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THE COHERENT PHOTON STATES

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1 The Coherent Photon States

The single-mode states of physical importance are not the indivisual number states $| n\rangle$ (because the electromagnetic wave generated by practical light source do not have definite numbers of photons), but the linear superposition of states $| n\rangle$. There is a wide variety of possible superposition states.

A superposition state can be constructed for which uncertainties in the expectation values of the phase operators $\hat{\cos}\phi$ and $\hat{\sin}\phi$ are both equal to zero. Such states have $\Delta n = \infty$. They cannot be excited in any real experiment. Another kind is the coherent state. A coherent state has equal amount of uncertainities in amplitude and phase. A field in coherent state is in a minimum uncertainity state. For coherent state an electric field variation approaches that of classical wave of stable amplitude and fixed phase, in the limit of high excitation. They are improtant because, they are the closest quantum mecahnical approach to a classical electromagnetic wave. A single mode laser operated well above threshold generates a coherent state excitation.

The coherent state $| \alpha\rangle$ is the eigen state of the positive frequency part of the electric field operator or the eigen state of the destruction operator of the field.

$$\hat{a} \mid \alpha\rangle = \alpha \mid \alpha\rangle,$$

where α is complex, $| \alpha\rangle$ in terms of linear superposition of number state $| n\rangle$ is given by

$$| \alpha\rangle = \sum_{n=0}^{\infty} C_n \mid n\rangle \tag{1}$$

$$\begin{aligned}
\hat{a} \mid \alpha\rangle &= \sum_{n=0}^{\infty} C_n \hat{a} \mid n\rangle \\
&= \sum_{n=0}^{\infty} C_n \sqrt{n} \mid n-1\rangle \\
&= 0 + C_1\sqrt{1} \mid 0\rangle + C_2\sqrt{2} \mid 1\rangle + \ldots \\
&= \sum_{n=0}^{\infty} C_{n+1}\sqrt{n+1} \mid n\rangle \qquad \text{(a)}
\end{aligned}$$

From eqn(1) multiplying with α we can write

$$\begin{aligned}
\alpha \mid \alpha\rangle &= \sum_{n=0}^{\infty} C_n \alpha \mid n\rangle \\
\hat{a} \mid \alpha\rangle &= \sum_{n=0}^{\infty} C_n \alpha \mid n\rangle \qquad \text{(b)}
\end{aligned}$$

comparing eqn(a) and (b)

$$\begin{aligned}
C_{n+1}\sqrt{n+1} &= C_n\alpha \\
C_n\sqrt{n} &= C_{n-1}\alpha \\
C_n &= \frac{\alpha}{\sqrt{n}}C_{n-1} \\
C_n &= \frac{\alpha}{\sqrt{n}}\frac{\alpha}{\sqrt{n-1}}\frac{\alpha}{\sqrt{n-2}},\ldots,\frac{\alpha}{\sqrt{1}}C_o \\
&= \frac{\alpha^n}{\sqrt{n!}}C_0
\end{aligned}$$

putting in Eqn(1),

$$|\,\alpha\rangle = C_0\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|\,n\rangle$$

the constant C_0 can be found by normalization

$$\begin{aligned}
\langle\alpha\ \ |\ \ \alpha\rangle &= C_0^*C_0\sum_{n=0}^{\infty}\sum_{n=0}^{\infty}\frac{\alpha^{*n}}{\sqrt{n!}}\frac{\alpha^m}{\sqrt{m!}}\langle n\,|\,m\rangle \\
\langle\alpha\ \ |\ \ \alpha\rangle &= |C_0|^2\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!}
\end{aligned}$$

using

$$\sum_{n=0}^{\infty}\frac{x^n}{n!} = e^x$$

we can write

$$\begin{aligned}
1 &= |C_0|^2\, e^{|\alpha|^2} \\
|C_0|^2 &= e^{-|\alpha|^2} \\
C_0 &= e^{-\frac{|\alpha|^2}{2}}
\end{aligned}$$

Therefore

$$|\,\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|\,n\rangle$$

Another way of proving the above relation which interpret $|\,\alpha\rangle$ as a superposition of number state is,

$$|\,\alpha\rangle = \sum_n |\,n\rangle\langle n\,||\,\alpha\rangle \tag{1}$$

where $\sum_n |\,n\rangle\langle n\,| = 1$ is the completeness relation for number state. As

$$|\,n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}\,|\,0\rangle \tag{2}$$

$$\langle n \mid = \langle 0 \mid \frac{(a)^n}{\sqrt{n!}} \tag{3}$$

Putting in Eqn(1) we can write as

$$\mid \alpha\rangle = \sum_n \mid n\rangle\langle 0 \mid \frac{(a)^n}{\sqrt{n!}} \mid \alpha\rangle$$

we know that

$$\begin{aligned}
\hat{a} \quad & \mid \quad \alpha\rangle = \alpha \mid \alpha\rangle \\
(\hat{a})^n \quad & \mid \quad \alpha\rangle = \alpha^n \mid \alpha\rangle \\
\Longrightarrow & \quad \mid \alpha\rangle = \sum_n \mid n\rangle \frac{(\alpha)^n}{\sqrt{n!}} \langle 0 \mid \alpha\rangle \qquad (4)
\end{aligned}$$

The value of $\langle 0 \mid \alpha\rangle$ is obtained by normalization i,e

$$\langle \alpha \mid \alpha\rangle = 1$$

$$\begin{aligned}
\langle \alpha \quad \mid \quad \alpha\rangle &= \sum_n \sum_m \frac{(\alpha^*)^m}{\sqrt{m!}} \frac{(\alpha)^n}{\sqrt{n!}} \langle m \mid n\rangle \mid\langle 0 \mid \alpha\rangle\mid^2 \\
&= \sum_n \frac{(\alpha^*\alpha)^n}{n!} \mid\langle 0 \mid \alpha\rangle\mid^2 \\
&= \sum_n \frac{(\mid\alpha\mid^2)^n}{n!} \mid\langle 0 \mid \alpha\rangle\mid^2 \\
&\Longrightarrow 1 = e^{\mid\alpha\mid^2} \mid\langle 0 \mid \alpha\rangle\mid^2 \\
\mid\langle 0 \mid \alpha\rangle\mid^2 &= e^{-\mid\alpha\mid^2} \\
\langle 0 \quad \mid \quad \alpha\rangle &= e^{-\frac{\mid\alpha\mid^2}{2}}
\end{aligned}$$

Putting in Eqn(4) we get

$$\mid \alpha\rangle = e^{-\frac{\mid\alpha\mid^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \mid n\rangle$$

hence proved.

Some other representations of the coherent state

$$| \alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} | n\rangle$$

$$| n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} | 0\rangle$$

$$| \alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{(a^\dagger)^n}{\sqrt{n!}} | 0\rangle$$

$$| \alpha\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger} | 0\rangle$$

since we know that $e^{-\alpha^* a} | 0\rangle = | 0\rangle$

$$\Longrightarrow \quad | \alpha\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger} e^{-\alpha^* a} | 0\rangle$$

$$D(\alpha) = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger} e^{-\alpha^* a}$$

where $D(\alpha)$ is called the displacement operator.

$$| \alpha\rangle = D(\alpha) | 0\rangle$$

1.1 Baker-Hausdorff identity

If

$$[[A, B], A] = [[A, B], B] = 0$$

then

$$e^{A+B} = e^{-[A,B]/2} e^A e^B$$

Let we have $A = \alpha a^\dagger$ and $B = -\alpha^* a$

$$\begin{aligned} e^{\alpha a^\dagger - \alpha^* a} &= e^{-\frac{1}{2}[-\alpha a^\dagger \alpha^* a + \alpha^* a \alpha a^\dagger]} e^{\alpha a^\dagger} e^{-\alpha^* a} \\ &= e^{-\frac{1}{2}|\alpha|^2[-a^\dagger a + a a^\dagger]} e^{\alpha a^\dagger} e^{-\alpha^* a} \\ &= e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger - \alpha^* a} \end{aligned}$$

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$$

another definition

$$\Longrightarrow | \alpha\rangle = e^{\alpha a^\dagger - \alpha^* a} | 0\rangle$$

The other equivalent antinormal form of $D(\alpha)$ is obtained by using $A = -\alpha^* a$ and $B = \alpha a^\dagger$, then we get

$$D(\alpha) = e^{\frac{|\alpha|^2}{2}} e^{-\alpha^* a} e^{\alpha a^\dagger}$$

or by using

$$\begin{aligned} e^{A+B} &= e^{\frac{1}{2}[A,B]}e^{B}e^{A} \\ D(\alpha) &= e^{\frac{|\alpha|^2}{2}}e^{-\alpha^* a}e^{\alpha a^\dagger} \end{aligned}$$

The operator $D(\alpha)$ is a unitary operator. i,e.

$$D^\dagger(\alpha) = D(-\alpha) = D^{-1}(\alpha)$$

It acts as a displacement operator upon the amplitudes a and $a^\dagger$ i,e.

$$\begin{aligned} D^{-1}(\alpha)\hat{a}D(\alpha) &= a+\alpha \\ D^{-1}(\alpha)\hat{a}^\dagger D(\alpha) &= a^\dagger+\alpha^* \end{aligned}$$

This can be proved by writing

$$\begin{aligned} D(\alpha) &= e^{\alpha a^\dagger - \alpha^* a} \\ D^\dagger(\alpha) &= e^{\alpha^* a - \alpha a^\dagger} = D^{-1}(\alpha) \end{aligned}$$

using these equations we get

$$D^{-1}(\alpha)\hat{a}D(\alpha) = e^{\alpha^* a}e^{-\alpha a^\dagger}\hat{a}e^{\alpha a^\dagger}e^{-\alpha^* a}$$

For any operators A and B we have

$$e^{-\alpha A}Be^{\alpha A} = B - \alpha[A,B] + \frac{\alpha^2}{2!}[A,[A,B]] + \ldots$$

For $A = \hat{a}^\dagger$ and $B = \hat{a}$

$$\begin{aligned} e^{-\alpha a^\dagger}\hat{a}e^{\alpha a^\dagger} &= a+\alpha \\ D^{-1}(\alpha)\hat{a}D(\alpha) &= a+\alpha \end{aligned}$$

Similarly for

$$D^{-1}(\alpha)\hat{a}^\dagger D(\alpha) = e^{\alpha^* a}\hat{a}^\dagger e^{-\alpha^* a}$$

here $A = \widehat{a}$ and $B = \hat{a}^\dagger$

$$\Longrightarrow D^{-1}(\alpha)\hat{a}^\dagger D(\alpha) = a^\dagger + \alpha^*$$

Prove

$$\begin{aligned} &e^{-\alpha A}Be^{\alpha A} = B - \alpha[A,B] + \frac{\alpha^2}{2!}[A,[A,B]] + \ldots \\ &\left(1 - \alpha A + \frac{\alpha^2 A^2}{2!} + \ldots\ldots\right)B\left(1 + \alpha A + \frac{\alpha^2 A^2}{2!} + \ldots\right) \\ &= B - \alpha(AB - BA) + \frac{\alpha^2}{2!}(\ldots) \\ &e^{-\alpha a^\dagger}ae^{\alpha a^\dagger} = a - \alpha\left[a^\dagger, a\right] + \frac{\alpha^2}{2!}\left[a^\dagger,\left[a^\dagger, a\right]\right] + \ldots \\ &= a^\dagger + \alpha \end{aligned}$$

1.2 Properties of coherent states

Properties of a cavity mode excited to a coherent state $|\alpha\rangle$ can be detrmined by the method applied to the number state $|n\rangle$.

1- The mean number of photon in the coherent state $|\alpha\rangle$ is given by

$$\begin{aligned}\langle n\rangle &= \langle\alpha \mid \hat{n} \mid \alpha\rangle = \langle\alpha \mid a^{\dagger}a \mid \alpha\rangle = |\alpha|^2 \\ \mid \alpha\rangle &= e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\mid n\rangle \\ \langle\alpha \mid &= e^{-\frac{|\alpha|^2}{2}}\sum_{m=0}^{\infty}\frac{(\alpha^*)^m}{\sqrt{m!}}\langle m\mid\end{aligned}$$

therefore

$$\begin{aligned}\langle n\rangle &= e^{-|\alpha|^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\frac{(\alpha^*)^m}{\sqrt{m!}}\langle m \mid a^{\dagger}a \mid n\rangle \\ &= e^{-|\alpha|^2}\sum_{n}\frac{(\alpha^*\alpha)^n}{n!}n \\ &= e^{-|\alpha|^2}\sum_{n}\frac{(|\alpha|^2)^n}{n!}n\end{aligned}$$

Let $x = |\alpha|^2$, and also

$$\begin{aligned}x\frac{\partial}{\partial x}\sum_{n}\frac{x^n}{n!} &= \sum_{n}\frac{x^n}{n!}n \\ x\frac{\partial}{\partial x}e^x &= \sum_{n}\frac{x^n}{n!}n\end{aligned}$$

therefore we can write

$$\begin{aligned}\langle n\rangle &= e^{-|\alpha|^2}|\alpha|^2\frac{\partial}{\partial|\alpha|^2}e^{|\alpha|^2} \\ &= |\alpha|^2\end{aligned}$$

Find

$$\begin{aligned}\langle\alpha \mid \hat{n}^2 \mid \alpha\rangle &= e^{-|\alpha|^2}\sum_{n}\frac{(\alpha^*\alpha)^n}{n!}n^2 \\ &= e^{-|\alpha|^2}\sum_{n}\frac{|\alpha|^{2n}}{n!}\{n(n-1)+n\}\end{aligned}$$

Let again $x = |\alpha|^2$, by the definition

$$\begin{aligned}\sum_{n}\frac{x^n}{n!}n(n-1) &= x^2\frac{\partial^2}{\partial x^2}\sum_{n}\frac{x^n}{n!} \\ &= x^2\frac{\partial^2}{\partial x^2}e^x\end{aligned}$$

so we can write

$$\langle \alpha \mid \hat{n}^2 \mid \alpha \rangle = e^{-|\alpha|^2} e^{|\alpha|^2} \left(|\alpha|^4 + |\alpha|^2 \right)$$

Root-mean-square deviation is

$$\Delta n = \sqrt{\langle n^2 \rangle - \langle n \rangle^2} = \sqrt{|\alpha|^2}$$

Where $|\alpha|^2$ is the mean number of photons in the cavity mode and uncertainty spread about the mean is equal to the square root of the mean number of photons.

ii)- Photon statistics: photon distribution function:

The probability of finding n-photons in the field $\mid \alpha \rangle$ is

$$p(n) = |\langle n \mid \alpha \rangle|^2$$

where

$$\langle n \mid \alpha \rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{m=0}^{\infty} \frac{(\alpha)^m}{\sqrt{m!}} \langle n \mid m \rangle$$

$$\begin{aligned} p(n) &= \frac{e^{-|\alpha|^2} |\alpha|^{2n}}{n!} \\ &= \frac{e^{-\langle n \rangle} \langle n \rangle^n}{n!} \end{aligned}$$

is a poisson distribution.

iii)- Coherent state is the minimum energy state: i,e

$$\Delta p \Delta q = \frac{\hbar}{2}$$

$\hat{a}$ and $\hat{a}^\dagger$ are not hermitian but their combinations are.

Let $m_j = 1$ at $t = 0$ $r = 0$

$$\begin{aligned} \hat{a} &= \frac{1}{\sqrt{2\hbar\omega}} \left(\omega\hat{q} + i\hat{p} \right) \\ \hat{a}^\dagger &= \frac{1}{\sqrt{2\hbar\omega}} \left(\omega\hat{q} - i\hat{p} \right) \end{aligned}$$

adding these two we get

$$\begin{aligned} \frac{1}{2}\left(\hat{a} + \hat{a}^\dagger \right) &= \sqrt{\frac{\omega}{2\hbar}} \hat{q} \\ \hat{a} + \hat{a}^\dagger &= \sqrt{\frac{2\omega}{\hbar}} \hat{q} \end{aligned}$$

and

$$
\begin{aligned}
\hat{a} - \hat{a^\dagger} &= i\sqrt{\frac{2}{\hbar\omega}}\hat{p} \\
\frac{1}{2i}(\hat{a} - \hat{a^\dagger}) &= \sqrt{\frac{1}{2\hbar\omega}}\hat{p}
\end{aligned}
$$

$\hat{p}$ and $\hat{q}$ are hermitain and represent observable quantities,

$$
\begin{aligned}
\left[\hat{q}, \hat{p}\right] &= i\hbar \\
\Delta\hat{q}\Delta\hat{p} &\geq \frac{\hbar}{2}
\end{aligned}
$$

for a coherent state we have to prove that

$$
\Delta p \Delta q = \frac{\hbar}{2}
$$

$$
\begin{aligned}
\langle\hat{p}\rangle &= \langle\alpha \mid \hat{p} \mid \alpha\rangle = \frac{\sqrt{2\hbar\omega}}{2i}\left(\langle a\rangle - \langle a^\dagger\rangle\right) \\
\langle a\rangle &= \langle\alpha \mid a \mid \alpha\rangle = \alpha\langle\alpha \mid \alpha\rangle
\end{aligned}
$$

$$
\begin{aligned}
\langle p\rangle &= \frac{\sqrt{2\hbar\omega}}{2i}\left(\alpha - \alpha^*\right) \\
\langle p^2\rangle &= \left(\frac{\sqrt{2\hbar\omega}}{2i}\right)^2 \langle(a - a^\dagger)(a - a^\dagger)\rangle \\
&= -\frac{2\hbar\omega}{4}\langle\alpha \mid a^2 + a^{\dagger 2} - aa^\dagger - a^\dagger a \mid \alpha\rangle \\
&= -\frac{2\hbar\omega}{4}(\alpha^2 + \alpha^{*2} - 2\alpha^*\alpha - 1)
\end{aligned}
$$

$$
\begin{aligned}
\Delta p^2 &= \langle p^2\rangle - \langle p\rangle^2 \\
&= -\frac{\hbar\omega}{2}(\alpha^2 + \alpha^{*2} - 2\alpha^*\alpha - 1) + \frac{\hbar\omega}{2}(\alpha^2 + \alpha^{*2} - 2\alpha^*\alpha) \\
&= \frac{\hbar\omega}{2}
\end{aligned}
$$

Similarly

$$
\Delta q^2 = \frac{\hbar}{2\omega}
$$

$$
\Delta q \Delta p = \frac{\hbar}{2}
$$

iv)- Coherent states are not orthagonal, but are normalized. i,e.

$$\langle\alpha \mid \alpha\rangle = e^{-|\alpha|^2}\sum_n \frac{(\alpha^*\alpha)^n}{n!} = 1$$

For two different complex numbers α and β.

$$\mid \alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_n \frac{\alpha^n}{\sqrt{n!}} \mid n\rangle$$
$$\mid \beta\rangle = e^{-\frac{|\beta|^2}{2}}\sum_m \frac{\beta^m}{\sqrt{m!}} \mid m\rangle$$

$$\langle\alpha \mid \beta\rangle = e^{-\frac{|\alpha|^2}{2}}e^{-\frac{|\beta|^2}{2}}e^{\alpha\times\beta} \neq 0$$
$$\Longrightarrow \quad |\langle\alpha \mid \beta\rangle|^2 = e^{-|\alpha-\beta|^2}.$$

The $\mid \alpha\rangle$ form an over complete set of states and lack of orthogonality is a consequence of this.i, j, k are orthogonal and independent of each other. If we divide space in 5 directions they would not be orthogonal and independent of each other. Therefore over complete. i,e. there are many more coherent states $\mid \alpha\rangle$ than there are states $\mid n\rangle$.

Completeness relation:

For number states

$$\sum_n \mid n\rangle\langle n \mid = 1$$

Similarly the set of all coherent states $\mid \alpha\rangle$ is a complete set and satisfy the completeness relation.

$$\frac{1}{\pi}\int d^2\alpha \mid \alpha\rangle\langle\alpha \mid = 1$$

Let

$$\begin{aligned} \alpha &= re^{i\theta} \\ d^2\alpha &= rdrd\theta \end{aligned}$$

$$\begin{aligned} \int d^2\alpha \mid \alpha\rangle\langle\alpha \mid &= \int e^{-\frac{|\alpha|^2}{2}}\sum_m \frac{\alpha^m}{\sqrt{m!}} \mid m\rangle e^{-\frac{|\alpha|^2}{2}}\sum_n \frac{\alpha^{*n}}{\sqrt{n!}}\langle n \mid d^2\alpha \\ &= \int_0^\infty\int_0^{2\pi} e^{-|\alpha|^2}\sum_{n,m}\frac{|\alpha|^{n+m+1}}{\sqrt{n!m!}}e^{i(m-n)\theta} \mid m\rangle\langle n \mid d|\alpha|\, d\theta \end{aligned}$$

as

$$\int_0^{2\pi} d\theta e^{i(m-n)\theta} = 2\pi\delta_{nm}$$

therefore

$$\int d^2\alpha \mid \alpha\rangle\langle\alpha \mid = 2\pi \int_0^\infty \sum_n \frac{1}{n!} |\alpha|^{2n+1} e^{-|\alpha|^2} \mid n\rangle\langle n \mid d(|\alpha|)$$
$$= \pi \sum_n \frac{1}{n!} \int_0^\infty 2\,|\alpha|\, d(|\alpha|)\, |\alpha|^{2n}\, e^{-|\alpha|^2} \mid n\rangle\langle n \mid$$

putting

$$x = |\alpha|^2$$
$$dx = 2\,|\alpha|\, d\,|\alpha|$$

$$\int d^2\alpha \mid \alpha\rangle\langle\alpha \mid = \pi \sum_n \frac{1}{n!} \int_0^\infty dx\; x^n e^{-x} \mid n\rangle\langle n \mid$$

$$\int_0^\infty dx\; x^n e^{-x} = n!$$
$$\int d^2\alpha \mid \alpha\rangle\langle\alpha \mid = \pi \sum_n \frac{1}{n!} n! \mid n\rangle\langle n \mid$$
$$= \pi \sum_n \mid n\rangle\langle n \mid$$
$$\frac{1}{\pi} \int d^2\alpha \mid \alpha\rangle\langle\alpha \mid = 1$$

The completeness property is essential for the utility of a set of states.$\mid \alpha\rangle$ is a complete but not orthogonal. As a result any coherent state can be expanded in terms of other states.

$$\mid \alpha\rangle = \frac{1}{\pi} \int d^2\alpha' \mid \alpha'\rangle\langle\alpha' \mid \alpha\rangle$$
$$= \frac{1}{\pi} \int d^2\alpha \mid \alpha'\rangle \exp\left[-\frac{1}{2}|\alpha|^2 + \alpha'\alpha^* - \frac{1}{2}|\alpha'|^2\right]$$

This shows that the coherent states are overcomplete.

2 Squeezed states of the radiation field

It is possible to generate states in which fluctuations are reduced below the symmetric quantum limit in one quadrature component, at the expense of enhanced fluctuations in the canonically conjugate quadrature such that the Heisenberg uncertainty principle is not voilated. Such states of the radiation field are called squeezed states.

Consider two hermitian operators A and B which satisfy the commutation relation

$$[A, B] = iC$$

According to the Heisenberg uncertainty principle, the product of the uncertainties in determining the expectation values of two variables A and B is given by.

$$\Delta A \Delta B \geq \frac{1}{2} |\langle C \rangle|$$

A state of the system is called a squeezed state if the uncertainty in one of the observables (say A) satisfies the relation

$$(\Delta A)^2 < \frac{1}{2} |\langle C \rangle|$$

If in addition to the above condition the variances satisfy the minimum uncertainty relation i,e.

$$\Delta A \Delta B = \frac{1}{2} |\langle C \rangle|$$

then the state is called an ideal squeezed state.

In a squeezed state , therefore the quantum fluctuations in one variable are reduced below their value in a symmetric minimum uncertainty state.

$$(\Delta A)^2 = (\Delta A)^2 = \frac{1}{2} |\langle C \rangle|$$

at the expense of the corresponding increased fluctuations in the conjugate variable such that the uncertainty relation is not voilated.

2.1 Quadrature amplitude operators

Let us define Hermitian amplitude operators.

$$\begin{aligned} X_1 &= \frac{1}{2}\left(a + a^\dagger\right) \\ X_2 &= \frac{1}{2i}\left(a - a^\dagger\right) \end{aligned}$$

X_1 and X_2 are dimensionless position and momentum operators

$$\begin{aligned} q &= \frac{\sqrt{2\hbar/\omega m}}{2}\left(a + a^\dagger\right) \\ p &= \frac{\sqrt{2\hbar\omega m}}{2i}\left(a - a^\dagger\right) \end{aligned}$$

$$\hat{a} = X_1 + iX_2$$
$$\hat{a}^\dagger = X_1 - iX_2$$

The operators X_1 and X_2 are Hermitian and satisfy the commutation relation

$$[X_1, X_2] = \frac{i}{2}$$

These operators are also called quadrature operators. In terms of X_1 and X_2 quantized single mode field can be written as

$$E(t) = 2\varepsilon\hat{\epsilon}\,(X_1 \cos\omega t + X_2 \sin\omega t)$$

The Hermitian operators X_1 and X_2 are the amplitudes of the two quadratures of the field having a phase difference $\pi/2$.

From the commutation relation of X_1 and X_2, we get the uncertainty relation for the two amplitudes i,e.

$$\Delta X_1 \Delta X_2 \geq \frac{1}{4}$$

A squeezed state of the rdiation field is obtained if

$$(\Delta X_i)^2 < \frac{1}{4}, \quad for \;\; i = 1 \; or \; 2$$

An ideal squeezed state is obtained if in addition to the above equation, the relation

$$\Delta X_1 \Delta X_2 = \frac{1}{4}$$

also holds . Example of an ideal squeezed state is the two-photon coherent state. The quadrature operators allow us to represent a beam of light graphically. These so called phasor diagrams are very popular in quantum optics. Any state of light can be represented on a phasor diagram of the operators , i,e. a plot of X_1 versus X_2. Unlike classical vacuum, quantum mechanical vacuum is represented by a circle at the origin. $(\langle X_1\rangle, \langle X_2\rangle)$ every point in this circle will represent a wave. In quadrature representation each point represents a wave.

Now consider how does different states look like pictorially

i)- Vaccum

$$\Delta X_1 = \sqrt{\langle X_1^2\rangle - \langle X_1\rangle^2}$$

$$\begin{aligned}
\langle X_1\rangle &= \langle 0 \mid \frac{a + a^\dagger}{2} \mid 0\rangle = 0 \\
\langle X_1^2\rangle &= \langle 0 \mid \frac{(a^2 + a^{\dagger 2} + aa^\dagger + a^\dagger a)}{4} \mid 0\rangle \\
&= \langle 0 \mid \frac{(a^2 + a^{\dagger 2} + 2a^\dagger a + 1)}{4} \mid 0\rangle \\
&= \frac{1}{4}
\end{aligned}$$

$$\Longrightarrow \Delta X_1^2 = \frac{1}{4}$$

Similarly

$$\Delta X_2^2 = \frac{1}{4}$$

$$\begin{aligned} \Delta X_1 &= \frac{1}{2}, \quad \Delta X_2 = \frac{1}{2} \\ \Delta X_1 \Delta X_2 &= \frac{1}{4} \end{aligned}$$

ii)- Coherent States:

$$\begin{aligned} \langle X_1 \rangle &= \frac{1}{2}\langle \alpha \mid a + a^\dagger \mid \alpha \rangle \\ &= \frac{1}{2}\left(\alpha + \alpha^*\right) \end{aligned}$$

$$\alpha = |\alpha|\, e^{i\phi}$$

Therefore

$$\begin{aligned} \langle X_1 \rangle &= |\alpha| \cos\phi \\ \langle X_2 \rangle &= \frac{1}{2i}\langle \alpha \mid a - a^\dagger \mid \alpha \rangle \\ &= \frac{1}{2i}\left(\alpha - \alpha^*\right) \\ &= |\alpha| \sin\phi \end{aligned}$$

$$\begin{aligned} \langle X_1^2 \rangle &= \frac{1}{4}\langle \alpha \mid a^2 + a^{\dagger 2} + a a^\dagger + a^\dagger a \mid \alpha \rangle \\ &= \frac{1}{4}\left(\alpha^2 + \alpha^{*2} + 2\alpha\alpha^* + 1\right) \\ \langle X_2^2 \rangle &= -\frac{1}{4}\left(\alpha^2 + \alpha^{*2} + 2\alpha\alpha^* + 1\right) \end{aligned}$$

$$\begin{aligned} \Longrightarrow \quad & \Delta X_1^2 = \langle X_1^2 \rangle - \langle X_1 \rangle^2 \\ = \quad & \frac{1}{4}\left(\alpha^2 + \alpha^{*2} + 2\alpha\alpha^* + 1\right) - \frac{1}{4}\left(\alpha^2 + \alpha^{*2} + 2\alpha\alpha^*\right) \\ = \quad & \frac{1}{4} \end{aligned}$$

Similarly

$$\Delta X_2^2 = \frac{1}{4}$$

which means that there is equal amount of uncertainty in both the quadratures. It represent a circle with displacement $|\alpha|$. Displacement operator displaced the vacuum by an amount α that creates a coherent state. Therefore coherent state is a displaced vacuum state.

$$\mid \alpha\rangle = D(\alpha) \mid 0\rangle$$

Coherent state has both amplitude and phase uncertainties.
Is Coherent and Fock state are squeezed states?
First consider the coherent state

$$\begin{aligned}
(\Delta \mathrm{X}_1)^2 &= \langle \alpha \mid X_1^2 \mid \alpha\rangle - (\langle \alpha \mid X_1 \mid \alpha\rangle)^2 \\
&= \frac{1}{4}\langle \alpha \mid a^2 + a^{\dagger 2} + aa^\dagger + a^\dagger a \mid \alpha\rangle - \frac{1}{4}\left(\langle \alpha \mid (a + a^\dagger) \mid \alpha\rangle\right)^2 \\
&= \frac{1}{4}
\end{aligned}$$

Similarly

$$(\Delta \mathrm{X}_2)^2 = \frac{1}{4}$$

$$\Delta \mathrm{X}_1 \Delta \mathrm{X}_2 = \frac{1}{4}$$

i,e. minimum uncertainty relation holds.
Coherent state is not a squeezed state.
Fock state:

$$\begin{aligned}
(\Delta \mathrm{X}_1)^2 &= \langle n \mid X_1^2 \mid n\rangle - (\langle n \mid X_1 \mid n\rangle)^2 \\
&= \frac{1}{4}(2n+1)
\end{aligned}$$

Simailarly

$$(\Delta \mathrm{X}_2)^2 = \frac{1}{4}(2n+1)$$

This implies that Fock states $\mid n\rangle$ are not squeezed states.

A coherent state with identical uncertainties in both X_1 and X_2 has a constant value for the variance of electric field. A squeezed state with reduced noise in X_1 has reduced uncertainty in the amplitude at the expense of large uncertainty in the phase of electric field.

A squeezed state with reduced noise in X_2 has reduced uncertainty in the phase at the expense of large uncertainty in the amplitude of the electric field. If we have a state such that ΔX_1^2 or $\Delta X_2^2 < \frac{1}{4}$ then the state is a squeezed state. According to heisenberg uncertainty principle, vacuum is a circle. Area of the circle should be the same, otherwise it will voilate Heisenberg uncertainty principle. In order to squeeze a coherent state we will define a squeezing operator.

2.1.1 Generation of squeezed state:

One example of generation of a squeezed state is the "Degenerate parametric process". The two photon Hamiltonian can be wirtten as

$$H = i\hbar(ga^{\dagger 2} - g^* a^2)$$

where g is the coupling constant . The state of the field is written as

$$\mid \Psi(t)\rangle = e^{(ga^{\dagger 2} - g^* a^2)} \mid 0\rangle$$

and this leads to define the unitary squeezed operator.

2.1.2 Squeeze operator:

The squeezed states can be generated by using the unitary squeezed operator.

$$S(\zeta) = \exp\left(\frac{1}{2}\zeta^* a^2 - \frac{1}{2}\zeta a^{\dagger 2}\right) \tag{1}$$

this is a unitary operator i,e.

$$S^{\dagger}(\zeta) = S(-\zeta) = S^{-1}(\zeta)$$

where $\zeta = re^{i\theta}$. We want to find the values of ΔX_1^2 and ΔX_2^2 or (ΔY_1^2 and ΔY_2^2 for a rotated frame). For this we first find

$$S^{\dagger}(\zeta)\hat{a}S(\zeta) = e^{\frac{1}{2}(\zeta a^{\dagger 2} - \zeta^* a^2)} a e^{\frac{1}{2}(\zeta^* a^2 - \zeta a^{\dagger 2})} \tag{2}$$

using

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2\perp}[A, [A, B]] + \frac{1}{3\perp}[A, [A, [A, B]]] + \ldots$$

Let we define

$$A = \frac{1}{2}\zeta a^{\dagger 2} - \frac{1}{2}\zeta^* a^2$$

and

$$B = \hat{a}$$

B will commute with the second term of A

$$[A, B] = \frac{1}{2}\zeta\left[a^{\dagger 2}, a\right] \tag{3}$$

using

$$\begin{aligned} \left[a, a^{\dagger n}\right] &= na^{\dagger(n-1)} \\ \left[a^{\dagger}, a^{n-1}\right] &= -na^{n-1} \end{aligned}$$

$$\Longrightarrow \qquad [A,B] = \frac{1}{2}\zeta\left(-2a^{\dagger}\right) = -\zeta a^{\dagger}$$

Now we have to find

$$[A,[A,B]] = ?$$

Ist term of A i,e. $\frac{1}{2}\zeta a^{\dagger 2}$ commutes with $[A,B]$

$$\begin{aligned} [A,[A,B]] &= \left[-\frac{1}{2}\zeta^{*}a^{2}, -\zeta a^{\dagger}\right] \\ &= \frac{1}{2}\zeta^{*}\zeta\left[a^{2}, a^{\dagger}\right] \\ &= \frac{|\zeta|^{2}}{2}(2a) \\ &= |\zeta|^{2} a \end{aligned} \tag{4}$$

Now for

$$[A,[A,[A,B]]] = ?$$

the second term of A will commute with $[A,[A,B]]$ therefore we have

$$\begin{aligned} [A,[A,[A,B]]] &= \left[\frac{1}{2}\zeta a^{\dagger 2}, |\zeta|^{2} a\right] \\ &= \frac{1}{2}\zeta|\zeta|^{2}\left[a^{\dagger 2}, a\right] \\ &= \frac{1}{2}\zeta|\zeta|^{2}\left[-2a^{\dagger}\right] \\ &= -\zeta|\zeta|^{2} a^{\dagger} \end{aligned} \tag{5}$$

Using these relations we get

$$\begin{aligned} S^{\dagger}(\zeta)\,\hat{a}S(\zeta) &= a - \zeta a^{\dagger} + \frac{1}{2!}|\zeta|^{2} a - \frac{\zeta|\zeta|^{2}}{3!}a^{\dagger} + \dots \\ &= a\left[1 + \frac{1}{2!}|\zeta|^{2} + \frac{1}{4!}|\zeta|^{4}\dots\right] - a^{\dagger}\left[\zeta + \frac{\zeta|\zeta|^{2}}{3!} + \dots\right] \end{aligned}$$

as

$$\zeta = re^{i\theta} \qquad \Longrightarrow |\zeta| = r$$

$$\begin{aligned} S^{\dagger}(\zeta)\,\hat{a}S(\zeta) &= a\left[1 + \frac{1}{2!}r^{2} + \frac{1}{4!}r^{4} + \dots\right] - a^{\dagger}\left[re^{i\theta} + \frac{r^{3}e^{i\theta}}{3!} + \dots\right] \\ &= a\cosh r - a^{\dagger}e^{i\theta}\sinh r \end{aligned}$$

Similarly

$$S^{\dagger}(\zeta)\,\hat{a}^{\dagger}S(\zeta) = a^{\dagger}\cosh r - e^{-i\theta}a\sinh r$$

As we defined

$$X_1 = \frac{1}{2}\left(a + a^\dagger\right)$$
$$X_2 = \frac{1}{2}\left(a - a^\dagger\right)$$

$$\Longrightarrow \qquad a = X_1 + iX_2$$
$$a^\dagger = X_1 - iX_2$$

Rotate the axes by an amount $\theta/2$. Rotated axes $Y_1$and Y_2 in terms of $X_1$and X_2 are written as

$$Y_1 = X_1 \cos\frac{\theta}{2} + X_2 \sin\frac{\theta}{2}$$
$$Y_2 = -X_1 \sin\frac{\theta}{2} + X_2 \cos\frac{\theta}{2}$$

$$\begin{aligned} Y_1 + iY_2 &= X_1\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right) + X_2\left(\sin\frac{\theta}{2} + i\cos\frac{\theta}{2}\right) \\ &= X_1 e^{-i\theta/2} + iX_2\left(-i\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right) \\ &= \left(X_1 + iX_2\right) e^{-i\theta/2} \end{aligned}$$

is the rotated complex amplitude at an angle $\theta/2$.

$$S^\dagger(\zeta)\left(X_1 + iX_2\right) S(\zeta) = S^\dagger(\zeta)\, a S(\zeta)$$
$$S^\dagger(\zeta)\left(X_1 + iX_2\right) S(\zeta) = a\cosh r - a^\dagger e^{i\theta}\sinh r$$

For rotated frame we have

$$S^\dagger(\zeta)\left(X_1 + iX_2\right) e^{-i\theta/2} S(\zeta) = a\cosh r e^{-i\theta/2} - a^\dagger e^{i\theta/2}\sinh r$$

$$S^\dagger(\zeta)\left(Y_1 + iY_2\right) S(\zeta) = ae^{-i\theta/2}\left(\frac{e^r + e^{-r}}{2}\right) - a^\dagger e^{i\theta/2}\left(\frac{e^r - e^{-r}}{2}\right)$$

$$= \left(ae^{-i\theta/2} - a^\dagger e^{i\theta/2}\right)\frac{e^r}{2}$$
$$+ \left(ae^{-i\theta/2} + a^\dagger e^{i\theta/2}\right)\frac{e^{-r}}{2}$$

As

$$\begin{aligned} Y_1 &= X_1 \cos\frac{\theta}{2} + X_2 \sin\frac{\theta}{2} \\ &= \frac{1}{2}\left(a + a^\dagger\right)\cos\frac{\theta}{2} + \frac{1}{2i}\left(a - a^\dagger\right)\sin\frac{\theta}{2} \\ &= \frac{1}{2}a\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right) + \frac{1}{2}a^\dagger\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \\ &= \frac{1}{2}\left(ae^{-i\theta/2} + a^\dagger e^{i\theta/2}\right) \qquad \text{(a)} \end{aligned}$$

$$\Longrightarrow S^{\dagger}(\zeta)(Y_1 + iY_2)S(\zeta) = Y_1 e^{-r} + iY_2 e^{r}$$

where

$$iY_2 = \frac{1}{2}\left(ae^{-i\theta/2} - a^{\dagger}e^{i\theta/2}\right) \quad \text{(b)}$$

The squeezed operatot attenuates one component of the (rotated) complex amplitude and it amplifies the other component. The degree of attenuation and amplification is determined by $r = |\zeta|$ which is called squeeze factor. The squeezed state $\mid \alpha, \zeta\rangle$ is obtained by first squeezing the vacuum and then displacing it.

$$\mid \alpha, \zeta\rangle = D(\alpha)S(\zeta) \mid 0\rangle$$

where α^2 is the intensity of the state, θ is the orientation of the squeezing axis and r the degree of squeezing. The reverse order of $D(\alpha)$ and $S(\zeta)$ in the above equation is also possible. This results in the so called two photon correlated state. A coherent state is generated by linear terms in a and $a^{\dagger}$ in the exponent

$$D(\alpha) \mid 0\rangle = \mid \alpha\rangle$$

where

$$D(\alpha) = e^{\alpha a^{\dagger} - \alpha^{*} a}$$

the squeezed coherent state requires quadratic terms.

2.1.3 Quadrature variance:

Now we will find $\Delta Y_1^2$and ΔY_2^2, in order to find the ΔY_1 we need to find $\langle a\rangle$, $\langle a^{\dagger}\rangle$, $\langle a^2\rangle$, $\langle a^{\dagger 2}\rangle$ and $\langle a^{\dagger}a\rangle$. We will find these one by one

$$\begin{aligned}
\langle a\rangle &= \langle \alpha, \zeta \mid a \mid \alpha, \zeta\rangle \\
&= \langle 0 \mid D^{\dagger}(\alpha)S^{\dagger}(\zeta)aS(\zeta)D(\alpha) \mid 0\rangle \\
&= \langle \alpha \mid (a\cosh r - a^{\dagger}e^{i\theta}\sinh r) \mid \alpha\rangle \\
&= \langle \alpha \mid a \mid \alpha\rangle \cosh r - \langle \alpha \mid a^{\dagger} \mid \alpha\rangle e^{i\theta}\sinh r \\
&= \alpha\cosh r - \alpha^{*}e^{i\theta}\sinh r
\end{aligned}$$

$$\begin{aligned}
&\langle a^2\rangle = \langle 0 \mid D^{\dagger}(\alpha)S^{\dagger}(\zeta)a^2S(\zeta)D(\alpha) \mid 0\rangle \\
&= \langle 0 \mid D^{\dagger}(\alpha)S^{\dagger}(\zeta)aS(\zeta)S^{\dagger}(\zeta)aS(\zeta)D(\alpha) \mid 0\rangle \\
&= \langle \alpha \mid (a\cosh r - a^{\dagger}e^{i\theta}\sinh r)^2 \mid \alpha\rangle \\
&= \langle \alpha \mid (a^2\cosh^2 r + e^{2i\theta}a^{\dagger 2}\sinh^2 r - aa^{\dagger}e^{i\theta}\cosh r\sinh r \\
&-a^{\dagger}ae^{i\theta}\cosh r\sinh r) \mid \alpha\rangle \\
&= \alpha^2\cosh^2 r + e^{2i\theta}\alpha^{*2}\sinh^2 r - 2|\alpha|^2 e^{i\theta}\cosh r\sinh r \\
&-e^{i\theta}\cosh r\sinh r \\
&= \langle (a^{\dagger})^2\rangle^{*}
\end{aligned}$$

Now

$$\langle a^\dagger a\rangle = \langle \alpha \mid S^\dagger(\zeta)\, a^\dagger S(\zeta)\, S^\dagger(\zeta)\, aS(\zeta) \mid \alpha\rangle$$
$$= \langle \alpha \mid \left(a^\dagger \cosh r - e^{-i\theta} a \sinh r\right)\left(a\cosh r - a^\dagger e^{i\theta} \sinh r\right) \mid \alpha\rangle$$
$$= |\alpha|^2 \left(\cosh^2 r + \sinh^2 r\right) - (\alpha^*)^2 e^{i\theta} \sinh r \cosh r$$
$$-\alpha^2 e^{-i\theta} \cosh r \sinh r + \sinh^2 r$$

As from Equ(a) and (b)

$$\begin{aligned} Y_1 &= \frac{1}{2}\left(ae^{-i\theta/2} + a^\dagger e^{i\theta/2}\right) \\ Y_2 &= \frac{1}{2i}\left(ae^{-i\theta/2} - a^\dagger e^{i\theta/2}\right) \end{aligned}$$

$$\begin{aligned} \langle Y_1\rangle &= \frac{1}{2}\left[\langle a\rangle e^{-i\theta/2} + \langle a^\dagger\rangle e^{i\theta/2}\right] \\ &= \frac{1}{2}\left(\alpha\cosh r - e^{-i\theta}\alpha^* \sinh r\right) e^{-i\theta/2} + \frac{1}{2}\left(\alpha^*\cosh r - e^{-i\theta}\alpha \sinh r\right) e^{i\theta/2} \end{aligned}$$

$$\langle Y_1^2\rangle = \frac{1}{4}\left[\langle a^2\rangle e^{-i\theta} + \langle a^{\dagger 2}\rangle e^{i\theta} + 2\langle a^\dagger a\rangle + 1\right]$$

Now

$$\begin{aligned} (\Delta Y_1)^2 &= \langle Y_1^2\rangle - \langle Y_1\rangle^2 \\ &= \frac{1}{4}e^{-2r} \end{aligned}$$

and

$$(\Delta Y_2)^2 = \frac{1}{4}e^{2r}$$

$$\Longrightarrow \quad \Delta Y_1 \Delta Y_2 = \frac{1}{4}$$

A squeezed coherent state is therefore an ideal squeezed state. In the complex amplitude plane the coherent state error circle is squeezed into an error ellipse of the same area. The principle axes of the ellipse lie along the Y_1 and Y_2 axes, and the principle radii are ΔY_1 and ΔY_2.

For figure see Scully and Zubairy (Quantum optics ch# 2)