbol{\rho}'', z)$, (6.70) by $v(\boldsymbol{\rho}', z)$ and subtracting the second equation from the first, we obtain

$$\begin{aligned} 2ik \frac{\partial}{\partial z} [v(\boldsymbol{\rho}', z)\, v^*(\boldsymbol{\rho}'', z)] + (\Delta' - \Delta'') \, [v(\boldsymbol{\rho}', z)\, v^*(\boldsymbol{\rho}'', z)] \\ + k^2\, h(\boldsymbol{\rho}', \boldsymbol{\rho}'', z) \, [v(\boldsymbol{\rho}', z)\, v^*(\boldsymbol{\rho}'', z)] = 0 \end{aligned} \tag{6.71}$$

with

$$h(\boldsymbol{\rho}', \boldsymbol{\rho}'', z) = \epsilon(\boldsymbol{\rho}', z) - \epsilon(\boldsymbol{\rho}'', z) \tag{6.72}$$

If we try to average (6.71), we face again an unclosed equation

$$2ik\,\frac{\partial\Gamma}{\partial z} + (\Delta' - \Delta'')\,\Gamma + k^2\ \langle h(\boldsymbol{\rho}', \boldsymbol{\rho}'', z)\, v(\boldsymbol{\rho}', z)\, v^*(\boldsymbol{\rho}'', z)\rangle = 0 \tag{6.73}$$

To calculate the correlation in the last term of this equation we can once again use the Furutsu-Novikov formula (6.30) and the diffusive Markov approximation with the correlation function (6.33,34). As a result the diffusive Markov approximation equation can be derived by analogy with eq. (6.40) on the form

$$\frac{\partial\Gamma}{\partial z} - \frac{i}{2k}\,(\Delta' - \Delta'')\,\Gamma + \frac{\pi k^2}{4}\ H_\epsilon(\boldsymbol{\rho}' - \boldsymbol{\rho}'')\ \Gamma(\boldsymbol{\rho}', \boldsymbol{\rho}'', z) = 0 \tag{6.74}$$

where

$$\pi\, H_\epsilon(\boldsymbol{\rho}' - \boldsymbol{\rho}'') = A_\epsilon(0) - A_\epsilon(\boldsymbol{\rho}' - \boldsymbol{\rho}'') \tag{6.75}$$

6.6.1 Solution of the equation for the coherence function

To solve eq. (6.74) let us introduce the new transverse variables

$$\boldsymbol{\rho} = \boldsymbol{\rho}' - \boldsymbol{\rho}''\ , \qquad 2\boldsymbol{\rho}_+ = \boldsymbol{\rho}' + \boldsymbol{\rho}'' \tag{6.76}$$

Then we have instead of (6.76)

$$\frac{\partial\Gamma}{\partial z} - \frac{i}{k}\,\frac{\partial^2\Gamma}{\partial\boldsymbol{\rho}\,\partial\boldsymbol{\rho}_+} + \frac{\pi k^2}{4}\ H_\epsilon(\boldsymbol{\rho})\ \Gamma(\boldsymbol{\rho}, \boldsymbol{\rho}_+, z) = 0 \tag{6.77}$$

Let us further represent the solution of (6.77) as a Fourier series in the variable $\boldsymbol{\rho}_+$:

$$\Gamma(\boldsymbol{\rho}, \boldsymbol{\rho}_+, z) = \int\limits_{-\infty}^{+\infty} \gamma(\boldsymbol{\rho}, \boldsymbol{\kappa}, z)\ \exp[i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}_+]\ \mathrm{d}\boldsymbol{\kappa} \tag{6.78}$$

Substitution of (6.78) into eq. (6.77) gives the following equation for the Fourier conjugate $\gamma(\boldsymbol{\rho}, \boldsymbol{\kappa}, z)$:

$$\frac{\partial\gamma}{\partial z} + \frac{\boldsymbol{\kappa}}{k}\,\frac{\partial\gamma}{\partial\boldsymbol{\rho}} + \frac{\pi k^2}{4}\ H_\epsilon(\boldsymbol{\rho})\ \gamma = 0 \tag{6.79}$$

Next we write the sum of the first two items in (6.79) in a symbolical operator form

$$\frac{\partial\gamma}{\partial z} + \frac{\boldsymbol{\kappa}}{k}\,\frac{\partial\gamma}{\partial\boldsymbol{\rho}} = \exp\left[-\frac{\boldsymbol{\kappa}z}{k}\,\frac{\partial}{\partial\boldsymbol{\rho}}\right]\ \frac{\partial}{\partial z}\ \exp\left[+\frac{\boldsymbol{\kappa}z}{k}\,\frac{\partial}{\partial\boldsymbol{\rho}}\right]\ \gamma \tag{6.80}$$

By means of this operator eq. (6.79) can be rewritten as

$$\frac{\partial}{\partial z}\ \exp\left[+\frac{\boldsymbol{\kappa}z}{k}\,\frac{\partial}{\partial\boldsymbol{\rho}}\right]\ \gamma(\boldsymbol{\rho}, \boldsymbol{\kappa}, z) = -\frac{\pi k^2}{4}\ \exp\left[+\frac{\boldsymbol{\kappa}z}{k}\,\frac{\partial}{\partial\boldsymbol{\rho}}\right]\ [H_\epsilon(\boldsymbol{\rho})\ \gamma(\boldsymbol{\rho}, \boldsymbol{\kappa}, z)] \tag{6.81}$$

If we take into account that for an arbitrary function $f(\boldsymbol{\rho})$ the relation

$$f(\boldsymbol{\rho}+\boldsymbol{\rho}_0) = \exp\left[\boldsymbol{\rho}_0 \, \frac{\partial}{\partial \boldsymbol{\rho}}\right] \; f(\boldsymbol{\rho}) \tag{6.82}$$

holds with $\exp[\boldsymbol{\rho}_0 \; \partial/\partial\boldsymbol{\rho}]$ being the formal notation for the full Taylor series expansion (translation operator), then we find from (6.81) the equation

$$\frac{\partial}{\partial z}\, \gamma\left(\boldsymbol{\rho}+\frac{\boldsymbol{\kappa} z}{k}, \boldsymbol{\kappa}, z\right) = -\frac{\pi k^2}{4}[H_\epsilon\left(\boldsymbol{\rho}+\frac{\boldsymbol{\kappa} z}{k}\right)\; \gamma\left(\boldsymbol{\rho}+\frac{\boldsymbol{\kappa} z}{k}, \boldsymbol{\kappa}, z\right) \tag{6.83}$$

The solution of this equation is easily found to be

$$\gamma\left(\boldsymbol{\rho}+\frac{\boldsymbol{\kappa} z}{k}, \boldsymbol{\kappa}, z\right) = \gamma\left(\boldsymbol{\rho}, \boldsymbol{\kappa}, 0\right)\; \exp\left[-\frac{\pi k^2}{4}\int\limits_0^z H_\epsilon\left(\boldsymbol{\rho}+\frac{\boldsymbol{\kappa}\zeta}{k}\right)\; \mathrm{d}\zeta\right] \tag{6.84}$$

or, if we substitute $\boldsymbol{\rho}-\boldsymbol{\kappa}z/k$ in the place of $\boldsymbol{\rho}$,

$$\gamma\left(\boldsymbol{\rho}, \boldsymbol{\kappa}, z\right) = \gamma\left(\boldsymbol{\rho}-\frac{\boldsymbol{\kappa} z}{k}, \boldsymbol{\kappa}, 0\right)\; \exp\left[-\frac{\pi k^2}{4}\int\limits_0^z H_\epsilon\left(\boldsymbol{\rho}-\frac{\boldsymbol{\kappa}(z-\zeta)}{k}\right)\; \mathrm{d}\zeta\right] \tag{6.84$'$}$$

Finally, taking the representation (6.78) into account, we obtain

$$\Gamma(\boldsymbol{\rho}, \boldsymbol{\rho}_+, z) = \int\limits_{-\infty}^{+\infty} \gamma\left(\boldsymbol{\rho}-\frac{\boldsymbol{\kappa} z}{k}, \boldsymbol{\kappa}, z\right)\; \exp\left[i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}_+ - \frac{\pi k^2}{4}\int\limits_0^z H_\epsilon\left(\boldsymbol{\rho}-\frac{\boldsymbol{\kappa}(z-\zeta)}{k}\right)\; \mathrm{d}\zeta\right]\; \mathrm{d}\boldsymbol{\kappa} \tag{6.85}$$

We note that $\gamma(\boldsymbol{\rho}, \boldsymbol{\kappa}, 0)$ is the Fourier transform of the initial distribution of the coherence function

$$\Gamma(\boldsymbol{\rho}, \boldsymbol{\rho}_+, 0) = \Gamma_0(\boldsymbol{\rho}, \boldsymbol{\rho}_+) \tag{7.86}$$

$$\gamma(\boldsymbol{\rho}, \boldsymbol{\kappa}, 0) = \frac{1}{4\pi^2}\int\limits_{-\infty}^{+\infty} \Gamma_0(\boldsymbol{\rho}, \boldsymbol{\rho}_+, 0)\; \exp[-i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}'_+]\; \mathrm{d}\boldsymbol{\rho}'_+ \tag{6.87}$$

Then putting together (6.85) and (6.87), we find the ultimate expression for the coherence function:

$$\Gamma(\boldsymbol{\rho}, \boldsymbol{\rho}_+, z) = \frac{1}{4\pi^2}\int\limits_{-\infty}^{+\infty} \Gamma_0\left(\boldsymbol{\rho}-\frac{\boldsymbol{\kappa} z}{k}, \boldsymbol{\rho}'_+\right)\; \exp\left[i\boldsymbol{\kappa}\cdot(\boldsymbol{\rho}_+-\boldsymbol{\rho}'_+) - \frac{\pi k^2}{4}\int\limits_0^z H_\epsilon\left(\boldsymbol{\rho}-\frac{\boldsymbol{\kappa}(z-\zeta)}{k}\right)\; \mathrm{d}\zeta\right] \mathrm{d}\boldsymbol{\kappa}\; \mathrm{d}\boldsymbol{\rho}'_+ \tag{6.88}$$

This is the most general form of the solution of the equation (6.77) for the initial distribution of the coherence function $\Gamma_0(\boldsymbol{\rho}, \boldsymbol{\rho}_+)$.

In the particular case when the incident field is a plane wave and

$$\Gamma_0 = \text{const.} = I_0 \tag{6.89}$$

eq. (6.81) yields after integration over $\boldsymbol{\rho}'_+$ first and then over $\boldsymbol{\kappa}$:

$$\Gamma(\boldsymbol{\rho}, \boldsymbol{\rho}_+, z) = I_0 \ \exp\left[-\frac{\pi k^2}{4} \ H_\epsilon(\boldsymbol{\rho}) \ z\right] \tag{6.90}$$

In particular, for the mean energy $W = \Gamma(0, \boldsymbol{\rho}_+, z)$ one obtains from (6.90)

$$W = I_0 \tag{6.91}$$

if we take the expression (6.75) for $H_\epsilon(\boldsymbol{\rho})$ into account.

6.7 Final remarks on Markov parabolic momenta equations

In the scientific literature one can find different points of view respectively the range of validity of the diffusive Markov approximation for the random field coherence functions of the different orders. Our derivation of the first two Markov equations (6.28) and (6.74, 75) was based on the Furustu-Novikov formula in the form (6.29). This form of the Furutsu-Novikov formula is only valid under the assumption of the normal (gaussian) distribution of fluctuations of the dielectric permittivity. This may lead to the conclusion that Markov momenta equations are only valid for normally distributed fluctuations of the dielectric permittivity. However, in a series of works, e.g., [Lee,1975], or some papers by Kljatskin the same form parabolic momenta equations were obtained without employing the Furutsu-Novikov relationship, which works in favour of a wider range of validity of the Diffusive Markov parabolic equations.

Furthermore, constructing the spaced position coherence function in the scope of both the geometrical optics approximation and Markov approximation gives the same result in the case of the plane incident wave and homogeneous background medium. This complicates distiguishing the range of validity of perturbation theories and Markov approximation in the description of weak and strong fluctuations. On the other hand, when dealing with the spaced position and frequency coherence functions, the geometrical optics approximation fails to properly describe the regime of strong fluctuations, whereas correct solution to Markov equation for the two-frequency, two-position coherence function (which is derived in the same fashion as for the single frequency case) shows substantial reduction of the frequency correlation radius as fluctuations increase.

Numerous attempts have been undertaken to construct the comprehensive analytical solution to the Markov equation for spaced position and frequency coherence function. [Liu and Yeh, 1975] solved the equation numerically. One of the first analytic solution to this equation was constructed by [Sreenivasiah et al., 1976] for the quadratic model of the structure function of fluctuations. In [1983] Knepp generalized this solution to the case of the spherical incident wave written in small angle approximation. Later on, different approaches to construct the solution to the case of more realistic models of the structure function of fluctuations of the medium of propagation have been suggested by

[Oz and Heyman, 1996, 1997a, 1997b, 1997c], by [Bronstein and Mazar, 2002]. To our understanding the most general results have been obtained here in the papers by [Bitjukov et al., 2002, 2003], who developed the quasi-classic method to solve the equation utilizing complex trajectories.As to our knowledge, all known earlier results follow from this technique.

The technique of Markov parabolic momenta equations was also extended to account for the inhomogeneous background channel of propagation. In [Mazar abd Beran, 1984] the intensity of the fluctuating field in a stratified acoustic channel of propagation was studied employing the the diffuse Markov equation for the spaced coherence function written in the rectangular co-ordinates. Appearance of the work by Hill [1985], who formulated appropriate Markov equations in courvilinear orthogonal variables of different types, gave rize to the attempts of solving these equations in ray-centred variables [Mazar and Felsen, 1987a, 1987b].

Finally, it should be emphasized once again that the comprehensive description of a random function is not confined by the definition of its second order coherence functions. The fourth moment enables description of the intensity fluctuations and, in particular, allows obtaining the spectral index, which is commonly used to quantify the level of scintillation. The Markov parabolic equations of the fourth order are less studied, and we can address those interested in more detail regarding the fourth order coherence function to the papers by Gozani [1985, 1993] and a substantional amount of references available in these papers.

Chapter 7

Conclusions

7.1 Remarks on pure numerical treatment in the problem of wave propagation in random media

We did not consider as our task discussing in detail pure numerical methods utilized when treating the problems of wave propagation in random media. However, it's of worth to say a few words on this subject. Among many of thenumerical methods the multiple phase screen technique (MPST) is the most commonly accepted and widely used by many authors to solve stochastic and deterministic parabolic equations. Knepp [1983b] studied the temporal behavior of stochastic waves by MPST. Kiang and Liu [1985] employed MPST to simulate of HF wave propagation in the turbulent stratified ionosphere. Similar problems of HF propagation in the ionospheric fluctuating reflection channel were considered in the scope of MPST by Rand and Yeh [1991], Wagen and Yeh [1986, 1989a, 1989b].

MPST has also been widely employed to solve various problems of transionospheric propagation of the fields of very high frequencies. In particular, Grimault [1998] essentially modified the classical scheme of MPST writing the appropriate parabolic equation and solving it by MPST in the spherical variables. Classical scheme of MPST was used by Beniguel [2002] in his global ionospheric propagation model of the field scintillation on transionospheric paths of propagation.

7.2 About other approaches in the theory of wave propagation in random media

The time provided for the discussion of the methods in the theory of wave propagation in random media is enormously insufficient even for a very brief review of the techniques available in the theory of wave propagation in random media. We made our choice focussing on the one hand on different perturbation theories, which obviously do not enable the description of the regime of strong scintillation, but work well in many cases when the regime of strong scintillation does not occur. On the other hand we found useful to also briefly outline formalizm of diffusive Markov momenta parabolic equations widely utilized in the problems of high and very high frequency propagation in the fluctuating ionosphere, which also enables the description of the regime of strong scintillation. We also paid special attention to the method of random (stochastic) screen and its particular case the method of phase screen. This standing alone method is remarkable in the sense that permits constructing the rigorous solution to the field generated by the screen for any given distribution of the field on the screen.

At the same time we were forced to leave beyond the scope of present consideration a wide variety of other approaches being also employed in wave propagation in random media. It's worth pointing out the technique using path integrals [Flatte, 1983, Dashen, 1979]], where Feinman integral [Feynman and Hibbs, 1965] is employed to construct the stochastic realizations of the "'solutions" of parabolic equation and its moments. Integral representations of the wave fields are also widely used in the problem of wave propagation in random media. Among them one will find Maslov's integral representation [Maslov, 1965], occillatory integral [Arnold, 1992], or interference integral [Tinin etal., 1992], integral representing of the wave field in terms of diffracting component waves [Zernov and Lundborg, 1996], or suggested by V.A.Fock integral representing the solution of the Helmholtz' equation in terms of the solutions of the parabolic equation [Tinin, 2004].

7.3 Occuring scintillation propagation models

As to our knowledge, at the time being there are two officially distributed scintillation propagation models. One of them is WBMOD, which is the model based on the theory of the phase screen with the inverse power law spatial spectrum of the phase fluctuations on the screen [Rino, 1979]. We are not aware of all the details of this model as having no direct access to its commercially distributed version. As far as we are aware of the model is valid to describe the case of weak scintillation, and two modifications of this model are available for the equatorial ionosphere [Secan et al., 1995] and for the high-latitude ionosphere [Secan et al., 1997].

Another scintillation propagation model was developed by Beniguel [Beniguel, 2002]. It is accepted by ITU as the officially recommended scintillation propagation model. This model is based on the multiple phase screen numerical technique. It works in several steps. On the first step the random distribution of the dielectric permittivity of the ionosphere is generated. Next, the random phase screen is generated on the Earth's surface by integrating the eikonal equation along the paths of propagation connecting the satellite with the points of observation on the Earth's surface. While integrating along a particular path it is accepted that the contribution into the phase advance along this path is formed by the layer of the width of 100 km centred respectively the height of the maximum of the electron density of the ionosphere. Then this phase screen is conveyed from the Earth's surface up to the level of the maximum of the electron density of the ionosphere. At the last step the formalizm of the classical problem of generating the field in vacuum by a given random phase screen is employed to obtain the random field on the surface of the Earth. Having random realizations of the field on the Earth's surface, different statistical moments of the field are constructed, e.g., correlation functions, power specrta, probability density functions, S_4, etc.

Along with the official scintillation propagation models mentioned we would like to briefly mention another model also based on the multiple phase screen technique [Grimault, 1998]. In this model the appropriate parabolic equation for the random field is written and then solved by the multiple phase screen technique in the spherical co-ordinate system, which is more appropriate for the real geometry.

Finally, we will briefly describe our own scintillation propagation model, which was developed in co-operation between the University of St.Petersburg, St.Petersburg, Russia and the University of Leeds, Leeds, United Kingdom with the participation of the Abdus Salam ICTP, Trieste, Italy. On the first stage the model was solely based on the Rytov's approximation [Gherm et al., 2000]. Statistical moments of the field radiated at the satellite and propagated through the 3-D inhomogeneous fluctuating ionosphere down to the Earth's surface were constructed by the Rytov's method technique, which together with the fact that Rytov's phase and log-amplitude fluctuations are normally distributed enabled also generating the random time series of the field. However, this model had an essential drawback expressed by the fact that the range of validity of this model was limited by the case of small values of the variance of the log-amplitude fluctuations. Alternatively, this meant that the case of strong fluctuations (scintillation) could not be described by this model. On the other hand, however, our numerous calculations in the scope of the complex phase method (Rytov's method) showed that for observation points lying inside the ionospheric layer, fluctuations of the field amplitude for frequencies of the order of 1 GHz and higher always have values which are within the range of validity of the Rytov's approximation. This is true even in the case of very large relative electron density fluctuations (up to 100 per cents) and high values of TEC. For smaller relative fluctuations and values of TEC this is also true for lower frequencies. This means that propagation in the ionospheric layer for the frequencies mentioned may always be well described in the scope of the complex phase method. In turn, this implies that at L band and higher frequencies the regime that results in strong scintillation does not normally occur inside the ionospheric layer, but may be formed in the region where the field propagates from the ionosphere down to the Earth's surface. This circumstance permits utilization of the complex phase method to properly introduce a physically substantiated random screen below the ionosphere, and then to employ the rigorous relationships of the random screen theory to correctly propagate the field down to the surface of the Earth, over which path the regime producing strong scintillation may well be found. This technique was termed as a hybrid method for scintillation on the transionospheric paths of propagation [Gherm et al., 2005b]. This technique permits constructing both the statistical characteristics of the field (correlation functions, probability density functions, power spectra of phase and amplitude fluctuations, S_4, etc.) and generate random time series of the field.

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HF propagation in a wideband ionospheric fluctuating reflection channel: Physically based software simulator of the channel

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[1] A wideband HF simulator has been constructed on the basis of a detailed physical model of propagation which can generate a time realization of the HF wideband channel for any HF carrier frequency, bandwidth, transmitter receiver path and background, and stochastic (irregularity) ionosphere models. To accomplish this, a comprehensive solution has been obtained on the basis of the complex phase method (Rytov's method) to the problem of HF wave propagation for the most general case of a three-dimensional (3-D) inhomogeneous ionosphere with time-varying electron density fluctuations. A simulation is presented for a 1000 km path for which E and low- and high-angle F mode paths exist. The time-varying field owing to each of these paths is summed at the receiving location, enabling the calculation of the scattering function and also the time realization of the received signal shown as a function of both fast and slow time.

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1. Introduction

[2] With the advent of digital HF broadcasting (e.g., DRM) and communications via the ionosphere, significantly higher data rates have become possible. However, the channel has not yet been well characterized for wideband (>8 kHz) digital signals. The ionospheric radio propagation channel is very complex and the ultimate success of new digital ionospheric radio systems will depend on a good understanding of important parameters of the channel such as Doppler shift, Doppler spread and multipath dispersion. The time variation of these parameters is also important, particularly the faster variations due to fluctuating ionospheric irregularities. Further complications arise from the geographical variations of the channel parameters with differences between equatorial, mid and high latitudes being particularly marked. To this must be added diurnal, seasonal, solar cycle and geomagnetic storm time variations. Because of the great variation of the ionosphere with different times and locations, it is also more difficult to adequately test out new HF communication systems. To cover all possible conditions, even for one fixed path, requires many trials to be performed. For a system that it is desired to deploy globally or for varying link distances and path locations, the necessary trials generally become prohibitively costly and time-consuming. Thus there is a need for a wideband simulator able to characterize the ionosphere response for any conditions, transmitter and receiver locations, transmission frequencies and bandwidths and taking into account not only the background ionosphere but all the fluctuating electron density irregularities. This should be able not only to generate realistic values of Doppler spread and delay spread for different paths, but also produce a time series output representative of the effect of the medium on the transmitted signal.

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[3] Further, since based purely on physical models and parameters, the simulator will enable the correspondence between the characteristics of the received field and the physical parameters of the model to be investigated. This permits fine-tuning of the model by comparison of received field and predicted output for a variety of conditions as well as providing a way of estimating the physical parameters from the characteristics of the received field. The theoretical basis of such a simulator, as outlined above, is described in section 2, the necessary steps and equations to construct it are explained in section 3 and the production of random time series employing it and some preliminary results are given in section 4.

2. Theoretical Basis for the Wideband HF Simulator

[4] The problem of HF propagation in the ionosphere is one of the classical issues in the theory of radio wave propagation in near-Earth space. When treating HF propagation in the real ionosphere, it should be considered that the medium of propagation is a 3-D smoothly inhomogeneous (in terms of wavelengths of the HF band) anisotropic dispersive background medium, which is additionally disturbed by local deterministic and random inhomogeneities of the ionospheric electron density over a wide range of scales. As far as propagation in the background smoothly inhomogeneous medium is concerned, this problem can be considered to be accomplished as the methods to construct the high-frequency asymptotic solutions to this sort of problem are fairly well known. These are the classical geometrical optics approximation [*Kravtsov and Orlov*, 1980] or appropriate integral representations of the wave field in terms of geometrical optics type component waves known as the interference integral [*Orlov*, 1972], or oscillatory integral [*Arnold*, 1982] (see also classical works on high-frequency asymptotic solutions in mathematical physics [*Ludwig*, 1966; *Maslov*, 1965; *Kravtsov*, 1968]).

[5] To treat the problem of the effects of local inhomogeneities of the ionosphere (including the effects of random inhomogeneities) on HF propagation, a solution to the scattering problem for the case of a 3-D inhomogeneous, dispersive and, strictly, anisotropic background medium with local inhomogeneities must be constructed. If the spatial scales of the local inhomogeneities are greater than the appropriate main Fresnel zones size, the scattering problem can still be solved in the geometrical optics approximation. However, as the ionospheric turbulence has a wide spatial spectrum characterized by an inverse power law, a reasonable fraction of the random inhomogeneities has spatial scales less than the appropriate Fresnel zone size. This means that the contribution of diffraction should be properly accounted for when treating the scattering problem. This together comprises a very complicated problem for which a comprehensive solution should be given for a variety of realistic models of the ionosphere and geometry of propagation. This explains why different empirical models have been developed [*Watterson*, 1981; *Vogler and Hoffmeyer*, 1993; *Mastrangelo et al.*, 1997; *Sudworth*, 1999; see also *Proakis*, 1983], which are widely employed [*Angling et al.*, 1998; *Messer*, 1999; *Nieto and Ely*, 1999] to characterize the HF fluctuating channel of propagation. By contrast to the empirical approach, we present here a rigorous treatment of the HF propagation in a 3-D inhomogeneous medium disturbed by fluctuations of the electron density of the ionosphere.

[6] Concerning the way to properly account for the wave polarization, when dealing with propagation in a smoothly inhomogeneous isotropic medium without fluctuations and considering power (quadratic) characteristics of the field, the wave polarization does not affect the result. However, when considering the scattering by local random inhomogeneities, then, for a completely rigorously treatment, the vector character of the scattered field should be taken into account. However, there is a physical reason to remain within the framework of the scalar approximation. It is well known that the differential scattering cross section of the same inhomogeneity is not the same for the scalar and vector field scattering, but the difference almost vanishes in the case of the scattering by large-scale inhomogeneities, in other words, in the case of forward scattering. The complex phase method we have employed just describes this case. All the inhomogeneities we consider are large scale in terms of the wavelength. Thus we consider that it is a reasonable basis to consider the problem in the scalar approximation, at least, to the zero-order approximation.

[7] The theoretical consideration of the problem of HF propagation in the disturbed ionosphere can be split into two parts. The first part is the HF propagation in the background 3-D smoothly inhomogeneous medium (section 2.1) and the second is the description of the effects of scattering of the HF field by local random inhomogeneities of the ionosphere (section 2.2). We will also present the description of the software simulator of the fluctuating channel of propagation, developed on the basis of rigorous treatment of the appropriate equations governing the propagation.

2.1. Propagation in the Background Ionosphere

[8] As mentioned above, the description of the HF propagation in the 3-D smoothly inhomogeneous medium is the simplest part of the problem. Characteristic scales of the background ionosphere in all the directions are sufficiently large to allow the geometrical optics approximation to be employed to describe the HF field.

Appropriate codes for calculation of ray paths and ray pencil divergences are available to enable the construction of simulated oblique sounding ionograms. These can then be employed, using the appropriate model of the background ionosphere, to determine possible paths (modes) connecting transmitter and receiver locations. Quantities such as divergences along the paths of propagation are also used when determining the scattering of the field by local random inhomogeneities along each actual mode of propagation between a transmitter and receiver.

2.2. Scattering of HF Field by Random Ionospheric Inhomogeneities

[9] This is the most complicated part of the propagation problem. The simultaneous presence of several scales of ionospheric density variations is very demanding when treating the scattering problem. Generally speaking, the solution should be obtained for the scattering problem for the case of a 3-D inhomogeneous dispersive background medium, accounting also for the contribution of diffraction effects in the scattering by local random inhomogeneities.

[10] The case of weak or moderate fluctuations of the amplitude of the field can be treated in the framework of perturbation theories. Among them the complex phase method (or the generalized Rytov's approximation) handles the scattering problem in the most comprehensive form as it can also account for diffraction by local random inhomogeneities and partly accounts for multiple scattering effects. Additionally, it enables construction of the appropriate two-position, two-frequency correlation and coherence functions of the random field for the condition of a strongly inhomogeneous and dispersive medium; a condition which is fully pertinent to the ionosphere. These functions are the core quantities when modeling the fluctuating channel of propagation both in terms of the statistical moments of the field propagated through the channel and random time sequences of the field. The method limitation is determined by the range of validity of the complex phase method, which can be roughly stated as that the variance of the fluctuations of the log-amplitude (level) of the field cannot be large. This is a well known limitation of the Rytov's approximation (or the complex phase approximation, which is its extension). In addition, for our application, puts certain limitations on the variance of the electron density fluctuations. The codes are arranged in a way that this is controlled for any given path and conditions of propagation. In turn, this means that there is no one particular universal limit for the fractional electron density fluctuations. However, for a typical one-hop path of propagation for a link distance of the order of 1000 km, it results in a limit of the order of 1% for the r.m.s. of the fractional electron density fluctuations. The same criteria needs to be applied for high and low latitudes as for midlatitude paths, but in the former cases the possible occurrence of strong scintillations can lead to a break down of the theory's validity.

[11] For the case of strong scintillation, such methods as Markov's parabolic equations for the statistical moments of the random field [*Ishimaru*, 1978; *Rytov et al.*, 1978] and the path integral technique [*Dashen*, 1979; *Flatte*, 1983] should be mentioned, which permit description of the effects of strong scintillation in some cases. Many problems of wave propagation in random media have been considered in the scope of these methods and we cannot here provide a complete bibliography. However, it is our current conviction that neither Markov's approximation, nor the path integral technique is yet capable of handling the problem of constructing spaced position and frequency coherency in the ionosphere-type medium, i.e., for the essentially inhomogeneous and dispersive background medium with local random inhomogeneities embedded. This led us to consider it best to confine the present treatment of the problem of scattering of the HF waves by local random ionospheric inhomogeneities within the framework of the complex phase method, at the same time taking account of the constraints of this method and its range of its validity as discussed above. Toward the end of the paper (in section 4.2) a numerical example is given for an ionosphere including the effect of the geomagnetic field. There is additional complexity for this case introduced by the anisotropy of the medium of propagation. In this paper we just present the theory for the isotropic case as we consider that the complexity of the anisotropic case requires special consideration. We intend to give a full description of this in a subsequent paper.

3. Complex Phase Method: General Case of 3-D Inhomogeneous Background Medium

[12] The complex phase method is the extension of the classic Rytov's approximation [*Rytov et al.*, 1978], dated back to 40 s, to the case of the point source field and the inhomogeneous background medium. The first extension of the method was performed by *Zernov* [1980], who considered the HF field in a stratified ionosphere, disturbed by local inhomogeneities. The extended Rytov approximation was further employed in a series of papers [*Gherm and Zernov*, 1995, 1998; *Gherm et al.*, 1997, 2001a] to construct and study the statistical moments of the random HF field in the plane-stratified ionosphere disturbed by fluctuations of the electron density.

[13] Obviously, the following extension of the method must include the general case of a 3-D inhomogeneous medium. In particular, this is necessary when characterizing the HF fluctuating ionospheric channel of propa-

gation, which is horizontally inhomogeneous (i.e., containing horizontal gradients of electron density). The appropriate generalization has been recently performed by *Gherm et al.* [2001b] in a paper written and issued in Russian. Here we will briefly reproduce the milestones of this extension.

[14] In the present consideration the scalar equation

$$\nabla^2 E + k^2[\varepsilon_0(\mathrm{r}) + \varepsilon(\mathrm{r})]E = A\delta\left(\mathrm{r} - \mathrm{r}_0'\right), \tag{1}$$

widely used to describe HF propagation, is employed, where k is the wave number in vacuum, $\varepsilon_0(\mathrm{r})$ is the dielectric permittivity of the background medium and $\varepsilon(\mathrm{r})$ is the dielectric permittivity of local inhomogeneities. r is the point of observation, r' the variable of integration and is the position of the source of the field (the transmitter). Quantity A characterizes, in some sense, the power of a source. In order to account for the time dependence of the electron density fluctuations function $\varepsilon(\mathrm{r})$ is also allowed to be a function of the slow time in the quasi-stationary approximation.

[15] Depending on the given model of the background medium $\varepsilon_0(\mathrm{r})$, the undisturbed (incident) field $E_0(\mathrm{r})$, which satisfies equation (1) with $\varepsilon(\mathrm{r}) = 0$, may have a multipath structure, i.e., several paths of propagation may occur, which connect the transmitter and receiver. The field propagating along each of m paths can be well described in the geometrical optics approximation, so that the full undisturbed field is represented by the sum of the geometrical optics type fields as follows:

$$E_0(\mathrm{r}) = \sum_m E_{0m}^{GO}(\mathrm{r}). \tag{2}$$

[16] Acceptance of the representation given by (2) for the undisturbed field implies limitation to the case when the observation points are far from any caustic (far from the skip distance, if the transmitter and receiver are located on the Earth's surface). This implies that the main Fresnel volumes for different paths of propagation do not overlap. In the same fashion the Green's function for the undisturbed equation (1) is also represented in a form similar to equation (2) by the sum of geometrical optics contributions

$$G(\mathrm{r},\mathrm{r}') = \sum_m G_m^{GO}(\mathrm{r},\mathrm{r}'), \tag{3}$$

providing r, r' are not near any caustic.

[17] To account for the effects of local random inhomogeneities of the ionosphere on every geometrical optics component E_{0m}^{GO} of the undisturbed field, its own complex phase ψ_m is introduced for each component, so that the full field disturbed by local ionospheric inhomogeneities is given as follows:

$$E(\mathrm{r}) = \sum_m E_{0m}^{GO}(\mathrm{r}) \exp[\psi_m(\mathrm{r})]. \tag{4}$$

According to the complex phase method each ψ_m is represented by the perturbation series in powers of the disturbances $\varepsilon(\mathrm{r})$, and the technique of the method permits solutions to the appropriate equations for different orders of ψ_m in the following invariant form [*Zernov*, 1980]:

$$\psi_{m1}(\mathrm{r}) = -k^2\left(E_{0m}^{GO}(\mathrm{r})\right)^{-1}\int \varepsilon(\mathrm{r}')E_{0m}^{GO}(\mathrm{r}')G_m^{GO}(\mathrm{r},\mathrm{r}')d\mathrm{r}', \tag{5}$$

$$\psi_{m2}(\mathrm{r}) = -\left(E_{0m}^{GO}(\mathrm{r})\right)^{-1}\int \left(\nabla\psi_{m1}(\mathrm{r}')\right)^2 E_{0m}^{GO}(\mathrm{r}') \cdot G_m^{GO}(\mathrm{r},\mathrm{r}')d\mathrm{r}'. \tag{6}$$

We have presented here only the disturbed complex phases of the first and second orders, which are employed in the following treatment.

3.1. Geometrical Optics Field

[18] It is convenient to specify the representations (5) and (6) in ray-centered variables (s, q_1, q_2), where the reference ray is a given m th curvilinear path connecting the communicating points in the 3-D inhomogeneous background medium, so that every path gives rise to its own ray-centered coordinate system. (From now on we omit subscript m referring to the m th path of propagation.) In these coordinates variable s is measured along the reference ray in the direction from the source to the receiver, and q_1 and q_2 lie in the plane perpendicular to the reference ray at each point. For this coordinate system, Lamé coefficients are as follows:

$$h_s(q_1,q_2,s) = 1 - q_1\frac{\partial}{\partial q_1}\ln n(s,0,0) - q_2\frac{\partial}{\partial q_2}\ln n(s,0,0); h_{q_1} = h_{q_2} = 1. \tag{7}$$

Here $n^2(s, q_1, q_2) = \varepsilon_0(s, q_1, q_2)$ and $n(s, 0, 0) = [\varepsilon_0(s, 0, 0)]^{1/2}$. In the following we denote $n(s, 0, 0) = n_0(s)$. In the introduced ray-centered variables we take the coordinates (0, 0, 0) for the transmitter and (s_0, 0, 0) for the receiver. The variable of integration r' in the integrals in (5) and (6) is now given by (s, q_1, q_2).

[19] To construct E_{0m}^{GO} and G_m^{GO} in equations (5) and (6) in the form of the geometrical optics approximation for

type such as A_0 exp $(ik\varphi)$, the appropriate eikonal equation

$$\left(\frac{\partial\varphi}{\partial q_1}\right)^2+\left(\frac{\partial\varphi}{\partial q_2}\right)^2+\left(\frac{1}{h_s}\frac{\partial\varphi}{\partial s}\right)^2=n^2(s,q_1,q_2) \quad (8)$$

for the phase function φ and the main transport equation for the amplitude A_0,

$$2\nabla A_0\nabla\varphi+A_0\nabla^2\varphi=0, \quad (9)$$

must be solved for each path of propagation. The solutions of equations (8) and (9) locally nearby the reference ray are sought for in the form of a series in the transverse variables q_1 and q_2 as follows:

$$\begin{aligned}\varphi(s,q_1,q_2)=&\int n_0(s)ds\\&+\frac{1}{2}\left[b_{11}(s)q_1^2+b_{22}(s)q_2^2+2b_{12}(s)q_1q_2\right]\\&+\ldots, \quad (10)\end{aligned}$$

$$A_0(s,q_1,q_2)=A_{00}(s)+\ldots. \quad (11)$$

The representation (10) means that the finite curvature of the front of the undisturbed (incident) field is accounted for to the accuracy of the main terms, which are given by the full quadratic form in the square brackets. The linear terms vanish here because the medium is isotropic so that the wave front must be orthogonal to the wave direction.

[20] Performing necessary expansions for n^2 and h_s in a series in the transverse plane to the reference ray variables and equating to zero coefficients at different powers of q_1, q_2 yields for the amplitude $A_{00}(s)$

$$A_{00}(s)=const\cdot n_0^{-1/2}(s)\cdot\exp\left[-\frac{1}{2}\int\frac{b_{11}(s)+b_{22}(s)}{n_0(s)}ds\right], \quad (12)$$

where the functions $b_{11}(s)$, $b_{22}(s)$, $b_{12}(s)$ satisfy a set of differential equations of Riccati type, which may be conveniently written in the matrix form as follows:

$$n_0\frac{\partial\hat{B}}{\partial s}+\hat{B}\cdot\hat{B}=\hat{C}. \quad (13)$$

[21] In the last equation,

$$\hat{B}=\{b_{ik}\},i,k=1,2,b_{12}=b_{21}, \quad (14)$$

$$\begin{aligned}\hat{C}=\{c_{ik}\},c_{ik}=&\frac{1}{2}\frac{\partial^2 n^2(s,0,0)}{\partial q_i\partial q_k}\\&-3n_0^2(s)\frac{\partial\ln n(s,0,0)}{\partial q_i}\frac{\partial\ln n(s,0,0)}{\partial q_k},i,k=1,2. \quad (15)\end{aligned}$$

When considering the equations in (13), the solution should reduce to the spherical wave near a source in the small-angle approximation (assuming that the source is in vacuum) as follows:

$$E_{00}(\vec{r})=-\frac{A}{4\pi s}\exp\left[iks+\frac{ik}{2s}\left(q_1^2+q_2^2\right)\right]. \quad (16)$$

This is also the recipe as to how to properly choose the constant and the limits of integration in (12). Then (10)–(12) finally yield the following expression for the undisturbed (incident) field

$$\begin{aligned}E_0(\mathrm{r})=&-\frac{A}{4\pi r_0}n_0^{-1/2}(s)\cdot\exp\left[-\frac{1}{2}\int_{r_0}^{s}\frac{b_{11}+b_{22}}{n_0}ds\right]\\&\cdot\exp\left[ik\int_0^s n_0ds+\frac{ik}{2}\left(b_{11}q_1^2+b_{22}q_2^2+2b_{12}q_1q_2\right)\right], \quad (17)\end{aligned}$$

where $b_{11}(s)$, $b_{12}(s)$, $b_{22}(s)$ are properly chosen solutions of equations (13)–(15). Formally this should be considered in the limit when the small quantity r_0 tends to zero. This equation reduces to the spherical wave (16), when $n_0(s)$ = 1. Small finite values of r_0 are employed when numerical solutions of equations are realized to properly specify equation (13)–(15).

[22] In the same manner, the representation for the Green's function $G(\mathrm{r}, \mathrm{r}')$ with r = $(s_0, 0, 0)$ and r$'$ = (s, q_1, q_2) may be written

$$\begin{aligned}G(\mathrm{r},\mathrm{r}')=&-\frac{1}{4\pi r_0}n_0^{-1/2}(s)\cdot\exp\left[-\frac{1}{2}\int_{r_0}^{s_0-s}\frac{b_{11}^g+b_{22}^g}{n_0}ds_1\right]\\&\cdot\exp\left[ik\int_{r_0}^{s_0-s}n_0ds_1+\frac{ik}{2}\left(b_{11}q_1^2+b_{22}q_2^2+2b_{12}q_1q_2\right)\right]. \quad (18)\end{aligned}$$

Here the variable s_1 is measured along the same reference ray, but in the direction from the receiver to the transmitter. Making use of this variable, the elements of the matrix $\hat{B}^g=\{b_{ik}^g\}$, b_{ik}^g = 1, 2, i, k = 1, 2 satisfy the same set of equations (13)–(15) as the matrix $\hat{B}$. If the substitution $s=s_0-s_1$ is performed under the sign of integration, equation (18) becomes

$$\begin{aligned}G(\mathrm{r},\mathrm{r}')=&-\frac{1}{4\pi r_0}n_0^{-1/2}(s)\cdot\exp\left[\frac{1}{2}\int_{s_0-r_0}^{s}\frac{b_{11}^g+b_{22}^g}{n_0}ds\right]\\&\cdot\exp\left[-ik\int_{s_0}^{s}n_0ds_1+\frac{ik}{2}\left(b_{11}q_1^2+b_{22}q_2^2+2b_{12}q_1q_2\right)\right]. \quad (19)\end{aligned}$$

When written by means of variable s, matrix $\hat{B}^g$ satisfies the set of equations

$$-n_0\frac{\partial \hat{B}^g}{\partial s}+\hat{B}^g\cdot\hat{B}^g=\hat{C}, \tag{20}$$

which differs from the set of equations (13)–(15) only by the sign at the first derivative.

3.2. First-Order Complex Phase

[23] Finally, putting together all the necessary representations gives the following equation for the first-order complex phase from equation (6):

$$\begin{aligned}\psi_1(s_0,0,0,)=&\frac{k^2}{4\pi r_0}\iiint dsdq_1dq_2\frac{\varepsilon(s,q_1,q_2)}{n_0(s)}\\&\cdot h_s(s,q_1,q_2)\\&\cdot\exp\left\{\frac{1}{2}\int\limits_s^{s_0-r_0}\frac{\left(b_{11}-b_{11}^g\right)+\left(b_{22}-b_{22}^g\right)}{n_0}ds\right.\\&+\frac{ik}{2}\left[\left(b_{11}+b_{11}^g\right)q_1^2+\left(b_{22}+b_{22}^g\right)q_2^2\right.\\&\left.\left.+2\left(b_{12}+b_{12}^g\right)q_1q_2\right]\right\}.\end{aligned}\tag{21}$$

To derive the last equation the relationship $E_0(s_0-r_0, 0, 0)\approx E_0(s_0, 0, 0)$ has been used.

[24] To further transform equation (21) some necessary relationships for the new matrix

$$\hat{B}^-=\hat{B}-\hat{B}^g \tag{22}$$

should be derived. Subtracting equation (20) from (13) and performing simple transformations yields

$$\left(\hat{B}^+\right)^{-1}\frac{\partial\hat{B}^+}{\partial s}=-\frac{\hat{B}^-}{n_0(s)}, \tag{23}$$

where

$$\hat{B}^+=\hat{B}+\hat{B}^g. \tag{24}$$

Then, the integral expression, which is the first term in the exponential in equation (21) is just:

$$\begin{aligned}\mathrm{Trace}\left[\frac{\hat{B}^-(s)}{n_0(s)}\right]&=-\mathrm{Trace}\left[\left(\hat{B}^+(s)\right)^{-1}\frac{\partial\hat{B}^+(s)}{\partial s}\right]\\&=-\frac{\partial}{\partial s}\ln\left[\det\hat{B}^+(s)\right].\end{aligned}\tag{25}$$

This allows us to finally write the quantity from equation (21) in the following form:

$$\begin{aligned}\psi_1(s_0,0,0,)=&-\frac{k^2}{4\pi}\iiint dsdq_1dq_2\frac{\varepsilon(s,q_1,q_2)}{n_0(s)}\\&\cdot h_s(s,q_1,q_2)\left[\det\hat{B}^+(s)\right]^{1/2}\\&\cdot\exp\left\{\frac{ik}{2}\left[\left(b_{11}+b_{11}^g\right)q_1^2+\left(b_{22}+b_{22}^g\right)\right.\right.\\&\left.\left.\cdot q_2^2+2\left(b_{12}+b_{12}^g\right)q_1q_2\right]\right\}.\end{aligned}\tag{26}$$

To obtain this expression, the relationship

$$\lim_{r_0\to 0} r_0\left[\det\hat{B}^+(s_0-r_0)\right]^{1/2}=1 \tag{27}$$

was used. Matrix $\hat{B}^+$, involved in calculations according to (26), is given by equation (24), where, in turn, elements of matrixes $\hat{B}$ and $\hat{B}^g$ satisfy sets of differential equations (13)–(15) and (20) respectively. When and where it is necessary to have the representation for the second-order complex phase ψ_2 the quantity $k^2\,\varepsilon$ in equation (26) should be replaced by $(\nabla\psi_1)^2$.

[25] Formula (26) is the final result, which extends the classic Rytov's method to the case of the point source in an arbitrary 3-D inhomogeneous medium. It permits different limiting cases. In particular, when the background medium is homogeneous it yields the known result for the complex phase of a spherical wave in a homogeneous background medium, disturbed by a local inhomogeneity ε(r) [*Tatarskii*, 1971; *Ishimaru*, 1978]. In this case $b_{11}=b_{22}=s^{-1}=x^{-1}$, $b_{12}=0$; $b_{11}^g=b_{22}^g=(s_0-s)^{-1}=(x_0-x)^{-1}$, $b_{12}^g=0$, and equation (26) yields:

$$\begin{aligned}\psi_1(x_0,0,0)=&\frac{k^2}{4\pi}\iiint dxdydz\,\varepsilon(x,y,z)\frac{x_0}{x(x_0-x)}\\&\cdot\exp\left\{\frac{ik_0[y^2+z^2]x_0}{2x(x_0-x)}\right\}.\end{aligned}\tag{28}$$

Another limiting case for the general representation (26) is when the quantities $(b_{11}+b_{11}^g)$, $(b_{22}+b_{22}^g)$, $(b_{12}+b_{12}^g)$ are large compared to the transversal characteristic scales of the inhomogeneities ε along all the path of integration in variable s. Then integration in q_1 and q_2 in equation (26) may be performed explicitly employing the steepest descent method to produce, to the first approximation, the first-order correction of the geometrical optics approximation as follows:

$$\psi_1(s_0,0,0)=\frac{ik}{2}\int\limits_0^{s_0}\frac{\varepsilon(s,0,0)}{n_0(s)}ds. \tag{29}$$

This is the case of local inhomogeneities with large spatial scales compared to the main Fresnel zone size along the path of propagation.

[26] To utilize the generalized complex phase given by (26), a special numerical code has been produced to solve the matrix equations (13) and (20), This was combined with a general ray-tracing code for the 3-D inhomogeneous background medium.

4. Simulation of Random Time Series and Statistical Moments of the HF Field

[27] When characterizing the ionospheric fluctuating HF reflection channel of propagation, both the random time series and statistical moments of the pulsed signal propagated through the channel are of interest.

4.1. Random Time Series

[28] A random realization of a pulsed signal propagated through the fluctuating ionosphere can be represented as the following Fourier integral in the frequency domain

$$U(\mathrm{r}, t, T) = \sum_m \int_{-\infty}^{+\infty} P(\omega) E_{om}^{GO}(\mathrm{r}, \omega) R_m(\mathrm{r}, \omega, T) e^{-i\omega t} d\omega. \tag{30}$$

Here $P(\omega)$ is the spectrum of a launched pulse, E_{0m}^{GO} represents the transfer function for a given m th path in the undisturbed channel, i.e., the functions from equation (2). Once the model of the 3-D background ionosphere is given, the quantities E_{0m}^{GO} are calculated employing the appropriate ray-tracing code, which also permits calculation of the ray tube divergence. A random phasor R_m (r, ω, T) is introduced in (30) to account for the effects of fluctuations of the electron density of the ionosphere. Variable t is the flight time of a pulse and T denotes slow time dependence of fluctuations, which can be treated in the quasi-stationary approximation. The background channel is assumed to be stationary that is time-independent. A corollary of this is the absence of slow time dependence in the transfer functions of the background channel in (30). The summation in (30) is performed over all paths of propagation from a transmitter to a receiver.

[29] According to the complex phase method, described above, random phasors R_m (r, ω, T) are represented utilizing complex phases as follows:

$$R_m(\mathrm{r}, \omega, T) = e^{\psi_m(\mathrm{r}, \omega, T)}, \tag{31}$$

where the first and second-order complex phases ψ_m in powers of the disturbances of the dielectric permittivity are given by equations (5) and (6) and specified in ray-centered variables by equation (26). Complex phases

$$\psi_m(\mathrm{r}, \omega, T) = \chi_m(\mathrm{r}, \omega, T) + iS_m(\mathrm{r}, \omega, T) \tag{32}$$

are random functions with the real part χ_m representing log-amplitude fluctuations and S_m giving the fluctuations of the phase of the field.

[30] To produce the time series of a pulsed signal given by the Fourier integral (30) for a point of observation r, two real random functions χ_m and S_m must be generated in the two-dimensional domain (ω, T) for a given value of r. This demands knowledge of the probability density functions for χ_m and S_m, as well as their autocorrelation and cross-correlation functions. In the scope of the complex phase method, these functions are given as follows:

$$B_\chi(\omega_1, \omega_2; T_1, T_2) = \langle \chi_1(\omega_1, T_1) \chi_1(\omega_2, T_2) \rangle, \tag{33}$$

$$B_S(\omega_1, \omega_2; T_1, T_2) = \langle S_1(\omega_1, T_1) S_1(\omega_2, T_2) \rangle, \tag{34}$$

$$B_{\chi S}(\omega_1, \omega_2; T_1, T_2) = \langle \chi_1(\omega_1, T_1) S_1(\omega_2, T_2) \rangle. \tag{35}$$

All these functions can be found making use of the two main autocorrelation functions of the first-order complex phase $B_{\psi 1} = \langle \psi_1(\omega_1, T_1) \psi_1{}^*(\omega_2, T_2) \rangle$ and $B_{\psi 2} = \langle \psi_1(\omega_1, T_1) \psi_1(\omega_2, T_2) \rangle$. Their explicit expressions will be presented below.

[31] As far as the probability density functions for the random functions χ_{m1} and S_{m1} are concerned, equation (26) shows that χ_{m1} and S_{m1} are represented by linear integrals over many random inhomogeneities. This guarantees, according to the central limit theorem, that both the random functions χ_{m1} and S_{m1} are normally distributed. When averaging in (33)–(35) it is also implied that the electron density fluctuations along different paths of propagation are not correlated. This is in a reasonable agreement with the requirement that the main Fresnel volumes of the neighboring rays do not overlap.

[32] If, additionally, the hypothesis of the "frozen drift" of random inhomogeneities in the ionosphere is adopted, slow time T is expressed through the position of the inhomogeneity structures, so that actually appropriate two-frequency, two-position autocorrelation and cross-correlation functions must be constructed. We have studied in detail these type of functions in the scope of the complex phase method for the case of a plane-layered background medium [*Gherm et al.*, 1997; *Gherm and Zernov*, 1998]. Having the representation (26), which extends the complex phase method to the case of a fully 3-D inhomogeneous medium, appropriate statistical moments of the complex phase can be constructed for an arbitrary 3-D inhomogeneous background

medium. In particular, the abovementioned correlation functions $B_{\psi 1}$ and $B_{\psi 2}$, which permit expressing the correlations (33)–(35), are of the following form

$$
\begin{aligned}
B_{\psi 1}(\omega_1,\omega_2,T_-) = \frac{\pi k_1 k_2}{2}\int_0^{s_0}\frac{ds}{\varepsilon_0(s)} \\
\cdot \int d\kappa_n d\kappa_\tau B_\varepsilon(s;0,\kappa_n,\kappa_\tau) \\
\cdot \exp\left[i\kappa_n(\Delta_n - v_n T_-) + i\kappa_\tau(\Delta_\tau - v_\tau T_-)\right] \\
\cdot \exp\left\{\frac{i(k_1-k_2)}{2k_1k_2}\left[\kappa_n^2 D_n(s) + \kappa_\tau^2 D_\tau(s)\right.\right. \\
\left.\left. + 2\kappa_n\kappa_\tau D_{n\tau}(s)\right]\right\}, \qquad (36)
\end{aligned}
$$

$$
\begin{aligned}
B_{\psi 2}(\omega_1,\omega_2,T_-) = -\frac{\pi k_1 k_2}{2}\int_0^{s_0}\frac{ds}{\varepsilon_0(s)} \\
\cdot \int d\kappa_n d\kappa_\tau B_\varepsilon(s;0,\kappa_n,\kappa_\tau)\exp \\
\cdot \left[i\kappa_n(\Delta_n - v_n T_-) + i\kappa_\tau(\Delta_\tau - v_\tau T_-)\right] \\
\cdot \exp\left\{-\frac{i(k_1+k_2)}{2k_1k_2}\left[\kappa_n^2 D_n(s)\right.\right. \\
\left.\left. + \kappa_\tau^2 D_\tau(s) + 2\kappa_n\kappa_\tau D_{n\tau}(s)\right]\right\}. \qquad (37)
\end{aligned}
$$

Here $k = \omega/c$ and $B_\varepsilon(s;\,0,\,\kappa_n,\,\kappa_\tau)$ is the three-dimensional spatial spectrum of the electron density fluctuations with zero value of the spectral variable, Fourier-conjugated to the difference variable along the path. It is also a function of the central variable along the reference ray. The spectral variables κ_n and κ_τ are Fourier-conjugated to the spatial variables q_1 and q_2 lying in the plane perpendicular to the reference ray at each point. The quantities Δ_n and Δ_τ are the components of the vector of distance between the rays corresponding to the frequencies ω_1 and ω_2, which also depend on s. Additionally, the hypothesis of the "frozen drift" of random inhomogeneities is utilized, so that v_n and v_τ are the components of the frozen drift velocity also depending on the point along the reference ray, and $T_- = T_1 - T_2$ is the difference in slow time. The central slow time T_+ is not involved in equations (36) and (37), because of the assumption of the statistical homogeneity of the fluctuations. The coefficients D_n, D_τ, and $D_{n\tau}$ are the elements of the matrix $\hat{D} = (\hat{B}^+)^{-1}$, which is the inverse of the matrix $\hat{B}^+$(24). These also depend on the variable s.

[33] In the numerical calculations, a turbulence model of the ionospheric fluctuations is considered having an anisotropic inverse power law spatial spectrum of the form

$$
B_\varepsilon(s,\boldsymbol{\kappa}) = C_N^2[1-\varepsilon_0(s)]^2\sigma_N^2(s)\left(1+\frac{\kappa_{tg}^2}{K_{tg}^2}+\frac{\boldsymbol{\kappa}_{tr}^2}{K_{tr}^2}\right)^{-\frac{p}{2}}. \qquad (38)
$$

Here C_N^2 is a known normalization coefficient. $K_{tg} = 2\pi l_{tg}^{-1}$, where l_{tg} is the outer scale of the turbulence along the geomagnetic field, and $K_{tr} = 2\pi l_{tr}^{-1}$, where l_{tr} is the outer scale of the turbulence across the magnetic field. Function $\varepsilon_0(s)$ is the distribution of the dielectric permittivity of the background ionosphere along the reference ray in the 3-D inhomogeneous background ionosphere and $\sigma_N^2\,(s)$ is the distribution of the variance of the relative fluctuations of the electron density of the ionosphere along the reference ray in the 3-D inhomogeneous ionosphere. As a result, functions (36) and (37) are, with a very high degree of generality, valid for arbitrary three-dimensional models of the background ionosphere and fluctuations of the ionospheric electron density.

[34] All the abovementioned results permit to uniquely produce random series of functions χ_m and S_m in the domain (ω, T), if, additionally, the cross correlation (35) is also properly accounted for. To generate the time series, spectra of the correlation functions of χ_m and S_m (power spectra) are calculated in the domain (τ, Ω), where τ is Fourier-conjugated to ω and Ω is Fourier-conjugated to T correspondingly. Complex valued Fourier spectra of random realizations of χ_m and S_m are assumed to have their absolute values equal to the square roots of the appropriate calculated power spectra and arguments uniformly distributed in the interval $0-2\pi$. A correct cross correlation of the χ_m and S_m realizations is then provided by the proper choice of two basic sequences of random numbers having their cross-correlation coefficient defined by the mutual correlation of χ_m and S_m [see, e.g., *Devroye*, 1986]. In turn, these permit generation of random values of the phasor R_m (r, ω, T) in the same domain, and finally to generate the random series of a signal that has propagated through the fluctuating ionosphere employing the appropriate methods of numerical calculation of the integrals in equation (30).

[35] Below we shall present some results of a simulation obtained using the developed technique and simulator. All the results have been calculated for a single-hop path of length 1000 km oriented to the west from St. Petersburg, Russia. The IRI model for July at 0700 LT was chosen for the transmitter site at St. Petersburg and for the receiver site 1000 km to the west of St. Petersburg. For this path, horizontal gradients of the electron density resulted in a difference of 0.5 MHz in foF2

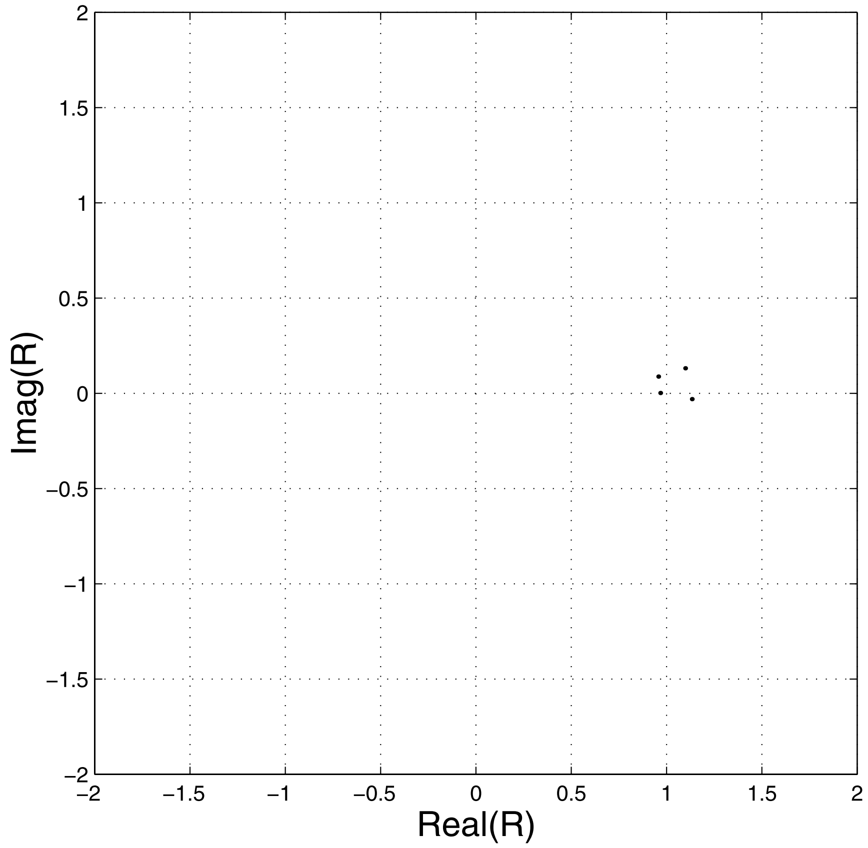

Figure 1. Random walk of the phasor R_m (r, ω, T) for the E mode. The fluctuations of the field are weak.

between the transmitter and receiver. The carrier frequency was 8.1 MHz.

[36] The fluctuations of the ionospheric electron density were characterized by the inverse power law anisotropic spatial spectrum with the spectral index of 3.7, the scale of random inhomogeneities across the geomagnetic field of 3 km and the aspect ratio of 5. The variance of relative fluctuations of the electron density was assumed to be uniform along the path of propagation and equal to 3×10^{-6}. The hypothesis of frozen drift of the random inhomogeneities was utilized with the same horizontal longitudinal and latitudinal velocity of 0.5 km/s. The bandwidth of the rectangular transmitted pulse was 20 kHz.

[37] In the first step, the oblique sounding ionogram was constructed for the chosen model of the background ionosphere, which indicated possible high- and low-angle F and E mode paths of propagation. In Figure 1, the random walk as a function of T is shown for the phasor corresponding to the E mode propagation path for a fixed frequency component ω, whereas Figure 2 demonstrates the same for the phasor of the high-angle F mode path. Clearly, the spread of possible random values of the phasor on the plot of Figure 2 is significantly wider due to the higher density of the background ionosphere at the altitude of the F layer, leading to the higher values of the absolute fluctuations of the electron density.

[38] In a similar way, phasors for all possible paths of propagation, connecting transmitter and receiver for the given conditions (the model of the background ionosphere and geometry), are produced. Then, calculating numerically the quantity U(r, t, T) according to the integral (30) random time sequences of a pulsed signal propagated through the fluctuating ionosphere are generated for different moments of slow time provided that the spectrum P(ω) of the transmitted signal is specified. In Figure 3 the results of generating the random sequences for a transmitted rectangular pulse are presented, as a function of the flight (fast) time, for different moments of slow time.

4.2. Scattering Functions

[39] The scattering function of a pulsed signal is introduced [*Proakis*, 1983; *Vogler and Hoffmeyer*, 1993; *Mastrangelo et al.*, 1997; *Gherm et al.*, 2001a] as the appropriate Fourier transform of the autocorrelation function of the random channel impulse response on

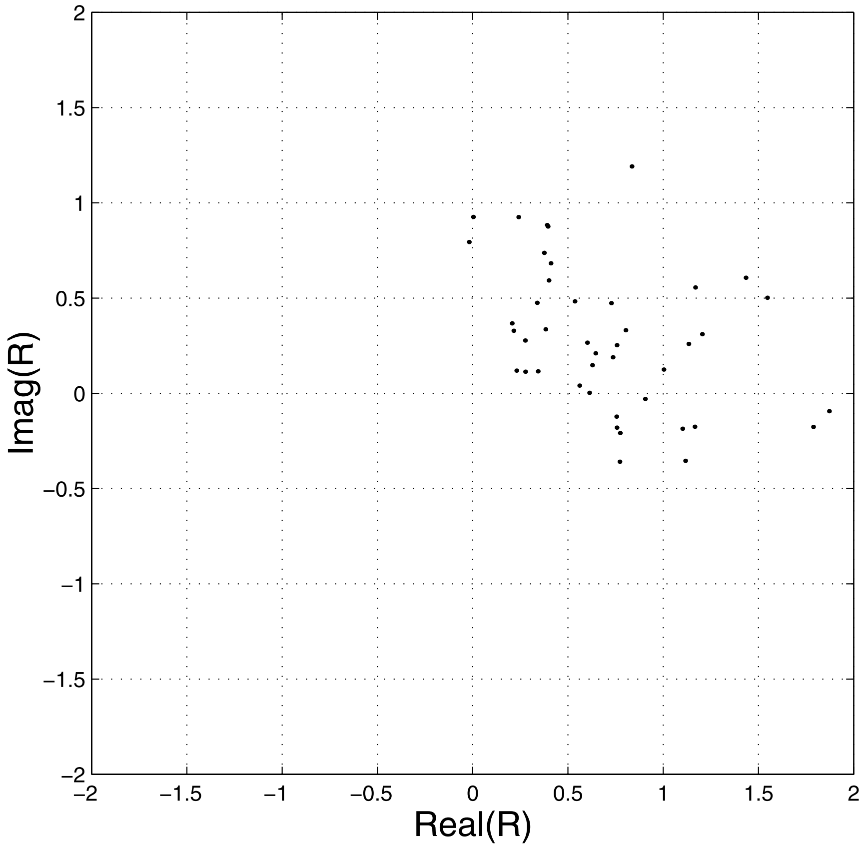

Figure 2. Random walk of the phasor R_m (r, ω, T) for the high-angle F mode. This shows stronger fluctuations of the field.

the difference slow time variable. Utilizing (30), the autocorrelation function of a pulsed signal on slow time T can be written as follows:

$$\begin{aligned}\Psi_U(t,T_1,T_2) \;=\; &\int P(\omega_1)P^*(\omega_2)\sum_m f_m(\omega_1)f_m^*(\omega_2)\\ &\cdot\Psi_{Rm}(\omega_1,\omega_2;T_1,T_2)\exp[ik_1\varphi_m(\omega_1)\\ &- ik_2\varphi_m(\omega_2) - i(\omega_1-\omega_2)t]d\omega_1 d\omega_2.\end{aligned} \tag{39}$$

Here the spatial variable r was suppressed and the following relationships have been introduced:

$$E_{0m}^{GO}(\omega) = f_m(\omega)\exp[ik\varphi_m(\omega)], \tag{40}$$

$$\Psi_{Rm}(\omega_1,\omega_2;T_1,T_2) = \langle R_m(\omega_1,T_1)R_m^*(\omega_2,T_2)\rangle. \tag{41}$$

Equation (40) shows explicitly the amplitude and phase of the field E_{0m}^{GO}, which are calculated for each possible mode of propagation, defined by the model of the background medium. Relationship (41) is the definition of the two-frequency two-time correlation function of the random phasor R_m (ω, T).

[40] It is convenient to work with the central and difference variables in the frequency and slow time domains

$$\omega_+ = \frac{1}{2}(\omega_1+\omega_2),\ \omega_- = \omega_1 - \omega_2, \tag{42}$$

$$T_+ = \frac{1}{2}(T_1+T_2),\ T_- = T_1 - T_2. \tag{43}$$

Utilizing new variables in (39) and performing Fourier transformation on difference slow time T_-, the following equation for the scattering function (which is sometimes termed the wideband scattering function) is obtained:

$$\begin{aligned}S(t,T_+,\omega_d) \;=\; &\frac{1}{2\pi}\int P\Big(\omega_+ + \frac{\omega_-}{2}\Big)P^*\Big(\omega_+ - \frac{\omega_-}{2}\Big)\\ &\cdot\sum_m f_m\Big(\omega_+ + \frac{\omega_-}{2}\Big)f_m^*\Big(\omega_+ - \frac{\omega_-}{2}\Big)\\ &\cdot\Psi_{Rm}(\omega_+,\omega_-;T_+,T_-)\\ &\cdot\exp\big[-i\big(t - t_{gm}(\omega_+)\big)\omega_- + i\omega_d T_-\big]\\ &\cdot d\omega_+ d\omega_- dT_-.\end{aligned} \tag{44}$$

Here the summation is performed over all paths of propagation from the source to the receiver. The

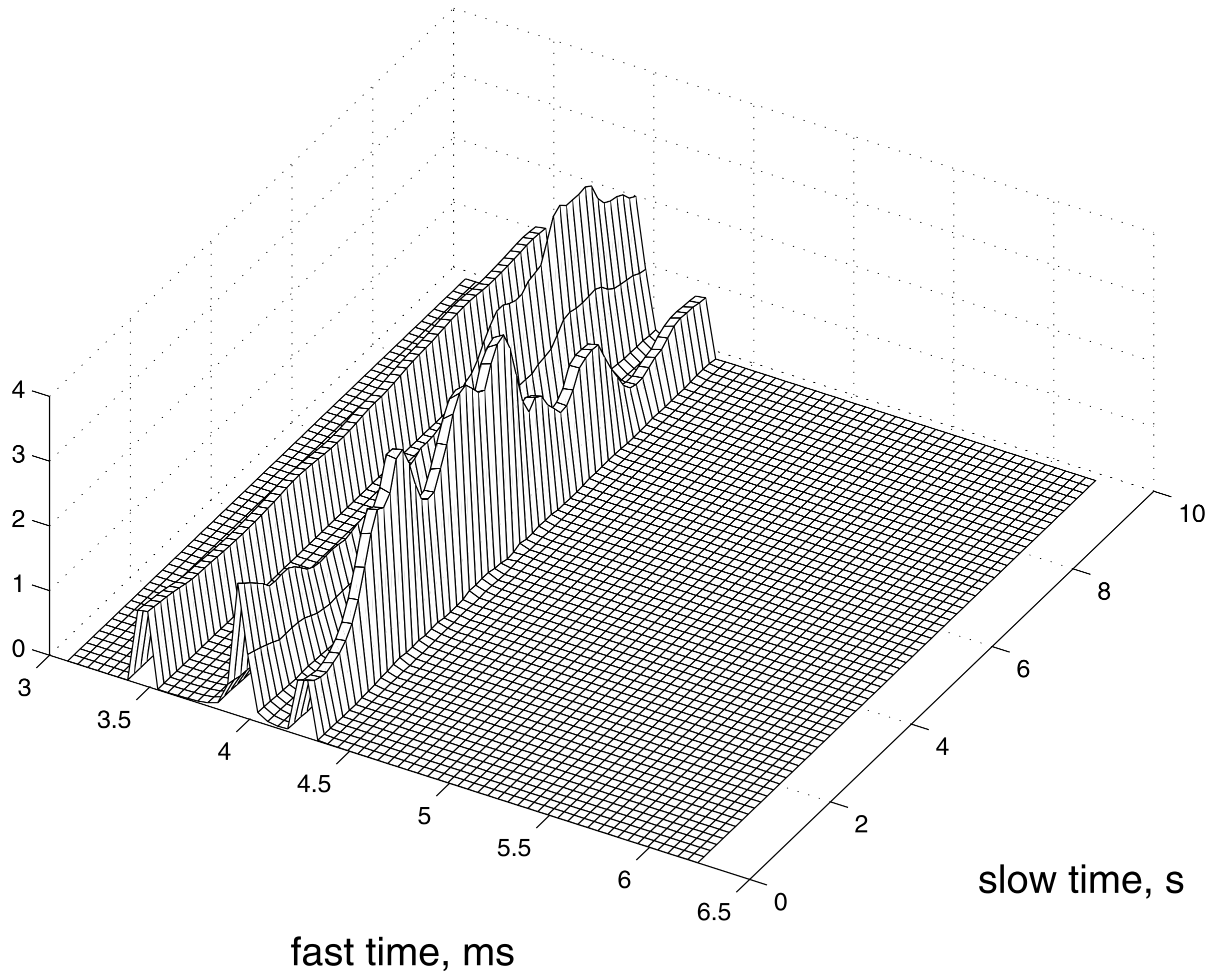

Figure 3. Realization of the received signal plotted in slow time and fast time variables.

scattering function of the channel $S(t, T_+, \omega_d)$ depends on the Doppler variable ω_d, (Fourier-conjugated to T_-), the group delay t and, generally, on the slow time T_+. The latter dependence vanishes when the random ionospheric fluctuations are assumed to be statistically stationary. Group delay time $t_{gm}(\omega_+)$ is given by the equation

$$t_{gm}(\omega_+) = \frac{\partial}{\partial\omega}\left[\frac{\omega}{c}\varphi_m(\omega)\right]_{\omega=\omega_+} \tag{45}$$

and is calculated for each mode of propagation.

[41] In the framework of the complex phase method, the frequency and time correlation functions of the random phasor Ψ_{Rm} in the integral (44) are expressed through the statistical moments of the complex phase [*Gherm and Zernov*, 1998; *Gherm et al.*, 2001a] as follows:

$$\Psi_{Rm}(\omega_+, \omega_-; T_+, T_-) = V\left(\omega_+ + \frac{\omega_-}{2}\right)V^*\left(\omega_+ - \frac{\omega_-}{2}\right) \cdot \left\{\exp\left[\Psi_\psi(\omega_+, \omega_-; T_+, T_-)\right] - 1\right\}, \tag{46}$$

$$V(\omega) = \exp\left[\langle\psi_2(\omega)\rangle + \frac{1}{2}\langle\psi_1^2(\omega)\rangle\right]. \tag{47}$$

The function Ψ_ψ denotes the autocorrelation function of the complex phase, with the main term being obtained employing the complex phase method within the relationship

$$\Psi_\psi(\omega_+, \omega_-; T_+, T_-) = \left\langle \psi_1\left(\omega_+ + \frac{\omega_-}{2}, T_+ + \frac{T_-}{2}\right) \cdot \psi_1^*\left(\omega_+ - \frac{\omega_-}{2}, T_+ - \frac{T_-}{2}\right)\right\rangle. \tag{48}$$

The functions (46)–(48) have been studied in detail in the work of *Gherm and Zernov* [1998] for the case of plane-layered background medium. The scattering function (44) has been studied in the work of *Gherm et al.* [2001a], also for the layered background medium. The extension of the complex phase method obtained above for the case of a

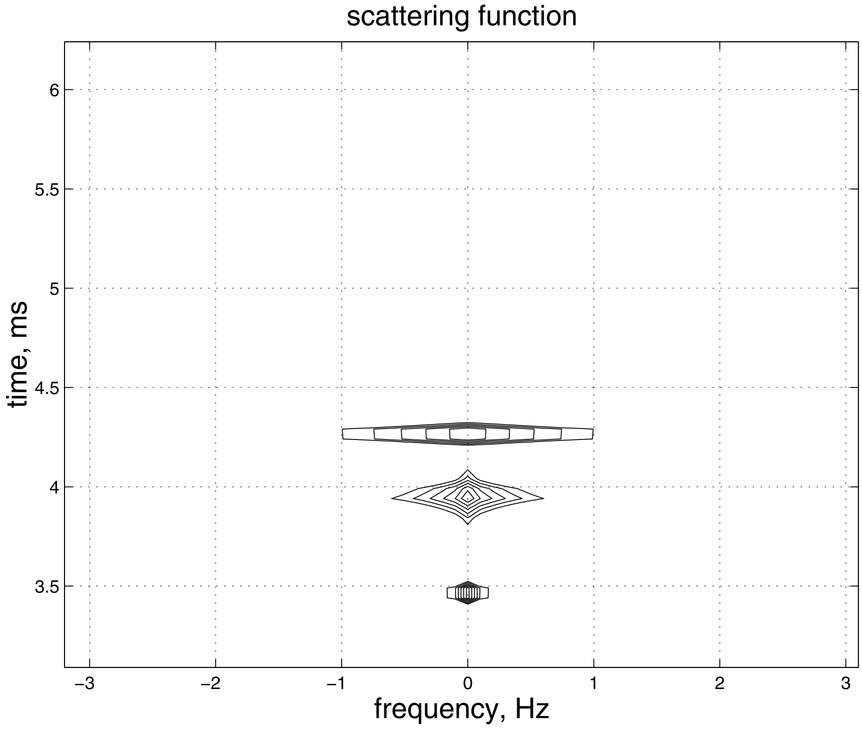

Figure 4. Scattering function calculated theoretically as a statistical moment of the field.

3-D inhomogeneous background medium (equation (26)) naturally permits the extension of the technique of calculation of the scattering functions to the general case of a 3-D background. The statistical moments of the complex phases involved in equations (44) and (46)–(48) are given through the representations (36) and (37). They have all been derived analytically for the general case of an arbitrary 3-D inhomogeneous background medium. The derivation is based on the general equation (26) for a random realization of the complex phase. The calculations are then performed numerically for a given model of the background ionosphere. In Figure 4, the scattering function is presented in the form of a contour plot, calculated according to the described technique as the appropriate statistical moment of the signal.

[42] Finally, there is also another possible method of obtaining the scattering function, namely from the random time series as represented in the plot shown in Figure 3. This is a numerical processing of the simulated time series of a signal analogous to what is really done to real experimental data. The result is presented in Figure 5 in the form of a contour plot. In both Figures 4 and 5, the adjacent contours are separated by 5 dB and range from 0 to −30 dB. Strictly speaking, these plotted values in Figure 5 are not statistical moments, but a sort of realization of the scattering function, obtained after averaging over a finite number of realizations of the received signal. If the period of this averaging is increased, then the number of random realizations will also increase and the resulting plot will converge to the true scattering function, which is the rigorous statistical moment presented in Figure 4.

[43] As far as the effects due to the magnetic field of the Earth are concerned, we have confined this consideration to the isotropic refractive index case. However, the appropriate extension of the theory (of the complex phase method) has also been developed to describe the effects of ordinary and extraordinary modes, so that the magnetoionic splitting can also be accounted for. It was not really practical to give the detailed description of the anisotropic version including the theory and simulator in the framework of a single paper. An additional paper is planned to be devoted to this subject. Some results for the anisotropic case have been recently reported in the work of *Gherm et al.* [2003]. To conclude this paper, in Figure 6 the anisotropic case is briefly presented for conditions analogous to the isotropic case in Figure 5. Again, the retrieved scattering function is represented in the form of a contour plot. It is clearly seen that the high-angle *F* mode (the uppermost local maximum) is split into o components and e components, whereas low-angle *F* mode and *E* mode are not resolved into o components and e components.

5. Practical Use of the Simulator

[44] The inputs required for the simulator are the geographic location of the transmitter and receiver, the

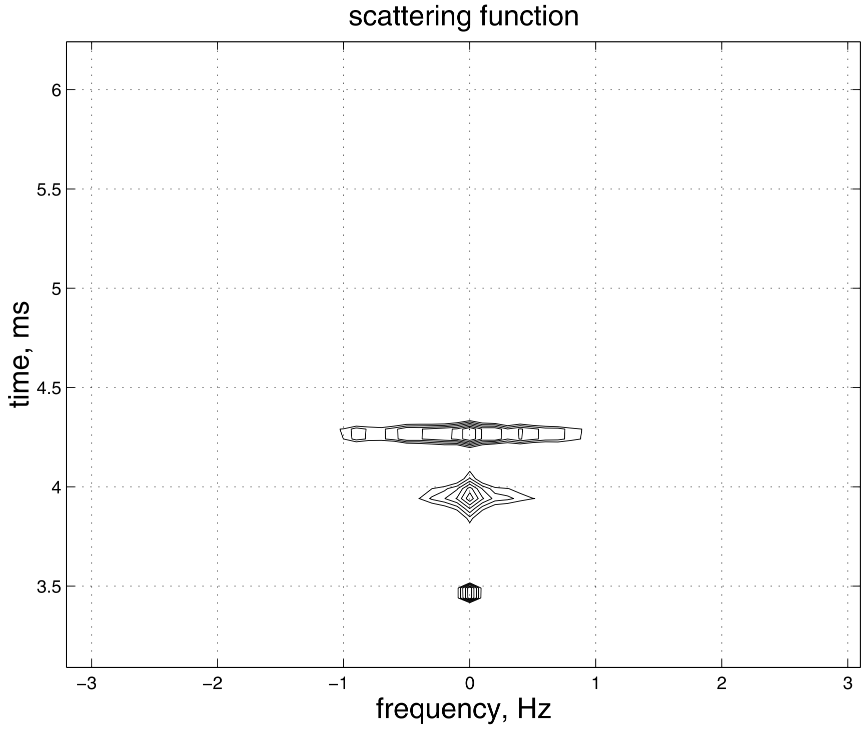

Figure 5. Scattering function retrieved from the field realization shown in Figure 3.

nature of the transmitted signal and the characteristics of the background and stochastic ionosphere (time-varying irregularities) components. The background ionosphere can be fully 3-D such as being represented as a fit to the IRI model over a given latitude and longitude range. Alternatively, it can be specified in terms of the parameters of a number of Chapman, parabolic or quasi-parabolic layers which can include linear latitudinal and/or longitudinal gradients of electron density and/or height of the electron density maximum. Slow time variation such a layer movement or TIDs can also be incorporated and will result in Doppler shift whereas the time-varying irregularities result in Doppler spread. The stochastic component of the ionosphere is specified in terms of the variance of the fractional electron density, the exponent of the inverse power law spatial spectrum, the outer scale of the irregularities along and transverse to the geomagnetic field direction and the direction and speed of the irregularities in three dimensions. The E and H field patterns of the transmitting and receiving antennas can be taken into account when determining the strength of the transmission at different azimuths and elevations and when summing the E fields of the different modes at the receiver. The initial azimuth and elevation angles of the signal for each multipath component are determined by the homing-in program and so are known. This information can also be obtained from the ray-tracing program for the end of the ray path at the receiver location. Either vertical or horizontal antennas can be used for the link.

6. Conclusions

[45] The general description of HF propagation in the ionosphere with 3-D inhomogeneous background and local random inhomogeneities embedded presented above comprises the physical basis for producing a software simulator for the wideband ionospheric fluctuating reflection HF channel. The simulator is capable of producing both random time sequences of a pulsed signal propagated through the fluctuating ionosphere and its statistical moments, e.g., scattering functions. The programs are arranged in the way that any given 3-D model of the background ionosphere can be utilized and multimode propagation can be included for any geometry of propagation. The software simulator utilizes the inverse power law spatial spectrum of fluctuations of the electron density of the ionosphere with given spectral index and different spatial scales of inhomogeneities along and across the magnetic field. Fluctuations are assumed to be statistically homogeneous in time (stationary). The simulator is capable of producing results for signals with bandwidths up to 0.5 MHz. A noise model, described by *Lemmon and Behm* [1991], has also been added. Bulk plasma motion of the background ionosphere can also be

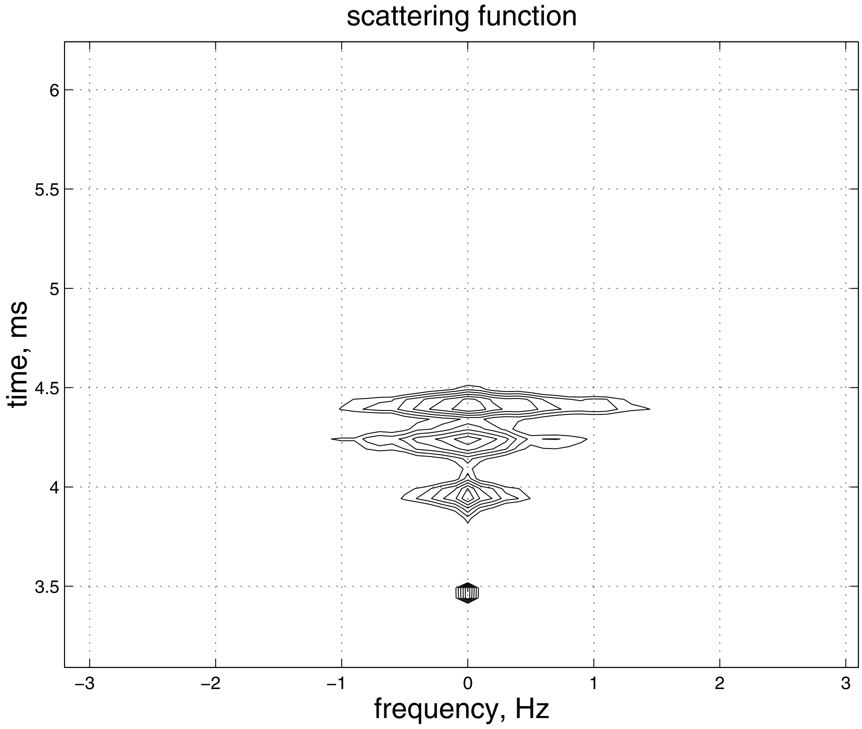

Figure 6. Scattering function retrieved from the field realizations for the case when the Earth's magnetic field is taken into account.

included, giving a Doppler shift in addition to the Doppler spread resulting from diffraction by the moving irregularities.

[46] This propagation model and simulator, since based purely on physical models and parameters, also enables the correspondence between characteristics of the received field and the physical parameters of the model to be investigated. This permits fine-tuning of the model by comparison of received field and predicted output for a variety of conditions as well as providing a way of estimating the physical parameters from the characteristics of the received field.

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Scattering function of the fluctuating ionosphere in the HF band

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Abstract. This paper is devoted to the investigation of the two-frequency, two-position, time coherence function and the ionospheric scattering function describing the HF ionospheric fluctuating radio channel. The complex phase method is applied to obtain the analytical expressions for the coherence and correlation functions, which are then calculated numerically for the realistic models of the fluctuating ionosphere. The numerical Fourier transformation of the correlation function gives the ionospheric scattering function. The numerical results obtained lead to the conclusion that in the general case the large variability of shapes of the scattering function of the fluctuating ionosphere exists depending on the concrete conditions of propagation. In particular, the well-known delay-Doppler coupling can be more or less pronounced in different propagation conditions. We have shown that the presence of the coupling is exclusively due to the nonzero imaginary part of the correlation function of the scattered field, which means that this effect has a purely diffractional nature and cannot be obtained in the geometrical optics approximation.

1. Introduction

It is commonly accepted to characterize HF ionospheric fluctuating radio channel in terms of two fundamental quantities, which are the transfer function and the scattering function of the ionosphere. While the transfer function describes a regular background channel, the scattering function accounts for the effects due to electron density fluctuations of the ionosphere. The latter may be produced through a spaced position, time and frequency coherence function of the HF field propagating in the fluctuating ionosphere by means of Fourier transformation in appropriate variables.

So far as the main properties of the undisturbed (regular) HF ionospheric channel may be considered as well studied, the interests of those going in for HF propagation are at the moment focused on the characterization of the fluctuating ionospheric channel. One may find in the scientific literature a lot of investigations of the ionospheric stochastic channel, experimental [*Proakis*, 1983; *Basler et al.*, 1988; *Wagner et al.*, 1988; *Cannon et al.*, 1995] as well as theoretical.

In theory, many approaches exist to construct a spaced time, frequency, and position coherence function of the HF field and the scattering function of the ionosphere. One of them, providing the description of strong fluctuations of a field amplitude as well, is the momenta parabolic equations for forward scattering, written in Markov's approximation [*Rytov et al.*, 1978; *Ishimaru*, 1978]. This method is well developed for the case of a homogeneous background medium [*Tatarskii*, 1971; *Knepp*, 1983a; *Nickisch*, 1992; *Gozani*, 1993]. For VHF, and especially HF range propagation, the inhomogeneous background medium, giving rise to rays bending and multipath effects, is of importance and has to be taken into account. To extend Markov's approximation technique to the case

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of inhomogeneous background medium, the necessary parabolic equations have been derived by *Hill* [1985] in ray-centered variables and the coordinate system of orthogonal trajectories (full ray variables). *Mazar and Felsen* [1987a, b] constructed analytical solutions of the appropriate equations in the scope of multiscale expansions. They have obtained the spaced frequency coherence function as well, but their results are restricted by the case of nondispersive media, when the fields of different frequencies propagate along the same paths, which the HF ionospheric propagation does not pertain to because of the dispersive nature of the ionosphere.

For practical calculations of the statistical effects of HF propagation in the fluctuating ionosphere with the inhomogeneous background, the numerical multiple phase-screen/diffraction method was developed [*Knepp*, 1983b; *Kiang and Liu*, 1985], and used, in particular, in the investigation of the coherence function of the HF field in the ionosphere [*Wagen and Yeh*, 1989a, b; *Rand and Yeh*, 1991]. The technique of the multiple phase-screen/diffraction method was also exploited to construct the analytical solution of Markov's parabolic equation for the spaced frequency, time, and position coherence function of a field in a medium with a homogeneous background [*Nickisch*, 1992].

To account for the regular refraction, together with the scattering by local ionospheric inhomogeneities of the ionosphere, including diffraction effects, we use Rytov's approximation generalized by *Zernov* [1980] to the case of an essentially inhomogeneous background medium. Although Rytov's method is invalid to describe strong-amplitude fluctuations of a field (with the variance of the logarithm of the amplitude of a field exceeding unity), it gives the most general description of the unsaturated regime of propagation in a random ionosphere and provides the possibility to advance significantly in the analytical and numerical investigation of HF propagation in the ionosphere with moderate fluctuations of the fractional electron density. The method produces in automatic fashion the geometrical optics perturbation theory as the limiting case when local inhomogeneities of the ionosphere are of the zero values of the wave (diffraction) parameter. The unsaturated regime of HF propagation corresponds to the conditions of the quiet midlatitude ionosphere. As for the high-latitude ionosphere, characterized by stronger fluctuations of the fractional electron density, it may give rise to the saturated regime of propagation, which requires the alternative treatment of the stochastic propagation problem. This may be, for instance, the path integral technique, outlined by *Dashen* [1979], *Flatte et al.* [1979], and *Flatte* [1983]. The saturated region of HF propagation lies beyond the scope of the present consideration.

We already used the generalized Rytov's method to describe HF field phase and level (logarithm of the amplitude) fluctuations [*Gherm and Zernov*, 1995] and HF pulse propagation through the fluctuating ionosphere [*Zernov and Lundborg*, 1995]. In the latter paper the analytical results have been derived, describing a shape of pulses, passed through the fluctuating ionosphere. At the same time, the paper by *Fridman et al.* [1995] has been released, where the generalized Rytov's approximation is used for studying of the two-frequency, time coherence function of the HF field. Below, we will discuss the results of this paper in more detail. **We should also like to point out that *Gherm et al.* [1997a, b] give recent results in pulse propa**gation, including the numerical simulation of the coherence and correlation functions. We present here our recent results of the analytical and numerical investigation of the scattering function of the fluctuating ionosphere, calculated for realistic models of the background ionosphere and ionospheric electron density fluctuations.

2. Coherence of a Field

In the investigation of propagation effects of the transient fields through the fluctuating ionosphere, the spaced time and position coherence function of the field received will be the subject of our interest. To describe this quantity, we use here the frequency domain technique and express the coherence function in the form of a double integral in the frequency domain as follows:

$$\Gamma_E(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2) = \iint_{-\infty}^{+\infty} \mathrm{d}\omega_1 \mathrm{d}\omega_2 \cdot P(\omega_1) P^*(\omega_2) f_0(\mathbf{r}_1, \omega_1) f_0^*(\mathbf{r}_2, \omega_2) \cdot \Gamma(\mathbf{r}_1, \mathbf{r}_2, \omega_1, \omega_2, t_1, t_2) \cdot \exp\{i[k_1\phi_0(\mathbf{r}_1, \omega_1) - k_2\phi_0(\mathbf{r}_2, \omega_2)] - i(\omega_1 t_1 - \omega_2 t_2)\} \qquad (1)$$

Here $k_1 = \omega_1/c$, $k_2 = \omega_2/c$, and c is the light velocity in vacuum. $P(\omega)$ represents the frequency spectrum of a nonmonochromatical signal launched. Functions $f_0(\mathbf{r}, \omega)$ and $\phi_0(\mathbf{r}, \omega)$ give the amplitude and phase of a harmonic component of a transient field, propagated through the undisturbed ionosphere, so that the function

$$E_0(\mathbf{r}, \omega) = f_0(\mathbf{r}, \omega) \exp[ik\phi_0(\mathbf{r}, \omega)] \qquad (2)$$

is the transfer function of the background ionospheric channel, given in the approximation of the dominant term of the ray (geometrical optics) expansion of a point source field in a smoothly inhomogeneous medium. The effects due to the ionospheric electron density fluctuations are taken into account through the two-time, two-frequency, two-position coherence function $\Gamma(\mathbf{r}_1, \mathbf{r}_2, \omega_1, \omega_2, t_1, t_2)$ in equation (1). If one makes use of the complex phase $\psi(\mathbf{r}, \omega, t)$ to account for the influence of the electron density fluctuations on the monochromatic component $E_0(\mathbf{r}, \omega)$, the disturbed component of a field $E_\omega(\mathbf{r}, \omega, t)$ is represented in the following form:

$$E_\omega(\mathbf{r}, \omega, t) = E_0(\mathbf{r}, \omega) \exp[\psi(\mathbf{r}, \omega, t)]. \qquad (3)$$

Then, in the scope of Rytov's approximation, the coherence function $\Gamma(\mathbf{r}_1, \mathbf{r}_2, \omega_1, \omega_2, t_1, t_2)$ may be expressed using the first- and second-order approximations ψ_1 and ψ_2 of the perturbation theory for the complex phase. After the decomposition of different orders of complex phases into real and imaginary parts has been performed

$$\psi_1 = \chi_1 + iS_1, \quad \psi_2 = \chi_2 + iS_2, \qquad (4)$$

the representation for the coherence function is given by the equation

$$\Gamma \approx \Gamma_2(\mathbf{r}_1, \mathbf{r}_2, \omega_1, \omega_2, t_1, t_2) = V_2(\mathbf{r}_1, \omega_1, t_1) V_2^*(\mathbf{r}_2, \omega_2, t_2) \cdot \exp[b(\mathbf{r}_1, \mathbf{r}_2, \omega_1, \omega_2, t_1, t_2) + iq(\mathbf{r}_1, \mathbf{r}_2, \omega_1, \omega_2, t_1, t_2)], \qquad (5)$$

where

$$V_2(\mathbf{r}, \omega, t) = \exp[\langle\chi_2(\mathbf{r}, \omega, t)\rangle + i\,\langle S_2(\mathbf{r}, \omega, t)\rangle + \tfrac{1}{2}\left\langle\chi_1^2(\mathbf{r}, \omega, t)\right\rangle + i\,\langle\chi_1(\mathbf{r}, \omega, t) S_1(\mathbf{r}, \omega, t)\rangle - \tfrac{1}{2}\left\langle S_1^2(\mathbf{r}, \omega, t)\right\rangle], \qquad (6)$$

$$b(\mathbf{r}_1, \mathbf{r}_2, \omega_1, \omega_2, t_1, t_2) = \langle\chi_1(\mathbf{r}_1, \omega_1, t_1)\chi_1(\mathbf{r}_2, \omega_2, t_2)\rangle + \langle S_1(\mathbf{r}_1, \omega_1, t_1) S_1(\mathbf{r}_2, \omega_2, t_2)\rangle,$$
$$q(\mathbf{r}_1, \mathbf{r}_2, \omega_1, \omega_2, t_1, t_2) = \langle\chi_1(\mathbf{r}_2, \omega_2, t_2) S_1(\mathbf{r}_1, \omega_1, t_1)\rangle - \langle\chi_1(\mathbf{r}_1, \omega_1, t_1) S_1(\mathbf{r}_2, \omega_2, t_2)\rangle. \qquad (7)$$

The subscripts to Γ_2 and V_2 mean that these quantities have been calculated with the first and second approximations of the complex phase taken into account. Representations (5)-(7) are essentially the same coherence function as derived by *Zernov and Lundborg* [1995, equations (32), (33a), and (33b)]. The difference here is that the time dependence is recovered and spaced position is introduced.

The dependence on time takes place in the relationships (3) – (7) in the sense of the slow time dependence of functions E_ω (and consequently the complex phases ψ) governed in quasi-stationary approximation by the equation

$$\nabla^2 E_\omega + k^2[\varepsilon_0(\mathbf{r}, \omega) + \varepsilon(\mathbf{r}, \omega, t)]\, E_\omega = \delta(\mathbf{r})\,. \qquad (8)$$

In this equation, $\delta(\mathbf{r})$ is the Dirac's delta function, $\varepsilon_0(\mathbf{r}, \omega)$ is a model of the undisturbed background ionosphere, and $\varepsilon(\mathbf{r}, \omega, t)$ represents local random inhomogeneities of the ionosphere. Time t stands to indicate possible slow time dependence of the properties of fluctuations in quasi-stationary approximation. The range of validity of the quasi-stationary approximation (8) for a dispersive plasma is given by the inequality

$$\sigma\nu \gg 1, \tag{9}$$

where σ is a characteristic timescale of random local inhomogeneities and ν is an effective collision frequency of the plasma electrons, so that ν^{-1} gives the timescale of relaxation of the ionospheric plasma.

To obtain the final expression for the coherence function, we introduce the center and difference variables as follows:

$$\mathbf{R} = \frac{\mathbf{r}_1 + \mathbf{r}_2}{2}, \qquad \boldsymbol{\rho} = \mathbf{r}_1 - \mathbf{r}_2,$$
$$T = \frac{t_1 + t_2}{2}, \qquad t = t_1 - t_2,$$
$$\Omega = \frac{\omega_1 + \omega_2}{2}, \qquad \delta = \omega_1 - \omega_2. \tag{10}$$

In the new variables, Γ_2 may be rewritten in the form

$$\Gamma_2(\mathbf{R}, \boldsymbol{\rho}; \Omega, \delta; T, t) = V_2\left(\mathbf{R} + \frac{\boldsymbol{\rho}}{2}, \Omega + \frac{\delta}{2}, T + \frac{t}{2}\right) \cdot V_2^*\left(\mathbf{R} - \frac{\boldsymbol{\rho}}{2}, \Omega - \frac{\delta}{2}, T - \frac{t}{2}\right) \cdot \exp[b(\mathbf{R}, \boldsymbol{\rho}; \Omega, \delta; T, t) + iq(\mathbf{R}, \boldsymbol{\rho}; \Omega, \delta; T, t)]. \tag{11}$$

Generally, the coherence function is not obliged to tend to zero as the difference argument tends to the infinity. As such, it is often convenient to split the coherence function into two items, extracting its behavior in the infinity in the explicit form. This may be performed by making use of the well-known relationship between the coherence function and the correlation function [*Rytov et al.*, 1978], which we denote as Ψ_2. then this relationship is as follows:

$$\Gamma_2(\mathbf{R}, \boldsymbol{\rho}; \Omega, \delta; T, t) = \Psi_2(\mathbf{R}, \boldsymbol{\rho}; \Omega, \delta; T, t) + V_2\left(\mathbf{R} + \frac{\boldsymbol{\rho}}{2}, \Omega + \frac{\delta}{2}, T + \frac{t}{2}\right) \cdot V_2^*\left(\mathbf{R} - \frac{\boldsymbol{\rho}}{2}, \Omega - \frac{\delta}{2}, T - \frac{t}{2}\right). \tag{12}$$

The last equation, together with representation (11), yields for the correlation function the result

$$\Psi_2(\mathbf{R}, \boldsymbol{\rho}; \Omega, \delta; T, t) = V_2\left(\mathbf{R} + \frac{\boldsymbol{\rho}}{2}, \Omega + \frac{\delta}{2}, T + \frac{t}{2}\right) \cdot V_2^*\left(\mathbf{R} - \frac{\boldsymbol{\rho}}{2}, \Omega - \frac{\delta}{2}, T - \frac{t}{2}\right) \cdot F(\mathbf{R}, \boldsymbol{\rho}; \Omega, \delta; T, t), \tag{13}$$

with

$$F(\mathbf{R}, \boldsymbol{\rho}; \Omega, \delta; T, t) = \exp[b(\mathbf{R}, \boldsymbol{\rho}; \Omega, \delta; T, t) + iq(\mathbf{R}, \boldsymbol{\rho}; \Omega, \delta; T, t)] - 1. \tag{14}$$

Equations (13) and (14) describe the two-position, two-frequency, two-time correlation function of the HF field in the fluctuating ionosphere. With the assumption of the statistical homogeneity of the ionospheric fluctuations in time (stationarity), Γ_2 and Ψ_2 depend only on the difference time variable t.

In the particular case $\mathbf{r} = \mathbf{r}_1 = \mathbf{r}_2$, that is, $\mathbf{R} = \mathbf{r}, \boldsymbol{\rho} = 0$, the correlation function from equations (13) and (14) gives the quantity $F(\mathbf{r}, \Omega, \delta, t)$, which was treated by *Fridman et al.* [1995]. Strictly speaking, they studied only the quantity $b(\mathbf{R}, \boldsymbol{\rho}; \Omega, \delta; T, t)$ in the domain (δ, t), and they had no the imaginary part in the exponent of equation (14). Below, we will derive the represenations for the functions V_2, b, q. One will see that different second-order moments involved in the description of these quantities through equations (6) and (7) are generally expressed as manyfold integrals. It will be shown that only in the case when the geometrical optics approximation is valid (i.e., in the case of spatial scales of the ionospheric local inhomogeneities being greater than the main Fresnel zone size for a given path of propagation) does the situation takes place that $q = 0$, and b is represented as onefold integral as it is by *Fridman et al.* [1995]. Generally, this is not the case. Moreover, as we will see later on, to obtain the parabolic shapes of the profiles of the scattering function of the ionosphere, the nonzero imaginary part q in the exponent in (14) is a crucial point. At the same time, we would like to point out that as our consideration shows, the approximation used by *Fridman et al.* [1995] is quite adequate for small values of the difference variable δ.

We will give here the general treatment of the two-position, two-frequency, time coherence and correlation functions for an arbitrary range of variables. This will provide the possibility to construct the ionospheric scattering function through the Fourier transformation in appropriate variables. We will also give the results of the numerical simulation of the scattering function and the distribution of the angles of arrival, constructed for arbitrary given models of the background ionosphere and the inverse power law of the spatial spectrum of the ionospheric electron density fluctuations.

To finish this section, we rewrite in new variables, introduced in (10), the quantity from equation (1), which is the two-time, two-position coherence function of a nonmonochromatic field, radiated by a point source, and passed through the fluctuating ionosphere, in the form as follows:

$$\begin{aligned}\Gamma_E(\mathbf{R},\boldsymbol{\rho},T,t) = \iint_{-\infty}^{+\infty} \mathrm{d}\Omega \mathrm{d}\delta \\ \cdot P\left(\Omega+\frac{\delta}{2}\right) P^*\left(\Omega-\frac{\delta}{2}\right) \\ \cdot f_0\left(\mathbf{R}+\frac{\boldsymbol{\rho}}{2},\Omega+\frac{\delta}{2}\right) f_0^*\left(\mathbf{R}-\frac{\boldsymbol{\rho}}{2},\Omega-\frac{\delta}{2}\right) \\ \cdot\Gamma_2(\mathbf{R},\boldsymbol{\rho},\Omega,\delta,t)\cdot\exp\left\{i\frac{\partial}{\partial\Omega}\left[K\phi_0(\mathbf{R},\Omega)\right]\delta\right. \\ \left.+i\frac{\partial}{\partial\mathbf{R}}\left[K\phi_0(\mathbf{R},\Omega)\right]\boldsymbol{\rho} - i(\Omega t+\delta T)\right\}. \quad (15)\end{aligned}$$

To obtain this representation the expansion

$$\begin{aligned}\frac{\Omega+\frac{\delta}{2}}{c}\phi_0\left(\mathbf{R}+\frac{\boldsymbol{\rho}}{2},\Omega+\frac{\delta}{2}\right) \\ -\frac{\Omega-\frac{\delta}{2}}{c}\phi_0\left(\mathbf{R}-\frac{\boldsymbol{\rho}}{2},\Omega-\frac{\delta}{2}\right) \\ = \frac{\partial}{\partial\Omega}\left[K\phi_0(\mathbf{R},\Omega)\right]\delta + \frac{\partial}{\partial\mathbf{R}}\left[K\phi_0(\mathbf{R},\Omega)\right]\boldsymbol{\rho} \quad (16)\end{aligned}$$

with $K=\Omega/c$ has been introduced and also the stationarity of the ionospheric electron density fluctuations has been assumed, which has resulted in the dependence of Γ_2 only on the difference time variable t. Of course, one must understand that when some expansion is used under the sign of integration, the expansion has to be valid in the domain giving the main contribution in the integral. This form is useful to derive some analytical results, which will be given below. At the same time, we would like to point out here that the expansion (16) does not pertain directly to the procedure of numerical calculation of correlation and scattering functions, the results of which will be also given in the following sections of the paper. In the next section we will use the last expression (15) to fulfil some analytical investigation of HF field propagation in the fluctuating ionosphere.

3. Field Coherence and the Ionospheric Scattering Function

Expression (15) is a rather general representation of a field coherence accounting for a lot of the effects of HF propagation in the ionosphere with the electron density fluctuations. While being interested in the mean energy $W_E(\mathbf{R},T)$ of a pulse propagating through the fluctuating ionosphere, one uses (1) and (15) with the arguments $T=t_1=t_2$ and $\mathbf{R}=\mathbf{r}_1=\mathbf{r}_2$, that is, as follows from (10), $\boldsymbol{\rho}=0, t=0$. Then the mean energy W_E is given by the equation

$$\begin{aligned}W_E(\mathbf{R},T) = \Gamma_E(\mathbf{R},0,T,0) = \iint_{-\infty}^{+\infty}\mathrm{d}\Omega\mathrm{d}\delta \\ \cdot P\left(\Omega+\frac{\delta}{2}\right)P^*\left(\Omega-\frac{\delta}{2}\right) \\ \cdot f_0\left(\mathbf{R},\Omega+\frac{\delta}{2}\right) f_0^*\left(\mathbf{R},\Omega-\frac{\delta}{2}\right) \\ \cdot\Gamma_2(\mathbf{R},0,\Omega,\delta,0) \\ \cdot\exp\left\{i\frac{\partial}{\partial\Omega}\left[K\phi_0(\mathbf{R},\Omega)\right]\delta - i\delta T\right\}. \quad (17)\end{aligned}$$

In the scope of Rytov's approximation the effects of pulse propagation through the fluctuating ionosphere have been investigated in detail by *Zernov and Lundborg* [1995] and *Gherm et al.* [1997a, b]. One understands from (17) that

in the studying of pulse propagation, the projection of the general coherence function $\Gamma_2(\mathbf{R}, \boldsymbol{\rho}, \Omega, \delta, t)$ is needed to the domain (Ω, δ), with the other arguments fixed as $\boldsymbol{\rho} = 0, t = 0$. This is the function appearing in equation (17). This two-frequency coherence function has been thoroughly investigated numerically by *Gherm et al.* [1997a] for realistic models of the background ionosphere and the ionospheric electron density fluctuations.

Another interesting effect can be described by representation (15) in the case of the field launched being the monochromatic. This is the case where the spectrum of an excited field is as follows:

$$P(\omega) = U_0 \delta(\omega - \omega_0), \tag{18}$$

so that

$$W_0 = U_0^2 \tag{19}$$

gives a full energy, radiated to the direction of a ray, connecting the communication points, and $\delta(\ )$ is the Dirac's function. With the spectrum (18), general expression (15) for a coherence function of a received field yields

$$\begin{aligned} \Gamma_E(\mathbf{R}, \boldsymbol{\rho}, T, t) &= \Gamma_E(\mathbf{R}, \boldsymbol{\rho}, 0, t) \\ = U_0^2 f_0(\mathbf{R} + \frac{\boldsymbol{\rho}}{2}, \omega_0) & f_0^*(\mathbf{R} - \frac{\boldsymbol{\rho}}{2}, \omega_0) \\ & \cdot \Gamma_2(\mathbf{R}, \boldsymbol{\rho}, \omega_0, 0, t) \\ \cdot \exp\left\{ i \frac{\partial}{\partial \mathbf{R}} \left[K \phi_0(\mathbf{R}, \omega_0) \right] \boldsymbol{\rho} - i \omega_0 t \right\}. \end{aligned} \tag{20}$$

This is the two-position, time coherence function of a monochromatic field of the frequency ω_0, corrupted while propagating due to the ionospheric electron density fluctuations. Obviously, this function is described by another projection of the general coherence function $\Gamma_2(\mathbf{R}, \boldsymbol{\rho}, \Omega, \delta, t)$, which is now the projection to the domain $(\boldsymbol{\rho}, t)$. This is the function $\Gamma_2(\mathbf{R}, \boldsymbol{\rho}, \omega_0, 0, t)$ with the arguments $\mathbf{R}, \Omega, \delta$, fixed as follows:

$$\mathbf{R} = \frac{\mathbf{r}_1 + \mathbf{r}_2}{2}, \qquad \Omega = \omega_0, \qquad \delta = 0. \tag{21}$$

Carrying out Fourier transformation on variables t and $\boldsymbol{\rho}$ of the function Γ_E in equation (20), one may find a frequency spectrum (conjugated to variable t) and two angle spectra (conjugated to the two displacements, which may be chosen, for instance, for a sky wave, as the displacements on the Earth's surface in the directions which are parallel and perpendicular to the plane of propagation). These spectra characterize the Doppler broadening and the distribution of the angles of arrival of a monochromatic field, propagated through the fluctuating ionosphere. In particular, if one makes use of the decomposition of the coherence function according to relationships (12)–(14) and performs Fourier transformation on t in equation (20), one obtains the Doppler broadening at the point of observation $\mathbf{r}$ in the following form:

$$\begin{aligned} \tilde{\Gamma}_E(\mathbf{r}, \omega_0, \tilde{\omega}) = U_0^2 |V_2(\mathbf{r}, \omega_0)|^2 |f_0(\mathbf{r}, \omega_0)|^2 \\ \cdot \left[G(\mathbf{r}, \omega_0, \tilde{\omega}) + \delta(\tilde{\omega} - \omega_0) \right]. \end{aligned} \tag{22}$$

Here the variable $\tilde{\omega}$ is a frequency, which is Fourier conjugated to t, and

$$\begin{aligned} G(\mathbf{r}, \omega_0, \tilde{\omega}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt F(\mathbf{r}, 0, \omega_0, 0, t) \\ \cdot \exp\left[i(\tilde{\omega} - \omega_0)t \right], \end{aligned} \tag{23}$$

where F is given by equation (14). The two items in the sum (equation (22)) represent the contributions of the fluctuational and coherent components of a field to the mean energy of a full field for each monochromatic component (strictly speaking, they are the contributions to the spectral density of the mean energy of a full field).

At last, the difference frequency variable δ of the coherence function Γ_2 in the integral in equation (15) is involved in the description of pulses stretching due to ionospheric electron density fluctuations studied by *Zernov and Lundborg* [1995] and *Gherm et al.* [1997a, b]. When one performs Fourier transformation of Γ_2 on the difference variable δ, one obtains the Fourier transform, being a function of the time variable, which is the additional delay time due to fluctuations. This Fourier transform may be written as follows:

$$\tilde{\tilde{\Gamma}}_2(\mathbf{R},\boldsymbol{\rho},\Omega,t_d,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\Gamma_2(\mathbf{R},\boldsymbol{\rho},\Omega,\delta,t) \cdot \exp(-i\delta t_d)\mathrm{d}\delta, \quad (24)$$

where t_d is a time variable conjugated to the frequency variable δ.

This way, to describe the different effects of HF propagation in the fluctuating ionosphere, a two-position, two-frequency, time coherence function of HF field or its Fourier transform in appropriate variables needs to be addressed. The last one is one of several possible definitions of scattering function, which we will follow after.

To have the quantitative description of different effects of HF propagation, numerical calculations of both the coherence and scattering functions have to be performed for concrete given models of the background ionosphere and ionospheric electron density fluctuations. The next section of the paper will deal with the regular procedure of the numerical calculations of the spaced time, frequency, and position coherence function and the scattering function of the ionosphere.

4. Algorithms for Numerical Calculations

It is convenient to express quantities V_2, b, and q involved in the representation of the correlation function from equation (13) through the moments of complex phases of the first and second orders ψ_1 and ψ_2. Then V_2, b, and q from equations (6) and (7) may be rewritten as follows:

$$V_2(\mathbf{r},\omega,t) = \exp\left[\langle\psi_2(\mathbf{r},\omega,t)\rangle + \tfrac{1}{2}\left\langle\psi_1^2(\mathbf{r},\omega,t)\right\rangle\right],$$

$$\begin{aligned} &b(\mathbf{r}_1,\mathbf{r}_2,\omega_1,\omega_2,t_1,t_2) \\ &+iq(\mathbf{r}_1,\mathbf{r}_2,\omega_1,\omega_2,t_1,t_2) \\ &= \langle\psi_1(\mathbf{r}_1,\omega_1,t_1)\psi_1^*(\mathbf{r}_2,\omega_2,t_2)\rangle \end{aligned} \quad (25)$$

If the stationarity of ionospheric electron density fluctuations is assumed, then single-point moments in equation (25) are no longer the functions of a time variable. As for the two-argument moment in (25), it is now a function of the difference time variable $t = t_1 - t_2$. Taking into account all the above mentioned, we finally represent the correlation function given by (13) and (14) in the following form, written making use of the center and difference variables (10):

$$\begin{aligned}\Psi_2(\mathbf{R},\boldsymbol{\rho};\Omega,\delta;t) = {} & V_2\left(\mathbf{R}+\frac{\boldsymbol{\rho}}{2},\Omega+\frac{\delta}{2}\right) \\ & \cdot V_2^*\left(\mathbf{R}-\frac{\boldsymbol{\rho}}{2},\Omega-\frac{\delta}{2}\right) \\ & \cdot F(\mathbf{R},\boldsymbol{\rho};\Omega,\delta;t), \end{aligned} \quad (26)$$

where

$$\begin{aligned} F(\mathbf{R},\boldsymbol{\rho};\Omega,\delta;t) = {} & \exp[b(\mathbf{R},\boldsymbol{\rho};\Omega,\delta;t) \\ & + iq(\mathbf{R},\boldsymbol{\rho};\Omega,\delta;t)] - 1. \end{aligned} \quad (27)$$

and V_2 is given by equation (25) with the time dependence supressed.

Below, we present a brief derivation of expressions for moments of complex phase involved in (25). More detailed derivation are given by *Gherm et al.* [1997a, b] with the only distinction that the derivation presented here extends the treatment to the case of two-position coherence and correlation functions.

To find the complex phase $\psi(\mathbf{r},\omega,t)$, we use here Rytov's method generalized to the case of inhomogeneous background ionosphere with local inhomogeneities [*Zernov*, 1980]. Substituting representation (3) for the disturbed monochromatic component of a field to the Helmholtz equation (8), one obtains the equation for the complex phase

$$(\nabla\psi)^2 + \nabla^2\psi + 2(\nabla \ln E_0 \cdot \nabla\psi) = -k^2\varepsilon, \quad (28)$$

which is then solved by a perturbation method assuming function ε to be a small quantity. Introducing Green's function $G(\mathbf{r}',\mathbf{r},\omega)$ of the undisturbed problem for equation (8), one can easily write the expressions for the first-order term of the perturbation series for the complex phase $\psi_1(\mathbf{r},\omega,t)$

$$\psi_1(\mathbf{r},\omega,t) = -k^2 E_0^{-1}(\mathbf{r},\omega) \cdot \int G(\mathbf{r}',\mathbf{r},\omega)\,\varepsilon(\mathbf{r}',\omega,t)\,E_0(\mathbf{r}',\omega)\,\mathrm{d}\mathbf{r}' , \qquad (29)$$

and the second-order term $\psi_2(\mathbf{r},\omega,t)$

$$\psi_2(\mathbf{r},\omega,t) = -E_0^{-1}(\mathbf{r},\omega) \cdot \int G(\mathbf{r}',\mathbf{r},\omega)\,[\nabla\psi_1(\mathbf{r}',\omega,t)]^2\,E_0(\mathbf{r}',\omega)\,\mathrm{d}\mathbf{r}' . \quad (30)$$

For the next consideration the functions $E_0(\mathbf{r},\omega)$ and $G(\mathbf{r}',\mathbf{r},\omega)$ are represented in the geometrical optics (GO) approximation, which is formally invalid near the caustic surfaces. However, as was shown by *Zernov* [1994], the expressions obtained from (29) and (30) by making use of these quasi-classical asymptotics are valid and uniform along the ray of the incident field, provided the condition $l_\varepsilon > l_c$ holds, where l_ε is a spatial scale of inhomogeneities and l_c is a near-caustic area size.

To analyze expressions (29) and (30), it is convenient [*Zernov*, 1980; *Gherm and Zernov*, 1995] to introduce a ray-centered coordinate system with transversal variables linked with a reference ray which connects the source and the point of observation $\mathbf{r}$ in the undisturbed ionosphere. Using these variables with s calculated along this ray, n perpendicular to the ray in the plane of propagation, and τ perpendicular to the plane of propagation, expressions (29) and (30) can be written in the Fresnel approximation for forward scattering as follows:

$$\psi_1(\mathbf{r},\omega,t) = \frac{ik^2}{4\pi}\iiint \mathrm{d}s\,\mathrm{d}n\,\mathrm{d}\tau \cdot \frac{\varepsilon(\mathbf{r}(s,n,\tau),\omega,t)}{\varepsilon_0^{\frac{1}{2}}(s)|D_n(s,s_0)D_\tau(s,s_0)|^{\frac{1}{2}}} \cdot \exp\left\{\frac{ik}{2}\left[\frac{n^2}{D_n(s,s_0)}+\frac{\tau^2}{D_\tau(s,s_0)}\right] - \frac{i\pi}{4}[\operatorname{sgn}(D_n(s,s_0))+\operatorname{sgn}(D_\tau(s,s_0))]\right\}, \quad (31)$$

$$\psi_2(\mathbf{r},\omega,t) = \frac{i}{4\pi}\iiint \mathrm{d}s\,\mathrm{d}n\,\mathrm{d}\tau \cdot \frac{[\nabla\psi_1(\mathbf{r}(s,n,\tau),\omega,t)]^2}{\varepsilon_0^{\frac{1}{2}}(s)|D_n(s,s_0)D_\tau(s,s_0)|^{\frac{1}{2}}} \cdot \exp\left\{\frac{ik}{2}\left[\frac{n^2}{D_n(s,s_0)}+\frac{\tau^2}{D_\tau(s,s_0)}\right] - \frac{i\pi}{4}[\operatorname{sgn}(D_n(s,s_0))+\operatorname{sgn}(D_\tau(s,s_0))]\right\} . \quad (32)$$

The integration over s in (31) and (32) is carried out along the ray from the point $s = 0$ to the point $s = s_0$, which correspond to the origin and the endpoint of the ray of reference, respectively. The integration domain in the plane (n,τ) should cover some Fresnel zones, so that the full integration volume covers the main Fresnel volume. Parameters $D_n(s,s_0)$ and $D_\tau(s,s_0)$ are defined as

$$D_n^{-1}(s,s_0) = \frac{\partial^2\phi_0[\mathbf{r}(s)]}{\partial n^2} + \frac{\partial^2\phi_1[\mathbf{r}(s),\mathbf{r}(s_0)]}{\partial n^2},$$
$$D_\tau^{-1}(s,s_0) = \frac{\partial^2\phi_0[\mathbf{r}(s)]}{\partial \tau^2} + \frac{\partial^2\phi_1[\mathbf{r}(s),\mathbf{r}(s_0)]}{\partial \tau^2}, \quad (33)$$

where $\phi_0(\mathbf{r})$ and $\phi_1(\mathbf{r},\mathbf{r}')$ are the eikonals of incident field and Green's function represented in GO approximation

$$G(\mathbf{r},\mathbf{r}') = f_1(\mathbf{r},\mathbf{r}')\exp[ik\phi_1(\mathbf{r},\mathbf{r}')], \qquad (34)$$

and $\mathbf{r}(s)$ is the ray trajectory. The derivatives in the right-hand sides of (33) are calculated at the point $\mathbf{r}(s)$ of the ray and are actually the main curvature radii of the wave fronts described by the corresponding eikonals.

For the further treatment, it is more convenient to introduce local Fourier-conjugated variables κ_s, κ_n, κ_τ instead of s, n, τ and to represent the fluctuations in the spectral domain. Thus, for the second-order moment of the first-order complex phase, we have

$$\left\langle\psi_1^2(\mathbf{r},\omega)\right\rangle = -\frac{k^2}{4}\int_0^{s_0}\frac{\mathrm{d}s}{\varepsilon_0(s)} \cdot \iint \mathrm{d}\kappa_n\,\mathrm{d}\kappa_\tau\; B_\varepsilon(0,\kappa_n,\kappa_\tau) \cdot \exp\left\{-\frac{i}{k}\left[\kappa_n^2 D_n(s,s_0)+\kappa_\tau^2 D_\tau(s,s_0)\right]\right\} , \quad (35)$$

where $B_\varepsilon(0, \kappa_n, \kappa_\tau)$ is the transverse spectrum of the correlation function for the dielectric permittivity fluctuations and the integration over s is performed along the ray for the frequency ω which connects the transmitter and the observation point with the coordinate $\mathbf{r} = \mathbf{r}(s_0)$.

For the calculation of the two-frequency, two-point, two-time moment $\langle \psi_1(\mathbf{r}_1, \omega_1, t_1)\psi_1^*(\mathbf{r}_2, \omega_2, t_2)\rangle$, three different rays should be used, which correspond to three different frequencies, ω_1, ω_2, and $\Omega = (\omega_1 + \omega_2)/2$, and connect the transmitter point to the points $\mathbf{r}_1, \mathbf{r}_2$, and $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$:

$$
\begin{aligned}
b(\mathbf{R}, \boldsymbol{\rho}; \Omega, \delta; t) + iq(\mathbf{R}, \boldsymbol{\rho}; \Omega, \delta; t) = \\
\frac{k_1 k_2}{4} \int_0^{s_0} \frac{\mathrm{d}s}{[\varepsilon_0(s_1)\varepsilon_0(s_2)]^{\frac{1}{2}}} \\
\cdot \iint \mathrm{d}\kappa_n \, \mathrm{d}\kappa_\tau \; B_\varepsilon(0, \kappa_n, \kappa_\tau) \\
\cdot \exp\{i\kappa_n[\Delta_n(s) - v_n t] + i\kappa_\tau[\Delta_\tau(s) - v_\tau t]\} \\
\cdot \exp\left\{ \frac{i(k_1 - k_2)}{2k_1 k_2} \left[\kappa_n^2 D_n(s, s_0) + \kappa_\tau^2 D_\tau(s, s_0) \right] \right\}.
\end{aligned}
\tag{36}
$$

Here the integration over s is performed along the ray for the center frequency Ω and reaching the point $\mathbf{R}$; s_1 and s_2 are the points on the rays for ω_1 to $\mathbf{r}_1$ and for ω_2 to $\mathbf{r}_2$, which are the intersection points of these rays with the plane perpendicular to the ray for the center frequency Ω and passing through the point s of the latter; $\Delta_n(s)$ and $\Delta_\tau(s)$ are the projections of the vector from point s_1 to s_2 in the n and τ directions; v_n and v_τ are the same projections of the drift speed vector.

The average of the second-order term $\langle \psi_2(\mathbf{r}, \omega)\rangle$, which is also needed for further calculations, may be written as follows:

$$
\begin{aligned}
\langle \psi_2(\mathbf{r}, \omega)\rangle = -\frac{ik}{8} \int_0^{s_0} \frac{\mathrm{d}s}{\varepsilon_0^{\frac{1}{2}}(s)} \int_0^{s} \frac{\mathrm{d}s'}{\varepsilon_0(s')} \\
\cdot \iint \mathrm{d}\kappa_n \, \mathrm{d}\kappa_\tau \; B_\varepsilon(0, \kappa_n, \kappa_\tau)(\kappa_n^2 + \kappa_\tau^2) \\
\cdot \exp\left\{ -\frac{i}{k} \left[\kappa_n^2 D_n(s', s) + \kappa_\tau^2 D_\tau(s', s) \right] \right\} .
\end{aligned}
\tag{37}
$$

Equations (35)-(37) solve the problem of constructing the two-frequency, two-position, time coherence and correlation functions for an arbitrary profile of the background ionosphere and an arbitrary spectrum of fluctuations. To obtain concrete results, one should specify the models of ionosphere and fluctuations spectrum, calculate ray trajectories and corresponding functions D_n and D_τ, and evaluate integrals (35)-(37). Generally, this procedure can be carried out only numerically, and the numerical evaluation of manyfold integrals of oscillating functions requires special methods to be applied here.

In some cases, when the fluctuation spectrum has specific form, for example, if it does not depend on the wave vector direction (isotropic spectrum), it becomes possible to reduce the number of integrals in (35)-(37). Let us consider such an isotropic power law spectrum model:

$$
\begin{aligned}
B_\varepsilon(\boldsymbol{\kappa}) &= \frac{C_\varepsilon^2(s)}{(1 + \kappa^2/\kappa_0^2)^{\frac{p}{2}}} , \\
C_\varepsilon^2(s) &= \frac{\Gamma(p/2)[1 - \varepsilon_0(s)]^2}{\pi^{\frac{3}{2}}\Gamma[(p-3)/2]\kappa_0^3} \, \sigma_N^2 ,
\end{aligned}
\tag{38}
$$

where Γ is the traditional gamma function; σ_N^2 is the variance of the fractional electron density fluctuations, which are considered as a homogeneous zero-mean random field; $\kappa_0 = 2\pi/\ell_\varepsilon$, where ℓ_ε is the outer scale size; and p is the spectral index. Slow spatial variations of the spectral parameters κ_0, p, σ_N are accounted for in the model by their dependence on the variable s. For the isotropic spectrum (38) one integration in the spectral domain can be performed analytically, resulting in the following expressions [*Gherm et al.*, 1997a]:

$$
\begin{aligned}
\left\langle \psi_1^2(\mathbf{r}, \omega) \right\rangle = -\frac{\pi k^2}{2} \int_0^{s_0} \frac{\mathrm{d}s}{\varepsilon_0(s)} \int_0^{\infty} \kappa \, \mathrm{d}\kappa \; B_\varepsilon(\kappa) \\
\cdot \exp\left\{ \frac{i\kappa^2}{2k} \left[D_n(s, s_0) + D_\tau(s, s_0) \right] \right\} \\
\cdot \mathrm{J}_0\left\{ \frac{\kappa^2}{2k} [D_n(s, s_0) - D_\tau(s, s_0)] \right\} ,
\end{aligned}
\tag{39}
$$

$$\langle\psi_2(\mathbf{r},\omega)\rangle = -\frac{\pi k}{4}\int_0^{s_0}\frac{\mathrm{d}s'}{\varepsilon_0(s')}\int_0^\infty \kappa^3\,\mathrm{d}\kappa\, B_\varepsilon(\kappa)$$
$$\cdot\int_{s'}^{s_0}\frac{\mathrm{d}s}{\varepsilon_0^{\frac{1}{2}}(s)}\exp\left\{\frac{i\kappa^2}{2k}\left[D_n(s',s)+D_\tau(s',s)\right]\right\}$$
$$\cdot \mathrm{J}_0\left\{\frac{\kappa^2}{2k}[D_n(s',s)-D_\tau(s',s)]\right\}, \tag{40}$$

$$\langle\psi_1(\mathbf{r}_1,\omega_1,t_1)\psi_1^*(\mathbf{r}_2,\omega_2,t_2)\rangle =$$
$$\frac{\pi k_1 k_2}{2}\int_0^{S_0}\frac{\mathrm{d}s}{[\varepsilon_0(s_1)\varepsilon_0(s_2)]^{\frac{1}{2}}}\int_0^\infty \kappa\mathrm{d}\kappa B_\varepsilon(\kappa)$$
$$\cdot\exp\left\{\frac{i(k_1-k_2)}{4k_1k_2}\kappa^2\left[D_n(s,s_0)+D_\tau(s,s_0)\right]\right\}$$
$$\cdot\mathrm{J}_0[\kappa\Delta(s)]$$
$$\cdot\mathrm{J}_0\left\{\frac{(k_1-k_2)}{4k_1k_2}\kappa^2\left[D_n(s,s_0)-D_\tau(s,s_0)\right]\right\}, \tag{41}$$

where $\kappa^2 = \kappa_n^2 + \kappa_\tau^2$ and J_0 is the zero-order Bessel function; $\Delta(s) = \{[\Delta_n(s)-v_n t]^2+[\Delta_\tau(s)-v_\tau t]^2\}^{-1/2}$.

5. Results and Discussion

In this section the results of numerical investigations of the two-frequency, two-point, time coherence function of the field propagating through the fluctuating ionosphere are presented. The ionospheric scattering function is connected to the coherence function (or to the correlation function) by means of Fourier transformation in the appropriate variables. Introducing the notation S for the scattering function, we may write

$$S(\mathbf{R},\Omega;\mathbf{k},t_d,\omega_d) =$$
$$\frac{1}{(2\pi)^4}\int\!\!\int\!\!\int\!\!\int \Psi(\mathbf{R},\boldsymbol{\rho},\Omega,\delta,t)$$
$$\cdot\exp[-i\mathbf{k}\boldsymbol{\rho}-it_d\delta+i\omega_d t]\mathrm{d}^2\boldsymbol{\rho}\,\mathrm{d}\delta\,\mathrm{d}t, \tag{42}$$

where $\Psi(\mathbf{R},\boldsymbol{\rho},\Omega,\delta,t)$ is a correlation function of two components of a point source field having frequencies $\Omega+\delta/2$ and $\Omega-\delta/2$, measured at the points $\mathbf{R}+\boldsymbol{\rho}/2$ and $\mathbf{R}-\boldsymbol{\rho}/2$ with a time shift t.

Functions Ψ and S depend on many variables. Namely, correlation function Ψ depends on the point of observation $\mathbf{R}$, on central frequency Ω, and, besides, on the displacement $\boldsymbol{\rho}$, difference frequency δ, and time shift t. According to this, the scattering function S is a function of wave vector $\mathbf{k}$, time t_d, and frequency ω_d, which are Fourier-conjugated variables to $\boldsymbol{\rho}$, δ, and t. Therefore the scattering function may be interpreted as the energy density distribution function of a scattered field in the space of the wave vectors ($\mathbf{k}$), time delays (t_d), and Doppler frequency shifts (ω_d). In fact, this function describes the spread of signal energy in angle, time, and frequency.

The technique presented here makes it possible to evaluate the coherence and correlation functions, as well as the scattering function for arbitrary values of their arguments. We shall present our results for the correlation function as different projections of this function, for example, $\Psi(\mathbf{R},0,\Omega,\delta,t)$, where $\boldsymbol{\rho}$ is made equal zero. The scattering function corresponding to this projection may be represented as the integral of the general scattering function (42):

$$\tilde{S}(t_d,\omega_d) = \int\!\!\int \mathrm{d}^2\mathbf{k}\; S(\mathbf{R},\Omega,\mathbf{k},t_d,\omega_d), \tag{43}$$

which describes the distribution of the energy density of a scattered field in the time delay-Doppler frequency domain. In the same way, the projection $\Psi(\mathbf{R},\boldsymbol{\rho},\Omega,\delta,0)$ of the correlation function generates another corresponding scattering function,

$$\tilde{S}(t_d,\mathbf{k}) = \int \mathrm{d}\omega_d\; S(\mathbf{R},\Omega,\mathbf{k},t_d,\omega_d), \tag{44}$$

describing the energy density distribution in the time delay-scattering angle domain. The single-variable one-dimensional distributions of scattered energy, such as Doppler broadening, the distribution of angles of arrival, and distribution in time, may be obtained by means of additional integrations of the two latter expressions.

The calculations have been performed for the model background ionosphere typical for summer daytime and midlattitude conditions with E and F layers. The propagation distance has been 900 km, and the center frequency has been

8 MHz. We performed numerical evaluations for the outer scale of the fluctuations $\ell_\varepsilon = 3$ km, spectral parameter $p = 3.7$, and drift speed $v = 300$ m/s, directed orthogonal to the plane of propagation. The variance of the relative electron density fluctuations is $\sigma_N^2 = 5\times10^{-6}$, which corresponds to the conditions of the quiet mid-latitude ionosphere.

We start with the case $\rho = 0$. The three-dimensional plots of the real and imaginary parts of the correlation function Ψ are represented in Figures 1a and 1b. As can be seen from Figure 1, the real part Ψ as a function of variables t and δ has its maximum at $t = \delta = 0$ and tends to zero as t and δ increase. The plot is symmetrical with respect to the inversion $t \to -t$

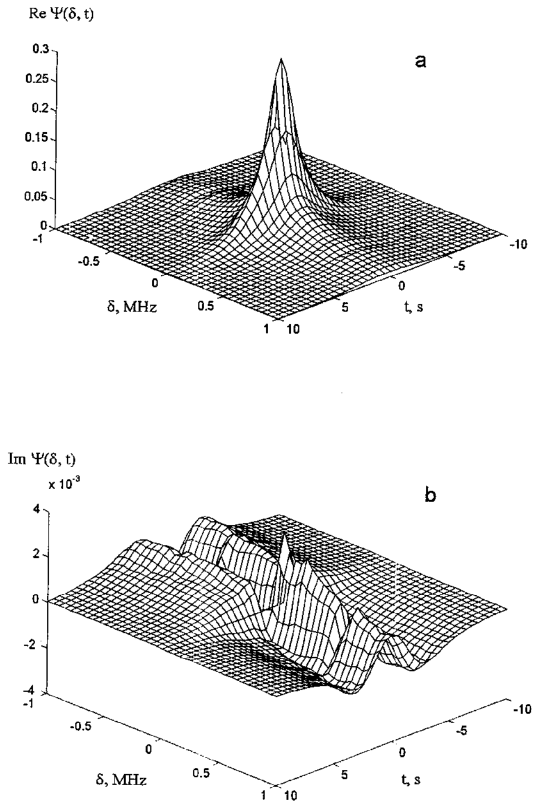

Figure 1. (a) The real part of the correlation function $\mathrm{Re}\Psi(\delta, t)$ in the domain of difference frequency and time, (b) The imaginary part of the correlation function $\mathrm{Im}\Psi(\delta, t)$ in the domain of difference frequency and time.

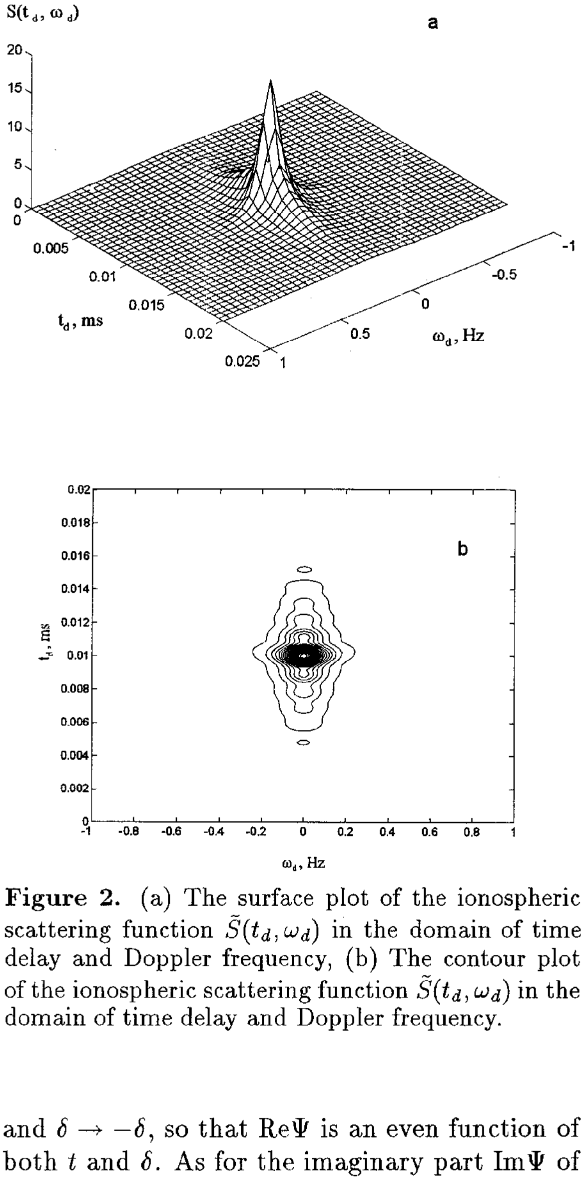

Figure 2. (a) The surface plot of the ionospheric scattering function $\tilde{S}(t_d, \omega_d)$ in the domain of time delay and Doppler frequency, (b) The contour plot of the ionospheric scattering function $\tilde{S}(t_d, \omega_d)$ in the domain of time delay and Doppler frequency.

and $\delta \to -\delta$, so that $\mathrm{Re}\Psi$ is an even function of both t and δ. As for the imaginary part $\mathrm{Im}\Psi$ of the correlation function, it is an even function of time t, but it is an odd function of frequency δ with $\mathrm{Im}\Psi(0, t) = 0$. The values of $\mathrm{Im}\Psi(\delta, t)$ are of the order of 10^{-3}, while $\mathrm{Re}\Psi(\delta, t)$ reaches 0.3 at its maximum. In spite of such a big difference between magnitudes of the real and imaginary parts of the correlation function, it would not be right to neglect its imaginary part when investigating the scattering function. As will be clear from the following treatment, the presence of the imaginary parts results in the appearance of specific parabolic-shaped structures on the plot of the scattering function $\tilde{S}(t_d, \omega_d)$.

The calculated scattering function is represented in Figure 2a as the three-dimensional (3-D) surface plot and in Figure 2b as the contour plot. The contour levels in Figure 2b are spaced linearly between the maximum value 18.3 and the minimum 0. The difference of the levels on the neighbouring contours is 0.9. One can note some asymmetry in the contour level map, which appears as a superposition of the symmetrical main shape and the parabolic-shaped structure, reflecting the well-known coupling between the time delay and Doppler frequency shift of the scattered signal; the time delay is proportional to the second power of the frequency shift. The width of the scattered energy distribution is about 20 μs in delays and about 0.4 Hz in Doppler shifts. This coupling has been observed experimentally [*Basler et al.*, 1988; *Cannon et al.*, 1995] and has been calculated by means of parabolic equation solved for the model situation by *Nickisch* [1992]. As can be seen from Figure 2, the coupling is not expressed significantly. This result is not in good agreement with that obtained by *Nickisch*, [1992], who had the well-pronounced parabolic shapes of the scattering function for the case of the frozen drift.

According to *Nickisch* [1992], this parabolic delay-Doppler coupling is unique to the case of frozen-in plasma drift; other delay-Doppler couplings are generated by nonuniform plasma motions (with completely turbulent motion producing absolutely no delay-Doppler coupling). Our calculations show that even in the scope of the homogeneous motion (frozen drift), the parabolic shape may be not represented significantly in the conditions of real propagation in the medium with the inhomogeneous background. The explanation of the fact lies in the following: In our case of an oblique propagation in the inhomogeneous ionosphere, every frequency component propagates along its own path, and the correlation falls off rapidly with the frequency separation. The correlation function (and hence the scattering function) of the real fluctuating ionosphere is formed by the contributions from each point of the ray trajectory. The parameters of the parabolic structures depend on the values of parameters D_n, D_τ and v_n, v_τ varying along the ray. The resulting effect is a superposition of many different shapes, so that the scattering function has a strongly diffused shape and the coupling is not pronounced significantly.

Formally, the weakness of the coupling is stipulated by the very small imaginary part of the correlation function compared with its real part. Indeed, if we put $\mathrm{Im}\Psi = 0$ and make the Fourier transformation of the $\mathrm{Re}\Psi$ only, we apparently get a pure symmetrical scattering function without any specific structures due to the symmetry of $\mathrm{Re}\Psi$. On the contrary, the odd-in-frequency imaginary part of Ψ gives after Fourier transformation the odd-in-time Fourier transform. The proper scattering function can be obtained only considering both real and imaginary parts of the correlation function, and the odd imaginary part does stipulate the asymmetry of the scattering function. As we already mentioned here, the correlation function obtained by *Fridman et al.* [1995] has no imaginary part, and hence the corresponding scattering function must be of purely symmetric shape. It is of importance to note here, that in the geometrical optics approximation the coherence and correlation functions are pure real functions, therefore the presence of the nonzero imaginary part is exclusively due to the diffractional effects.

To understand it better, we have performed some calculations for the different conditions of propagation which provide a higher contribution of diffraction in a full field. This is, for instance, the case of a longer distance of propagation. We have performed calculations for the distance of 2000 km. The scattering function calculated for this case is represented in Figure 3, where the coupling is much more pronounced than in Figure 2.

Finally, we have tried another numerical experiment to find out the influence of the spatial separation of the rays corresponding to different frequencies. Namely, we calculated the scat-

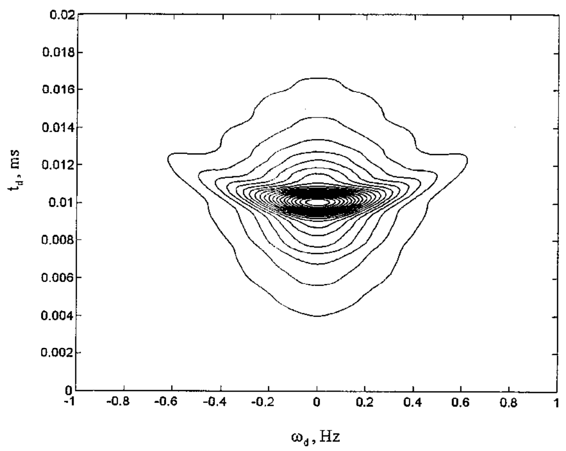

Figure 3. The contour plot of the ionospheric scattering function $\tilde{S}(t_d, \omega_d)$ in the domain of time delay and Doppler frequency for the path 2000 km.

tering function for the same propagation conditions as for the results represented in Figures 1 and 2 but neglecting the ray separation by artificially forcing the parameters Δ_n and Δ_τ of equation (36) to be equal to zero. The result is represented in Figure 4, where one can see a well-resolved parabolic structure. This result explains the clear parabolic shapes obtained by *Nickisch* [1992], where the components having different carrier frequencies propagate along the same straight path. Our conclusion then is that even in a scope of a homogeneous drift a large variety of shapes of the scattering function may be produced by different concrete conditions of real propagation.

We have also investigated the scattering function in the domain of time delay-angle of arrival. This is the case of $t = 0$ (the absence of the time shift). The real and imaginary parts of another projection of the correlation function $\Psi(\mathbf{R}, \boldsymbol{\rho}, \Omega, \delta, 0)$ are represented in Figures 5a and 5b; the spacing vector $\boldsymbol{\rho}$ is orthogonal to the plane of propagation. The contour map of the respective scattering function surface $\tilde{S}(t_d, k_y)$

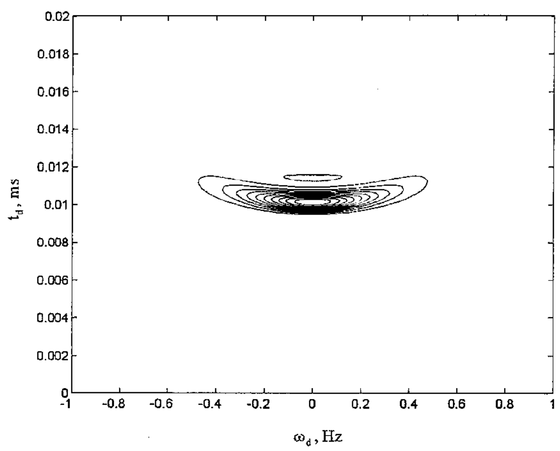

Figure 4. The contour plot of the ionospheric scattering function $\tilde{S}(t_d, \omega_d)$ in the domain of time delay and Doppler frequency for the artificial situation when the spatial separation of the rays due to the frequency separation is neglected.

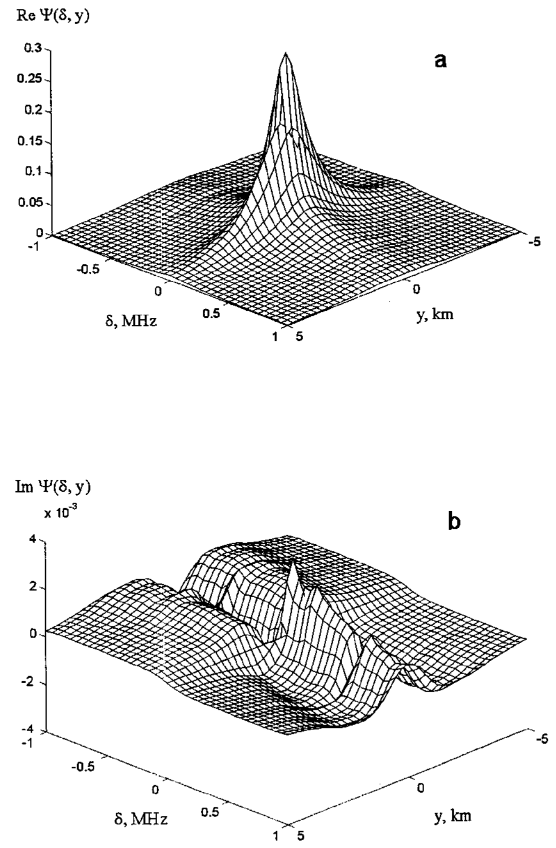

Figure 5. (a) The real part of the correlation function $\mathrm{Re}\Psi(\delta, y)$ in the domain of difference frequency and transversal spatial separation, (b) The imaginary part of the correlation function $\mathrm{Im}\Psi(\delta, y)$ in the domain of difference frequency and transversal spatial separation.

is plotted in Figure 6, where the similar weak parabolic-shaped structure can be observed. In this case, the presence of such structure reflects the coupling between time delay and scattering wave vector, or angle of arrival of the scattered signal. The angle of arrival β is connected with the scattering wave vector through the relation $\beta = k_y/k$. The width of the scattered energy distribution is about 20 μs in delays (as in the previous case) and about $4 \cdot 10^{-3} \simeq 0.23^0$ in angles of arrival. All the ideas pointed out for the previous case may be addressed also for this time delay-angle of arrival scattering function, so we shall not discuss the results in detail.

6. Conclusion

We have performed an investigation of the two-frequency, two-position, time coherence function and the ionospheric scattering function, which are the most general fundamental quantities describing the HF ionospheric fluctuating radio channel. The quantitative description has been based upon numerical calculations in the scope of the complex phase method for realistic models of the fluctuating ionosphere.

The numerical results obtained lead to the conclusion that in the general case the large variability of shapes of the scattering function of the fluctuating ionosphere exists depending on the concrete conditions of propagation. In particular, the well-known delay-Doppler coupling can be more or less pronounced in different propagation conditions. There are two competitive effects contributing to the final shape of the scattering function. These are diffraction, which results in highly pronounced parabolic shapes, and spatial separation of the rays due to the ionospheric dispersion, which leads to the spreading out of the parabolic structures. We have shown that the presence of the coupling is exclusively due to the nonzero imaginary part of the correlation function of the scattered field, which means that this effect has a purely diffractional nature and cannot be obtained in the geometrical optics approximation.

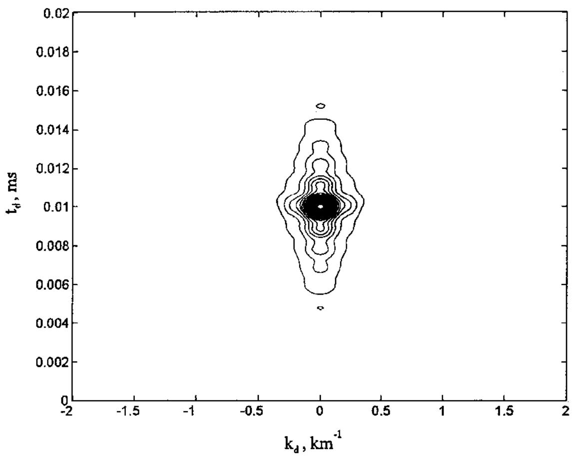

Figure 6. The contour plot of the ionospheric scattering function $\tilde{S}(t_d, k_d)$ in the domain of time delay and wave vector.

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On the solution of Markov's parabolic equation for the second-order spaced frequency and position coherence function

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[1] An analytic technique has been developed to construct the asymptotic representation of the spaced position and frequency coherence function in Markov's diffusive approximation. The technique employs the formalism of the quasi-classic complex paths in an extended complex-valued coordinate space. It allows the construction of the coherency for arbitrary realistic models of the structure function of the fluctuations of the refractive index of the medium of propagation. The technique has been employed to obtain explicit analytic asymptotic solutions for some realistic models of the structure function. For the quadratic structure function the method produces the known rigorous solution in an automatic fashion. *INDEX TERMS:* 0659 Electromagnetics: Random media and rough surfaces; 0669 Electromagnetics: Scattering and diffraction; 0689 Electromagnetics: Wave propagation (4275); *KEYWORDS:* Markov's approximation, coherence function, complex trajectories

1. Introduction

[2] The technique of Markov's parabolic equations for the moments of a stochastic field is one of the classical approaches in the problem of wave propagation in random media [*Rytov et al.*, 1978; *Ishimaru*, 1978]. Despite many years of exploitation of the method, there still exist unsolved problems in the scope of Markov's approximation. In particular, a general analytic solution has not yet been constructed for the fourth moment, and some questions arise with respect to the two-frequency, two-position coherence function.

[3] As far as the second-order coherence function is concerned, the exhaustive solution has been constructed for the single-frequency (pure spatial) coherence function [*Rytov et al.*, 1978]. The spaced position and frequency coherence function was studied numerically by *Lin and Yeh* [1975], who investigated this function in a fading ionospheric communication channel numerically for power-law and Gaussian spectra of ionospheric fluctuations. For the two-frequency coherence function the closed form exact analytic solution was only constructed in the case of a plane wave propagating in a random medium with the quadratic structure function of fluctuations [*Sreenivasiah et al.*, 1976]. *Knepp* [1983] generalized this solution to the case of spherical wave propagation. The technique of separation of variables was employed by *Oz and Heyman* [1996, 1997a, 1997b, 1997c] to construct the second-order coherence function for an arbitrary structure function of the fluctuations of the properties, but in a medium with a homogeneous background. It implies the expansion of the solution into the series of the transversal eigenfunctions of the problem, and, in this way, requires additional quantification of the number of terms needed to achieve the necessary accuracy for a given distance of propagation and spaced frequency. The series fails to converge for some types of initial conditions (e.g., an incident plane wave) as the distance and difference frequency tend to zero. Additional constraints in the separation variables technique may also arise when considering fluctuations with the structure function tending to a constant as the difference variable tends to infinity. In this case the continuous spectrum may likely occur in the spectrum of the transversal operator of the problem, which makes expansion of the solution in terms of the transversal eigenfunctions much more complicated.

[4] In the present paper the asymptotic technique is developed to construct the solution to Markov's parabolic equation for the two-frequency, two-position coherence function in the case of an arbitrary structure function of the refractive index of fluctuations and an inhomogeneous stratified background medium. The technique employs quasi-classic representation in terms of complex trajectories. It has no constraints pertinent to the initial conditions in the form of an incident plane wave,

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and is also valid beginning with the zero distance from a boundary surface and zero spaced frequency. In the case of the quadratic structure function it produces the known exact solution of the problem [*Sreenivasiah et al.*, 1976] in automatic fashion.

2. Statement of the Problem

[5] We consider the background medium, stratified in z direction, with local random inhomogeneities embedded, which is characterized by the dielectric permittivity of the form

$$\varepsilon(z,t,\omega)=\varepsilon_0(z,\omega)[1+\varepsilon_1(\mathbf{r},t,\omega)]. \tag{1}$$

Here $\varepsilon_0(z,\omega)$ is the dielectric permittivity of the background medium, and $\varepsilon_1(\mathbf{r},\omega,t)$ represents relative space and time varying fluctuations of the permittivity. Variable t indicates time dependence of fluctuations in quasi-stationary approximation.

[6] We consider an incident plane wave propagating in the positive z direction. The field random realization is searched for in the following form:

$$E(z,\rho,t,\omega)=\frac{\varepsilon_0^{1/4}(0,\omega)}{\varepsilon_0^{1/4}(z,\omega)}U(z,\rho,t,\omega)\cdot\exp\left[-i\omega t+ik\int\varepsilon_o^{1/2}(z',\omega)dz'\right], \tag{2}$$

provided that $\varepsilon_0(z,\omega)$ is finite and not equal zero at any z, ω. In equation (2) $k=\omega/c$ is the vacuum wave number corresponding to the circle frequency ω, and $U(z,\rho,t,\omega)$ is the random complex amplitude of the field. The two-frequency, two-position, two-time coherence function of the field is defined as

$$\Gamma(z,\rho_1,\rho_2,t_1,t_2,\omega_1,\omega_2)=\langle U(z,\rho_1,t_1,\omega_1)U^*(z,\rho_2,t_2,\omega_2)\rangle. \tag{3}$$

Employing a standard averaging procedure of Markov's technique for a medium described by equation (1) results in the following parabolic equation for the coherence function Γ:

$$\begin{aligned}&\frac{\partial\Gamma}{\partial z}-\frac{i}{2}\left(\frac{1}{k_1\sqrt{\varepsilon_0(z,\omega_1)}}\nabla_1^2-\frac{1}{k_2\sqrt{\varepsilon_0(z,\omega_2)}}\nabla_2^2\right)\Gamma\\&+\frac{1}{8}k_1^2\varepsilon_0(z,\omega_1)A_{11}(z,\mathbf{0},0)\Gamma+\frac{1}{8}k_2^2\varepsilon_0(z,\omega_2)A_{22}(z,\mathbf{0},0)\Gamma\\&-\frac{1}{4}k_1k_2\sqrt{\varepsilon_0(z,\omega_1)\varepsilon_0(z,\omega_2)}A_{12}(z,\boldsymbol{\rho}_1-\boldsymbol{\rho}_2,t_1-t_2)\Gamma=0,\end{aligned} \tag{4}$$

Here k_1 and k_2 are the vacuum wave numbers corresponding to the frequencies ω_1 and ω_2 in the background medium, and ∇_1^2 and ∇_2^2 are the Laplacians with respect to coordinates ρ_1 and ρ_2. Equation (4) was derived in the approximation of the fluctuations delta-correlated in the direction of propagation. This means that functions A_{mn} are determined through the following relationship for the correlation functions of the relative dielectric permittivity:

$$\begin{aligned}&B_{mn}(z,z',\boldsymbol{\rho}_1-\boldsymbol{\rho}_2,t_1-t_2)\\&=\langle\varepsilon_1(z+z'/2,\rho_1,t_1,\omega_m)\varepsilon_1(z-z'/2,\boldsymbol{\rho}_2,t_2,\omega_n)\rangle\\&=\delta(z')A_{mn}(z,\boldsymbol{\rho}_1-\boldsymbol{\rho}_2,t_1-t_2),\end{aligned} \tag{5}$$

m, n = 1,2, so that

$$A_{mn}(z,\boldsymbol{\rho}_1-\boldsymbol{\rho}_2,t_1-t_2)=\int B_{mn}(z,z',\boldsymbol{\rho}_1-\boldsymbol{\rho}_2,t_1-t_2)dz'. \tag{6}$$

Here $z=(z_1+z_2)/2$, $z'=z_1-z_2$ are longitudinal central and difference variables.

[7] Introducing also central and difference transversal variables $\mathbf{R}=(\boldsymbol{\rho}_1+\boldsymbol{\rho}_2)/2$, $\rho=\boldsymbol{\rho}_1-\boldsymbol{\rho}_2$, as well as difference time $t=t_1-t_2$ in the assumption of the stationarity of fluctuations in time, and substituting

$$\begin{aligned}&\Gamma(z,\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,t_1,t_2,\omega_1,\omega_2)=\Gamma_1(z,\mathbf{R},\rho,t,\omega_1,\omega_2)\\&\cdot\exp\left\{-\frac{1}{8}\int_0^z[k_1^2\varepsilon_0(z',\omega_1)A_{11}(z',\mathbf{0},0)+k_2^2\varepsilon_0(z',\omega_2)A_{22}(z',\mathbf{0},0)]dz'\right\}\\&\cdot\exp\left\{\frac{1}{4}\int_0^z\left[k_1k_2\sqrt{\varepsilon_0(z',\omega_1)\varepsilon_0(z',\omega_2)}\cdot A_{12}(z',\mathbf{0},0)\right]dz'\right\}\end{aligned} \tag{7}$$

results in the following equation for function Γ_1

$$\begin{aligned}&\frac{\partial\Gamma_1}{\partial z}+\frac{i}{2k_1k_2\sqrt{\varepsilon_0(z,\omega_1)\varepsilon_0(z,\omega_2)}}\cdot\left[k_d\nabla_d^2+\frac{1}{4}k_d\nabla_s^2-2k_s\nabla_s\nabla_d\right]\Gamma_1\\&+\frac{k_1k_2\sqrt{\varepsilon_0(z,\omega_1)\varepsilon_0(z,\omega_2)}}{4}\cdot[A_{12}(z,\mathbf{0},0)-A_{12}(z,\rho,t)]\Gamma_1=0\end{aligned} \tag{8}$$

with

$$k_d = k_1\sqrt{\varepsilon_0(z,\omega_1)} - k_2\sqrt{\varepsilon_0(z,\omega_2)},$$
$$k_s = \left(k_1\sqrt{\varepsilon_0(z,\omega_1)} + k_2\sqrt{\varepsilon_0(z,\omega_2)}\right)/2.$$

∇_s and ∇_d are the operators of gradient with respect to the sum and difference coordinates $\mathbf{R}$ and ρ.

[8] Until now, the assumption of the statistical homogeneity of the fluctuations of the dielectric permittivity in space was not implied. When introducing fluctuations of the dielectric permittivity in the form of a product in the second item in equation (1) (e.g., as given by *Sreenivasiah et al.* [1976]), it is convenient to consider the relative fluctuations of the dielectric permittivity ε_1 as a stationary and homogeneous zero mean random function. In the case where the background medium is homogeneous and nondispersive and the relative fluctuations are statistically homogeneous, equations (4) and (8) become the same as those treated by *Sreenivasiah et al.* [1976] and *Oz and Heyman* [1996, 1997a, 1997b, 1997c]. Alternatively, for the cold ionospheric plasma the statistically homogeneous relative fluctuations of the electron density are introduced (e.g., as given by *Lin and Yeh* [1975] or *Knepp* [1983]) instead of homogeneous fluctuations of the relative dielectric permittivity. In this case, equations (4) and (8) lead to those considered by *Knepp* [1983]. We shall follow our formulation of the problem (equations (2), (4), (7), (8)).

[9] To complete the statement of the problem, equation (8) should be complemented by the boundary condition at $z = 0$

$$\Gamma_1(0,\mathbf{R},\rho,t,\omega_1,\omega_2) = \Gamma_0(\mathbf{R},\rho,\omega_1,\omega_2), \qquad (9)$$

which is determined by the incident field.

3. Solution to the Plane Wave

[10] Here we confine the consideration by the case where the incident field is a plane wave propagating in the z direction. In this case, $\Gamma_0 = 1$, and equation (8) can be simplified. Indeed, the coefficients and boundary conditions do not depend on the central variable $\mathbf{R}$; therefore the solution Γ_1 should not depend on $\mathbf{R}$, and a simplified equation can be considered as follows:

$$\frac{\partial\Gamma_1}{\partial z} + \frac{ik_d}{2k_1k_2\sqrt{\varepsilon_0(z,\omega_1)\varepsilon_0(z,\omega_2)}}\nabla_d^2\Gamma_1 + \frac{k_1k_2\sqrt{\varepsilon_0(z,\omega_1)\varepsilon_0(z,\omega_2)}}{4} \cdot [A_{12}(z,\mathbf{0},0) - A_{12}(z,\rho,t)]\Gamma_1 = 0 \qquad (10)$$

When considering this simplified equation, the technique of solving based on quasi-classic representation in terms of complex trajectories can be most transparently outlined. The extension of the technique to the general case of a full equation (8) and the incident field not necessarily propagating along the z axis is the next step that will be considered separately.

[11] To construct the asymptotic solution to equation (10), first, the dimensionless variables should be introduced. Let us denote as l_ε the scale of the random inhomogeneities in the z direction and substitute $z = \zeta l_\varepsilon$ and $\rho = \mathbf{r}l_\varepsilon$. In new dimensionless variables (ζ, $\mathbf{r}$) the last equation can be rewritten as follows:

$$K\frac{\partial\Gamma_1}{\partial\zeta} + \frac{i\tilde{k}_d}{2\sqrt{\varepsilon_0(\zeta,\omega_1)\varepsilon_0(\zeta,\omega_2)}}\nabla_{\mathbf{r}}^2\Gamma_1 + \frac{K^2\sqrt{\varepsilon_0(\zeta,\omega_1)\varepsilon_0(\zeta,\omega_2)}}{4} \cdot \left[\tilde{A}_{12}(\zeta,\mathbf{0},0) - \tilde{A}_{12}(\zeta,\mathbf{r},t)\right]\Gamma_1 = 0, \qquad (11)$$

where $\tilde{A}_{mn}(\zeta,\mathbf{r},t) = l_\varepsilon^{-1}A_{mn}(z,\mathbf{r},t)$ and in the equation (6) for A_{mn} transfer to the dimensionless variables was performed, $\tilde{k}_d = k_d l_\varepsilon$. $\nabla_{\mathbf{r}}^2$ is the transversal Laplacian written in the dimensionless difference variables, and $K = k_1k_2l_\varepsilon^2$ is the dimensionless parameter, which is assumed to be the large parameter of the problem. Physically, this means that random inhomogeneities of a medium are of large spatial scale in terms of vacuum wavelengths for both frequencies ω_1 and ω_2. At the same time this is one of the limitations of Markov's diffusive approximation.

[12] Formally, provided $K \to \infty$, the solution of equation (11) is sought for in the form of the following asymptotic series:

$$\Gamma_1(\mathbf{r},\zeta,t) = \exp\left[K\psi(\mathbf{r},\zeta,t)\right] \cdot \sum_{j=0}^{\infty}\frac{U_j(\mathbf{r},\zeta,t)}{K^j}. \qquad (12)$$

The dependencies of the functions newly introduced here, ψ and U_n, on frequencies ω_1, ω_2 are not indicated explicitly in representation (12).

[13] Series (12) is almost the traditional Debye series for constructing the high-frequency asymptotic. The distinction is that the exponent function does not obey the imaginary unity i in its power and that the expansion is carried out into inverse powers of the real parameter K rather than in powers of (iK). The reason to do so is that in the case of a single frequency the equation (11) for the pure space coherency, evidently, has the solution in the form of a real exponential function (it will be shown that in this case $U_0 = 1$, $U_j = 0$, $j > 0$). Additionally, in the general case of a spaced frequency we shall be dealing with complex "eikonals" ψ, so that it is no matter whether or not it is introduced with i in the exponent's power in (12).

[14] The standard asymptotic procedure of substituting (12) into equation (11) results in the following "eikonal" equation for ψ

$$\frac{\partial\psi}{\partial\zeta}+\frac{i\tilde{k}_d}{2\sqrt{\varepsilon_0(\zeta,\omega_1)\varepsilon_0(\zeta,\omega_2)}}(\nabla_{\mathbf{r}}\psi)^2+\frac{\sqrt{\varepsilon_0(\zeta,\omega_1)\varepsilon_0(\zeta,\omega_2)}}{4}\cdot\left[\tilde{A}_{12}(\zeta,\mathbf{0},0)-\tilde{A}_{12}(\zeta,\mathbf{r},t)\right]=0, \quad (13)$$

and transport equations for amplitudes U_j

$$\frac{\partial U_0}{\partial\zeta}+\frac{i\tilde{k}_d}{\sqrt{\varepsilon_0(\zeta,\omega_1)\varepsilon_0(\zeta,\omega_2)}}(\nabla_{\mathbf{r}}\psi\cdot\nabla_{\mathbf{r}}U_0)+\frac{i\tilde{k}_d}{2\sqrt{\varepsilon_0(\zeta,\omega_1)\varepsilon_0(\zeta,\omega_2)}}U_0\nabla^2_{\mathbf{r}}\psi=0, \quad (14)$$

$$\frac{\partial U_j}{\partial\zeta}+\frac{i\tilde{k}_d}{\sqrt{\varepsilon_0(\zeta,\omega_1)\varepsilon_0(\zeta,\omega_2)}}(\nabla_{\mathbf{r}}\psi\cdot\nabla_{\mathbf{r}}U_j)+\frac{i\tilde{k}_d}{2\sqrt{\varepsilon_0(\zeta,\omega_1)\varepsilon_0(\zeta,\omega_2)}}U_j\nabla^2_{\mathbf{r}}\psi=-\frac{i\tilde{k}_d}{2\sqrt{\varepsilon_0(\zeta,\omega_1)\varepsilon_0(\zeta,\omega_2)}}\nabla^2_{\mathbf{r}}U_{j-1},\quad j>0. \quad (15)$$

[15] To solve equations (13), (14), and (15), the general method of characteristics can be employed. Equation (13) is a Hamilton-Jacobi type equation, so that the appropriate Hamilton equations may be written in the following form:

$$\frac{d\zeta}{d\tau}=1, \quad (16)$$

$$\frac{d\mathbf{r}}{d\tau}=\frac{i\tilde{k}_d}{\sqrt{\varepsilon_0(\zeta,\omega_1)\varepsilon_0(\zeta,\omega_2)}}\mathbf{p}, \quad (17)$$

$$\frac{dp_\zeta}{d\tau}=-\frac{1}{4}\frac{\partial}{\partial\zeta}\left(\sqrt{\varepsilon_0(\zeta,\omega_1)\varepsilon_0(\zeta,\omega_2)}\cdot\left[\tilde{A}_{12}(\zeta,\mathbf{0},0)-\tilde{A}_{12}(\zeta,\mathbf{r},t)\right]\right), \quad (18)$$

$$\frac{d\mathbf{r}}{d\tau}=-\frac{1}{4}\nabla_{\mathbf{r}}\left(\sqrt{\varepsilon_0(\zeta,\omega_1)\varepsilon_0(\zeta,\omega_2)}\cdot\left[\tilde{A}_{12}(\zeta,\mathbf{0},0)-\tilde{A}_{12}(\zeta,\mathbf{r},t)\right]\right). \quad (19)$$

In this set, three-dimensional vectors (ζ, $\mathbf{r}$) and (p_ζ, $\mathbf{p}$) are coordinates and moments, respectively, along the complex trajectory, which is parameterized by variable τ. Due to equation (16) it may be accepted $\tau = \zeta$, where ζ is a real variable.

[16] Equations (16)–(19) determine complex trajectories $\mathbf{r} = \mathbf{r}(\zeta)$, $p_\zeta = p_\zeta(\zeta)$, $\mathbf{p} = \mathbf{p}(\zeta)$, which arrive at real points of observation (ζ, $\mathbf{r}$) and are subject to the initial conditions (from $\Gamma_0 = 1$ on the initial surface $\zeta = 0$):

$$\mathbf{p}(0)=\mathbf{0}, \quad (20)$$

$$p_\zeta(0)=\frac{1}{4}\sqrt{\varepsilon_0(0,\omega_1)\varepsilon_0(0,\omega_2)}\cdot\left[\tilde{A}_{12}(0,\mathbf{r}(0),t)-\tilde{A}_{12}(0,\mathbf{0},0)\right] \quad (21)$$

Value $\mathbf{r}(0) = \mathbf{r}_0$ is a complex coordinate of the point in the initial plane $\zeta = 0$ that should be determined to provide a complex trajectory to arrive at the real point of observation (ζ, $\mathbf{r}$).

[17] Once equations (16)–(19) with proper initial conditions (20) and (21) have been solved and complex trajectories have been determined, the function ψ in representation (12) can then be calculated as the integral

$$\psi(\zeta,\mathbf{r})=\int_0^\zeta\left(\frac{i\tilde{k}_d\mathbf{p}^2(\zeta)}{\sqrt{\varepsilon_0(\zeta,\omega_1)\varepsilon_0(\zeta,\omega_2)}}+p_\zeta(\zeta)\right)d\zeta. \quad (22)$$

As for solving the transport equations (14) and (15), the solution of the main equation is given by

$$U_0(\zeta,r)=\exp\left[-\frac{i}{2}\int_0^\zeta\frac{\tilde{k}_d\nabla^2_{\mathbf{r}}\psi d\zeta}{\sqrt{\varepsilon_0(\zeta,\omega_1)\varepsilon_0(\zeta,\omega_2)}}\right]=\left[\frac{J(0)}{J(\zeta)}\right]^{1/2}, \quad (23)$$

where Jacobian $J(\zeta)$ is calculated as follows:

$$J(\zeta)=\frac{\partial(\zeta,x_1,x_2)}{\partial(\zeta,x_{01},x_{02})}=\frac{\partial(x_1,x_2)}{\partial(x_{01},x_{02})} \quad (24)$$

with $\mathbf{r} = (x_1, x_2)$ and $\mathbf{r}_0 = (x_{01}, x_{02})$. Coordinates $\mathbf{r}_0$ are two-dimensional orthogonal complex trajectories transversal to the ζ axis; $J(0) = 1$. To calculate the determinant in equation (24), quantities $\alpha_{lk} = \partial x_l/\partial x_{0k}$ should be calculated directly after the ray equations have been solved, or additional differential equations for α_{lk} may be obtained. In particular, in the case of the homogeneous background medium, the following equations may be derived for α_{lk} differentiating (17) in variables $\mathbf{r}_0 = (x_{01}, x_{02})$ and making use of (19):

$$\frac{\partial^2\alpha_{lk}}{\partial\zeta^2}+\frac{i\tilde{k}_d}{4}\sum_q\alpha_{qk}\frac{\partial^2}{\partial x_l\partial x_q}\cdot\left[\tilde{A}_{12}(\zeta,\mathbf{0},0)-\tilde{A}_{12}(\zeta,\mathbf{r},t)\right]=0,\quad l,k,q=1,2. \quad (25)$$

In this way, the general scheme of constructing the asymptotic solution of Markov's parabolic equation (11) in terms of the representation (12) is completed. It is clear that this scheme in its most general form, when the background medium is inhomogeneous and dispersive, can only be realized numerically. When doing this, some general problems arise. In particular, when solving ray equations (17)–(19), the homing problem to construct the complex trajectories will be the crucial point. Additionally, the problem of the analytic continuation of the correlation function of fluctuations into the complex domain of its argument is also a nontrivial one. It should be considered independently for each accepted model of the correlation function of fluctuations.

[18] To demonstrate more transparently how the developed technique works, below the problem is discussed under the simplifying assumption of a homogeneous background medium characterized by $\varepsilon_0 = 1$. This is just the problem treated by *Sreenivasiah et al.* [1976] and *Oz and Heyman* [1996, 1997a, 1997b]. Here we can proceed further analytically.

4. Plane Wave Propagating in a Homogeneous Background Medium With Fluctuations

[19] In this case, $\widetilde{A}$ does not depend on ζ and, according to equations (13) and (18), $p_\zeta(\zeta) = p_\zeta(0) = (1/4)[\widetilde{A}(\mathbf{r}_0, t) - \widetilde{A}(0,0)]$ ($\widetilde{A}_{mn} = \widetilde{A}$, if a medium is nondispersive). Provided, additionally, that the fluctuations are isotropic in the transversal planes $\left(\widetilde{A}(\mathbf{r}, t) = \widetilde{A}(r, t)\right)$ and cylindrical coordinates are introduced to express vector $\mathbf{p}$, the explicit relationship for a complex trajectory can be written (ray or Hamilton equations can be solved analytically) as follows:

$$\sqrt{\frac{i\widetilde{k}_d}{2}}\zeta = \int_{r_0}^{r} \frac{dr}{\sqrt{\widetilde{A}(r,t) - \widetilde{A}(r_0,t)}}. \tag{26}$$

At the same time, equation (26) is the transcendent equation, which allows us to find the initial complex point $r_0 = r_0(\zeta, r)$, where the complex trajectory comes out at $\zeta = 0$, to arrive at the real point of observation (ζ, r).

[20] Once r_0 was found from equation (26), function ψ in the asymptotic representation (12) is expressed through

$$\psi(\zeta, r) = \frac{1}{2}\int_0^{\zeta} \left(\widetilde{A}(r(\zeta), t) - \widetilde{A}(r_0, t)\right) d\zeta + \frac{1}{4}\int_0^{\zeta} \left(\widetilde{A}(r_0, t) - \widetilde{A}(0,0)\right) d\zeta. \tag{27}$$

When calculating the main amplitude U_o according to equations (23)–(25) after introducing cylindrical coordinates for isotropic fluctuations, equation (25) is reduced to a simpler form

$$J(\zeta) = \frac{r}{r_0}\alpha, \tag{28}$$

with $\alpha = \partial r/\partial r_0$. Then instead of a general set of equations (25), it is now one equation for the divergence α, as follows:

$$\frac{\partial^2\alpha}{\partial\zeta^2} + \frac{i\widetilde{k}_d\alpha}{4}\frac{\partial^2}{\partial r^2}\left[\widetilde{A}(0,0) - \widetilde{A}(r(\zeta), t)\right] = 0. \tag{29}$$

This way, the general scheme of constructing the asymptotic solution to the second-order Markov's equation for the space-frequency coherency has now, in the conditions of a homogeneous background medium with isotropic fluctuations, been reduced to a very simple procedure of considering equations (26)–(29). This allows the construction of the asymptotic solution for an arbitrary given correlation (or structure) function of fluctuations.

[21] Some results may even be obtained without specifying a model of the structure function of fluctuations. In particular, for the case of pure frequency coherency (no spaced position and time, $r = 0$, $t = 0$) equation (26) yields $r = r_0$; therefore $\psi = 0$, as follows from equation (27). Finally, equation (29) for the divergence $\alpha = \alpha_\omega$ is simplified to the form

$$\frac{\partial^2\alpha_\omega}{\partial\zeta^2} - \frac{i\widetilde{k}_d\alpha_\omega}{4}\frac{\partial^2}{\partial r^2}\left[\widetilde{A}(0,0)\right] = 0, \tag{30}$$

so that

$$\alpha_\omega(\zeta) = \cos\left[\sqrt{-\frac{i\widetilde{k}_d}{2}\frac{\partial^2}{\partial r^2}\left[\widetilde{A}(0,0)\right]}\,\zeta\right], \tag{31}$$

and the main term of the high-frequency asymptotic solution for the two-frequency coherence function is given by

$$\Gamma\left(\zeta, 0, \widetilde{k}_d\right) = \cos^{-1}\left[\sqrt{-\frac{i\widetilde{k}_d}{2}\frac{\partial^2}{\partial r^2}\left[\widetilde{A}(0,0)\right]}\,\zeta\right] \cdot \exp\left[-\frac{\widetilde{k}_d^2\widetilde{A}(0,0)\zeta}{8}\right]. \tag{32}$$

This is quite a fundamental result. Actually, it is straightforward to obtain the two-frequency coherence function in the approximation of geometrical optics, which only describes the case of weak fluctuations (unsaturated regime). The second factor in (32) just stands for the geometrical optics approximation. A nontrivial thing is the derivation of the first factor, which may only account for a possible saturated regime of propagation (strong fluctuations). It was first obtained when the strict coherence function was constructed for the parabolic structure function of fluctuations [*Sreenivasiah et al.*, 1976]. The same type of results were also obtained for the quadratic structure function in the scope the technique of path integrals (functional integrals) [*Dashen*, 1979; *Flatte*, 1983]. We have obtained (32) for an arbitrary model of the structure function. This was just a particular result of the general theory being developed here.

[22] It should also be shown how the technique works to produce well-known results for a single-frequency case $\tilde{k}_d = 0$. In this case, according to equation (17), $r_0 = r$, and according to (28) and (29), $J = 1$. As to the main amplitude U_0 in representation (12), this results in $U_0 =$ const. As far as the higher-order amplitudes U_j, ($j = 1,2\ldots$) are concerned, $U_j = 0$, ($j = 1,2\ldots$) may be chosen because of the zero initial conditions and homogeneous equations (15) in the case of $U_0 =$ const. Finally, integrating equation (13) or (27) along straight lines $r_0 = r$ yields the known solution to the single-frequency problem (see equation (45.20) from *Rytov et al.* [1978]).

[23] Below the general case of a spaced frequency and position coherency in the medium with a structure function of fluctuations other than quadratic will be considered. Before doing this, it will be shown in the next subsection how equations (26)–(29) produce the exact solution for the space-frequency coherence function in the case of the quadratic structure function of fluctuations obtained by *Sreenivasiah et al.* [1976].

4.1. Quadratic Structure Function

[24] For the structure function of the form

$$\tilde{D}_\varepsilon(r) = 2\left(\tilde{A}(0) - \tilde{A}(r)\right) = 2\sigma_\varepsilon^2 r^2 \tag{33}$$

integral in equation (26) is calculated analytically to yield

$$\frac{r}{r_0} = \cos\left[\sqrt{\frac{i\tilde{k}_d}{2}}\sigma_\varepsilon\zeta\right]. \tag{34}$$

This expression explicitly relates the initial complex point r_0 (at $\zeta = 0$) and a real point of observation (ζ, r). Alternatively, when r_0 was determined for a given real point (ζ, r), the equation explicitly describes a complex trajectory arriving at a real given point in the form $r = r(\zeta, r_0)$.

[25] Once a complex trajectory has been defined by (34), integration in equation (27) along this complex ray gives the following complex eikonal:

$$\psi(\zeta, r) = -\frac{\sigma_\varepsilon r^2}{\sqrt{8i\tilde{k}_d}} tg\left(\sqrt{\frac{i\tilde{k}_d}{2}}\sigma_\varepsilon\zeta\right). \tag{35}$$

Equation (29) for the divergence becomes of the form

$$\frac{\partial^2\alpha}{\partial\zeta^2} + \frac{i\tilde{k}_d\sigma_\varepsilon^2}{2}\alpha = 0, \tag{36}$$

which allows the explicit analytic solution, satisfying necessary initial conditions, as follows:

$$\alpha(\zeta) = \cos\left[\sqrt{\frac{i\tilde{k}_d}{2}}\sigma_\varepsilon\zeta\right]. \tag{37}$$

Finally, putting together relationships (7), (12), (23), (28), (34), (35), and (37), the following representation for the space-frequency coherence function U_0 can be written

$$\Gamma(\zeta, r) = \cos^{-1}\left[\sqrt{\frac{i\tilde{k}_d}{2}}\sigma_\varepsilon\zeta\right]\exp\left[-\frac{K\sigma_\varepsilon r^2}{\sqrt{8i\tilde{k}_d}} tg\left(\sqrt{\frac{i\tilde{k}_d}{2}}\sigma_\varepsilon\zeta\right)\right] \cdot \exp\left[\frac{-\tilde{k}_d^2\sigma_\varepsilon^2\zeta}{8}\right]. \tag{38}$$

In corresponding notations, this is exactly the function derived by *Sreenivasiah et al.* [1976]. It is worth pointing out that in our technique the higher-order transport equations (15) give identically zero solutions for the quadratic structure function of fluctuations (33). As a result, in this case the asymptotic theory produces the rigorous solution.

4.2. Structure Function of Fluctuations Other Than Quadratic

[26] We have developed the method of constructing the space-frequency coherency, enabling investigation of the problem with the models of the structure function of fluctuations more realistic than those considered in the literature. Namely, both the quadratic structure function [*Sreenivasiah et al.*, 1976] and the structure function of the form r^ν, investigated by *Oz and Heyman* [1996, 1997a, 1997b] do not allow limiting transition to the case of moderate or weak fluctuations, at least, in the case of spaced position. Both tending to infinity as $r \to \infty$ are good models in the case of strong fluctuations, whereas for the opposite case the structure function should tend to a positive constant as $r \to \infty$. As already mentioned in section 1, the method of separation of variables employed by *Oz and Heyman* [1996, 1997a, 1997b] is also formally valid for this type of structure function. However, it will

likely face additional constraints, stipulated by a more complicated structure of the spectrum of the transversal operator of the problem. When the structure function is a constant at infinity, the continuous spectrum may likely occur in the spectrum of a transversal operator. This makes expansion of the solution in terms of the transversal eigenfunctions much more complicated. We use another type representation of the coherence function, and our approach is free of the mentioned difficulties.

[27] The central point of the paper is to demonstrate how the technique developed works for realistic structure functions of fluctuations others than quadratic and, in this way, to investigate both cases of weak and strong fluctuations together. Some models of the structure function (correlation function) allow explicit evaluation or direct calculation of the integrals in equations (26)–(28). Below we shall consider two model cases. Before doing this, it should be pointed out that the key point in realizing the developed technique is the analytic continuation of the model correlation function into the complex domain of its argument. As was already mentioned, for each model chosen this should be particularly discussed, but any model should possess the property of positive Fourier spectrum as representing the energy spectrum of fluctuations.

4.2.1. Inverse Power Law Correlation Function

[28] This is the case where the correlation function of fluctuations is modeled by

$$\widetilde{A}(r) = \frac{\sigma_\varepsilon^2}{1 + \alpha^\nu r^\nu}. \tag{39}$$

Recalling the definition of the dimensionless variable $r = \rho/l_\varepsilon$, (39) means that the correlation function, in some sense, has the effective spatial scale $l_\varepsilon\alpha^{-1}$ in the plane perpendicular to the z axis. In particular, in the case of 3-D isotropic fluctuations, one obtains $\alpha = 1$. The variance of fluctuations is σ_ε^2. Model (39) is smooth at $r = 0$ in the sense that $d\widetilde{A}(0)/dr = 0$, if $\nu > 1$. Additionally, parameter ν should be chosen to provide the property of positive Fourier spectrum of the correlation function.

[29] The structure function

$$\widetilde{D}_\varepsilon(r) = \frac{2\sigma_\varepsilon^2\alpha^\nu r^\nu}{1 + \alpha^\nu r^\nu} \tag{40}$$

corresponds to the correlation function (39). As r increases, $\widetilde{D}_\varepsilon(r)$ tends to the positive constant $2\sigma_\varepsilon^2$ (if $\nu > 1$).

[30] For the correlation function (39), equation (26) has the form

$$\sigma_\varepsilon\sqrt{\frac{i\widetilde{k}_d}{2}}\zeta = \frac{\sqrt{1+\beta^\nu}}{\alpha^{\nu/2} r_0^{\nu/2-1}} \int_1^{r/r_0} \frac{\sqrt{1+\beta^\nu x^\nu}}{\sqrt{1-x^\nu}}dx, \ \beta = \alpha r_0. \tag{41}$$

The integral in the last equation can be expressed asymptotically through the full hyper-geometric function with different sets of parameters for the cases $\beta \ll 1$ and $\beta \gg 1$. The comprehensive analysis of equation (41) for an arbitrary ν is not trivial. In particular, when speaking of the analytic continuation of the function (39) into the complex domain r, it should be noted that, on the one hand, it is straightforward, because the function is given explicitly, and the complex-valued r can be employed. On the other hand, all the singularities of this analytic function should be carefully accounted for, which are the appropriate poles and possible cut at $r = 0$, and the manifold Riemann surface should be properly introduced [see, e.g., *Shabat*, 1976] with a finite, or, possibly, infinite number of folds (if ν is an irrational fraction).

[31] However, the transparent explicit asymptotic results may be obtained in the particular case of $\nu = 2$. For this case the spatial spectrum of (39) can be easily calculated to be positive. As far as its analytical continuation into the complex r-domain is concerned, it is defined on a single-fold Riemann surface and it is the even function of its argument on the real axis. It has no branch points, but only two poles at $r = \pm i\alpha^{-1}$, $\alpha > 0$. The circle $|r| = \alpha^{-1}$ separates the complex plane r into two domains $|r| < \alpha^{-1}$ and $|r| > \alpha^{-1}$. The appropriate Loran series in the domain $|r| < \alpha^{-1}$ coincides with the Tailor series of function (39) as follows:

$$\widetilde{A}(r) = \sigma_\varepsilon^2\left(1 - \alpha^2 r^2 + \ldots\right). \tag{42}$$

In the domain $|r| > \alpha^{-1}$, the Loran series for (39) is given by the expansion into inverse powers of r:

$$\widetilde{A}(r) = \frac{\sigma_\varepsilon^2}{\alpha^2 r^2}\left(1 - \frac{1}{\alpha^2 r^2} + \ldots\right). \tag{43}$$

The explicit formula (39) just shows how the series for $|r| < \alpha^{-1}$ should be analytically continued into the domain $|r| > \alpha^{-1}$, and vice versa.

[32] Employing series (42) for a small spaced position results in the coherence function, coinciding with that for the quadratic model of the structure function. When using the second series (43) for the large spaced position in equations (26)–(28) and (41), this yields the following explicit asymptotic form of the complex path

$$r_0^2 = \frac{r^2}{2}\left(1 + \sqrt{\frac{1 + 2i\widetilde{k}_d\sigma_\varepsilon\zeta^2}{r^4\alpha^2}}\right), \tag{44}$$

and the following coherence function [*Bitjukov et al.*, 2001]

$$\Gamma(\zeta, r) = \left(\frac{r^2\alpha + \sqrt{r^4\alpha^2 + 2i\tilde{k}_d\sigma_\varepsilon\zeta^2}}{2\sqrt{r^4\alpha^2 + 2i\tilde{k}_d\sigma_\varepsilon\zeta^2}} \right)^{1/2}$$
$$\cdot \exp\left[-\frac{K\sigma_\varepsilon^2}{4}\left(1 + \frac{1}{r_0^2\alpha^2}\right)\zeta \right]$$
$$\cdot \exp\left[\frac{K\sigma_\varepsilon^2}{2\alpha\sqrt{2i\tilde{k}_d\sigma_\varepsilon}} \ln\left(\frac{r_0^2\alpha\sqrt{2} + \sqrt{i\tilde{k}_d\sigma_\varepsilon\varsigma}}{r_0^2\alpha\sqrt{2} - \sqrt{i\tilde{k}_d\sigma_\varepsilon\varsigma}} \right) \right]$$
$$\cdot \exp\left[\frac{-\tilde{k}_d^2\sigma_\varepsilon^2\zeta}{8} \right], \qquad (45)$$

where $r_0(\zeta, r)$ is given by equation (44). Equations (44) and (45) describe the coherency for all spaced frequencies and large transversal spaced position r. In contrast to (38), where in the case of $\tilde{k}_d = 0$ and any finite ζ function $\Gamma(\zeta, r) \to 0$ as $r \to \infty$, coherency (equations (44) and (45)) yields in the single-frequency case

$$\Gamma^{sf}(\zeta, r) = \exp\left[-\frac{K\sigma_\varepsilon^2}{4}\left(1 - \frac{1}{r^2\alpha^2}\right)\zeta \right]. \qquad (46)$$

This is exactly what the single-frequency two-position coherence function should be [see *Rytov et al.*, 1978; equation (45.20)] as $r \to \infty$ in the case of the structure function of fluctuations (40) with $\nu = 2$. For finite ζ it gives a nonzero constant as r tends to infinity. The value of the constant depends on the intensity of fluctuations of dielectric permittivity σ_ε^2.

4.2.2. Exponential Correlation Function

[33] Finally, one more model of the correlation function of fluctuations, allowing analytic assessment, is the exponential correlation function as follows:

$$\tilde{A}(r) = \sigma_\varepsilon^2 e^{-r}. \qquad (47)$$

Here the consideration is confined by the isotropic model of fluctuations with the correlation radius l_ε (recall $r = \rho/l_\varepsilon$). The spatial spectrum of (47) is positive. When formally continuing (47) evenly to the negative r, $\tilde{A}(r) = \sigma_\varepsilon^2 e^r$ if $r < 0$ should be accepted. Therefore, this model has the finite-step first derivative at $r = 0$ and turns out to be nonanalytic. This results in the fact that the analytic continuation is not possible in the vicinity of $r = 0$, and no proper solution for the coherency can be constructed for small spaced positions, including the pure frequency coherence function, for the model (47).

[34] On the other hand, when considering the large positive r the analytic continuation of (47) can be performed into a limited complex domain around the real axis for large r, which provides the explicit form to equation (26) as follows:

$$\sigma_\varepsilon\sqrt{\frac{i\tilde{k}_d}{2}}\,\zeta = -e^{r_0/2}\arccos(2e^{r-r_0} - 1). \qquad (48)$$

This is valid for large r and any $\tilde{k}_d$, so that the coherency can be constructed for the large value of the spaced position and any value of the mistuning frequency $\tilde{k}_d$. This is of importance in the case of the inverse power law spatial spectrum of fluctuations, when the spatial correlation function, expressed through the modified Bessel function, has the exponential asymptotic at large r.

[35] Employing (48), the appropriate analysis of equations (27) and (28) can then be performed and the representation for the spaced position (large r) and frequency (any $\tilde{k}_d$) coherence function can be written. It is fairly space consuming and is not presented here, but in the limiting case of $\tilde{k}_d = 0$ (i.e. for a single-frequency spaced position coherency) it yields the following expected [*Rytov et al.*, 1978, equation (45.20)] rigorous result:

$$\Gamma^{sf}(\zeta, r) = \exp\left[-\frac{\sigma_\varepsilon^2}{4}(1 - e^{-r})\zeta \right]. \qquad (49)$$

If comparing (46) and (49), both functions tend to the same constant when the spaced position increases, but the rate of decay is different as the different models were employed in these two cases.

[36] To conclude this section, for both power law (39) with $\nu = 2$ and exponential model (47) of the correlation function of fluctuations considered here, the general solutions constructed for the space-frequency coherency in the limiting case of single frequency (pure spatial coherency) produce in automatic fashion the results, which are in agreement with the general theory of the pure spatial coherence functions [*Rytov et al.*, 1978].

5. Conclusion

[37] An analytic technique has been developed to construct the asymptotic representation of the two-position, two-frequency coherence function in Markov's diffusive approximation. The technique employs the formalism of the quasi-classic complex paths. It allows the construction of the coherency for a wide range of realistic models of the structure function of fluctuations of dielectric permittivity of the medium of propagation, which tend to a finite value as the spaced transversal variable tends to infinity. For some models the final result can even be achieved analytically;

others need numerical calculations. For the quadratic structure function the method produces the known rigorous solution in an automatic fashion.

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Quasi-classic approximation in Markov's parabolic equation for spaced position and frequency coherency

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[1] An asymptotic technique to solve Markov's parabolic equation for the second-order spaced position and frequency coherence function is discussed. Rather than employing separation of variables, the technique is based on the quasi-classic representation in terms of complex trajectories and is also valid in the case of a nonhomogeneous background medium and does not demand the statistical homogeneity of fluctuations. It has no constraints relevant to the initial conditions in the form of an incident plane wave and produces in automatic fashion different known rigorous solutions, in particular, to the case of quadratic structure function. *INDEX TERMS:* 0659 Electromagnetics: Random media and rough surfaces; 0669 Electromagnetics: Scattering and diffraction; 0689 Electromagnetics: Wave propagation (4275); *KEYWORDS:* quasi-classics, space-frequency coherency

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1. Introduction

[2] In the paper the technique to solve Markov parabolic equation to the spaced position and frequency second-order coherence function [*Rytov et al.*, 1978, *Ishimaru*, 1978] is developed, which is based on the quasi-classic approximation in terms of complex trajectories. Rather than separation of variables employed by *Oz and Heyman* [1996, 1997a, 1997b, 1997c], this technique is valid including the case of a nonhomogeneous background medium and does not demand the statistical homogeneity of fluctuations. It has no constraints relevant to the initial conditions in the form of an incident plane wave and produces in automatic fashion known solutions [*Sreenivasiah et al.*, 1976; *Knepp*, 1983; *Bronstein and Mazar*, 2002; *Bitjukov et al.*, 2002].

[3] The present paper further extends the method of the investigation of the two-position, two-frequency coherence function, previously developed by the authors [*Bitjukov et al.*, 2002] for a particular case of plane wave, to the case of an incident field of a general type. In that paper there was considered a simple case of an incident plane wave propagating along the axis of the parabolic equation such that the appropriate second-order Markov moment equation did not contain differential operations in the central transversal variable and the solution did not depend on this variable. In this case the technique of complex trajectories (complex geometrical optics) has the most transparent form. The technique allowed constructing the solution to the cases of realistic behavior of the structure function of fluctuations tending to a positive constant rather than to the infinity as the difference argument increases. The extension of the technique to the general case (where the dependence on the central variable also occurs) that will be considered here employs more complicated complex trajectories in five-dimensional complex space.

2. Main Equations and Relationships

[4] Classic two-frequency second-order Markov's parabolic equation [*Ishimaru*, 1978] is considered for the homogeneous background medium, where possible statistical nonhomogeneity of fluctuations in the longitudinal direction is allowed as follows:

$$\frac{\partial \Gamma_1}{\partial z} + \frac{i}{2k_1k_2}\left[k_d\nabla_d^2 + \frac{1}{4}k_d\nabla_s^2 - 2k_s\nabla_s\nabla_d\right]\Gamma_1 + \frac{k_1k_2}{4}\left[A(z,\mathbf{0}) - A(z,\mathbf{r}_-)\right]\Gamma_1 = 0. \qquad (1)$$

Here ∇_s and ∇_d are the operators of gradient with respect to the sum and difference transversal co-ordinates $\mathbf{r}_+$ and $\mathbf{r}_-$, $k_d = k_1 - k_2$, $k_s = (k_1 + k_2)/2$, $k_{1,2}$ are the vacuum wave numbers for frequencies $\omega_{1,2}$. It was accepted that the background medium had the unity dielectric permittivity $\varepsilon_0 = 1$. Quantity $A(z, \mathbf{r}_-)$ is the effective transversal correlation function of the fluctuations δ-correlated in z-direction.

[5] If complex amplitude $U(z, \boldsymbol{\rho}, \omega)$ of the field $E(z, \boldsymbol{\rho}, \omega)$ is introduced according to

0048-6604/03/2002RS002714$11.00

$$E(z, \boldsymbol{\rho}, \omega) = U(z, \boldsymbol{\rho}, \omega) \exp[-i\omega t + ikz], \quad (2)$$

the coherence function of the amplitudes $U(z, \boldsymbol{\rho}, \omega)$

$$\Gamma(z, \boldsymbol{\rho}_1, \boldsymbol{\rho}_2, t_1, t_2, \omega_1, \omega_2) = \langle U(z, \boldsymbol{\rho}_1, \omega_1) U^*(z, \boldsymbol{\rho}_2, \omega_2)\rangle \quad (3)$$

is expressed through the solution Γ_1 of equation (1) as follows:

$$\Gamma(z, \mathbf{r}_+, \mathbf{r}_-, \omega_1, \omega_2) = \Gamma_1(z, \mathbf{r}_+, \mathbf{r}_-, \omega_1, \omega_2) \cdot \exp\left[-\frac{k_d^2 z}{8} A(z, \mathbf{0})\right]. \quad (4)$$

To further treat equation (1) for function Γ_1, the dimensionless variables are introduced according to: $z = \zeta l_\varepsilon$, $\mathbf{r}_- = \mathbf{r} l_\varepsilon$ and $\mathbf{r}_+ = \mathbf{R} l_\varepsilon$. Here l_ε is the effective spatial scale of fluctuations. Employing these dimensionless variables ς, $\mathbf{r}$, $\mathbf{R}$ equation (1) can be rewritten as

$$K\frac{\partial \Gamma_1}{\partial \varsigma} + \frac{i}{2}\left[\tilde{k}_d \nabla_{\mathbf{r}}^2 + \frac{\tilde{k}_d}{4}\nabla_{\mathbf{R}}^2 - 2\tilde{k}_s \nabla_{\mathbf{R}}\nabla_{\mathbf{r}}\right]\Gamma_1 + \frac{K^2}{4}[\tilde{A}(z, \mathbf{0}) - \tilde{A}(z, \mathbf{r})]\Gamma_1 = 0. \quad (5)$$

In equation (5) parameter $K = k_1 k_2 l_\varepsilon^2$ is the dimensionless parameter, which is assumed to be the large parameter of the problem; $\tilde{k}_d$ is the following dimensionless mistuning $\tilde{k}_d = l_\varepsilon(k_1 - k_2)$, and the dimensionless central wave number is given by $\tilde{k}_s = 2^{-1} l_\varepsilon (k_1 + k_2)$. Dimensionless correlation function of fluctuations $\tilde{A}(\zeta, \mathbf{r}) = l_\varepsilon^{-1} A(z, \mathbf{r}_-)$ is expressed through the effective transversal correlation function of δ-correlated fluctuations $A(z, \mathbf{r}_-)$.

[6] To finally rewrite equation (5) in the form enabling asymptotic solution at large K, additional rescaling of the central transversal dimensionless variable $\mathbf{R}$ should be performed. If taking account of the relationship between quantities K, $\tilde{k}_d$ and $\tilde{k}_s$:

$$\tilde{k}_s^2 = K + \frac{\tilde{k}_d^2}{4}, \quad (6)$$

rescaled central transversal variable $\boldsymbol{\rho}$ is introduced according to

$$\left(K + \frac{\tilde{k}_d^2}{4}\right)^{1/2} \boldsymbol{\rho} = \mathbf{R}. \quad (7)$$

Employing (7), equation (5) is finally rewritten as

$$K\frac{\partial \Gamma_1}{\partial \varsigma} + \frac{i}{2}\left[\tilde{k}_d \nabla_{\mathbf{r}}^2 + \frac{\tilde{k}_d}{4}\left(K + \frac{\tilde{k}_d^2}{4}\right)^{-1}\nabla_\rho^2 - 2\nabla_\rho \nabla_{\mathbf{r}}\right]\Gamma_1 + \frac{K^2}{8}\tilde{D}(\varsigma, \mathbf{r})\Gamma_1 = 0. \quad (8)$$

Here

$$\tilde{D}(\varsigma, \mathbf{r}) = 2[\tilde{A}(\varsigma, 0) - \tilde{A}(\varsigma, \mathbf{r})] \quad (9)$$

is the effective transversal structure function of fluctuations δ-correlated in z-direction.

[7] Equation (8) allows asymptotic solution at $K \to \infty$. Physically, the large K means that random inhomogeneities of a medium are of large spatial scale in terms of vacuum wavelengths for both frequencies ω_1 and ω_2. At the same time this is one of the limitations of Markov's diffusive approximation. When constructing the asymptotic solution at large K, it should be accepted that for any finite $\tilde{k}_d$, but $K \to \infty$, quantity $(K + 4^{-1}\tilde{k}_d^2)^{-1}$ in equation (8) can be expanded into a series as follows:

$$\left(K + \frac{\tilde{k}_d^2}{4}\right)^{-1} = \frac{1}{K}\left[1 - \frac{\tilde{k}_d^2}{4K} + \left(\frac{\tilde{k}_d^2}{4K}\right)^2 + \ldots + (-1)^j\left(\frac{\tilde{k}_d^2}{4K}\right)^j + \ldots\right]. \quad (10)$$

Analysis of possible forms of the coherence function of the high frequency incident field indicates the following most general form of the solution to equation (8):

$$\Gamma_1(\mathbf{r}, \boldsymbol{\rho}, \varsigma, \tilde{k}_d, K) = \exp[K\psi(\mathbf{r}, \boldsymbol{\rho}, \boldsymbol{\zeta}, \tilde{k}_d) + \sqrt{K}\Psi(\mathrm{r}, \rho, \zeta, \tilde{k}_d)] \cdot \sum_{n=0}^{\infty} K^{-n/2} U_n(\mathbf{r}, \boldsymbol{\rho}, \zeta, \tilde{k}_d), \; n = 0, 1, \ldots. \quad (11)$$

Standard asymptotic procedure of substituting (11) into (8) and taking account of (10) results in the "eikonal" equations for "phase" functions ψ and Ψ

$$\frac{\partial \psi}{\partial \zeta} + \frac{i\tilde{k}_d}{2}(\nabla_{\mathbf{r}}\psi)^2 - i(\nabla_{\boldsymbol{\rho}}\psi \cdot \nabla_{\mathbf{r}}\psi) + \frac{1}{8}D(\varsigma, \mathbf{r}) = 0, \quad (12)$$

$$\frac{\partial \Psi}{\partial \zeta} + i\tilde{k}_d(\nabla_{\mathbf{r}}\psi \cdot \nabla_{\mathbf{r}}\Psi) - i(\nabla_\rho \psi \cdot \nabla_{\mathbf{r}}\Psi) - i(\nabla_r \psi \cdot \nabla_\rho \Psi) = 0, \quad (13)$$

and the main transport equation for the amplitude U_0

$$\frac{\partial U_0}{\partial \varsigma} + \frac{i\tilde{k}_d}{2}[2(\nabla_r U_0 \cdot \nabla_r \psi) + U_0 \nabla_r^2 \psi + U_0(\nabla_r \Psi)^2] + \frac{i\tilde{k}_d^2}{8}U_0(\nabla_\rho \psi)^2 - i[(\nabla_\rho U_0 \cdot \nabla_r \psi) + (\nabla_r U_0 \cdot \nabla_\rho \psi) + U_0 \nabla_r \nabla_\rho \psi + U_0(\nabla_r \Psi \cdot \nabla_\rho \Psi)] = 0. \quad (14)$$

As far as higher-order transport equations for amplitudes U_n, $n = 1, 2, \ldots$ in the series (11) are concerned, they can be also written taking account of the full expansion

(10), but they are fairly space consuming and are not exposed here in order to save the space.

[8] Equations (12–14) can be solved by the method of characteristics, which, according to *Kravtsov and Orlov* [1980], is also applied to the case of complex characteristics. Within this method, the Hamilton-Jacobi equation [see *Kravtsov and Orlov*, 1980, equations (2.1') and (2.1")] in our case has the form given by (12). Then, according to equations (2.3)–(2.5) from *Kravtsov and Orlov* [1980], presenting the appropriate Hamilton equations for characteristics, complex characteristics corresponding to the equation (12), are given by the following set of the first-order equations:

$$\frac{d\varsigma}{d\tau} = 1, \tag{15}$$

$$\frac{d\mathbf{r}}{d\tau} = i\tilde{k}_d \mathbf{p}_r - i\mathbf{p}_\rho, \tag{16}$$

$$\frac{d\boldsymbol{\rho}}{d\tau} = -i\mathbf{p}_r, \tag{17}$$

$$\frac{dp_\varsigma}{d\tau} = -\frac{1}{8}\frac{\partial \tilde{D}(\varsigma, \mathbf{r})}{\partial \varsigma}, \tag{18}$$

$$\frac{d\mathbf{p}_r}{d\tau} = -\frac{1}{8}\nabla_r \tilde{D}(\varsigma, \mathbf{r}), \tag{19}$$

$$\frac{d\mathbf{p}_\rho}{d\tau} = 0. \tag{20}$$

These equations define complex trajectories in 5-dimensional space (ς, $\mathbf{r}$, $\boldsymbol{\rho}$). According to equation (15) it can be accepted that the points along a trajectory are parameterised by the real parameter $\tau = \varsigma$. The trajectories start at initial complex points ($\mathbf{r}_0$, $\boldsymbol{\rho}_0$) at $\varsigma = 0$ and arrive at real points of observation (ς, $\mathbf{r}$, $\boldsymbol{\rho}$), so that ($\mathbf{r}_0$, $\boldsymbol{\rho}_0$) are the initial conditions to equations (15–17). In fact, ($\mathbf{r}_0$, $\boldsymbol{\rho}_0$) are being determined as the function of the real point of observation (ς, $\mathbf{r}$, $\boldsymbol{\rho}$) when the "inverse" homing problem is considered.

[9] Initial conditions to the moments ($\boldsymbol{\rho}_r$, $\mathbf{p}_\rho$) from the set of equations (18–20) are defined by a given initial distribution of the eikonal function $\psi_0(\mathbf{r}_0, \boldsymbol{\rho}_0, \tilde{k}_d) = \psi(\mathbf{r}_0, \boldsymbol{\rho}_0, 0, \tilde{k}_d)$ of the coherence function of the incident field at $\varsigma = 0$ (see equation (11)) as follows:

$$\mathbf{p}_{r0} = \nabla_r \psi_0, \tag{21}$$

$$\mathbf{p}_{\rho 0} = \nabla_\rho \psi_0. \tag{22}$$

Finally, the initial condition to p_ς is obtained from equation (12) employing also relationships (21, 22) at $\varsigma = 0$, which yield

$$p_\varsigma = -\frac{i\tilde{k}_d}{2}(\nabla_r \psi_0)^2 + i(\nabla_{\boldsymbol{\rho}}\psi_0 \cdot \nabla_{\mathbf{r}}\psi_0) - \frac{1}{8}D(0, \mathbf{r}_0). \tag{23}$$

When dealing with the complex trajectories the analytical continuation of a structure function to the complex domain of its argument is an important point, and it should be specially discussed in each particular case. However, this continuation is straightforward when the structure function of fluctuations is given explicitly by the analytic function. Then the appropriate Loran series can be written for different domains of convergence in the complex domain.

[10] Once characteristic equations (15–20) with the initial conditions outlined above have been solved eikonal ψ from equation (12) is then obtained by integrating along the appropriate trajectory as follows:

$$\psi(\mathbf{r}, \boldsymbol{\rho}, 0, \tilde{k}_d) = \psi_0(\mathbf{r}_0, \boldsymbol{\rho}_0, \tilde{k}_d) + \int_0^\varsigma [p_\varsigma + i\tilde{k}_d \mathbf{p}_r^2 - 2i(\mathbf{p}_\rho \cdot \mathbf{p}_r)]d\varsigma. \tag{24}$$

[11] When considering the second eikonal Ψ governed by equation (13), it turns out that $d\Psi/d\varsigma = 0$ along the trajectories satisfying equations (15 - 20) with the appropriate initial conditions, so that along any trajectory

$$\Psi(\mathbf{r}, \boldsymbol{\rho}, \zeta, \tilde{k}_d) = \Psi(\mathbf{r}_0, \boldsymbol{\rho}_0, 0, \tilde{k}_d) = \Psi_0(\mathbf{r}_0, \boldsymbol{\rho}_0, \tilde{k}_d), \tag{25}$$

and $\Psi_0(\mathbf{r}_0, \boldsymbol{\rho}_0, \tilde{k}_d)$ is defined by a given initial distribution of the second eikonal of the coherence function of the incident field at $\varsigma = 0$ (see equation (11)). In particular, if this eikonal is not presented in the incident field ($\Psi_0 = 0$), then it is also not present in the solution of the problem given by representation (11). As will be seen, this holds in the case of an incident field of a spherical wave that will be considered below. However, Ψ may be present in other cases, for instance, when the incident field is a plane wave propagating in the direction different than z-direction.

[12] Finally, the solution to the main transport equation (14) is also obtained by integrating along the constructed trajectories as follows:

$$U_0(\mathbf{r}, \boldsymbol{\rho}, \varsigma) = U_{00}(\mathbf{r}_0, \boldsymbol{\rho}_0, \tilde{k}_d)\left\| \frac{\partial(\mathbf{r}, \boldsymbol{\rho})}{\partial(\mathbf{r}_0, \boldsymbol{\rho}_0)} \right\|^{-\frac{1}{2}} \cdot \exp\left[-\frac{i}{2}\int_0^\varsigma \left(\frac{\tilde{k}_d}{4}\mathbf{p}_\rho^2 + \tilde{k}_d^2(\nabla_r \Psi)^2 - (\nabla_r \Psi \cdot \nabla_\rho \Psi) \right) \right] d\varsigma. \tag{26}$$

In equation (26), $U_{00}(\mathbf{r}_0, \boldsymbol{\rho}_0, \tilde{k}_d)$ is the distribution of the "amplitude" of the coherence function of the incident field, and the quantity under the sign of the square root is the determinant of the appropriate matrix of the derivatives of the points along a trajectory by the initial conditions to this trajectory.

3. Spaced Position and Frequency Coherence Function to the Spherical Wave

[13] To demonstrate how the outlined technique works, the case where the incident field is a spherical wave written in the small-angle approximation respectively z-direction is considered here. Its coherence function is given by

$$\Gamma_0(\mathbf{r}_0, \boldsymbol{\rho}_0, K, \tilde{k}_d) = U_{00}(\mathbf{r}_0, \boldsymbol{\rho}_0, \tilde{k}_d)\exp[K\psi(\mathbf{r}_0, \boldsymbol{\rho}_0, \tilde{k}_d)] \quad (27)$$

with

$$\psi_0(\mathbf{r}_0, \boldsymbol{\rho}_0, \tilde{k}_d) = \frac{i\boldsymbol{\rho}_0}{\varsigma_s}\left(\mathbf{r}_0 + \frac{\tilde{k}_d\boldsymbol{\rho}_0}{2}\right), \quad (28)$$

$$U_{00}(\mathbf{r}_0, \boldsymbol{\rho}_0, \tilde{k}_d) = \varsigma_s^{-2}\exp\left[\frac{i\tilde{k}_d}{8\varsigma_s}(\mathbf{r}_0 + \tilde{k}_d\boldsymbol{\rho}_0)^2\right]. \quad (29)$$

Here ς_s is the dimensionless distance from the source of the spherical wave to the plane $\varsigma = 0$. According to (27) $\Psi_0 = 0$ in the coherence function of the incident spherical wave, then, according to (25), eikonal Ψ at $\sqrt{K}$ identically equals zero

$$\Psi = 0, \quad (30)$$

so that Ψ is not present in the solution to the spherical wave (27–29).

[14] With equations (28, 29) characterizing the incident spherical wave the initial conditions (21–23) for ray equations (15–20) become as follows:

$$\mathbf{p}_{r0} = \frac{i\boldsymbol{\rho}_0}{\varsigma_s}, \quad (31)$$

$$\mathbf{p}_{\rho 0} = \frac{i}{\varsigma_s}(\mathbf{r}_0 + \tilde{k}_d\boldsymbol{\rho}_0), \quad (32)$$

$$p_\varsigma = -\frac{i\boldsymbol{\rho}_0}{\varsigma_s^2}\left(\mathbf{r}_0 + \frac{\tilde{k}_d\boldsymbol{\rho}_0}{2}\right) - \frac{1}{8}D(0, \mathbf{r}_0). \quad (33)$$

If also the statistical homogeneity of fluctuations along ς-direction as well as the statistical isotropy of fluctuations in the transversal planes is accepted such that $\tilde{D}(\varsigma, \mathbf{r}) = \tilde{D}(r)$ ray equations (15–20) with the initial conditions (31–33) can be solved in the closed form. With it all, it turns out that vector $\mathbf{r}(\varsigma)$ is always in the plane of the initial vector $\mathbf{r}_0$ ($\mathbf{r}(\varsigma)$ is collinear to $\mathbf{r}_0$), so that in cylindrical variables for $\mathbf{r}$ its absolute value is given by the relationship

$$\varsigma = \int_{r_0}^{r}\left[\frac{r_0^2}{\varsigma_s^2} + \frac{i\tilde{k}_d}{4}(\tilde{D}(r_0) - \tilde{D}(r))\right]^{-\frac{1}{2}}dr, \quad (34)$$

whereas its polar angle φ is the same as the angle φ_0 for the initial vector $\mathbf{r}_0$ that is $\varphi = \varphi_0$. As far as the central variable $\boldsymbol{\rho} = \boldsymbol{\rho}(\varsigma)$ is concerned, it is expressed through $\mathbf{r}(\varsigma)$ as follows:

$$\boldsymbol{\rho}(\varsigma) = \left(1 + \frac{\varsigma}{\varsigma_s}\right)\boldsymbol{\rho}_0 + \frac{1}{\tilde{k}_d}\left(1 + \frac{\varsigma}{\varsigma_s}\right)\mathbf{r}_0 - \frac{\mathbf{r}(\varsigma, \mathbf{r}_0)}{\tilde{k}_d}. \quad (35)$$

To further proceed in consideration of the two-frequency, two-position coherence function to the spherical wave, the model of the structure function of fluctuations $\tilde{D}(r)$ should be specified. In the next subsection a simple power law model of fluctuations will be considered which allows further analytic assessment.

3.1. Inverse Power Law Model of Fluctuations

[15] The following model of the effective transversal structure function of fluctuations is considered below:

$$D(r) = \frac{2\sigma^2 r^2}{1 + r^2}. \quad (36)$$

It has a positive spatial spectrum, and also has a realistic behavior at $r \to \infty$, tending to a positive constant $2\sigma^2$.

[16] Analytical continuation of (36) into the complex domain of its argument is straightforward. In particular, its Loran series in the circle $|r| < 1$ (here this is the same as its Tailor series) is

$$D(r) = 2\sigma^2 r^2(1 - r^2 + \ldots), \quad (37)$$

and for the circle $|r| > 1$ it is given by

$$D(r) = 2\sigma^2(1 - r^{-2} + \ldots). \quad (38)$$

The rigorous form (36) shows how Loran series (37) can be analytically continued into the domain $|r| > 1$, or, how the series (38) can be continued into $|r| < 1$.

[17] Employing main terms of representations (37, 38) explicit calculation of the integral in (34) can be carried out so that the final explicit representations for the coherence function can be obtained for the cases of small ($|r| < 1$) and large ($|r| > 1$) values of the difference variable r.

[18] By means of representation (37), complex trajectories are constructed which are launched and landed at $|r|, |r_0| < 1$. Using the main term in (37) (that is equivalent of considering the model of the quadratic structure function) results in the following explicit form of the equation (34):

$$\frac{r}{r_0} = \cos\left[\sqrt{\frac{i\tilde{k}_d\sigma^2}{2}}\,\varsigma\right] + \frac{1}{\varsigma_s}\sqrt{\frac{2}{i\tilde{k}_d\sigma^2}}\sin\left[\sqrt{\frac{i\tilde{k}_d\sigma^2}{2}}\,\varsigma\right]. \quad (39)$$

Equations (35, 39) describe in the explicit form the complex trajectories $\mathbf{r} = \mathbf{r}(\varsigma, \mathbf{r}_0)$, $\boldsymbol{\rho} = \boldsymbol{\rho}(\varsigma, \boldsymbol{\rho}_0, \mathbf{r}, \mathbf{r}_0)$. In other words, they also show how the initial complex values $\mathbf{r}_0$, $\boldsymbol{\rho}_0$ at $\varsigma = 0$ should be chosen for the trajectory to come at the real point of observation (ς, $\boldsymbol{\rho}$, $\mathbf{r}$). The limiting case of an incident plane wave corresponds to $\varsigma_s \to \infty$. In this limiting case, equation (39) becomes exactly equation (34) from *Bitjukov et al.* [2002], which describes complex trajectories in the difference variable in the case of the incident field of a plane wave.

[19] Having (39) and taking account of (31–33, 35) allows explicit calculation of the integral for the eikonal function given by (24), as well as the amplitude expressed through (26). Then, when putting all together including (30), this yields the main term of the asymptotic representation of the two-frequency, two-position coherence function for small $|r|,|r_0|$ in the model (36) as follows:

$$\Gamma_1(\mathbf{r}, \boldsymbol{\rho}, \varsigma) = U_0\,(\mathbf{r}, \boldsymbol{\rho}, \varsigma)\exp[\boldsymbol{K}\psi(\mathbf{r}, \boldsymbol{\rho}, \varsigma)], \quad (40)$$

where

$$\psi(\mathbf{r}, \boldsymbol{\rho}, \varsigma) = \frac{i(\mathbf{r} + \tilde{k}_d\boldsymbol{\rho})^2}{2\tilde{k}_d(\varsigma + \varsigma_s)} - \frac{\sigma^2\mathbf{r}^2}{\sqrt{8i\tilde{k}_d\sigma^2}} \cdot tg\left(\sqrt{\frac{i\tilde{k}_d\sigma^2}{2}}\,\varsigma - \arccos A\right), \quad (41)$$

$$A = \sqrt{\frac{i\tilde{k}_d\sigma^2}{2}}\left(\frac{i\tilde{k}_d\sigma^2}{2} + \frac{1}{\varsigma_s^2}\right)^{-1/2}; \quad (42)$$

$$U_0(\mathbf{r}, \boldsymbol{\rho}, \varsigma) = \varsigma_s^{-1}\,(\varsigma + \varsigma_s)^{-1}\exp\left[\frac{i\tilde{k}_d(\mathbf{r} + \tilde{k}_d\boldsymbol{\rho})^2}{8(\varsigma + \varsigma_s)}\right] \cdot \left[\cos\left(\sqrt{\frac{i\tilde{k}_d\sigma^2}{2}}\,\varsigma\right) + \frac{1}{\varsigma_s}\sqrt{\frac{2}{i\tilde{k}_d\sigma^2}}\sin\left(\sqrt{\frac{i\tilde{k}_d\sigma^2}{2}}\,\varsigma\right)\right]^{-1} \quad (43)$$

Being the solution to the spherical wave in the medium with fluctuations having the structure function (36) at small $|r|,|r_0|$ (the main term in (37)), representation of the coherence function (40–43) at the same time gives the rigorous solution to the case of a spherical wave in the medium with quadratic structure function of fluctuations considered by *Knepp* [1983]. Moreover, in the limiting case of a plane incident wave ($\varsigma_s = \infty, A = 1$) representation (40–43), properly renormalized, gives equation (38) from *Bitjukov et al.* [2002] that is the solution to the coherence function of a plane wave propagating in the medium with the quadratic structure function of fluctuations initially obtained by *Sreenivasiah et al.*, [1976]. As can be shown, for the quadratic structure function and the incident field of a plane wave propagating along z-axis, the higher-order amplitudes in (11) identically equal zero, so that in this case our asymptotic technique produces the rigorous solution in automatic fashion.

[20] However, the most interesting is the case with trajectories located at $|r|,|r_0| > 1$, when the structure function of fluctuations is given by representation (38). This is the case of realistic behavior of the structure function of fluctuations at large values of its spaced position r. Considering trajectory equations (34, 35) with the model of the structure function of fluctuations given by the two terms in the series (38) yields the following explicit form for trajectories instead of (34):

$$\varsigma\left(\frac{r_0^2}{\varsigma_s^2} - \frac{i\tilde{k}_d\sigma^2}{2r_0^2}\right) = \sqrt{\left(\frac{r_0^2}{\varsigma_s^2} - \frac{i\tilde{k}_d\sigma^2}{2r_0^2}\right)r^2 + \frac{i\tilde{k}_d\sigma^2}{2}} - \frac{r_0^2}{\varsigma_s}. \quad (44)$$

Equation (44) together with (35) explicitly describe complex trajectories $\mathbf{r} = \mathbf{r}(\varsigma, \mathbf{r}_0)$, $\boldsymbol{\rho} = \boldsymbol{\rho}(\varsigma, \boldsymbol{\rho}_0, \mathbf{r}, \mathbf{r}_0)$, which now start and finish at large $|r|$, $|r_0|$. In the limiting case of the incident field of a plane wave ($\varsigma_0 \to \infty$) equation (44) is reduced to equation (44) from *Bitjukov et al.* [2002].

[21] Having (44) and again taking account of (31–33, 35) allows explicit calculation of the integral for the eikonal function ψ given by (24), as well as the amplitude expressed through (26), now for the case of large $|r|$, $|r_0|$. Then again, when putting all together including (30), this yields the main term of the asymptotic representation of the two-frequency, two-position coherence function now for large $|r|$, $|r_0|$ in the model (36) as follows:

$$\Gamma_1(\mathbf{r}, \boldsymbol{\rho}, \varsigma) = U_0(\mathbf{r}, \boldsymbol{\rho}, \varsigma)\exp[\boldsymbol{K}\psi(\mathbf{r}, \boldsymbol{\rho}, \varsigma)], \quad (45)$$

where

$$\psi(\mathbf{r}, \boldsymbol{\rho}, \varsigma) = \frac{i\boldsymbol{\rho}_0}{\varsigma_s}\left(\mathbf{r}_0 + \frac{\tilde{k}_d\boldsymbol{\rho}_0}{2}\right)\left(1 + \frac{\varsigma}{\varsigma_s}\right) + \frac{\sigma^2(1 - r^{-2})\varsigma}{4} - \frac{\sigma^2\varsigma}{2} - \frac{\sigma^2}{\sqrt{8i\tilde{k}_d\sigma^2}}\left[\ln\left(1 + \frac{\varsigma}{\varsigma_s} - \frac{\varsigma}{r_0^2}\sqrt{\frac{i\tilde{k}_d\sigma^2}{2}}\right) - \ln\left(1 + \frac{\varsigma}{\varsigma_s} + \frac{\varsigma}{r_0^2}\sqrt{\frac{i\tilde{k}_d\sigma^2}{2}}\right)\right], \quad (46)$$

$$U_0(\mathbf{r}, \boldsymbol{\rho}, \varsigma) = \varsigma_s^{-1}(\varsigma + \varsigma_s)^{-1}\left|\frac{r}{r_0}\frac{dr}{dr_0}\right|^{-1/2} \cdot \exp\left[\frac{i\tilde{k}_d(\mathbf{r} + \tilde{k}_d\boldsymbol{\rho})^2}{8(\varsigma + \varsigma_s)}\right]. \quad (47)$$

In equation (47) for amplitude U_0 trajectory $r = r(\varsigma, r_0)$ is defined implicitly by equation (44). In the limiting case of plane incident field ($\varsigma_s = \infty$) and with proper renormalization equations (45 – 47) are reduced to the coherence function presented by equation (45) from *Bitjukov et al.* [2002].

3.2. Single-Frequency Spaced Position Coherence Function to the Spherical Wave

[22] To conclude the consideration, it is discussed here how the general technique outlined in Sections 2 and 3 works in the limiting single-frequency case, when $\tilde{k}_d = 0$. This is the case of the pure spatial coherency comprehensively studied by Tatarskii [see *Rytov et al.*, 1978, chap. VII]. Equation (45.18) from this book represents the general solution of the single frequency problem for any given incident field in the form of appropriate Fourier integral in the central transversal variable. When applied to the problem of a spherical wave in the small angle approximation whose coherence function is given by (27–29) with $\tilde{k}_d = 0$, Fourier transform of the coherence function of the incident spherical wave is proportional to the appropriate δ-function, so that integration in spectral parameter in (45.18) from *Rytov et al.* [1978, chap. VII] can be easily performed. This finally yields the following solution to the single-frequency coherence function (in our notations):

$$\Gamma(z, \mathbf{r}_-, \mathbf{r}_+) = \frac{l_\varepsilon^2}{(z+z_s)^2} \exp\left[\frac{ik(\mathbf{r}_- \cdot \mathbf{r}_+)}{z+z_s} - \frac{k^2}{8} \cdot \int_0^z D\left(\mathbf{r}_- - \frac{\mathbf{r}_-}{z+z_s}(z-z')\right) dz'\right]. \quad (48)$$

On the other hand, when considering the same problem by means of the technique discussed in this paper, in the single frequency case ray equations (15–20) give the following trajectories:

$$\mathbf{r}_- = (\mathbf{r}_-)_0 \left(1 + \frac{z}{z_s}\right), \quad (49)$$

$$\mathbf{r}_+ = (\mathbf{r}_+)_0 \left(1 + \frac{z}{z_s}\right). \quad (50)$$

Then, performing necessary integrations for eikonal function ψ given by (24) along trajectories (49) and calculating the appropriate determinant in the amplitude factor U_0 given by (26) for trajectories (49, 50) results exactly in representation (48) for the coherence function, if (4) and (11) (with $\Psi = 0$) have been also taken into account.

4. Conclusion

[23] Asymptotic technique to solve Markov's parabolic equation for the second-order spaced position and frequency coherence function has been developed, which employs quasi-classic approximation with complex paths, or complex geometrical optics with complex ray trajectories. In the most general case of fluctuations, which are not statistically homogeneous, appropriate complex ray equations should be solved numerically, however, the method allows explicit analytic representations of the spaced position and frequency coherency for a series of realistic models of the structure function of statistically homogeneous fluctuations. In automatic fashion, it produces the solutions to the space-frequency coherence function known in the scientific literature.

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Propagation model for signal fluctuations on transionospheric radio links

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Abstract. The complex phase method has been further extended to the problem of electromagnetic (EM) field scintillations on Earth-satellite GPS paths of propagation. The numerical and analytic technique based on the method has been developed to characterize the transionospheric channel of propagation. The effects of additional range errors due to the ionospheric electron density fluctuations in space and time have been studied taking into account the ray bending due to the inhomogeneous background ionosphere and the diffraction on local random ionospheric inhomogeneities. In the method developed, the impact of the Earth's magnetic field is accounted for by the anisotropic spatial spectrum of the ionospheric turbulence with different outer scales along and across the magnetic field lines. The variances of the EM field phase (yielding range errors) and level (log amplitude) fluctuations have been calculated for different models of the background ionospheres characterized by different height electron density profiles and total electron content. The conditions of the saturated regime of propagation, which will likely result in the degradation of a GPS navigation system, have been discussed. In addition, the scattering function of the GPS transionospheric channel of propagation has been constructed and simulated for a wideband signal.

1. Introduction

Despite many years of research effort there continue to be many new publications on the effects of the ionospheric electron density fluctuations on electromagnetic wave propagation through the ionosphere. The early attention to this problem was mainly stimulated by ionospheric studies and by the need to interpret data of radiation from natural radio sources, whereas recent interest in this problem has been fueled by the intensive development of satellite-to-satellite and Earth-satellite communication systems. In order to increase the accuracy of range measurements the effects due to the inhomogeneous regular background ionosphere and troposphere were first studied. Consequently, errors arising out of an assumed isotropic inhomogeneous ionosphere, which are proportional to the total electron content (TEC), have been taken into account, and then finer effects due to the Earth's magnetic field have been investigated. A review of these results is given by, for example, *Dieminger et al*, [1985] and *Leitinger* [1998]. Recent advances on the effect of the Earth's magnetic field are given, in particular, by *Ashmanets et al.* [1996] and *Strangeways and Ioannides* [1999]. *Altshuler* [1998] has described tropospheric range errors.

Statistical effects of VHF and UHF propagation through the fluctuating ionosphere have been studied in many papers. In their review paper, *Yeh* and *Liu* [1982] have given an exhaustive description of the results obtained on this subject by that time. In recent years a lot of publications have appeared dealing with experimental data on fluctuations on transionospheric paths (see, for

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instance, *Basu et al.* [1988] and *Aarons* [1997]). Different theories accounting for the effects of the ionospheric electron density fluctuations have been developed and further extended. A power law phase screen approach [*Rino*, 1979] has been used by *Secan et al.* [1995, 1997] to model equatorial and high-latitude scintillation effects. The multiple-phase-screen method, initially introduced by *Knepp* [1983], has been slightly generalized, with the parabolic equation written in a cylindrical coordinate system and for a cylindrical incident wave instead of in rectangular co-ordinates for a plane incident wave. It was then used in the numerical simulation of the effects of transionospheric propagation in the gigahertz band [*Grimault*, 1998]. Amplitude scintillation effects on GPS links due to tropospheric turbulence have been considered by *Marzano and d'Auria* [1998]. When dealing with scintillation effects, attention is paid to the behavior of phase and amplitude fluctuations, the scintillation index S_4, the type of statistics of the field fading, the time coherence range, etc. Attention is also paid to the relationship between GPS amplitude scintillation and TEC variations referred to as GPS phase fluctuations [*Beach and Kintner*, 1999]. A simple propagation model of a single one-dimensional phase screen is employed in this paper in order to interpret the experimental data.

From the point of view of the assessment of the error of the regular range measurements the statistical properties of the field phase are important for highly accurate phase measurements, and the group delay time spread is significant for the GPS spread spectrum technique. The latter is characterized by the time scale of the GPS channel scattering function.

The analytical-numerical technique for characterisation of the HF ionospheric fluctuating channel of propagation, which is based on the complex phase method, or generalized Rytov's approximation, has been developed by *Zernov* [1980, 1992] and *Gherm and Zernov* [1995, 1998]. It deals with the point source field, rather than with plane wave propagation, and any orientation of the path of propagation with respect to the Earth's surface. It takes into account the ray bending due to the inhomogeneous background ionosphere and the diffraction effects on local random ionospheric inhomogeneities. The method provides the possibility to simulate statistical characteristics of the channel for different geophysical conditions of propagation, and in this way, to study, in particular, their variability with the form of the height electron density profile [*Radicella et al.*, 1998].

In the present paper, this approach is extended to the case of a transionospheric channel of propagation, and the range of its validity for this problem is discussed. Such a fine effect as the dependence of the curved path of propagation on transmission frequency due to the frequency dispersion of the ionosphere is taken into account, which may be of importance, in particular, for calculation of the scattering functions. The effects due to Earth's magnetic field, resulting in longitudinally extended forms of the ionospheric inhomogeneities, are also discussed. Additionally, it should be pointed out that the propagation model developed provides rigorous numerical results for both phase and amplitude fluctuations for realistic models of the background ionosphere and ionospheric electron density fluctuations, which is of importance, in particular, in the analysis of TEC fluctuations.

Our attention will be focused on the assessment of the intrinsic additional errors in the pseudorange measurements in the case of single-frequency measurements resulting from the fluctuations of the electron density of the ionosphere. The assessment of the fluctuational range error will be performed under the condition that the variance σ_χ^2 of the field log amplitude (level) fluctuations is such that $\sigma_\chi^2 < 1$. This inequality will be also used to forecast the ionospheric conditions, which would result in the saturated regime of propagation, characterized by strong fluctuations of the field amplitude and stochastic multipath effect contribution.

2. Propagation Model

A complete statement of the propagation problem and the details of the analytical technique used can be found in *Zernov* [1980, 1992] and *Gherm and Zernov* [1995; 1998]. To give a brief description, the field propagating through the fluctuating ionosphere is assumed to have the following form:

$$E(\mathbf{r},\omega,t)=E_0(\mathbf{r},\omega)\exp[\psi(\mathbf{r},\omega,t)], \quad (1)$$

where $E_0(\mathbf{r},\omega)$ is the field in the background regularly inhomogeneous ionosphere and the effects due to ionospheric electron density fluctuations are taken into account by means of complex phase $\psi(\mathbf{r},\omega,t)$. As ψ and E are random functions, the statistical characteristics of the level (log amplitude) χ and phase S of the field (1), which are the real and imaginary parts of ψ, respectively ($\psi=\chi+iS$), are of interest, as well as the statistical moments of the full field E.

For the complex phase ψ, variances of the real (log amplitude) and imaginary (phase) parts $\sigma_\chi^2=<\chi^2>, \sigma_S^2=<S^2>$ and mutual correlation $<\chi S>$ are expressed through the second-order moments $<\psi\psi>$ and $<\psi\psi^*>$ as follows:

$$< \chi^2 >= \frac{1}{2}\left[< \psi\psi^* > + \mathrm{Re} < \psi\psi >\right], \qquad (2)$$

$$< S^2 >= \frac{1}{2}\left[< \psi\psi^* > - \mathrm{Re} < \psi\psi >\right], \qquad (3)$$

$$< \chi S >= \mathrm{Im} < \psi\psi > . \qquad (4)$$

We will also be interested in the two-position, two-frequency, two-time correlation function Ψ of the full field E, which is necessary for constructing the scattering function of the transionospheric channel of propagation. This is expressed through the first and second order approximations ($\psi_1 = \chi_1 + iS_1, \psi_2 = \chi_2 + iS_2$) of the perturbation series for the complex phase as follows:

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \omega_1, \omega_2, t_1, t_2) = V(\mathbf{r}_1, \omega_1, t_1) V^*(\mathbf{r}_2, \omega_2, t_2) \cdot \{\exp\left[\langle \psi_1(\mathbf{r}_1, \omega_1, t_1) \psi_1^*(\mathbf{r}_2, \omega_2, t_2) \rangle\right] - 1\} \qquad (5)$$

Here quantity V is the mean field in the fluctuating ionosphere, given by the equation

$$V(r, \omega, t) = \exp\left[\langle \psi_2(r, \omega, t) \rangle + \tfrac{1}{2}\langle \psi_1^2(r, \omega, t) \rangle\right] \qquad (6)$$

As shown by *Zernov* [1992], in the case of transionospheric propagation through the inhomogeneous ionosphere, the following relationships hold asymptotically on the parameter ω_{pl}^4 / ω^4

$$< \chi_1^2 > = - < \chi_2 >, \; < \chi_1 S_1 > = - < S_2 > ; \qquad (7)$$

ω_{pl} is the maximal plasma frequency of the ionospheric layer and ω is a working frequency. Relations (7) are of the same type as in the classical problem of wave propagation in a random medium with a homogeneous background [*Tatarskii,* 1971] when they hold identically. Direct substitution of (7) into (6) results in the simpler expression for the mean field V :

$$V(\vec{r}, \omega) = \exp\left[-\frac{1}{2} < \psi_1(\vec{r}, \omega) \psi_1^*(\vec{r}, \omega) >\right]. \qquad (8)$$

In principle, the relationships (7) allow us to avoid constructing the second order approximation in the complex phase method. However, they are of use as criteria for validation of the results of numerical calculations performed. When first order phase and log amplitude fluctuations and second order fluctuations have been independently calculated a reasonable validation procedure is provided by determining if they satisfy these relationships. This procedure is employed in our numerical codes in order to ensure rigorous numerical results.

A full description of the procedures to construct all the necessary quantities is given by *Gherm and Zernov* [1995; 1998]. To give here just a brief description, we first point out that analysis is performed in the local ray-centered co-ordinate system. The undisturbed field E_0 and the Green's function of the undisturbed problem are represented in the geometrical optics approximation, and the curved path of propagation of E_0 through the background ionosphere is considered as the reference ray of the ray-centered variables. It traverses the background ionosphere, connecting the corresponding points under and above the ionosphere rather than refracting in the ionosphere, as was the case in the work by [*Gherm and Zernov* [1998] for HF sky wave propagation.

Employing the ray-centered variables, we can determine the complex phase ψ_1 which is given by equation (31) from the paper [*Gherm and Zernov*, 1998], which is as follows:

$$\psi_1(r, \omega, t) = \frac{ik^2}{4\pi} \iiint ds\, dn\, d\tau \cdot \frac{\varepsilon(r(s, n, \tau), \omega, t)}{\varepsilon_0^{1/2}(s) [D_n(s, s_0) D_\tau(s, s_0)]^{1/2}} \cdot \exp\left\{\frac{ik}{2}\left[\frac{n^2}{D_n(s, s_0)} + \frac{\tau^2}{D_\tau(s, s_0)}\right] - \frac{i\pi}{4}[\mathrm{sgn}\,[D_n(s, s_0)] + \mathrm{sgn}\,[D_\tau(s, s_0)]]\right\}. \qquad (9)$$

This corresponds to a random function ψ_1 if ε is random. In equation (9) variable s is measured along the reference ray, and the integration over s is carried out from the point $s = 0$ to the point $s = s_0$ which correspond to the origin and the end point of the reference ray respectively. Variables n and τ are the orthogonal variables pertinent to this ray, so that n is in the plane of propagation, and τ is perpendicular to this plane. Parameters $D_n(s, s_0)$ and $D_\tau(s, s_0)$ are defined as

$$D_n^{-1}(s, s_0) = \frac{\partial^2 \phi_0[\mathbf{r}(s)]}{\partial n^2} + \frac{\partial^2 \phi_1[\mathbf{r}(s), \mathbf{r}(s_0)]}{\partial n^2},$$
$$D_\tau^{-1}(s, s_0) = \frac{\partial^2 \phi_0[\mathbf{r}(s)]}{\partial \tau^2} + \frac{\partial^2 \phi_1[\mathbf{r}(s), \mathbf{r}(s_0)]}{\partial \tau^2} \qquad (10)$$

where $\phi_0(\mathbf{r})$ and $\phi_1(\mathbf{r}, \mathbf{r}')$ are the eikonals of the incident field and the Green's function, represented in the geometrical optics approximation.

The expression for ψ_2 can be written in similar fashion. This has been given by equation (32) of *Gherm and Zernov* [1998] and is not repeated here. The necessary statistical moments involved in relationships (6) and (8) are constructed employing representation (9) and the appropriate formula for ψ_2, so that variance $\langle \psi_1^2(\mathbf{r}, \omega) \rangle$ is expressed by formula (35) of *Gherm and Zernov* [1998] and the correlation function $\langle \psi_1(\mathbf{r}_1,\omega_1,t_1)\psi_1^*(\mathbf{r}_2,\omega_2,t_2)\rangle$ is given by equations (25), (36), and (33) in the same paper. To save space, we do not repeat all these equations here. The assumptions of the "frozen drift" and the statistical stationarity of the ionospheric random inhomogeneities have also been incorporated and used where necessary.

Once the technique to calculate the correlation function (5) and (6) of the transionospheric propagation channel is introduced, another important quantity, characterizing the signal spread in the group delay time and Doppler frequency, may be also constructed, namely the scattering function of the channel. Different types of scattering functions have been considered in the scientific literature [see, e.g., *Nickisch*, 1992; *Mastrangelo et al.*, 1997; *Lundborg and Zernov*, 1998; *Gherm and Zernov*, 1998]. The definition of the scattering function which is used in problems of the channel simulation system design involves the correlation function of the channel impulse response [*Mastrangelo et al.*, 1997; *Lundborg and Zernov*, 1998]. We shall also adopt this definition, which defines the scattering function as the Fourier-transform of the appropriate auto-correlation function of the channel impulse response on the difference time variable t of the form as given by [*Lundborg and Zernov*, 1998];

$$P_s(\tau,\omega_d,\omega_c)=\iiint P\left(\Omega+\frac{\delta}{2}-\omega_c\right)P^*\left(\Omega-\frac{\delta}{2}-\omega_c\right)$$
$$\cdot f_0\left(\Omega+\frac{\delta}{2}\right)f^*{}_0\left(\Omega-\frac{\delta}{2}\right)\Psi(\Omega,\delta,t)$$
$$\cdot\exp\left\{-i[\tau-t_g(\Omega)]\delta+i\omega_d t\right\}d\Omega\, d\delta\, dt. \qquad (11)$$

In (11) function f_0 is the known slowly varying function in the representation for the transfer function of the background channel, P is the spectrum of a launched pulse with a carrier frequency ω_c, Ω and δ are the center and difference frequencies and t_g and ω_d are the group delay time and Doppler shift, respectively. Function Ψ is the two-time, two-frequency correlation function of the monochromatic component of the field, introduced by (5) and (6). Scattering function P_s in (11) does not depend on the center time variable if stationarity of the ionospheric fluctuations is assumed. According to (11) the time-frequency correlation function of the monochromatic components of the field is the central point in calculation of the scattering functions. These functions have been investigated in different approximations for the sky wave and transionospheric channels (see, for instance, *Liu and Yeh* [1975], *Fridman et al.* [1995], and *Gherm and Zernov* [1998]).

The technique outlined here will be used in the consideration of the transionospheric propagation effects, resulting from the ionospheric electron density fluctuations. The background ionosphere will be assumed isotropic, but the presence of the Earth's magnetic field will be taken into account by means of the anisotropic model of the ionospheric electron density fluctuations.

We now discuss briefly the range of validity of our approach. It is commonly accepted (see, for instance, *Tatarskii* [1971], *Rytov et al.* [1978] and *Ishimaru* [1978]) that the Rytov-type approximation is valid for weak fluctuations of the field amplitude, resulting in the inequality

$$\sigma_\chi^2 < 1. \qquad (12)$$

Then, when performing any numerical simulation to obtain different moments of the random field, which has propagated through the fluctuating ionosphere, the validity of that inequality should be checked. However, it is also known that fluctuations of the phase of a field are well described by this approximation even beyond the formal scope of its validity [*Barabanenkov et al.*, 1971; *Ishimaru*, 1978]. The latter provides the possibility to study phase fluctuations for a wider range of the ionospheric fluctuation parameters than strictly limited by inequality (12). Then the condition $\sigma_\chi^2=1$ can be considered as only indicating a minimal order of magnitude of the log amplitude fluctuations up to which the results in phase fluctuation assessment are correct. In other words, phase errors obtained for σ_χ^2 reasonably greater than unity may also be considered valid.

From the point of view of the scintillation index S_4 the case of weak amplitude fluctuations, or the unsaturated regime of propagation, is outlined [*Yeh and Liu*, 1982] by the condition $S_4 < 0.5-0.6$. The latter, taking into account the relationship $S_4^2=4\sigma_\chi^2$ for small log amplitude fluctuations, results in an approximately 1 order of magnitude stricter condition than the limitation given by (10) for the amplitude fluctuations

$$\sigma_\chi^2 < 0.1. \qquad (13)$$

In principle, the severer condition (13) should also be taken into account when discussing the moments of the full field in the saturated regime of propagation.

To conclude this section it should be pointed out that our propagation model employs (according to equations (9) and (10) and equations cited from *Gherm and Zernov* [1998]) integration along the path of propagation of the effects of ionospheric irregularities with the properties of the background ionosphere being included, in particular, through the term in $\varepsilon_0(s)$ in the denominator of (9). It also accounts for the diffraction effects on local ionospheric inhomogeneities. As a result, this propagation model is free of a series of limitations inherent in propagation models based on a power law phase screen method [*Rino*, 1979; *Secan et al.*, 1995]. The latter assume a uniform propagation geometry and irregularity structure along the path of propagation and do not take into account diffraction effects in the ionospheric layer.

3. Model of the Electron Density Fluctuations

For the calculation of the effects due to the ionospheric electron density fluctuations the following model of the spatial spectrum of fluctuations will be utilized

$$B_\varepsilon(\kappa, s) = C_N^2 [1-\varepsilon_0(s)]^2 \sigma_N^2 f\left(\frac{\kappa_{tg}^2}{K_{tg}^2} + \frac{\kappa_{tr}^2}{K_{tr}^2}\right). \quad (14)$$

Here the values of the background ionosphere dielectric permittivity $\varepsilon_0(s)$ are given at the points along the reference ray as a function of variable s. Quantity σ_N^2 in (14) is the variance of the fractional electron density fluctuations of the ionosphere. The $f(\kappa)$ with $\kappa = \{\kappa_{tg}, \kappa_{tr}\}$ is a dimensionless spatial spectrum of fluctuations, normalized so that $f(0)=1$, and C_N^2 is the normalization coefficient of the dimension L^3. To characterize the anisotropy of the ionospheric inhomogeneities, the wave vector κ was introduced in (14) with projections along (κ_{tg}) and across (κ_{tr}) the force lines of the magnetic field of the Earth, $K_{tg} = 2\pi/L_{tg}$ and $K_{tr} = 2\pi/L_{tr}$. Quantity L_{tg} is the outer scale of the random ionospheric inhomogeneities along the Earth's magnetic field and L_{tr} is their transversal outer scale, so that the aspect ratio of the inhomogeneities is given by $a = L_{tg}/L_{tr}$. Variables (κ_{tg}, κ_{tr}) and the variables of the wave vector, conjugated with ray-centered variables defined by the reference ray, are linked by the transformation of rotation, so that the rotation matrix elements depend on the latitude and longitude of the path of propagation. This way, both the propagation model and the model of the ionospheric electron density fluctuations take into account the Earth's magnetic field and the orientation of the path of propagation, as well as the geophysical conditions.

The most adequate model of the ionospheric fluctuations spatial spectrum $f(\kappa)$ in equation (14) is known to be the inverse power law spectrum as follows:

$$f(\kappa) = \left(1 + \frac{\kappa_{tg}^2}{K_{tg}^2} + \frac{\kappa_{tr}^2}{K_{tr}^2}\right)^{-p/2}, \quad (15)$$

where p is the spectral index. This model is used in the numerical simulation of the propagation effects. The normalization coefficient C_N^2 for the inverse power law spectrum (15) is given by

$$C_N^2 = \frac{\Gamma(p/2)}{\pi^{3/2}\Gamma[(p-3)/2]K_{tr}^2 K_{tg}}. \quad (16)$$

However, an anisotropic gaussian model with the spectrum of the form

$$f(\kappa) = \exp\left(-\frac{\kappa_{tg}^2}{K_{tg}^2} - \frac{\kappa_{tr}^2}{K_{tr}^2}\right) \quad (17)$$

is also employed in the analysis, permitting analytical assessment.

4. Results

To preface the presentation of the results, we would like to point out that as previously mentioned, in particular, by *Rino* [1979], the numerical value of the outer scale of the ionospheric plasma turbulence, which divides the developed structures modeled by a statistically homogeneous random process and the evolving fairly large scale structures, is not well defined. A series of sources [*Dyson et al.*, 1974; *Basu et al.*, 1976; *Yeh and Liu*, 1982; *Alimov and Erukhimov*, 1995] indicate that the value of the outer scale is on the order of tens of kilometers. The same order of magnitude of the outer scale is also employed by *Beach and Kintner* [1999] to give an interpretation of some effects of field scintillation on transionospheric paths.

However, the models of ionospheric inhomogeneities with quasi power law spatial spectrum out to very large outer scales (fairly more than 10 km) have also been discussed and employed. In such a treatment, very low frequency components of the ionospheric irregularities contribute to the phase non diffractive variations associated with TEC variations [*Bhattacharyya et al.*, 2000]. Alternatively, very large scale variations of the electron density of the ionosphere may be treated in a quasi-deterministic statement as wavelike traveling ionospheric disturbances resulting in quasi-regular phase trends.

It should also be emphasized that the numerical value of the outer scale of the ionospheric turbulence is of importance in the interpretation of results of measurements of the field phase variance. This interpretation depends on the correlation of the data interval (or the integration time) and the temporal interval corresponding to the outer scale of the inhomogeneities [*Myers et al.*, 1979; *Rino*, 1979]. *Rino* [1979] developed the technique to calculate the rms phase based on the phase screen approximation, valid for integration times both smaller than and larger than the appropriate cutoff time (inverse of cutoff frequency). Our propagation model also allows analysis of both these cases. In the present paper, we give the results of calculations pertinent to the case where the integration time is large enough to exceed the appropriate cut-off time. We used values for the outer scale of the ionospheric turbulence of the order of tens to hundreds of kilometers based on values given by *Dyson et al.* [1974], *Basu et al.* [1976], *Yeh and Liu* [1982] and *Alimov and Erukhimov* [1995], but any other values could be utilized within the scope of our model.

4.1. Analytical Assessments

As has been mentioned, the Gaussian model (17) of the spatial spectrum of the ionospheric electron density fluctuations allows analytical calculations. In particular, the effects due to the Earth's magnetic field may easily be described. Calculations of the variance $\sigma_S^2 = < S_1^2 >$ of phase fluctuations for frequencies of interest and the conditions of the real ionosphere and satellite height in the range of 2000–20,000 km show the variance to be proportional to the scale of inhomogeneity along the path of propagation. Consequently, phase effects from propagation along and across the magnetic field force lines are proportional to the aspect ratio of the ionospheric turbulence.

4.2. Phase and Log Amplitude Fluctuations: Numerical Results and Discussion

The propagation model is able to account for a wide variety of effects of propagation for realistic conditions. To demonstrate this, we present here a series of results, obtained in the scope of this model. All the calculations were performed numerically for an anisotropic inverse power law spatial spectrum (15) and (16) and the spherical geometry of the background ionosphere. Calculations were carried out taking into account the dispersive properties of the ionosphere, for curved paths of propagation. In particular, the frequency dependence of the ray trajectories was taken into account when constructing the scattering functions of the transionospheric channel of propagation.

The results of the simulation of the effects of transionospheric electromagnetic (EM) wave propagation through the fluctuating ionosphere presented below have been obtained for different ionospheric height profiles with the values of vertical TEC of 31.8, 69, and 153 TEC units (1 TEC unit is equal to 10^{16} el m^{-2}). The values of TEC of 31.8 and 69 are in the range of typical vertical TEC values. As for the last largest TEC value of 153, although fairly high, it is nevertheless within the range of observable values (see, for instance, *Jursa* [1985] indicating TEC values as high as 180 TEC units). Using this value in calculations helps to quantify maximum possible errors in GPS range-finding.

The profiles used are characterized by the following values of f_0F2 and h_mF2: 9 MHz and 270 km for TEC = 31.8, 13 MHz and 325 km for TEC = 69, and 16.4 MHz and 420 km for TEC = 153. A value of $p = 3.7$ has been used in (15) to determine the spectrum index of the ionospheric electron density turbulence. Additionally, different values of longitudinal (to the Earth's magnetic field) scale L_{tg} and transversal scale L_{tr} and, consequently, of the aspect ratio a, as well as different values of the variance σ_N^2 of fluctuations of the fractional electron density of the ionosphere have been employed. Calculations have been carried out for the frequencies of L1 (1575.42 MHz) and L2 (1227.60 MHz), as well as for 137 and 2000 MHz. Vertical and oblique propagation have been considered for the satellite height of 2000 km above the Earth. To assess the spread of the effects due to the zenith angle of the oblique path of propagation, the fairly extreme cases of vertical propagation and propagation with a zenith angle of 85° have been considered. In the latter case, the value of the effective TEC is near maximal. The effects pertinent to the Earth's magnetic field have been taken into account by means of the anisotropic model of the ionospheric random inhomogeneities (15) and (16), and the limiting cases of propagation along and across magnetic field lines have been considered. Of course, all the intermediate situations may be also considered in the scope of our model, if desired.

4.2.1. Midlatitude ionosphere. To perform calculations for the midlatitude ionosphere we used, as an option, the transversal outer scale of the ionospheric turbulence $L_{tr} = 5\,\mathrm{km}$ and the aspect ratio $a = 4$. This results in the longitudinal outer scale of 20 km. These scales are in agreement with values given by *Dyson et al.* [1974], *Basu et al.*, [1976], *Yeh and Liu* [1982] and *Alimov and Erukhimov* [1995]; e.g. *Umeki et al.* [1977] employ scales of the same order. The results of simulation for this case are presented in Tables 1–3. They are arranged so that each table contains results of calculations for a given vertical TEC of the background ionosphere. Different rows in each table correspond to different conditions of propagation, which are the following: the frequency, the zenith angle of a path of propagation α in degrees and the angle between the path of propagation and the Earth's magnetic field β in degrees. Although β might vary appreciably over the total path of propagation, the range error will be dominated by its value at ionospheric heights, where the electron density is maximal, so that β is specified at this level. To derive absolute values of the range error in centimeters, quantities given in tables 1-3 in the columns "Range Error" should be multiplied by the rms of the fractional electron density fluctuations $\sqrt{\sigma_N^2} = \sigma_N$. The variance of the field level fluctuations σ_χ^2 is expressed in the units of the variance of the fractional electron density fluctuations σ_N^2. The value $\sigma_N^2 = 10^{-4}$ means 1% fluctuations of the fractional electron density, $\sigma_N^2 = 10^{-2}$ implies 10% fluctuations, and $\sigma_N^2 = 1$ corresponds to 100% fluctuations of the fractional electron density. Thus tables 1-3 actually give in the columns the range error in centimeters for 100% fluctuations.

Table 1. Results of Calculations for TEC = 31.8

Frequency, MHz	α^o	β^o	Range Error	σ_x^2
137	0	0	2565	863
137	0	90	1332	109
137	85	0	4008	5980
137	85	90	2155	977
1230	0	0	34	0.4
1230	0	90	17	0.04
1230	85	0	58	15
1230	85	90	30	1.7
1575	0	0	21	0.15
1575	0	90	10	0.015
1575	85	0	36	6.5
1575	85	90	18	0.68
2000	0	0	13	0.06
2000	0	90	6	0.006
2000	85	0	22	2.7
2000	85	90	11	0.28

Here α is the zenith angle of the path of propagation in degrees, and β is the angle between the path of propagation and the Earth's magnetic field. The σ_x^2 is the variance of the field level fluctuations expressed in units of the variance of the fractional electron density fluctuations σ_N^2. To derive absolute values of the range error in centimeters, quantities given in the column "Range Error" should be multiplied by the rms of the fractional electron density fluctuations $\sqrt{\sigma_N^2} = \sigma_N$

Table 2. Results of Calculations for TEC = 69.0

Frequency, MHz	α^o	β^o	Range Error	σ_x^2
137	0	0	5418	4250
137	0	90	2822	551
137	85	0	8264	26,500
137	85	90	4452	4370
1230	0	0	73	2.2
1230	0	90	36	0.22
1230	85	0	120	72
1230	85	90	61	7.9
1575	0	0	44	0.83
1575	0	90	22	0.082
1575	85	0	74	31
1575	85	90	38	3.3
2000	0	0	28	0.32
2000	0	90	14	0.032
2000	85	0	46	13
2000	85	90	23	1.4

Table 3. Results of Calculations for TEC=153.0

Frequency, MHz	α^o	β^o	Range Error	σ_x^2
137	0	0	10^4	1.8×10^4
137	0	90	5242	2440
137	85	0	1.4×10^4	8.9×10^4
137	85	90	7830	1.5×10^4
1230	0	0	136	13
1230	0	90	109	41
1230	85	0	212	274
1230	85	90	106	31
1575	0	0	83	4.9
1575	0	90	42	0.48
1575	85	0	131	121
1575	85	90	67	13
2000	0	0	52	1.9
2000	0	90	26	0.19
2000	85	0	82	53
2000	85	90	41	5.5

When beginning the discussion of the results, it is to be remembered that the absolute range of validity of the technique developed is stated by inequality (12). This means that figures in the last columns of tables 1-3 together with (12) indicate maximal values of the variance σ_N^2 of the fractional electron density fluctuations for which the results are valid. It also needs to be considered that the phase calculations are actually valid reasonably far beyond the formal scope of validity of the method.

As the numerical results show, the worst case in the sense of maximal phase fluctuations is obviously the case of very oblique ($\alpha = 85^o$) propagation along the direction of Earth's magnetic field lines ($\beta = 0^o$) with the densest background ionosphere (highest vertical TEC). It is characterized as well by the maximal restraints on the range of validity. Even this worst case is almost within the scope of the method validity for the gigahertz band and for fractional electron density fluctuations up to 10% (higher, sometimes even 100% fluctuations of the fractional electron density for less bad conditions). In this case the range errors of up to 6, 12, and 21 cm occur for TEC values of 31.8, 69, and 153 units, respectively. On the other hand, for a frequency of 137 MHz the method works only for very weak fractional fluctuations of the order of a few percent and yields, for example, for 2% electron density fractional fluctuations, a maximal range error of ~3 m for TEC = 153.

Additionally, the dependencies of the single-point moments, represented in Tables 1-3, on the aspect ratio have also been investigated. We will not present here the detailed results of these calculations, obtained for greater values of the aspect ratio, but point out that these only critically depend on the value of the aspect ratio for the special conditions when propagation is very close to the direction of the geomagnetic field lines. Thus calculations repeated for the conditions of propagation specified in line 5 of the Table 2, but performed instead for $a = 10$, result in a range error of $115\sigma_N$ cm, and $\sigma_\chi^2 = 5.5\sigma_N^2$. We see that at least for the parameters here considered, the variance of the field log amplitude is approximately proportional to the aspect ratio and the range error is approximately proportional to its square root. On the other hand, calculations performed when the line of sight is not parallel to geomagnetic field demonstrate fairly weak dependence of the results on the aspect ratio, providing that the aspect ratio exceeds 3-5 and the cross-field outer scale exceeds a value of order 1 km. For instance, the calculations performed for TEC = 69, $\alpha = \beta = 45^0$, frequency 1230 MHz, $L_{tr} = 5$ km, and aspect ratio $a = 4$ give the range error $49.3\sigma_N$ cm, and $\sigma_\chi^2 = 0.76\sigma_N^2$, whereas for the case of $a = 10$ the figures are $49.9\sigma_N$ cm, and $\sigma_\chi^2 = 0.72\sigma_N^2$, respectively showing a negligible difference.

4.2.2. Low-latitude ionosphere. In the low-latitude ionosphere, characteristic scales of local ionospheric inhomogeneities are reported of the order of tens, hundreds and even thousands of kilometers. On the other hand, the point of view exists that transversal scales close to the main Fresnel zone size should be employed to model the saturated regime of propagation. Within the scope of our propagation model, parameters of local ionospheric inhomogeneities like the outer scale and aspect ratio may easily be chosen which result in considerable amplitude variations for a given model of the background ionosphere and geometry and orientation of the propagation path and for the inverse power law spectrum of the electron density fluctuations.

To start with, calculations have been performed for the case of vertical TEC of 69 units, frequency 1230 MHz, and $L_{tr} = 10$ km, $L_{tg} = 500$ km (aspect ratio $a = 50$). Then for slant propagation with $\alpha = 85^o$ and $\beta = 0^o$ (propagation along magnetic field lines), the range error is given by the quantity $620\sigma_N$ cm, and $\sigma_\chi^2 = 168\sigma_N^2$. For transversal propagation ($\beta = 90^0$) the range error is $88\sigma_N$ cm and $\sigma_\chi^2 = 1.3\sigma_N^2$. According to these figures, 100% fluctuations of the fractional electron density evidently result in the regime of strong amplitude scintillation, both for the longitudinal and transversal propagation. However, more typical 10% fluctuations still result in the saturated regime for the longitudinal propagation but do not give rise to the strong scintillation for the transversal propagation.

The calculations performed for the same parameters of the background ionosphere and fluctuations, but for the propagation geometry with $\alpha = \beta = 45^0$ give the following results: $71\sigma_N$ cm for the range error and $\sigma_\chi^2 = 0.09\sigma_N^2$. The latter demonstrates, even for 100% fluctuations of the fractional electron density, too small a value of the amplitude fluctuations to be considered as the saturated regime.

When searching for the parameters of the ionospheric fluctuations resulting in the saturated regime of propagation for the same conditions as previously, we consider smaller values of the outer scale of the turbulence. For instance, for the same geometry, background ionosphere, and aspect ratio, but $L_{tr} = 1$ km (this corresponds to the longitudinal outer scale of 50 km) our propagation model yields the range error $20\sigma_N$ cm and $\sigma_\chi^2 = 6.2\sigma_N^2$. Then a value of the rms of the fractional electron density of 0.4 (40%) results in $\sigma_\chi^2 = 1$ which indicates the strong amplitude fluctuation occurrence.

Finally, as in section 4.2.1, we would like to point out the fairly weak dependence of the scintillation effects on the aspect ratio if the direction of propagation is not close to longitudinal (along the geomagnetic field line direction). For transversal outer scales of both 1 and 10 km, calculations have been performed for different smaller aspect ratios such as 20 or even 10 with the rest of the parameters the same as in the previous case. These calculations demonstrate approximately the same results irrespective of the aspect ratio.

4.2.3. Fraunhoffer diffraction. For the paths of propagation and frequencies under consideration the main Fresnel zone size is on the order of 300 m. This means that in the case of ionospheric turbulence with the outer scale considerably less than this, the propagation model, if correctly constructed, should give the same values for the variance of the phase and log amplitude fluctuations. This was investigated as a check of the validity of our model. For an outer scale of 0.1 km, $a = 1$ and the rest of the propagation conditions are the same as previously, our model yields $\sigma_S^2 = \sigma_\chi^2 = 8.1\sigma_N^2$. The latter demonstrates that this limiting case of Fraunhoffer diffraction is properly accounted for in the scope of the propagation model developed.

4.3. Scattering Function

In this section the results of simulation of the scattering function for the Earth-satellite channel are presented. The calculations were performed for the frequency 1230 MHz, a vertical path of propagation, and a satellite height of 20,000 km above the Earth. The profile with vertical TEC of 69 units was employed, and anisotropic ionospheric fluctuations with $L_{tr} = 10$ km and $L_{lg} = 500$ km (aspect ratio $a = 50$) were employed. As our estimates show, for a geostationary satellite the main contribution to the "frozendrift" velocity is given by the ionospheric wind, and so the drift velocity of 300 ms^{-1} has been employed. The two limiting cases of the drift direction perpendicular and parallel with respect to magnetic field lines are presented.

In Figure 1 the three-dimensional plots of the time-frequency correlation functions of the field are given as a function of difference time and frequency variables for both cases (Figure 1a is for the transversal drift direction, and Figure 1b is for the longitudinal drift). Of note are the different scales in the time domain (correlation time), which arise because of the anisotropic spectrum of the electron density fluctuations. As can be seen, the correlation functions depend very slowly on the frequency separation variable δ. This means that their Fourier

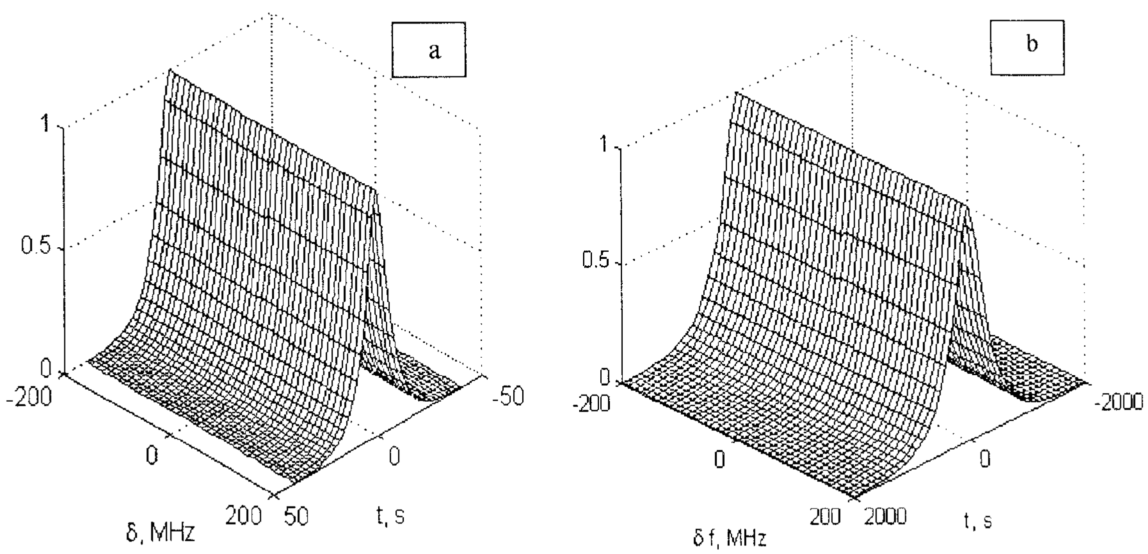

Figure 1. Time-frequency coherence functions of the GPS channel for the two limiting directions of the frozen drift with respect to geomagnetic field lines: (a) transverse and (b) parallel.

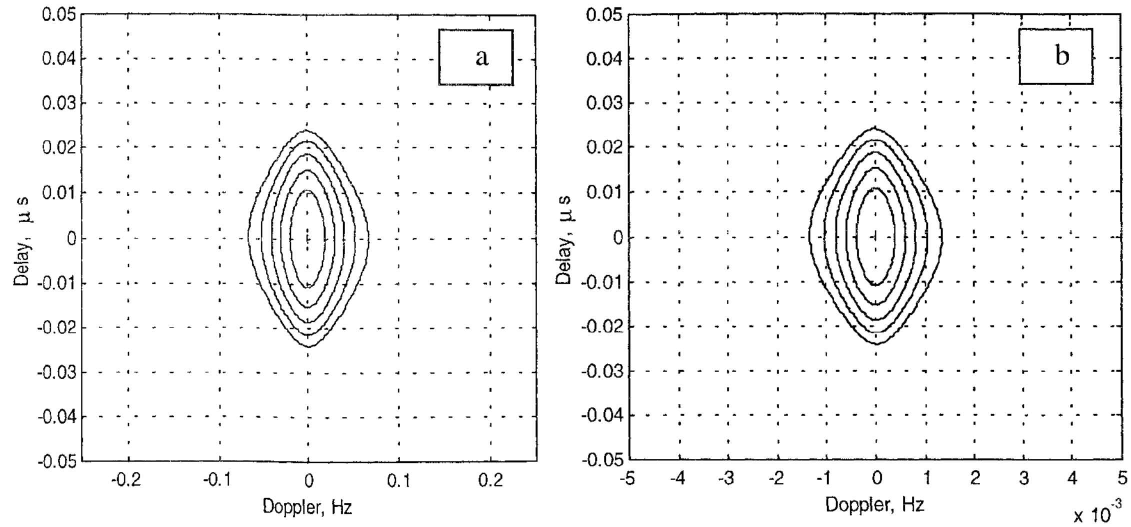

Figure 2. Scattering functions of the GPS channel for the two limiting directions of the frozen drift with respect to geomagnetic field lines: (a) transverse and (b) parallel. Contours are shown at 2 dB intervals.

transforms on δ are "almost" delta functions in the domain of the group delay time spread, which is Fourier-conjugated to δ. In other words, the effects of the group delay time spread due to the ionospheric electron density fluctuations are negligibly small for GPS links.

The main factor governing these effects is given by the bandwidth of the GPS transmitter-receiver system. This factor is involved in the calculation of the scattering function $P_s(\tau, \omega_d, \omega_c)$ of the GPS channel given by (11) through the spectral function $P(\omega - \omega_c)$. In our calculations the Gaussian spectral window used was of the form

$$P(\omega - \omega_c) = \exp\left[- \frac{(\omega - \omega_c)^2}{\Delta^2} \right]. \tag{18}$$

In (18), ω_c is the carrier frequency of the transmitter and Δ is the half-bandwidth of the system. The value of $\Delta = 10$ MHz was used in the calculations. In Figure 2 the contour plots of the scattering function of the GPS channel are shown with contours separated by 2 dB for the same cases of the drift direction. According to these graphs the characteristic scales of the scattering function at the level -10 dB with respect to the maximum are as follows: 0.025 ms in group delay for both plots, 0.075 Hz in Doppler spread for the case of transversal drift, and 0.0015 Hz for the longitudinal drift direction. As can be seen, the values of Doppler spread differ from each other by a value of the order of the aspect ratio of the fluctuation spectrum model.

5. Conclusion

The numerical and analytic technique based on the complex phase method has been further extended to the problems of EM field scintillations on Earth-satellite GPS paths of propagation. The evaluation of the additional range errors due to the ionospheric electron density fluctuations has been performed, and the assessment of the conditions of the saturated regime of propagation, which may likely result in the degradation of a GPS navigation system, has been given. In addition, by means of the method developed, the scattering function of the GPS transionospheric channel of propagation has been constructed and simulated for a wideband signal. It has been shown that the propagation model developed is capable of describing a wide variety of conditions of propagation, including different models of the background ionosphere, ionospheric fluctuation parameters, and geometry and orientation of the path of propagation. In this way, a versatile tool has been created to assess the effects of signal scintillations on real GPS paths of propagation.

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Propagation model for transionospheric fluctuating paths of propagation: Simulator of the transionospheric channel

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[1] A new model for scintillation on transionospheric links, based on a hybrid method and valid for strong scintillations, has been developed and used to construct a software transionospheric channel simulator. The method is a combination of the complex phase method and the random screen technique. The parameters of the random screen are determined as the result of a rigorous solution to the problem of propagation inside the ionosphere using the extended Rytov approximation (the complex phase method). The random two-dimensional spatial spectrum at the screen is then transferred down to the Earth's surface employing the rigorous relationships of the random screen theory. Thus the complex phase method can adequately introduce a random screen below the ionosphere for L-band frequencies. The technique is capable of producing statistical characteristics and simulating time realizations of the field for a wide range of input parameters. Preliminary results are presented for both weak and strong scintillations.

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1. Introduction

[2] There has been a significant amount of work over the last 40 years considering the problem of wave propagation in random media and, in particular, into investigation of the effects of fluctuations of the electron density of the ionosphere on signals propagating through the ionosphere. The latter has been partly motivated by the desire to quantify and assess scintillation effects on satellite navigation systems such as GPS and Galileo. Among the widely known models for the transionospheric propagation are WBMOD, which is based on the phase screen approximation and works for weak scintillation [*Rino*, 1979] and GISM which employs a numerical solution using the multiple phase screen technique [*Béniguel*, 2002]. Another propagation model also valid to describe the regime of weak scintillation employs the complex phase method, or generalized Rytov's approximation [*Gherm et al.*, 2000, 2002a, 2002b].

[3] Many attempts have been made to study the transionospheric fluctuating channel of propagation in the framework of the diffusive Markov's approximation, or approximation of Markov's parabolic equations for the statistical moments of the field. When characterizing the fluctuating channel of propagation, the second-order spaced frequency and position coherence functions are of particular interest, where spaced position can be transformed into a spaced time variable within the assumption of "frozen drift" of local random inhomogeneities of the ionosphere. The equation for the second-order space and frequency coherency was treated numerically [*Liu and Yeh*, 1975] (see also the review paper by *Yeh and Liu* [1982]). Many authors suggested different approximate

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analytic solutions to the second-order two-frequency two-position diffusive Markov's parabolic equation. Among recent results, the method developed in the papers by *Bitjukov et al.* [2002, 2003] to asymptotically solve the two-frequency two-position Markov's parabolic equation, based on the quasi-classic approximation with complex trajectories, seems to be the most general. It produces automatically other known solutions to this equation [*Sreenivasiah et al.*, 1976; *Knepp*, 1983a; *Oz and Heyman*, 1996, 1997a, 1997b, 1997c; *Bronshtein and Mazar*, 2002]. However, even the exhaustive knowledge of the two-frequency two-position coherence functions (both, the first and second) is not sufficient to comprehensively characterize the transionospheric fluctuating channel of propagation. The characterization procedure also requires the appropriate algorithms to generate random series of a signal that has propagated through the fluctuating channel, which, in turn, requires the appropriate probability density functions (PDFs) of the field's components. The problem of finding PDFs has not yet been rigorously solved for conditions pertaining to propagation through the fluctuating ionosphere. Although knowledge of the scintillation index S_4, obtained when solving the fourth-order Markov's parabolic equation, can be of use in choosing the approximate PDF in the family of Nakagami distributions, this is only one of the possible approximate approaches. Additionally, it should be said that constructing the rigorous solution to the fourth-order Markov's parabolic equation is an even more complicated problem than solving the second-order equation [see, e.g., *Gozani*, 1993] and fairly comprehensive list of references on this topic given there). This means that a rigorous derivation of the scintillation index is also fairly complicated.

[4] An alternative approach in the description of the transionospheric fluctuating channel of propagation is a pure numerical solution of the appropriate parabolic equations governing the propagation by the split step technique, or multiple phase screen method [*Knepp*, 1983b; *Grimault*, 1998; *Béniguel*, 2002]. The constraints of this technique lie in the problem of a correct generation of the random inhomogeneities of the medium of propagation. Additionally, this approach is fairly time consuming due to the purely numerical solution.

[5] The method to characterize the fluctuating transionospheric channel of propagation, which will be presented in this paper, can be termed a hybrid method. It is a combination of the complex phase method and the technique of a random screen. Preliminary assessments show that, for observation points lying inside the ionospheric layer, fluctuations of the field amplitude for frequencies of the order of 1 GHz and higher always have values which are within the range of validity of the complex phase method. This is true even in the case of very large relative electron density fluctuations (up to 100%) and high values of TEC. For smaller relative fluctuations and values of TEC this is also true for lower frequencies. This means that propagation in the ionospheric layer for the frequencies mentioned may always be well described in the scope of the complex phase method. In turn, this implies that, at L band and higher frequencies, the regime that results in strong scintillation does not normally occur inside the ionospheric layer, but may occur in the region where the field propagates from the ionosphere down to the Earth's surface. This circumstance permits utilization of the complex phase method to properly introduce the random screen below the ionosphere, and then to employ the rigorous relationships of the random screen method to correctly propagate the field down to the surface of the Earth, over which path the regime producing strong scintillation may well be found. This hybrid technique will be outlined in more detail below.

2. Propagation in the Ionospheric Layer

[6] Propagation in the ionosphere is described in the scalar approximation by the equation

$$\nabla^2 E + k^2[\varepsilon_0(\mathrm{r}) + \varepsilon(\mathrm{r}, T)]E = A\delta(\mathrm{r} - \mathrm{r}_0), \qquad (1)$$

widely used when considering very high frequencies. Here k is the wave number in vacuum, $\varepsilon_0(\mathbf{r})$ is the dielectric permittivity of the background medium and $\varepsilon(\mathrm{r}, \mathrm{T})$ is the dielectric permittivity of local random inhomogeneities. The quantity A characterizes the power of a source. The variable T denotes the possible slow time dependence of the ionospheric random inhomogeneities, which can be described in the quasi-stationary approximation.

[7] The complex phase method will be employed to describe the disturbed field in the ionospheric layer. According to this method the disturbed field is represented as follows:

$$E(\mathbf{r}, \omega, T) = E_0(\mathbf{r}, \omega) \exp[\psi(\mathbf{r}, \omega, T)], \qquad (2)$$

where the undisturbed field E_0 is the solution to the equation (1) with $\varepsilon_0(\mathbf{r})$ only. The complex phase ψ can account for the effects of random inhomogeneities on the undisturbed field and is sought as a perturbation series,

$$\psi = \psi_1 + \psi_2 + \dots, \qquad (3)$$

in powers of the disturbances of the ionosphere $\varepsilon(\mathbf{r})$. According to the technique of the complex phase method [see, e.g., *Zernov*, 1980] the solution to the functions ψ_1, ψ_2, ... can be written in the invariant form

$$\psi_1(\mathbf{r}) = -k^2(E_0(\mathbf{r}))^{-1}\int \varepsilon(\mathbf{r}')E_0(\mathbf{r}')G(\mathbf{r},\mathbf{r}')d\mathbf{r}', \quad (4)$$

$$\psi_2(\mathbf{r}) = -(E_0(\mathbf{r}))^{-1}\int (\nabla\psi_1(\mathbf{r}'))^2 E_0(\mathbf{r}')G(\mathbf{r},\mathbf{r}')d\mathbf{r}'. \quad (5)$$

In (4) and (5) the function $G(\mathbf{r}, \mathbf{r}')$ is the Green's function for the undisturbed problem and the ω dependence is suppressed as well as the dependence on slow time T.

[8] In the classical Rytov's approximation, the case was considered of a homogeneous background medium and the incident field in the form of a plane wave. The extended Rytov's approximation, or the complex phase method must be capable of constructing and describing complex phases $\psi(\mathbf{r}, T)$ also in the case of an inhomogeneous background medium and a point source of the field.

[9] For a slowly varying medium, like the background ionosphere, both the quantities E_0 and G may be well represented by the main term of their geometrical optics expansions (if the points of observation are not located in the vicinity of any caustic).

$$E_0(\mathbf{r}) = E_0^{GO}(\mathbf{r}), \quad (6)$$

$$G(\mathbf{r},\mathbf{r}') = G^{GO}(\mathbf{r},\mathbf{r}'). \quad (7)$$

[10] Then (5) and (6) together with (3) and (4) yield

$$\psi_1(\mathbf{r}) = -k^2\left(E_0^{GO}(\mathbf{r})\right)^{-1}\int \varepsilon(\mathbf{r}')E_0^{GO}(\mathbf{r}')G^{GO}(\mathbf{r},\mathbf{r}')d\mathbf{r}', \quad (8)$$

$$\psi_2(\mathbf{r}) = -\left(E_0^{GO}(\mathbf{r})\right)^{-1}\int (\nabla\psi_1(\mathbf{r}'))^2 E_0^{GO} \cdot (\mathbf{r}')G^{GO}(\mathbf{r},\mathbf{r}')d\mathbf{r}'. \quad (9)$$

Employing (8) and (9), *Zernov* [1980] has accomplished the extension of the classical Rytov's method, initially to the case of a plane-stratified background medium and a point source of the field. In a series of papers, this extension was used, interalia, to study the statistical properties of the HF field in the fluctuating ionosphere [e.g., *Gherm and Zernov*, 1995, 1998; *Gherm et al.*, 2001a]. It was also utilized to describe the transionospheric fluctuation channel of propagation [*Gherm et al.*, 2000, 2002a, 2002b].

[11] Recently, the complex phase method has been further extended to the most general case of a 3-D inhomogeneous background medium [*Gherm et al.*, 2001c]. The main point of the extension was to construct the complex phases ψ_1, ψ_2 for the case when the incident field is the field of a point source and the background medium is an arbitrary 3-D slowly varying inhomogeneous medium. This appeared to be possible and convenient by making use of ray-centered curvilinear variables introduced for an arbitrary 3-D inhomogeneous background medium. The reference ray for this type of coordinate system is the path of propagation, which connects the transmitter and receiver in the background medium.

[12] In the work of *Gherm et al.* [2001c] the explicit expressions were derived for ψ_1, ψ_2 using these curvilinear ray-centered variables with variable s directed along the reference ray and $\mathbf{q}$ = (q_1, q_2) lying in the plane perpendicular to the reference ray at each point. In this coordinate system complex phases are represented in the following form (further details are given in the work of *Gherm et al.* [2005]):

$$\psi_1(s_0,0,0) = -\frac{k^2}{4\pi}\iiint ds dq_1 dq_2 \frac{\varepsilon(s,q_1,q_2)}{n_0(s)} h_s(s,q_1,q_2) \cdot \left[\det\hat{B}^+\right]^{1/2} \exp\left\{\frac{ik}{2}\left[\left(b_{11}+b_{11}^g\right)q_1^2 + \left(b_{22}+b_{22}^g\right)q_2^2 + 2\left(b_{12}+b_{12}^g\right)q_1q_2\right]\right\}. \quad (10)$$

Here $n_0(s) = n(s, 0, 0) = [\varepsilon_0(s, 0, 0,)]^{1/2}$. $\hat{B}^+ = \hat{B} + \hat{B}^g$ where the matrices $\hat{B}$ and $\hat{B}^g$, with the elements b_{ik}, i, k = 1, 2 and b_{ik}^g, i, k = 1, 2 respectively, satisfy the matrix Riccati equations

$$n_0\frac{\partial\hat{B}}{\partial s} + \hat{B}\cdot\hat{B} = \hat{C}, \quad (11)$$

$$-n_0\frac{\partial\hat{B}^g}{\partial s} + \hat{B}^g\cdot\hat{B}^g = \hat{C}. \quad (12)$$

The matrix equations (11) and (12) are added to the ray-tracing code to construct the field of the rays in the undisturbed background medium.

[13] It is worth pointing out that the derived general representation for the complex phase ψ_1 permits known limiting cases to be obtained. In particular, if the background medium is homogeneous, it produces automati-

cally the complex phase for the case of a spherical incident field [*Tatarskii*, 1961],

$$\psi_1(x_0,0,0)=\frac{k^2}{4\pi}\iiint dxdydz\,\varepsilon(x,y,z)\frac{x_0}{x(x_0-x)}\cdot\exp\left\{\frac{ik_0[y^2+z^2]x_0}{2x(x_0-x)}\right\}, \quad (13)$$

and for large-scale local inhomogeneities with scales greater than the appropriate main Fresnel zone sizes, it gives the geometrical optics limit as follows:

$$\psi_1(s_0,0,0)=\frac{ik}{2}\int_0^{s_0}\frac{\varepsilon(s,0,0)}{n_0(s)}ds. \quad (14)$$

[14] Finally, the disturbed field inside the ionospheric layer is represented in the following form:

$$E(\mathbf{r},\omega)=E_0^{GO}(\mathbf{r})\exp[\psi_1(\mathbf{r},\omega)], \quad (15)$$

where $\psi_1(\mathbf{r},\omega)$ is given by equation (10). To also obtain $\psi_2(s_0, 0, 0)$, the product $k^2\varepsilon(s, q_1, q_2)$ in equation (10) must be substituted by $(\nabla\psi_1(s, q_1, q_2))^2$.

3. Generation of Random Realizations

[15] The first goal of the investigation is to obtain the random time series of the field $E(\mathbf{r})$ at the bottom of the ionosphere in order to be able to further convey it down to the level of the Earth's surface. To produce this random series, both autocorrelation and cross-correlation functions of the real and imaginary parts of ψ_1, where

$$\psi_1(\mathbf{r},\omega)=\chi_1(\mathbf{r},\omega)+iS_1(\mathbf{r},\omega), \quad (16)$$

are needed, i.e.,

$$B_\chi(\omega;\mathbf{q_1},\mathbf{q_2})=\langle\chi_1(\omega,\mathbf{q_1})\chi_1(\omega,\mathbf{q_2})\rangle, \quad (17)$$

$$B_S(\omega;\mathbf{q_1},\mathbf{q_2})=\langle S_1(\omega,\mathbf{q_1})S_1(\omega,\mathbf{q_2})\rangle, \quad (18)$$

$$B_{\chi S}(\omega;\mathbf{q_1},\mathbf{q_2})=\langle\chi_1(\omega,\mathbf{q_1})S_1(\omega,\mathbf{q_2})\rangle. \quad (19)$$

However, this is not sufficient. In addition, the probability density functions (PDF) for the distributions of the random functions χ_1 (level, or log-amplitude) and S_1 (phase) of the random field are also necessary. When generating χ_1 and S_1 spatial distributions, the appropriate spatial spectra of the correlation functions (17)–(19) are employed. Having derived the representation (10), these spectra can be obtained in the following form

$$B_S(\kappa_n,\kappa_\tau)=\frac{\pi k^2}{2}\int_0^{S_0}\frac{ds}{\varepsilon_0(s)}B_\varepsilon(s,0,\eta_n,\eta_\tau)\cdot\cos^2\left\{\frac{1}{2k}\left[\eta_n^2D_n(s)+\eta_\tau^2D_\tau(s)+2\eta_n\eta_\tau D_{n\tau}(s)\right]\right\}, \quad (20)$$

$$B_\chi(\kappa_n,\kappa_\tau)=\frac{\pi k^2}{2}\int_0^{S_0}\frac{ds}{\varepsilon_0(s)}B_\varepsilon(s,0,\eta_n,\eta_\tau)\cdot\sin^2\left\{\frac{1}{2k}\left[\eta_n^2D_n(s)+\eta_\tau^2D_\tau(s)+2\eta_n\eta_\tau D_{n\tau}(s)\right]\right\}, \quad (21)$$

$$B_{S\chi}(\kappa_n,\kappa_\tau)=\frac{\pi k^2}{4}\int_0^{S_0}\frac{ds}{\varepsilon_0(s)}B_\varepsilon(s,0,\eta_n,\eta_\tau)\cdot\sin\left\{\frac{1}{k}\left[\eta_n^2D_n(s)+\eta_\tau^2D_\tau(s)+2\eta_n\eta_\tau D_{n\tau}(s)\right]\right\}, \quad (22)$$

where η_n, η_τ depend on variable s along the reference ray and are expressed through spectral variables κ_n, κ_τ, conjugated to vector $\mathbf{q}$, by means of the following set of linear relationships

$$\eta_n\frac{\partial n}{\partial n_0}+\eta_\tau\frac{\partial\tau}{\partial n_0}=\kappa_n,\qquad \eta_n\frac{\partial n}{\partial\tau_0}+\eta_\tau\frac{\partial\tau}{\partial\tau_0}=\kappa_\tau. \quad (23)$$

Coefficients of the variables η_n and η_τ, on the left-hand side of the equation (23), are the matrix elements of the Jacobian $\partial(n,\tau)/\partial(n_0,\tau_0)$, being the partial derivatives of the components of the deviation vector at the current point s with respect to the appropriate deviations at the final point s_0, calculated for the bundle of rays originating at the source point. The vacuum wave number is $k = \omega/c$. The function $B_\varepsilon(s; 0, \kappa_n, \kappa_\tau)$ is the three-dimensional spatial spectrum of the electron density fluctuations with zero value of the spectral variable Fourier conjugated to the variable s along the path. The coefficients D_n, D_τ, and $D_{n\tau}$ are the elements of the matrix $\hat{D} = (\hat{B}^+)^{-1}$, i.e., the inverse to the matrix $\hat{B}^+$, introduced in the work of *Gherm et al.* [2001c]. The matrix $\hat{B}^+$ is actually a sum of curvature matrices of the incident field and of the Green's function. These are obtained by integrating the corresponding differential equations along the reference ray and also depend on the variable s.

[16] In the case when particular accuracy of calculation of scintillation effects is not of a very high importance, the curvature of the line of sight ray can be neglected and the appropriate equations (20)–(22) can be simplified to the final form of

$$B_S(\kappa_n,\kappa_\tau)=\frac{\pi k^2}{2}\int_0^{S_0}\frac{ds}{\varepsilon_0(s)}B_\varepsilon(0,\eta_n,\eta_\tau,s)\cdot\cos^2\left\{\frac{1}{2k}\left(\eta_n^2+\eta_\tau^2\right)\frac{s(s_0-s)}{s_0}\right\},\quad(24)$$

$$B_\chi(\kappa_n,\kappa_\tau)=\frac{\pi k^2}{2}\int_0^{S_0}\frac{ds}{\varepsilon_0(s)}B_\varepsilon(0,\eta_n,\eta_\tau,s)\cdot\sin^2\left\{\frac{1}{2k}\left(\eta_n^2+\eta_\tau^2\right)\frac{s(s_0-s)}{s_0}\right\},\quad(25)$$

$$B_{S\chi}(\kappa_n,\kappa_\tau)=\frac{\pi k^2}{4}\int_0^{S_0}\frac{ds}{\varepsilon_0(s)}B_\varepsilon(0,\eta_n,\eta_\tau,s)\cdot\sin\left\{\frac{1}{k}\left(\eta_n^2+\eta_\tau^2\right)\frac{s(s_0-s)}{s_0}\right\},\quad(26)$$

with

$$\eta_n=\frac{\kappa_n}{s/s_0},\qquad\eta_\tau=\frac{\kappa_\tau}{s/s_0}.\quad(27)$$

[17] Here the integration is carried out along the straight line of sight, but functions $\varepsilon_0(s)$ and $\sigma_N^2(s)$, involved in B_ε (see below), are taken from real 3-D distributions of the background ionosphere and fluctuations of the electron density of the ionosphere.

[18] In the numerical calculations, a turbulence model of the ionospheric fluctuations is considered with an anisotropic inverse power law spatial spectrum of the form:

$$B_\varepsilon(s,\kappa)=C_N^2[1-\varepsilon_0(s)]^2\sigma_N^2(s)\left(1+\frac{\kappa_{tg}^2}{K_{tg}^2}+\frac{\kappa_{tr}^2}{K_{tr}^2}\right)^{-\frac{p}{2}}.\quad(28)$$

Here C_N^2 is a known normalization coefficient. $K_{tg}=2\pi l_{tg}^{-1}$ and $K_{tr}=2\pi l_{tr}^{-1}$ where l_{tg} and l_{tr} are the outer scales of the turbulence along and perpendicular to the geomagnetic field direction respectively. The function $\varepsilon_0(s)$ is the distribution of the dielectric permittivity of the background ionosphere along the reference ray in the 3-D inhomogeneous background ionosphere and $\sigma_M^2(s)$ is the distribution of the variance of the relative fluctuations of the electron density of the ionosphere, again along the reference ray in the 3-D inhomogeneous ionosphere.

[19] As far as the PDFs for χ_1 and S_1 are concerned, both these quantities, according to equation (10), are represented as a sum of a large number of statistically independent random contributions (the integral in (10) is over a large number of random inhomogeneities). According to the central limit theorem, this means that both the random functions χ_1 and S_1 are normally distributed.

[20] The spectra (20)–(22) of the appropriate correlation functions (17)–(19) and the known PDFs permit the proper generation of the random two-dimensional distributions of χ_1 and S_1 below the ionosphere, and, in this way, to introduce the random screen with a given random field. Then, its random spatial Fourier transform is conveyed down to the Earth's surface by the propagator

$$\tilde{E}(z,\boldsymbol{\kappa},t)=e^{ikz}\tilde{E}(0,\boldsymbol{\kappa},t)\exp\left(-\frac{i\kappa^2z}{2k}\right).\quad(29)$$

[21] Finally, performing inverse Fourier transformation of the random spectra of χ_1 and S_1 at the Earth's surface yields, at this location, the desired random spatial distributions of χ_1 and S_1. The latter, with the assumption of "frozen drift" of the inhomogeneities in the ionosphere, can be transformed to random time series of the field log amplitude χ_1 and the phase S_1 of the field.

4. Applications

[22] The hybrid technique, outlined above, for describing the transionospheric fluctuating channel of propagation permits the determination of different characteristics of the field propagated from satellite altitudes down to the Earth's surface. The results, which will be presented below, have been obtained for the following conditions of propagation: (1) the NeQuick model profile of the background ionosphere for the low-latitude ionosphere with a TEC value of 90 TECU; (2) the value of the spectral index (p) of the spatial spectrum of the electron density fluctuation in (28) was 3.7; (3) the variance of the relative electron fluctuations σ_N^2 was 10^{-2}; (4) the cross-field outer scale was 10 km and the aspect ratio a was 5; (5) the elevation angle of the path of propagation was 60° and its azimuth was 180°; (6) the effective velocity of the horizontal frozen drift was 300 m/s; and (7) the transmission frequency was 430 MHz (for strong scintillation) and 1545 MHz (for weak scintillation).

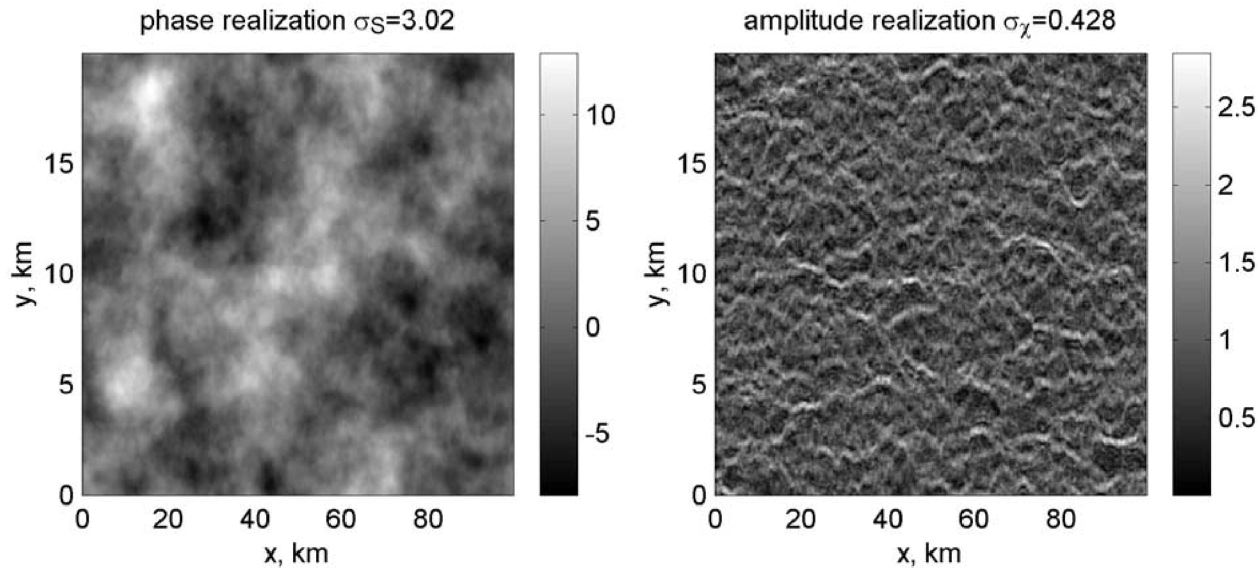

Figure 1. Two-dimensional realizations of random distributions of (left) phase and (right) log amplitude at the Earth's surface. See color version of this figure in the HTML.

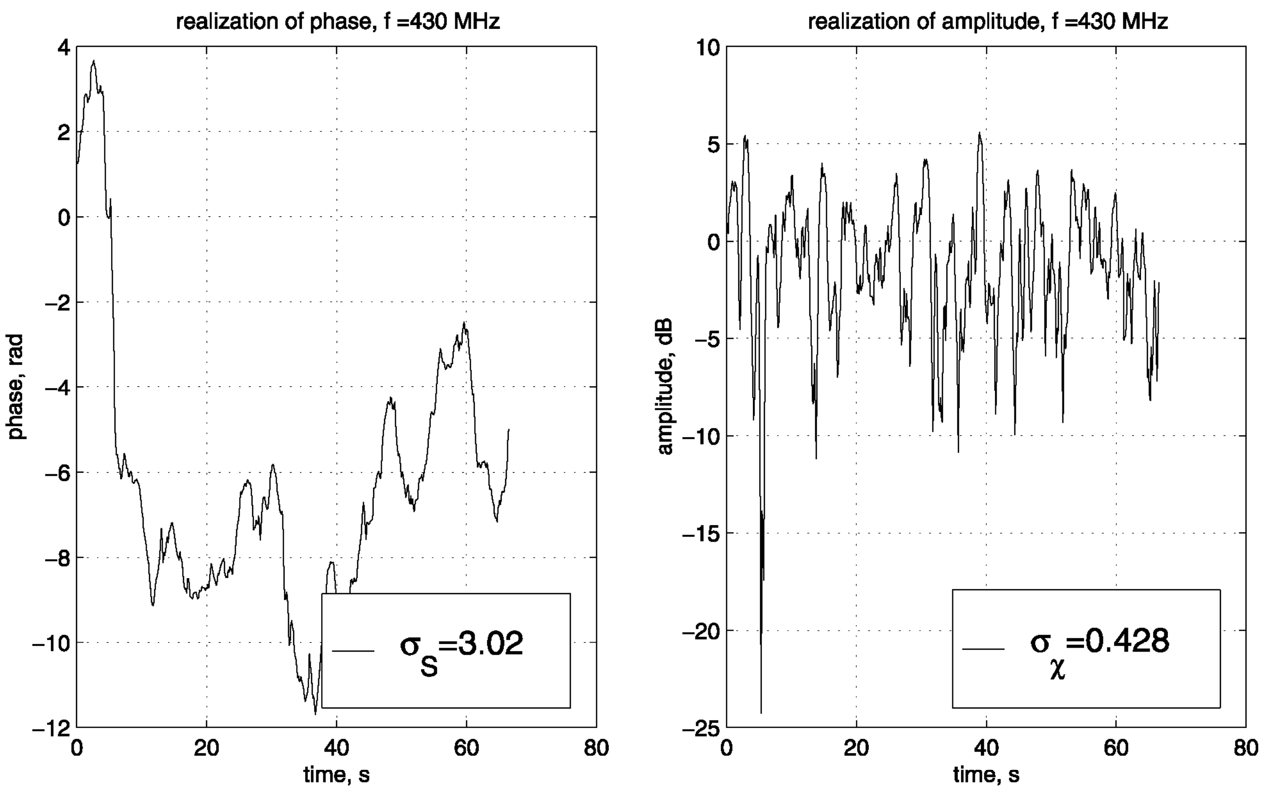

Figure 2. Random time series for the (left) phase (radians) and (right) amplitude of the field. See color version of this figure in the HTML.

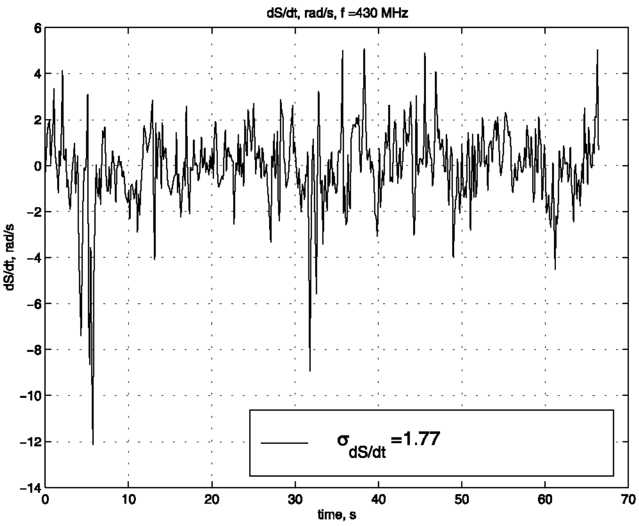

Figure 3. Rate of phase change obtained by taking the time derivative of the phase variation shown in Figure 2. See color version of this figure in the HTML.

[23] The following series of figures illustrate how the hybrid technique works. Figure 1 shows the two-dimensional realizations of the random distributions of phase (left panel) and log amplitude (right panel) at the Earth's surface. When the hypothesis of the frozen drift is assumed the distributions represented in Figure 1 are transformed into random time series for the phase (left panel in Figure 2) and amplitude (right panel in Figure 2) of the field. Then the rate of phase change, shown in Figure 3, is determined by taking the time derivative of the phase variation as shown in Figure 2. This is an important parameter when assessing the probability of phase lock loss at the moments of the deepest fading of the signal [c.f. *Gherm et al.*, 2003]. In Figure 4 the frequency spectra of the phase and log amplitude fluc-

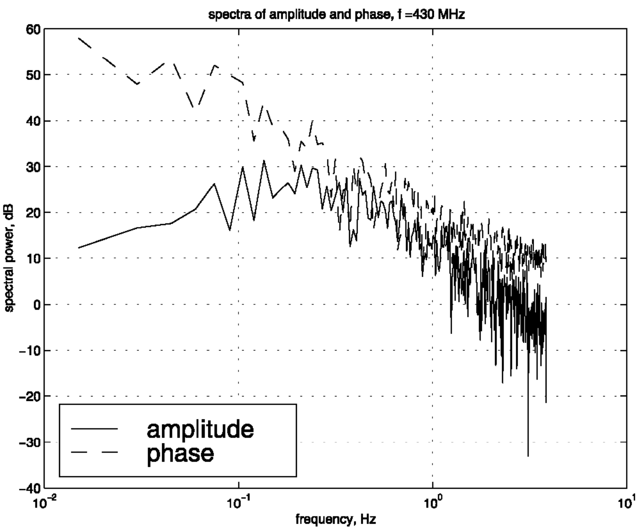

Figure 4. Frequency spectra of the phase and log-amplitude fluctuations of the field simulated for the regime of strong scintillation (S_4 = 0.727). See color version of this figure in the HTML.

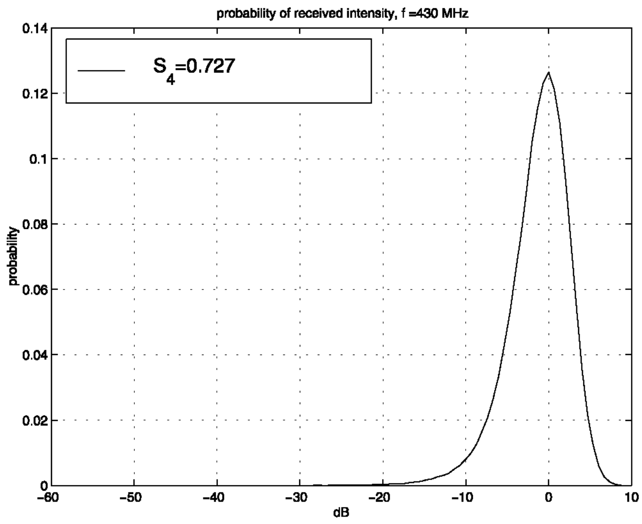

Figure 5. The probability density function for the amplitude fluctuations showing an asymmetric form for the regime of strong scintillation (S_4 = 0.727). See color version of this figure in the HTML.

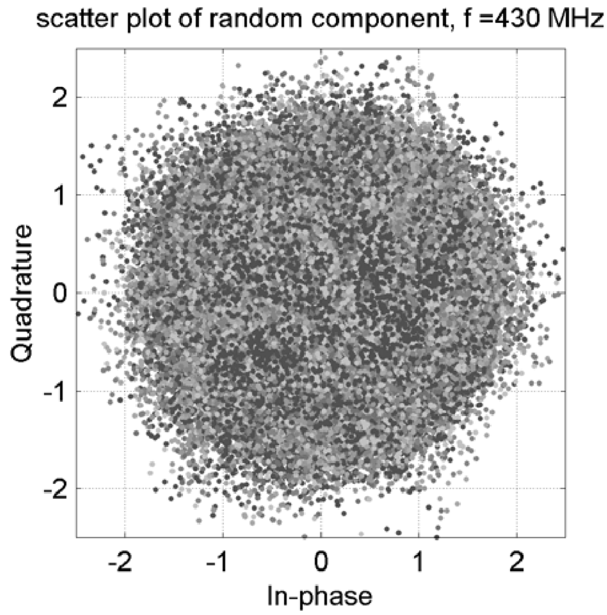

Figure 6. Scatterplot of the random walk of the phasor $R(\mathbf{r}\ , \omega, T)$, representing the field propagated through the fluctuating ionosphere for the case of strong scattering. See color version of this figure in the HTML.

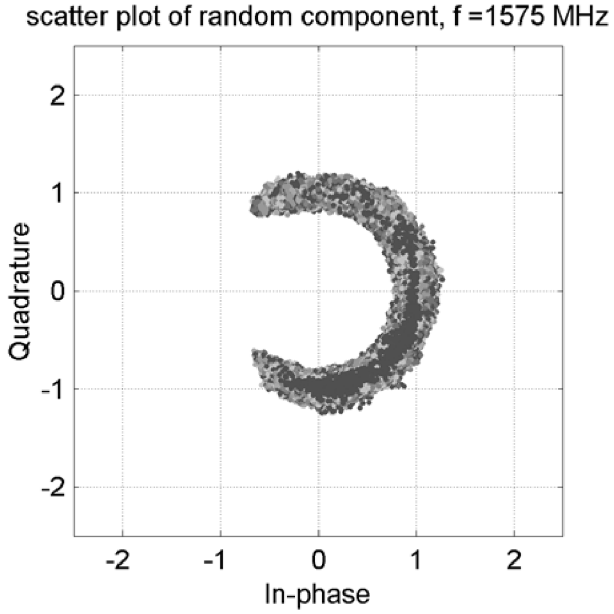

Figure 7. Scatterplot of the random walk of the phasor $R(\mathbf{r}\ , \omega, T)$ for the case of weak scattering (transmission frequency = 1.545 GHz and S_4 = 0.158). See color version of this figure in the HTML.

tuations of the field are presented, simulated for the regime of strong scintillation $S_4 = 0.727$. Considering the forms of the log amplitude and phase spectra, the regime of strong scintillation is characterized by the fact that both curves do not merge at the high-frequency tail. This distinguishes the case of strong scintillation from the case of weak scintillation, when the curves of the phase and log amplitude spectra merge at high frequencies. Additionally, in the case of strong scintillation, the probability density function for the amplitude fluctuations is of asymmetric form, as seen in Figure 5. Finally, Figure 6 demonstrates the random walk of the phasor $R(\mathbf{r}, \omega, T)$ representing the field propagated through the fluctuating ionosphere. By contrast the scatterplot of the random component is shown in Figure 7 for the case of weak scattering (transmission frequency 1.545 GHz and S_4 = 0.158).

5. Conclusions

[24] The presented technique is capable of producing statistical characteristics and simulating time realisations of the field (including regime of strong amplitude fluctuations) for a wide range of the input parameters: (1) coordinates of the satellite and point of observation; (2) slant electron density profile along a given path; (3) zenith angle of a satellite; (4) magnetic azimuth of the plane of propagation; (5) magnetic field dip angle at the pierce point; (6) the parameters of the random irregularities: their spectral index, their outer scale along and perpendicular to the geomagnetic field direction, their effective drift velocity and the variance of the fractional electron density fluctuations. Additionally, the developed technique allows investigation of the problems of the spatial correlation of the signals on transionospheric links and, in particular, for differential GPS [cf. *Gherm et al.*, 2001b].

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