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QUANTIZATION OF FREE ELECTROMAGNETIC FIELD

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1 Quantization of the Free Electromagnetic Field

Dirac combined the wave and particle like aspects of light. Wave nature shows all the interference phenomena. Particle nature shows the exitation of a specific atom absorbing one photon of energy.

Classical field fails to explain

1. Spontaneous emission
2. Atomic decay
3. Lamb shift
4. Photon statistics

An interesting consequences of the quantization of the radiation is the fluctuations associated with the zero-point energy or so called vacuum fluctuations. These fluctuations have no classical analog and are responsible for many interacting phenomena in quantum optics.

1.1 Spontaneous Emission and Atomic Decay

A phenomena which we described phenomenologically in our treatment of semi-classical theory requires a quantum field. Spontaneous emission is often said to be the result of stimulating the atom by vacuum fluctuations.

1.2 Lamb Shift

According to the classical description of the field (*Dirac theory*) the $2S_{\frac{1}{2}}$ and $2P_{\frac{1}{2}}$ states in the hydrogen atom should have equal energies. Experimentally the two levels differ by approximately 1057 MHz. a fully quantized treatment of the field and atomic systems gives impressive agreement with the experimentally observed shift, because of the radiative correction due to the interaction between the atomic electron and the vacuum shift the $2S_{\frac{1}{2}}$ level higher in energy by around 1057 MHz relative to the $2P_{\frac{1}{2}}$ level.

1.3 Photon Statistics

In order to explain the photon statistics the concept of a particle, the photon is either necessary or convenient. For the quantization of the electromagnetic field in free space, it is convenient to begin with the classical description of the field based on Maxwell's equations. In MKS system

$$\begin{aligned} \nabla . D &= 0 \\ \nabla . B &= 0 \\ \nabla \times E &= -\frac{\partial B}{\partial t} \\ \nabla \times H &= \frac{\partial D}{\partial t} \end{aligned}$$

where in free space

$$\begin{aligned} D &= \epsilon_0 E \\ B &= \mu_0 H \end{aligned}$$

here ϵ_0 and μ_0 are the free space permitivity and permeability respectively and

$$\mu_0 \epsilon_0 = \frac{1}{c^2}$$

where c is the speed of light in vacuum. Using these Maxwell,s equations we know that $E(r,t)$ and also $B(r,t)$ satisfies the wave equation

$$\nabla^2 E - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$$

Following the Dirac approach we associate each mode of the radiation field with a quantized simple Harmonic oscillator. Energy of the Harmonic oscillator (classically) is given by the hamiltonian

$$H = \frac{1}{2}m\omega^2 x^2 + \frac{p^2}{2m}$$

and quantum mechanically it is written as

$$H = \frac{1}{2}m\omega^2 \hat{x}^2 + \frac{\hat{p}^2}{2m}$$

2 Mode Expansion of The Field

2.1 Quantization of Field Inside the Cavity of Length L

Electric field is linearly polarized in the x direction. Expanding the field in the normal modes of the cavity

$$E_x(z,t) = \sum_j A_j q_j(t) \sin(k_j z)$$

j corresponds to different modes such that

$$L = j\frac{\lambda}{2};\ \lambda = \frac{2\pi}{k} \text{ and } L = \frac{j\pi}{k_j}, \text{where } j = 1, 2, 3, ...$$

Where q_j is the normal mode amplitude with the dimensions of length (position) and

$$\begin{aligned} A_j &= \left(\frac{2\omega_j^2 m_j}{V\epsilon_0}\right)^{1/2} \\ \text{where } \omega_j &= ck_j = \frac{j\pi c}{L} \end{aligned}$$

is the cavity eigen frequency. $V = LA$ is the volume (A is the transverse area of the optical resonator) m_j is a constant with the dimensions of mass, included to make an anology with SHO nothing to do with mass of photon. The E.M.F is assumed to be transverse with electric field polarized in the x-direction. Such field satisfies

$$\nabla . E = 0$$

The nonvanishing component of the magnetic field in the cavity is obtained by using Maxwell's 4rth equation i.e,

$$\nabla \times H = \frac{\partial D}{\partial t} = \epsilon_0 \frac{\partial E}{\partial t}$$

$$\begin{aligned} \nabla \times H &= \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{pmatrix} \\ &= \hat{i}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) + \hat{j}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) + \hat{k}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) \end{aligned}$$

x-component of $(\nabla \times H)_x$ is written as

$$\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)_x = \epsilon_0 \frac{\partial E_x}{\partial t}$$

$H_z = 0$, z is the direction of propagation

$$\begin{aligned} -\frac{\partial H_y}{\partial z} &= \epsilon_0 \frac{\partial}{\partial t}\left(\sum_j A_j q_j(t) \sin(k_j z)\right) \\ &= \epsilon_0 \sum_j A_j \dot{q_j}(t) \sin(k_j z). \end{aligned}$$

As

$$\sin(k_j z) = -\frac{1}{k_j}\frac{\partial}{\partial z}\cos(k_j z),$$

putting the value of $\sin(k_j z)$ in the above equation we can write

$$\begin{aligned} -\frac{\partial H_y}{\partial z} &= -\frac{\partial}{\partial z}\left(\sum_j A_j \left(\frac{\dot{q_j}(t)\epsilon_0}{k_j}\right)\cos(k_j z)\right), \\ H_y &= \sum_j A_j \left(\frac{\dot{q_j}(t)\epsilon_0}{k_j}\right)\cos(k_j z). \end{aligned}$$

The classical Hamiltonian of the field i.e, the (total energy of the field) is

$$H = \frac{1}{2}\int_v d\tau \left(\epsilon_0 E_x^2 + \mu_0 H_y^2\right)$$

where the integration is over the volume of the cavity. Substituting the values of E_x and H_y in the above equation and performing the integration we get,

$$\begin{aligned} H &= \frac{1}{2}\sum_j\left(m_j\omega_j^2q_j^2+m_j\dot{q}_j^2\right) \\ &= \frac{1}{2}\sum_j\left(m_j\omega_j^2q_j^2+\frac{p_j^2}{m_j}\right) \end{aligned}$$

where $p_j = m_j\dot{q}_j$ is the canonical momentum of the jth mode. The above equation expresses the hamiltonian of the radiation field as a sum of independent oscillator energies. Each mode of the field is dynamically equivalent to a mecahanical harmonic oscillator.

3 Quantization

The present dynamical problem can be quantized by identifying q_j and p_j as operators, which obey the commutation relations

$$\begin{aligned} \left[\hat{q}_j,\hat{p}_{j'}\right] &= i\hbar\delta_{jj'}. \\ \left[\hat{q}_j,\hat{q}_{j'}\right] &= \left[\hat{p}_j,\hat{p}_{j'}\right]=0. \end{aligned}$$

It can be transformed as

$$\widehat{a}_j=\frac{1}{\sqrt{2m_j\hbar\omega_j}}\left(m_j\omega_j\widehat{q}_j+i\widehat{p}_j\right)\exp\left(i\omega_jt\right)$$

and

$$\widehat{a}_j^{\dagger}=\frac{1}{\sqrt{2m_j\hbar\omega_j}}\left(m_j\omega_j\widehat{q}_j-i\widehat{p}_j\right)\exp\left(-i\omega_jt\right)$$

$$\begin{aligned} \widehat{q}_j &= \left(\widehat{a}_j\exp\left(-i\omega_jt\right)+\widehat{a}^{\dagger}{}_j\exp\left(i\omega_jt\right)\right)\sqrt{\frac{\hbar}{2m_j\omega_j}} \\ \widehat{p}_j &= -i\sqrt{\frac{m_j\omega_j\hbar}{2}}\left(\widehat{a}_j\exp\left(-i\omega_jt\right)-\widehat{a}^{\dagger}{}_j\exp\left(i\omega_jt\right)\right) \end{aligned}$$

The commutation relations between $\widehat{a}_j$ and $\widehat{a}^{\dagger}$ follow from those between $\widehat{q}_j$ and $\widehat{p}_j$,

$$\begin{aligned} \left[\hat{a}_j,\hat{a}_j^{\dagger}\right] &= \frac{1}{2m_j\hbar\omega_j}\left[-im_j\omega_j\left[\hat{q}_j,\hat{p}_j\right]+im_j\omega_j\left[\hat{p}_j,\hat{q}_j\right]\right] \\ &= \frac{1}{2m_j\hbar\omega_j}\left[-im_j\omega_j\left(i\hbar\right)+im_j\omega_j\left(-i\hbar\right)\right] \\ &= 1. \end{aligned}$$

Similarly

$$\left[\hat{a}_j, \hat{a}_{j'}\right] = \left[\hat{a}^\dagger{}_j, \hat{a}^\dagger{}_{j'}\right] = 0$$

$$\left[\hat{a}_j, \hat{a}^\dagger{}_{j'}\right] = \delta_{jj'}$$

The operators $\hat{a}$ and $\hat{a}^\dagger$ are referred to as the destruction and creation operators, they are not hermitian. Substituting the value of $\widehat{q}_j$ and $\widehat{p}_j$ in the equation for Hamiltonian we get

$$\begin{aligned} H &= \sum_j \frac{1}{2} m_j \omega_j^2 \left(\frac{\hbar}{2m_j\omega_j}\right)\left(a_j^2 \exp\left(-2i\omega_j t\right) + a^{\dagger 2}{}_j \exp\left(2i\omega_j t\right) + \hat{a}_j \hat{a}_j^\dagger + \hat{a}_j^\dagger \hat{a}_j\right) \\ &\quad + \frac{1}{2m_j}\left(-\frac{m_j \hbar\omega_j}{2}\right)\left(a_j^2 \exp\left(-2i\omega_j t\right) + a^{\dagger 2}{}_j \exp\left(2i\omega_j t\right) - \hat{a}_j \hat{a}_j^\dagger - \hat{a}_j^\dagger \hat{a}_j\right) \\ &= \sum_j \left(\frac{\hbar\omega_j}{2}\right)\left(\hat{a}_j \hat{a}_j^\dagger + \hat{a}_j^\dagger \hat{a}_j\right) \end{aligned}$$

As

$$\begin{aligned} \left[\hat{a}_j, \hat{a}_j^\dagger\right] &= 1 \\ \left(\hat{a}_j \hat{a}_j^\dagger - \hat{a}_j^\dagger \hat{a}_j\right) &= 1 \\ \hat{a}_j \hat{a}_j^\dagger &= \hat{a}_j^\dagger \hat{a}_j + 1 \end{aligned}$$

thus we can write

$$H = \sum_j \hbar\omega_j \left(\hat{a}_j^\dagger \hat{a}_j + \frac{1}{2}\right)$$

In terms of $\hat{a}_j$and $\hat{a}_j^\dagger$, the electric and magnetic fields can be written as

$$\begin{aligned} E_x\left(z,t\right) &= \sum_j \varepsilon_j \left(\widehat{a}_j \exp\left(-i\omega_j t\right) + \widehat{a}^\dagger{}_j \exp\left(i\omega_j t\right)\right)\sin\left(k_j z\right) \\ H_y\left(z,t\right) &= -i\epsilon_0 c \sum_j \varepsilon_j \left(\widehat{a}_j \exp\left(-i\omega_j t\right) - \widehat{a}^\dagger{}_j \exp\left(i\omega_j t\right)\right)\cos\left(k_j z\right) \end{aligned}$$

where the quantity

$$\varepsilon_j = \left(\frac{\hbar\omega_j}{\epsilon_0 V}\right)^{1/2}$$

has the dimensions of an electric field.

3.1 Quantization of Field Inside a Large Cavity of Finite Length L

Consider the field in a large but finite cubic cavity of side L. We consider the running wave solutions instead of the standing wave solutions. The classical electric and magnetic field can be expanded in terms of plane waves.

$$E(r,t) = \sum_k \hat{\epsilon}_k \varepsilon_k \alpha_k \exp(-i\omega_k t + ik.r) + c.c$$

using Maxwell's equation i.e,

$$\nabla \times H = \frac{\partial D}{\partial t}$$

we get

$$H(r,t) = \frac{1}{\mu_0} \sum_k \frac{k \times \hat{\epsilon}_k}{\omega_k} \varepsilon_k \alpha_k \exp(-i\omega_k t + ik.r) + c.c$$

where the summation is taken over an infinite discrete set of values of wave vector $k = (k_x, k_y, k_z)$, ϵ_k is a unit polarization vector, α_k is a dimensionless amplitude and

$$\varepsilon_k = \left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right)^{1/2}$$

Periodic boundary conditions require that

$$k_x = \frac{2\pi n_x}{L}, k_y = \frac{2\pi n_y}{L}, k_z = \frac{2\pi n_z}{L}$$

where n_x, n_y, n_z are integers $0, \pm 1, \pm 2, \cdots$. A set of numbers (n_x, n_y, n_z) defines a mode of electromagnetic field. For transverse field

$$\nabla.D = 0$$

which requires

$$\vec{k}.\hat{\epsilon}_k = 0$$

There are two independent polarization directions of $\hat{\epsilon}_k$ for each k. Changing from a discrete distribution of modes to a continous distribution i.e,

$$\sum_k \Longrightarrow 2\left(\frac{L}{2\pi}\right)^3 \int d^3k$$

where factor of 2 accounts for two possible states of polarization. The number of modes available in a cavity is infinite, however the number of modes whose frequency lies between ω and $\omega + d\omega$ is finite. This is the same number of field modes having the magnitude of k, between k and $k + dk$. Making transformation from rectangular coordinates (k_x, k_y, k_z) to the polar coordinates $(k \sin\theta \cos\phi, k \sin\theta \sin\phi, k \cos\theta)$, the volume element in k-space is written as

$$
\begin{aligned}
d^3k &= k^2 dk \sin\theta d\theta d\phi \\
&= \frac{\omega^2}{c^3} d\omega \sin\theta d\theta d\phi.
\end{aligned}
$$

The total number of modes in the volume L^3 in the range between ω and $\omega + d\omega$ is given by

$$
\begin{aligned}
dN &= 2\left(\frac{L}{2\pi}\right)^3 \frac{\omega^2 d\omega}{c^3} \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi \\
&= \frac{L^3\omega^2}{\pi^2 c^3} d\omega
\end{aligned}
$$

Radiation field is quantized by identifying α_k and α_k^* by the harmonic oscillator operators $\hat{a}_k$and $\hat{a}_k^\dagger$ respectively, which satisfiy the commutation relation

$$
\left[\hat{a}_k, \hat{a}_k^\dagger\right] = 1
$$

The quantized electric and magnetic fields takes the form

$$
\begin{aligned}
E(r,t) &= \sum_k \hat{\epsilon}_k \varepsilon_k \hat{a}_k \exp(-i\omega_k t + ik.r) + H.C \\
H(r,t) &= \frac{1}{\mu_0} \sum_k \frac{k \times \hat{\epsilon}_k}{\omega_k} \varepsilon_k \hat{a}_k \exp(-i\omega_k t + ik.r) + H.C
\end{aligned}
$$

where H.C is Hermetian conjugate. Seperating positive and negative frequency parts of these field operators

$$
\begin{aligned}
E^+(r,t) &= \sum_k \hat{\epsilon}_k \varepsilon_k \hat{a}_k \exp(-i\omega_k t + ik.r) \\
E^-(r,t) &= \sum_k \hat{\epsilon}_k \varepsilon_k \hat{a}_k^\dagger \exp(i\omega_k t - ik.r)
\end{aligned}
$$

where $E^+(r,t)$ is the annihilation operator and $E^-(r,t)$ is the creation operator.

4 Fock or Number States of Radiation Field

Consider a single mode of the field of frequency ω having creation and annihilation operators $\widehat{a}^\dagger$ and $\widehat{a}$ respectively. Let $|n\rangle$ be the energy eigen state corresponding to the energy eigen value E_n, i,e.

$$\begin{aligned} H|n\rangle &= \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)|n\rangle \\ &= E_n|n\rangle \end{aligned} \tag{1}$$

applying operator $\hat{a}$ from the left of the eigenstates we have

$$Ha|n\rangle = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)a|n\rangle \tag{2}$$

$$\begin{aligned} \left[a, a^\dagger\right] &= aa^\dagger - a^\dagger a = 1 \\ \Longrightarrow & \quad aa^\dagger - 1 = a^\dagger a \end{aligned}$$

Putting in Eq. (2) we get

$$\begin{aligned} Ha|n\rangle &= \hbar\omega\left(aa^\dagger - 1 + \frac{1}{2}\right)a|n\rangle \\ &= \hbar\omega\left(aa^\dagger a - a + \frac{a}{2}\right)|n\rangle \\ &= a\hbar\omega\left(a^\dagger a + \frac{1}{2} - 1\right)|n\rangle \\ &= a(\hbar\omega(a^\dagger a + \frac{1}{2}) - \hbar\omega)|n\rangle \\ &= a(E_n - \hbar\omega)|n\rangle \\ &= (E_n - \hbar\omega)a|n\rangle \end{aligned}$$

where $a|n\rangle$ is an energy eigen state with eigen value $(E_n - \hbar\omega)$. Operator a lowers the energy and therefore it is called annihilation, destruction or absorption operator.

$$\Longrightarrow |n-1\rangle = \frac{a}{\alpha_n}|n\rangle,$$

is an energy eigen state but with the reduced eigen value i,e.

$$\begin{aligned} E_{n-1} &= (E_n - \hbar\omega) \\ H|n-1\rangle &= E_{n-1}|n-1\rangle, \end{aligned}$$

and α_n is a constant which will be determined from the normalization condition,

$$\langle n-1 \mid n-1\rangle = 1.$$

If we repeat this procedure n times we move down the energy ladder in steps of $\hbar\omega$ until we obtain

$$Ha|0\rangle = (E_0 - \hbar\omega)\,a|0\rangle$$

E_0 is the ground state energy . $E_n - \hbar\omega$ is smaller than E_0 i,e, $E_n - \hbar\omega$ is negative. Since energy eigen value cannot be negative

$$a|0\rangle = 0$$

The state $|0\rangle$ is called the vaccum state. (in which no photon is excited).

$$\begin{aligned} H|0\rangle &= \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)|0\rangle \\ &= \frac{1}{2}\hbar\omega|0\rangle \\ \Longrightarrow & \; E_0 = \frac{1}{2}\hbar\omega \end{aligned}$$

is the energy of the ground state. Now we go step by step up as

$$\begin{aligned} E_{n-1} &= E_n - \hbar\omega \\ E_n &= E_{n-1} + \hbar\omega \end{aligned}$$

For $n = 1$ we can write

$$\begin{aligned} E_1 &= E_0 + \hbar\omega \\ &= \frac{1}{2}\hbar\omega + \hbar\omega = \frac{3}{2}\hbar\omega \end{aligned}$$

Similarly

$$\begin{aligned} E_2 &= E_1 + \hbar\omega \\ &= \frac{3}{2}\hbar\omega + \hbar\omega \\ &= \frac{5}{2}\hbar\omega \end{aligned}$$

It can also be written as

$$\begin{aligned} E_2 &= (2 + \frac{1}{2})\hbar\omega, \\ &\vdots \\ E_n &= \left(n + \frac{1}{2}\right)\hbar\omega \end{aligned}$$

$$\Longrightarrow H \mid n\rangle = E_n \mid n\rangle$$

$$\begin{aligned} \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)|n\rangle &= \left(n + \frac{1}{2}\right)\hbar\omega|n\rangle \\ &\Longrightarrow a^\dagger a|n\rangle = n|n\rangle \end{aligned}$$

$|n\rangle$ is also an energy eigen state of the number operator

$$n = a^\dagger a$$

The normalization constant can be now calculated

$$\langle n-1 \mid n-1\rangle = 1$$

as

$$\begin{aligned}
|n-1\rangle &= \frac{\hat{a}}{\alpha_n}|n\rangle \\
\langle n-1| &= \frac{\langle n| a^\dagger}{\alpha_n^*} \\
\langle n-1 \quad | \quad n-1\rangle &= \frac{\langle n| a^\dagger a |n\rangle}{\alpha_n^* \alpha_n} \\
&= \frac{1}{|\alpha_n|^2}\langle n| a^\dagger a |n\rangle \\
1 &= \frac{n}{|\alpha_n|^2}\langle n \mid n\rangle = |\alpha_n|^2 = n \\
\alpha &= \sqrt{n}e^{i\phi}
\end{aligned}$$

If we take the phase of the normalization constant α_n to be zero then $\alpha_n = \sqrt{n}$

$$\begin{aligned}
a|n\rangle &= \alpha_n \mid n-1\rangle \\
&= \sqrt{n} \mid n-1\rangle
\end{aligned}$$

now for operator $a^\dagger$

$$\begin{aligned}
Ha^\dagger|n\rangle &= \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)a^\dagger|n\rangle \\
&= \hbar\omega\left(a^\dagger a a^\dagger + \frac{a^\dagger}{2}\right)|n\rangle
\end{aligned}$$

using $aa^\dagger = a^\dagger a + 1$,

$$\begin{aligned}
Ha^\dagger \quad | \quad n\rangle &= \hbar\omega\left(a^\dagger a^\dagger a + a^\dagger + \frac{a^\dagger}{2}\right)|n\rangle \\
&= a^\dagger\left(E_n + \hbar\omega\right)|n\rangle \\
(\hbar\omega\left(a^\dagger a + \frac{1}{2}\right)a^\dagger|n\rangle) &= \left(E_n + \hbar\omega\right)a^\dagger|n\rangle
\end{aligned}$$

Thus $a^\dagger|n\rangle$ is also an energy eigen state of the field with eigen value $E_n+\hbar\omega$. We define

$$\begin{aligned}
|n+1\rangle &= \frac{\hat{a}^\dagger}{\beta_n}|n\rangle \\
E_{n+1} &= E_n + \hbar\omega \\
&\implies H|n+1\rangle = E_{n+1}|n+1\rangle
\end{aligned}$$

using the same procedure we get

$$a^{\dagger}|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

A repeated use of the above equation gives,

$$|n\rangle = \frac{\left(a^{\dagger}\right)^{n}}{\sqrt{n!}}\;|\,0\rangle$$

The energy eigen states $|n\rangle$ are called fock states or photon number states. They form a complete set of state i,e.

$$\sum_{n=0}^{\infty}|n\rangle\langle n| = 1$$

The energy eigen value are discrete in contrast to classical electromagnetic theory where energy can have any value. State vector is written as the superposition of energy eigen states. i,e

$$|\Psi\rangle = \sum_{n} c_n\,|n\rangle$$

where c_n are complex coefficients. The energy $E_0 = \frac{1}{2}\hbar\omega$ is called zero-point energy. The energy levels for Q.M oscillations associated with the electromagnetic field are given as

$$\begin{aligned}
-\,-\,-\,\underset{a^{\dagger}\uparrow}{-}\,-\,-\,-\,-\,-E_{n+1} &= \left(n+\frac{3}{2}\right)\hbar\omega \\
-\,-\,-\,\underset{a\downarrow}{-}\,-\,-\,-\,-\,-\,-\,-\;E_{n} &= \left(n+\frac{1}{2}\right)\hbar\omega \\
-\,-\,-\,-\,-\,-\,-\,-\,-\,-E_{n-1} &= \left(n-\frac{1}{2}\right)\hbar\omega \\
&\;\vdots \\
-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\;E_{2} &= \frac{5}{2}\hbar\omega \\
-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\;E_{1} &= \frac{3}{2}\hbar\omega \\
-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\;E_{0} &= \frac{1}{2}\hbar\omega
\end{aligned}$$

The operators a and $a^{\dagger}$ are not hermitian but some of the combinations are Hermitian such as,

$$\begin{aligned}
a_1 &= \left(a+a^{\dagger}\right)/2, \\
a_2 &= (a-a^{\dagger})/2i.
\end{aligned}$$

Different energy eigen states of the field are orthagonal. The only non-vanishing matrix elements of a and $a^{\dagger}$are of the types;

$$\begin{aligned}\langle n-1 \quad | \quad a \mid n\rangle &= \sqrt{n}\\ \langle n+1 \quad | \quad a^{\dagger} \mid n\rangle &= \sqrt{n+1}\end{aligned}$$

An important property of $\mid n\rangle$ is that the expectation value of the single mode linearly polarized field operator vanishes. Using

$$E_x\left(z,t\right) = \varepsilon(ae^{-i\omega t} + a^{\dagger}e^{i\omega t})\sin kz,$$

or

$$E\left(r,t\right) = \varepsilon a e^{-i\omega t+ik.r} + \varepsilon^{*}a^{\dagger}e^{i\omega t-ik.r}$$

$$\langle n \mid E\left(r,t\right) \mid n\rangle = \varepsilon\langle n \mid a \mid n\rangle e^{-i\omega t+ik.r} + \varepsilon^{*}\langle n \mid a^{\dagger} \mid n\rangle e^{i\omega t-ik.r} = 0$$

Now in order to find the average value of $\langle E^2\rangle$, we write

$$\langle n \quad | \quad E^2 \mid n\rangle = \left|\varepsilon\right|^2\langle n \mid aa^{\dagger} + a^{\dagger}a \mid n\rangle + \varepsilon^2 e^{-2i\omega t+2ik.r}\langle n \mid a^2 \mid n\rangle$$
$$+\varepsilon^2 e^{2i\omega t-2ik.r}\langle n \quad | \quad \overset{\dagger 2}{a} \mid n\rangle.$$

$$\begin{aligned}\langle E^2\rangle &= (2n+1)\left|\varepsilon\right|^2 = 2\left(n+\frac{1}{2}\right)\left|\varepsilon\right|^2\\ \Delta E^2 &= \langle E^2\rangle - \langle E\rangle^2\\ &= 2\left(n+\frac{1}{2}\right)\left|\varepsilon\right|^2,\end{aligned}$$

as $\langle E\rangle^2 = 0$. For $n = 0$ i,e. in vaccum

$$\Delta E^2 \neq 0,$$

but is equal to

$$\Delta E^2 = \left|\varepsilon\right|^2$$

From these equations we conclude that the mean value is zero but fluctuations are present. These fluctuations are considered to be responsible for spontaneous emission, Lamb shift etc.

5 Multimode Field

The wave function of a single mode field can be written as a linear superposition of photon number states $\mid n\rangle$.

$$\mid \Psi\rangle = \sum_n c_n \mid n\rangle$$

For multimode field

$$H = \sum_k H_k$$

where

$$H_k = \hbar\omega_k \left(a_k^\dagger a_k + \frac{1}{2} \right)$$

The energy eigen state of H_k is

$$H_k \mid n_k\rangle = \hbar\omega_k \left(n_k + \frac{1}{2} \right) \mid n_k\rangle$$

The general eigen state of $H = \sum_k H_k$ can therefore have n_{k1} photon in the 1st mode, n_{k2} in the 2nd mode, n_{kl} in the lth mode and so on and can be written as

$$\mid n_{k1}, n_{k2}, \cdots, n_{kl}, \cdots\rangle$$

Different cavity modes are independent, the state of the total field can be written as the product of the individual modes i,e.

$$\mid n_{k1}, n_{k2}, \cdots, n_{kl}, \cdots\rangle = \mid n_{k1}\rangle \mid n_{k2}\rangle \mid n_{k3}\rangle \cdots$$

As the states of the individual cavity modes are normalized, this implies that the total state of the field is also normalized. This state can be written in short as

$$\mid n_{k1}\rangle \mid n_{k2}\rangle \mid n_{k3}\rangle \cdots = \{\mid n_k\rangle\}$$

The symbols $\{n_k\}$ denote the complete set of numbers that specify the excitation levels of the harmonic oscillators with the cavity modes. The annihilation and creation operators a_{kl} and $a_{kl}^\dagger$ lower and raise the n_lth entry alone.

$$\begin{aligned} a_{kl} \quad &\mid \quad n_{k1}, n_{k2}, \cdots, n_{kl}, \cdots\rangle = \sqrt{n_{kl}} \mid n_{k1}, n_{k2}, \cdots, n_{kl}, \cdots\rangle \\ a_{kl}^\dagger \quad &\mid \quad n_{k1}, n_{k2}, \cdots, n_{kl}, \cdots\rangle = \sqrt{n_{k(l+1)}} \mid n_{k1}, n_{k2}, \cdots, n_{k(l+1)}, \cdots\rangle \end{aligned}$$

The general state vector of the field is a linear superposition of these states.

$$\begin{aligned} \mid \quad &\Psi\rangle = \sum_{nk_1} \sum_{nk_2}, \cdots \sum_{nk_l} \cdots C_{nk_1, nk_2, \cdots, nk_l \cdots} \mid n_{k1}, n_{k2}, \cdots, n_{kl} \cdots\rangle \\ = \quad &\sum_{\{nk\}} C_{\{nk\}} \mid \{n_k\}\rangle \end{aligned}$$

It can also be written as

$$\mid \Psi\rangle = \mid \Psi_{k1}\rangle \mid \Psi_{k2}\rangle .... \mid \Psi_{kl}\rangle$$

6 State of the Field

6.1 The Mode Phase Operators

Classically an electromagnetic field consists of waves with well defined amplitude and phase

$$E(r,t) = \varepsilon_0 \exp i\,(k.r - \omega t + \phi) + c.c$$

Quantized fields has fluctuations associated with both amplitude and phase of the field. An elctromagnetic field in a number state $\mid n\rangle$ has a well defined amplitude but completely uncertain phase

$$\hat{n} \mid n\rangle = n \mid n\rangle$$

where

$$\mid n\rangle = \frac{\left(a^{\dagger}\right)^{n}}{\sqrt{n!}} \mid 0\rangle$$

The single mode quantum mechanical electric field operator is

$$\hat{E} = \overset{\wedge}{\epsilon}\left(\frac{\hbar\omega}{2\epsilon_0 V}\right)^{1/2}\left(a\exp\left(-i\omega t + ik.r\right) + a^{\dagger}\exp\left(i\omega t - ik.r\right)\right)$$

where ϵ is a unit polarization vector. In an analogy to classical field seperate a and $a^{\dagger}$ into a product of amplitude and phase operators. In quantum mechanics there is no unique way or definition of the quantum mechanical phase operators, But there are some conditions on the phase operator.

1- It should have same significance as classical phase in the appropriate limit.

2- It should be associated with hermitian operators (observable quantity).

Consider the phase operator $e^{i\phi}$ defined by the relation (Ref: Susskind and Glogower (1964))

$$a = \sqrt{\widehat{n} + 1 e^{i\phi}}$$

The hermitian conjugate relation is

$$a^{\dagger} = \overset{-i\phi}{e}\sqrt{\widehat{n} + 1}$$

The exponential operator defined, describe the quantum mechanical phase of electromagnetic field,

$$\begin{aligned} e^{i\phi} &= (\widehat{n} + 1)^{-1/2} a \\ and,\ e^{-i\phi} &= a^{\dagger}(\widehat{n} + 1)^{-1/2} \end{aligned}$$

Now from the relation

$$\begin{aligned} \left[a, a^{\dagger}\right] &= aa^{\dagger} - a^{\dagger}a = 1 \\ &\Longrightarrow e^{i\phi}e^{-i\phi} = 1 \end{aligned}$$

The reverse order product of exponential operator is not equal to unity. Now we have to check whether these phase operators are hermitian or not. Find $e^{i\phi} \mid n\rangle$ and $e^{-i\phi} \mid n\rangle$

$$\begin{aligned} e^{i\phi} \quad \mid \quad n\rangle &= (\widehat{n}+1)^{-1/2} a \mid n\rangle \\ &= (\widehat{n}+1)^{-1/2}\sqrt{n} \mid n-1\rangle \end{aligned}$$

As we know that $f(\widehat{n}) \mid n\rangle = f\,(n) \mid n\rangle$, we can write

$$\begin{aligned} e^{i\phi} \quad \mid \quad n\rangle &= (n-1+1)^{-1/2}\sqrt{n} \mid n-1\rangle = \mid n-1\rangle \\ e^{-i\phi} \quad \mid \quad n\rangle &= a^{\dagger}(\widehat{n}+1)^{-1/2} \mid n\rangle = \mid n+1\rangle \end{aligned}$$

The non-vanishing matrix elements of phase operators are

$$\begin{aligned} \langle n-1 \quad &\mid \quad e^{i\phi} \mid n\rangle = 1 \\ \langle n+1 \quad &\mid \quad e^{-i\phi} \mid n\rangle = 1 \end{aligned}$$

and all other types of matrix elements are zero. It implies that $e^{i\phi}$and $e^{-i\phi}$do not satisfy the relation.

$$\langle i \mid \hat{O} \mid j\rangle = \langle j \mid \hat{O} \mid i\rangle^{*}$$

where $O = O^{\dagger}$. They are not hermitain operators but can be combined to produce another pair of operators.

$$\begin{aligned} \overset{\wedge}{\cos}\phi &= \frac{1}{2}\{e^{i\phi}+e^{-i\phi}\} \\ \overset{\wedge}{\sin}\phi &= \frac{1}{2}\{e^{i\phi}-e^{-i\phi}\} \end{aligned}$$

whose non-vanishing matrix elements are

$$\langle n-1 \mid \overset{\wedge}{\cos}\phi \mid n\rangle = \langle n \mid \overset{\wedge}{\cos}\phi \mid n-1\rangle = \frac{1}{2}$$

and

$$\langle n-1 \mid \overset{\wedge}{\sin}\phi \mid n\rangle = -\langle n \mid \overset{\wedge}{\sin}\phi \mid n-1\rangle = \frac{1}{2i}$$

These matrix elements do satisfy equation

$$\langle i \mid \hat{O} \mid j\rangle = \langle j \mid \hat{O} \mid i\rangle^{*}$$

and operators $\overset{\wedge}{\cos}\phi$ and $\overset{\wedge}{\sin}\phi$ are hermitain. They represent the observable phase properties of the electromagnetic field.

$$\begin{aligned} \left[n, \overset{\wedge}{\cos}\phi\right] &= -i\overset{\wedge}{\sin}\phi \\ \left[n, \overset{\wedge}{\sin}\phi\right] &= i\overset{\wedge}{\cos}\phi \end{aligned}$$

Number and phase operators do not commute therefore it is not possible to setup states of the radiation field that are simulataneous eigen states of the operator. The amplitude of an electromagnetic wave associated with $\hat{n}$ and phase associated with $\hat{\cos}\phi$ or $\hat{\sin}\phi$ cannot both be precisely specified. The uncertainity relation

$$\begin{aligned} [q,p] &= i\hbar \\ \Delta q \Delta p &\geq \frac{1}{2}\left|\langle \hbar \rangle\right| \\ &\geq \frac{\hbar}{2} \end{aligned}$$

The uncertainity relation between $\hat{n}$ and phase operators are

$$\begin{aligned} \Delta n \Delta \cos\phi &\geq \frac{1}{2}\left|\langle \hat{\sin}\phi \rangle\right| \\ \Delta n \Delta \sin\phi &\geq \frac{1}{2}\left|\langle \hat{\cos}\phi \rangle\right| \end{aligned}$$

where Δ indicates the root mean square deviation. The uncertainity in the photon number is zero for the state $\mid n\rangle$ i,e.

$$\begin{aligned} \Delta n &= \sqrt{\langle n^2\rangle - \langle n\rangle^2} \\ \langle n &\mid \hat{n} \mid n\rangle = \langle n \mid a^\dagger a \mid n\rangle = n\langle n \mid n\rangle = n \\ \Delta n &= \sqrt{n^2 - n^2} = 0 \end{aligned}$$

For the phase operators,

$$\langle n \mid \hat{\cos}\phi \mid n\rangle = \langle n \mid \hat{\sin}\phi \mid n\rangle = 0$$

Diagonal elements of the phase operators are zero. Now the expectation values of phase operators are

$$\begin{aligned} \langle n &\mid \hat{\cos}\phi \mid n\rangle = \langle \hat{\cos}\phi\rangle = 0 \\ \langle n &\mid \hat{\sin}\phi \mid n\rangle = \langle \hat{\sin}\phi\rangle = 0 \end{aligned}$$

and

$$\begin{aligned} \langle n &\mid \hat{\cos}^2\phi \mid n\rangle = \langle n \mid \hat{\sin}^2\phi \mid n\rangle = \frac{1}{2}, \qquad \text{for } n \neq 0 \\ &= \frac{1}{4} \qquad\qquad\qquad \text{for } n = 0 \end{aligned}$$

The root mean square deviation or uncertanities in phase are

$$\Delta\cos\phi = \Delta\sin\phi = \frac{1}{\sqrt{2}} \qquad \text{for} \;\; \mathrm{n} \neq 0$$

This value shows that phase angle can have any value between 0 to 2π. The expectation value of field operator is zero

$$\langle E \rangle = \langle n \mid E \mid n \rangle = 0$$

and the expectation value of E^2 is given as

$$\begin{aligned} \langle n \quad &| \quad E^2 \mid n \rangle = 2 \left| \varepsilon \right|^2 (n + 1/2) \\ \varepsilon \quad &= \quad \left(\frac{\hbar\omega}{2\epsilon_0 V} \right)^{1/2} \end{aligned}$$

$$\begin{aligned} \Delta E \quad &= \quad \sqrt{\langle E^2 \rangle - \langle E \rangle^2} \\ \Delta E \quad &= \quad \left(\frac{\hbar\omega}{\epsilon_0 V} \right)^{1/2} \left(n + \frac{1}{2} \right)^{\frac{1}{2}} \end{aligned}$$

Field amplitude of electromagnetic wave is given by

$$E_0 = \left(\frac{\hbar\omega}{\epsilon_0 V} \right)^{1/2} \left(n + \frac{1}{2} \right)^{\frac{1}{2}}$$

The actual position of the wave along the horizontal axis is completely undetermined owing to the complete uncertainity in the phase angle. Field oscillates like a sine wave of known frequency ω.

Number states form a useful repersentation for high energy photons, e.g, $\gamma-$rays where the number of photons is very small. They are not the most suitable representation for optical fields where the total number of photons is large. Experimental difficulties have prevented the generation of photon number states with more than a small number of photons.

Most optical fields are either a superposition of number states (pure state) or a mixture of number states (mixed state). Despite this the number states of the electromagnetic field have been used as a basis for several problems in quantum optics including some laser theories. A more appropriate basis for many optical fields are the coherent states. The coherent states have indefinite number of photons which allow them to have a more precisely defined phase than a number state where the phase is completely random.