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THE RADIATIVE DENSITY OPERATOR

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1 The radiative density operator

The number and coherent state excitations are pure states of the radiation field. The number state is given by a particular state $| n\rangle$ and coherent state is expressed as a linear combination of the basic number states $| n\rangle$. The state of the total electromagnetic field is formed by a product of the indivisual modes.

There exists mode excitations that are not expressable as definite linear superpositions of complete set of basic states $| n\rangle$. Some times in quantum mechanics we do not make a definite predictions of the state of the emitted field. We used the probabilistic discription that the radiation will be found in a range of states, each corresponding to a linear combination of the basic number states, then the state of the field is called a statistical mixture.

Statistical distributions are introduced into quantum mecahnics by means of the density operator. Consider a cavity electromagnetic field for which there is a known probability P_Ψ, that the field is in a state $| \Psi\rangle$. Here Ψ is a label that runs over a set of pure states sufficient to describe the field.

For a single cavity mode the states $| \Psi\rangle$ could be the number states $| n\rangle$ or the coherent state $| \alpha\rangle$ or they could be some other type of pure state. For complete cavity field, the states $| \Psi\rangle$ would be all possible products of the single-mode states, with one state for each mode of the cavity included in each of the basic states $| \Psi\rangle$. The state described by the probability P_Ψ is a statistical mixture , the magnitudes of the P_Ψ for a given set of pure states $| \Psi\rangle$ contains all the available information about the state.

Consider some observable that is represented by a quantum mechanical operator $\hat{O}$. The average value of $\hat{O}$ for a pure state $| \Psi\rangle$ is given by

$$\langle \hat{O} \rangle = \langle \Psi \mid \hat{O} \mid \Psi\rangle$$

The expectation value for a mixed case is obtained by taking an ensemble average as well.

$$\langle\langle \hat{O} \rangle\rangle = \sum_{\Psi} P_\Psi \ \langle \Psi \mid \hat{O} \mid \Psi\rangle$$

Now using completeness relation

$$\sum_n | n\rangle\langle n |= 1$$

$$\begin{aligned}
\langle\langle \hat{O} \rangle\rangle &= \sum_n \sum_{\Psi} P_\Psi \ \langle \Psi \mid \hat{O} \mid n\rangle\langle n \mid \Psi\rangle \\
&= \sum_n \sum_{\Psi} P_\Psi \ \langle n \mid \Psi\rangle\langle \Psi \mid \hat{O} \mid n\rangle \\
&= \sum_n \langle n \mid \sum_{\Psi} P_\Psi \mid \ \Psi\rangle\langle \Psi \mid \hat{O} \mid n\rangle \\
&= \sum_n \langle n \mid \rho \hat{O} \mid n\rangle
\end{aligned}$$

where

$$\rho = \sum_{\Psi} P_{\Psi} \mid \Psi\rangle\langle\Psi \mid$$

is the density operator for a mixed case. Thus the radiation is in general described by the density operator.

$$\rho = \sum_{\Psi} P_{\Psi} \mid \Psi\rangle\langle\Psi \mid$$

where P_{Ψ} is the probability of being in the state $\mid \Psi\rangle$. The expectation value of any field operator $\hat{O}$ is then given by

$$\langle\hat{O}\rangle = Tr\left(\hat{O}\rho\right)$$

where Tr stands for trace. The trace of an operator is the sum of its diagonal matrix elements for any complete set of states.

1.1 Density operators for pure states

A pure state can be regarded as a special case of a statistical mixture in which one of the probabilities P_{Ψ} is equal to unity, and all the remaining are zero.

$$\hat{\rho} = \mid \Psi\rangle\langle\Psi \mid$$

The radiation field is definitely in a particular pure states $\mid \Psi\rangle$ in this case. For pure state density operator only

$$\hat{\rho}^2 = \hat{\rho}$$

For a field in one of the number states $\mid n\rangle$, where n photons are definitely present, the density operator is

$$\hat{\rho} = \mid n\rangle\langle n \mid$$

The only non-vanishing matrix elements for the number states is

$$\langle n \mid \hat{\rho} \mid n\rangle = 1$$

and average value of an observable represented by operator $\hat{O}$ is

$$\begin{aligned}\langle\hat{O}\rangle &= Tr\left(\mid n\rangle\langle n \mid \hat{O}\right) \\ &= \langle n \mid \hat{O} \mid n\rangle\end{aligned}$$

The density operator for of the coherent states $\mid \alpha\rangle$ is written as

$$\hat{\rho} = \mid \alpha\rangle\langle\alpha \mid$$

From the normalization of the coherent states

$$\langle \alpha \mid \hat{\rho} \mid \alpha \rangle = 1$$

Since different coherent states are not orthogonal, $\langle \alpha \mid \hat{\rho} \mid \alpha \rangle$ is not the only non-zero matrix element of $\hat{\rho}$. Indeed every coherent state matrix element of $\hat{\rho}$ is non-vanishing.

Consider a general matrix element of the pure coherent-state density operator for the number states.

$$\langle n \mid \hat{\rho} \mid n' \rangle = e^{-|\alpha|^2} \frac{\alpha^n \alpha^{*n'}}{(n!n'!)^{1/2}}$$

The density operator for the coherent state has non-vanishing off-diagonal matrix elements for the number states. This is an example of the importance of choosing the appropraite state $\mid \Psi \rangle$ in which to express the density operator, so that no information about the state of the system is lost.

It is not possible to describe the pure coherent state $\mid \alpha \rangle$ fully in terms of a diagonal density operator based on number state$\mid n \rangle$. Such a density operator would have zero off diagonal matrix elements $\langle n \mid \hat{\rho} \mid n' \rangle$ and information contained in above equation for $n \neq n'$ could not be reproduced. The off diagonal matrix elements of the density operator are particularly important in the calculation of average values of operators $\langle \hat{O} \rangle$, which themselves have non-zero off diagonal matrix elements $\langle n \mid \hat{O} \mid n \rangle$, .for example expection value of electric-field operator $\langle E \rangle$ A state of the radiation field can have a non-zero vanishing average electric vector only if the density operator has non-zero off-diagonal matrix elements. The diagonal number-state matrix element of the density operator $\langle n \mid \hat{\rho} \mid n \rangle$, is the probability that n photons are excited in the state of the field described by $\hat{\rho}$.

1.2 Pure states of the complete radiation field

If each cavity mode has a definite number of photons excited the state $|\Psi\rangle$ is one of the states $|\{n_k\}\rangle$ and density operator is

$$\begin{aligned} \hat{\rho} &= |\{n_k\}\rangle \langle\{n_k\}| \\ &= \mid n_{k_1}\rangle \mid n_{k_2}\rangle \mid n_{k_3}\rangle, \ldots, \langle n_{k_3} \mid \langle n_{k_2} \mid \langle n_{k_1} \mid \end{aligned}$$

If each cavity mode is excited to a definite coherent state, then the state of the total field is

$$\mid \{\alpha_k\}\rangle = \mid \alpha_{k_1}\rangle \mid \alpha_{k_2}\rangle \mid \alpha_{k_3}\rangle, \ldots$$

and corresponding density operator for the multi-mode coherent state is

$$\hat{\rho} = \mid \{\alpha_k\}\rangle\langle\{\alpha_k\} \mid$$

other pure states are similarly treated.

1.3 Statistical mixture states of the radiation field:

Consider the thermal excitation of the photons in a single mode of a cavity maintained at temperature T. The density operator based on the number state is

$$\hat{\rho} = \sum_n P_n \mid n\rangle\langle n \mid$$

where P_n is the probability that n photons are excited in the state of the field described by $\hat{\rho}$.

$$\hat{\rho} = \left(1 - \exp\left(-\frac{\hbar\omega}{k_B T}\right)\right) \sum_n \exp\left(-\frac{n\hbar\omega}{k_B T}\right) \mid n\rangle\langle n \mid$$

The number state $\mid n\rangle$ is the correct basis for the density operator in this case because the thermal probability distribution gives information only on the probabilities of finding a system in its various energy eigen states. The density operator for the thermal photon distribution has only diagonal number state matrix elements, thus the average electric field is always zero.

In terms of $\langle n\rangle$, $\hat{\rho}$ is written as

$$\hat{\rho} = \sum_n \frac{\langle n\rangle^n}{(1 + \langle n\rangle)^{n+1}} \mid n\rangle\langle n \mid$$

Consider the thermal excitation of all the cavity modes. Since the different field modes are independent, the combeined density operator is a product of the contributions of the different modes. The density operator

$$\hat{\rho} = \sum_{\{n_k\}} P_{\{n_k\}} \mid \{n_k\}\rangle\langle\{n_k\} \mid$$

$$P_{\{n_k\}} = \prod_k \frac{\langle n_k\rangle^{n_k}}{(1 + \langle n_k\rangle)^{n_k+1}}$$

$\langle n_k\rangle$ is the mean number of photons excited in mode k. The density operator for the radiation field in thermal cavity is

$$\hat{\rho} = \sum_{\{n_k\}} \mid \{n_k\}\rangle\langle\{n_k\} \mid \prod_k \frac{\langle n_k\rangle^{n_k}}{(1 + \langle n_k\rangle)^{n_k+1}}$$

The above equation applies not only to the thermal photon distribution but also to a wide range of excitations in which the statistical properties of the light generation are suitably random. This density operator applies in particular to the light beam emitted by chaotic source.

2 REPRESENTATION OF THE ELECTROMAGNETIC FIELD

Full description of the electromagnetic field requires a quantum statistical treatment. The electromanetic field has an infinite number of modes and each mode requires a statistical description in terms of its allowed quantum states. There are number of possible representations for the density operator of the electromagnetic field. One represenation is to expand the density operator in terms of the number states. The coherent states allow a number of possible represenation via the P-function, the Q-function and the Wigner function.

2.1 Number state representation

$$\begin{aligned}
\hat{\rho} &= I\hat{\rho}I = \sum_n \sum_m \mid n\rangle\langle n \mid \hat{\rho} \mid m\rangle\langle m \mid \\
&= \sum_{n,m} \rho_{nm} \mid n\rangle\langle m \mid
\end{aligned}$$

where

$$I = \sum_n \mid n\rangle\langle n \mid = 1$$

The expectation value of an operator is defined as

$$\langle \hat{O} \rangle = Tr\left(\rho\hat{O}\right)$$

The expectation value of any normal order operator, for number state is

$$\langle a^\dagger a\rangle = Tr\left(\rho a^\dagger a\right)$$

$$\rho = \sum_{n,m} \rho_{nm} \mid n\rangle\langle m \mid$$

therefore

$$\begin{aligned}
\langle a^\dagger a\rangle &= \sum_n \sum_m \rho_{nm}\ Tr\ (a^\dagger a \mid n\rangle\langle m \mid) \\
&= \sum_n \sum_m \rho_{nm}\ \langle m \mid a^\dagger a \mid n\rangle \\
&= \sum_n \sum_m \rho_{nm}\ \ (n\delta_{nm}) \\
&= \sum_n n\rho_{nn}
\end{aligned}$$

2.2 P-reprsentation of Coherent state

$$\hat{\rho} = I\hat{\rho}I = \frac{1}{\pi^2}\int\int d^2\alpha \mid \alpha\rangle\langle\alpha \mid \hat{\rho} \mid \beta\rangle\langle\beta \mid d^2\beta$$

Following Glauber,s convention, we define the R-representation as

$$R(\alpha^*, \beta) = \langle\alpha \mid \hat{\rho} \mid \beta\rangle e^{\left(|\alpha|^2+|\beta|^2\right)}$$

the density operator may be written as

$$\hat{\rho} = \int \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi} \mid \alpha\rangle\langle\beta \mid R(\alpha^*, \beta)\, e^{-\left(|\alpha|^2+|\beta|^2\right)}$$

The diagonal coherent state representation is written as

$$\hat{\rho} = \int P(\alpha, \alpha^*) \mid \alpha\rangle\langle\alpha \mid d^2\alpha$$

The function $P(\alpha, \alpha^*)$ is called the P-representation or coherent state representation.

2.2.1 Properties

1) The function $P(\alpha, \alpha^*)$ can be used to evaluate the expectation value of the any normal ordered function of $\hat{a}$ and $\hat{a}^\dagger$ (normal order means all the creation operators $\hat{a}^\dagger$ on the left-hand side and all the annihilation perators $\hat{a}$ on the right-hand side).

2) $P(\alpha, \alpha^*)$ is real.

Due to the hermiticity of the density operator ρ the distribution function $P(\alpha, \alpha^*)$ is real.

3) $P(\alpha, \alpha^*)$ is normalized

Since we know

$$\begin{aligned} Tr(\rho) &= 1 \\ Tr(\rho) &= \int P(\alpha, \alpha^*)\ \langle\alpha \mid \alpha\rangle\ d^2\alpha \\ 1 &= \int P(\alpha, \alpha^*)\ d^2\alpha \end{aligned}$$

The expectation value of an operator is defined as

$$\langle\hat{O}\rangle = Tr\left(\rho\hat{O}\right)$$

In P-reprsentaion, expectation value of any normal ordered operator is given by

$$\begin{aligned}\langle a^{\dagger}a\rangle &= Tr\left(a^{\dagger}a\rho\right) \\ &= Tr\ (a^{\dagger}a\int P(\alpha,\alpha^{*})\ \ |\ \alpha\rangle\langle\alpha\ |\ \ d^{2}\alpha \\ &= \int d^{2}\alpha\ P(\alpha,\alpha^{*})\ \langle\alpha\ |\ a^{\dagger}a\ |\ \alpha\rangle \\ &= \int d^{2}\alpha\ P(\alpha,\alpha^{*})\ \ |\alpha|^{2}\end{aligned}$$

$$\langle O(a^{\dagger}a)\rangle = \int P(\alpha,\alpha^{*})\ O(\alpha,\alpha^{*})\ d^{2}\alpha$$

The function $P(\alpha,\alpha^{*})$ can be used to evaluate the expectation value of any normal ordered function of $\hat{a}$ and $\hat{a}^{\dagger}$. Simply we have to replace $\hat{a}^{\dagger}$ by α^{*} and $\hat{a}$ by α.

4) $P(\alpha,\alpha^{*})$ is not necessarily non-negative definite.

The P-representation forms a correspomdence between the quantum and classical coherence theory. This distribution function does not have all the properties of the classical distribution functions for certain states of the field, it can be negative. The study of interface between classical and quantum physics is a facinating subject. This fact is better illustrated in quantum optics, where we often faced with the problem of characterizing fields which are nearly classical but have important quantum features. The coherent states are well suited to such studies.

2.3 Procedure of finding P(α,α^{*})

In order to write $P(\alpha,\alpha^{*})$ in terms of ρ, consider two coherent states $|\beta\rangle$ and $|-\beta\rangle$.These are the eigen states of $\hat{a}$ with eigen value β and $-\beta$ respectively.

$$\langle-\beta\,|\rho|\,\beta\rangle = \int P(\alpha,\alpha^{*})\,\langle-\beta|\alpha\rangle\,\langle\alpha|\beta\rangle\,d^{2}\alpha$$

where

$$\rho = \int P(\alpha,\alpha^{*})\,|\alpha\rangle\,\langle\alpha|\,d^{2}\alpha$$

using

$$\langle\alpha|\beta\rangle = e^{-\frac{|\alpha|^{2}}{2}}e^{-\frac{|\beta|^{2}}{2}}e^{\alpha^{*}\beta}$$

and

$$
\begin{aligned}
\langle -\beta|\alpha\rangle &= e^{-\frac{|\alpha|^2}{2}}e^{-\frac{|\beta|^2}{2}}e^{-\beta^*\alpha} \\
&\implies \langle -\beta\left|\rho\right|\beta\rangle = \int e^{-|\beta|^2}e^{-|\alpha|^2}e^{\beta\alpha^*-\beta^*\alpha}d^2\alpha \\
&= e^{-|\beta|^2}\int d^2\alpha e^{-(\alpha\beta^*-\alpha^*\beta)}e^{-|\alpha|^2}d^2\alpha
\end{aligned}
$$

let

$$
\begin{aligned}
\alpha &= x+iy \\
\beta &= a+ib \\
\beta\alpha^*-\beta^*\alpha &= 2iya-2ixb
\end{aligned}
$$

$$
\implies \langle -\beta\left|\rho\right|\beta\rangle e^{|\beta|^2} = \int\int P(x,y)e^{-(x^2+y^2)}e^{2iya-2ixb}dxdy
$$

$\langle -\beta|\rho|\beta\rangle e^{|\beta|^2}$ is the two dimensional fourier transform of $P(\alpha,\alpha^*)e^{-|\alpha|^2}$. The inverse fourier transform gives $P(\alpha,\alpha^*)$ in term of the density operator ρ.

$$
\begin{aligned}
P(\alpha,\alpha^*) &= \frac{e^{-(x^2+y^2)}}{\pi^2}\int \langle -\beta\left|\rho\right|\beta\rangle e^{(a^2+b^2)}e^{2ixb-2iya}dadb \\
P(\alpha,\alpha^*) &= \frac{e^{|\alpha|^2}}{\pi^2}\int \langle -\beta\left|\rho\right|\beta\rangle e^{|\beta|^2}e^{(\alpha\beta^*-\alpha^*\beta)}d^2\beta
\end{aligned}
$$

2.4 Examples of the Coherent state representation

2.5 Thermal field

A field emitted by a source in thermal equilibrium at temperature T is described by a canonical ensemble.

$$
\rho = \frac{e^{\frac{-H}{k_BT}}}{Tr(e^{\frac{-H}{k_BT}})}
$$

Where k_B is Boltzmann constant,and H is the free field hamiltonian.

$$
\implies \rho = \frac{e^{-\gamma(a^\dagger a+\frac{1}{2})}}{Tr[e^{-\gamma(a^\dagger a+\frac{1}{2})}]}
$$

where $\gamma = \frac{\hbar\omega}{k_BT}$

$$\rho = \frac{e^{-\gamma\hat{n}}}{Tr[e^{-\gamma\hat{n}}]}$$

Inserting unity

$$\begin{aligned}
\rho &= \frac{\sum_n e^{-\gamma\hat{n}} |n\rangle \langle n|}{Tr(\sum_n e^{-\gamma\hat{n}} |n\rangle \langle n|)} \\
&= \frac{\sum_n e^{-\gamma\hat{n}} |n\rangle \langle n|}{\sum_n \langle n| e^{-\gamma\hat{n}} |n\rangle} \\
&= \frac{\sum_n e^{-\gamma\hat{n}} |n\rangle \langle n|}{\sum_n e^{-\gamma n}}
\end{aligned}$$

as $\sum_n x^n = \frac{1}{1-x}$

$$\begin{aligned}
\implies \quad & \rho = \left(1 - e^{-\gamma}\right) \sum_n e^{-\gamma n} |n\rangle \langle n| \\
= \quad & (1 - e^{\frac{-\hbar\omega}{k_B T}}) \sum_n e^{\frac{-\hbar\omega n}{k_B T}} |n\rangle \langle n|
\end{aligned}$$

This is the density operator for a single mode thermal field, ρ has zero off-diagonal matrix element. The photon distribution function for thermal state can be calculated using,

$$\begin{aligned}
\left\langle n^{/} |\rho| m \right\rangle &= (1 - e^{\frac{-\hbar\omega}{k_B T}}) \sum_n e^{\frac{-\hbar\omega n}{k_B T}} \langle n^{/} |n\rangle \langle n| m\rangle \\
&= (1 - e^{\frac{-\hbar\omega}{k_B T}}) e^{\frac{-\hbar\omega n}{k_B T}} \delta_{n^{/} m} \\
P(n) &= \langle n |\rho| n\rangle = (1 - e^{\frac{-\hbar\omega}{k_B T}}) e^{\frac{-\hbar\omega n}{k_B T}}
\end{aligned}$$

The average value of photons in thermal field is

$$\begin{aligned}
\langle n\rangle &= Tr(a^{\dagger} a \rho) \\
&= \left(1 - e^{-\gamma}\right) \sum_n e^{-\gamma n} Tr(a^{\dagger} a |n\rangle \langle n|) \\
&= \left(1 - e^{-\gamma}\right) \sum_n n e^{-\gamma n}
\end{aligned}$$

as

$$\sum_n n e^{-\gamma n} = (\frac{-\partial}{\partial\gamma}) \sum_n e^{-\gamma n}$$

and

$$\begin{aligned}
\sum_n x^n &= \frac{1}{1-x} \\
&\Longrightarrow \sum_n e^{-\gamma n} = \frac{1}{1-e^{-\gamma}} \\
&\Longrightarrow \langle n\rangle = (1-e^{-\gamma})(\frac{-\partial}{\partial\gamma})\frac{1}{1-e^{-\gamma}} \\
\langle n\rangle &= (1-e^{-\gamma})\frac{e^{-\gamma}}{(1-e^{-\gamma})^2} \\
&= \frac{e^{-\gamma}}{(1-e^{-\gamma})} \\
&= \frac{1}{e^{\gamma}-1}
\end{aligned}$$

$$\begin{aligned}
\langle n\rangle &= \frac{1}{e^{\frac{\hbar\omega}{k_BT}}-1} \\
e^{\frac{\hbar\omega}{k_BT}} &= \frac{1}{\langle n\rangle}+1 \\
e^{-\frac{\hbar\omega}{k_BT}} &= \frac{\langle n\rangle}{1+\langle n\rangle}
\end{aligned}$$

as

$$\begin{aligned}
\rho &= (1-e^{-\frac{\hbar\omega}{k_BT}})\sum_n e^{-\frac{\hbar\omega\bar{n}}{k_BT}}\left|n\right\rangle\left\langle n\right| \\
&= (1-\frac{\langle n\rangle}{1+\langle n\rangle})\sum_n\left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^n\left|n\right\rangle\left\langle n\right| \\
&= \sum_n\frac{\langle n\rangle^n}{(1+\langle n\rangle)^{n+1}}\left|n\right\rangle\left\langle n\right|
\end{aligned}$$

$\Longrightarrow$The probability of finding n-photons in the field is

$$P\left(n\right) = \left\langle n\left|\rho\right|n\right\rangle = \rho_{nn} = \frac{\langle n\rangle^n}{(1+\langle n\rangle)^{n+1}}$$

$\Longrightarrow$The photon distribution in a thermal field is described by the Bose Einstein distribution.

2.5.1 P(α,α^*) for thermal field

As $P(\alpha,\alpha^*)$ in terms of ρ is given by

$$P(\alpha,\alpha^*) = \frac{e^{|\alpha|^2}}{\pi^2}\int \langle -\beta\,|\rho|\,\beta\rangle\, e^{|\beta|^2} e^{-\beta\alpha^*+\beta^*\alpha} d^2\beta$$

For thermal field

$$\begin{aligned}\langle -\beta\,|\rho|\,\beta\rangle &= \sum_n \frac{\langle n\rangle^n}{(1+\langle n\rangle)^{n+1}}\langle -\beta\,|n\rangle\,\langle n|\,\beta\rangle \\ |\beta\rangle &= e^{-\frac{|\beta|^2}{2}}\sum_n \frac{(\beta)^n}{(n!)^{\frac{1}{2}}}\end{aligned}$$

Putting in the above equation, the matrix element of ρ can be written as

$$\begin{aligned}\langle -\beta\,|\rho|\,\beta\rangle &= \sum_n \frac{\langle n\rangle^n}{(1+\langle n\rangle)^{n+1}} e^{-\frac{|\beta|^2}{2}}\frac{(-\beta^*)^n}{(n!)^{\frac{1}{2}}} e^{-\frac{|\beta|^2}{2}}\frac{(\beta)^n}{(n!)^{\frac{1}{2}}} \\ &= \frac{e^{-|\beta|^2}}{1+\langle n\rangle}\sum_{n=0}^{\infty}\frac{(-|\beta|^2)^n}{n!}\left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^n \\ &= \frac{e^{-|\beta|^2}}{1+\langle n\rangle} e^{\left(e^{-|\beta|^2}/(1+1/<n>)\right)}\end{aligned}$$

Putting this value in equation for $P(\alpha,\alpha^*)$

$$\begin{aligned}P(\alpha,\alpha^*) &= \frac{e^{|\alpha|^2}}{\pi^2(1+\langle n\rangle)}\int e^{e^{-|\beta|^2}/(1+1/<n>)} e^{-|\beta|^2} e^{|\beta|^2} e^{-\beta\alpha^*+\alpha\beta^*} d^2\beta \\ &= \frac{e^{|\alpha|^2}}{\pi^2(1+\langle n\rangle)}\int e^{e^{-|\beta|^2}/(1+1/<n>)} e^{-\beta\alpha^*+\alpha\beta^*} d^2\beta\end{aligned}$$

This is a 2-dimensional Fourier transform of Gaussian funtion. Fourier transform of Gaussian is a Gaussian.

Putting

$\beta = x+iy$: $\alpha = a+ib$; we get $-\beta\alpha^*+\beta^*\alpha = -2i(xa+yb)$ and $d^2\beta = dxdy$

Using all these relations the above integration will reduce to

$$P(\alpha,\alpha^*) = \frac{1}{\pi\,\langle n\rangle} e^{-|\alpha|^2/\,\langle n\rangle}$$

This is a perfect classical thermal distribusion. It gives a broad peak. The $P(\alpha,\alpha^*)$ gives the probability distribution for the field amplitude and the P-representation of the thermal distribution is given by Gaussian distribution.To describe white light completely we take a product over all k.

$$P(\alpha,\alpha^*) = \prod_k \frac{1}{\pi\langle n_k\rangle} e^{-\frac{|\alpha_k|^2}{\langle n_k\rangle}}$$

2.5.2 P-representation of a Coherent state

$$\begin{aligned}
\rho &= |\alpha_\circ\rangle\langle\alpha_\circ| \\
P(n) &= \langle n|\rho|n\rangle = \langle n|\alpha_\circ\rangle\langle\alpha_\circ|n\rangle = Poisson\ distribution
\end{aligned}$$

$$\begin{aligned}
\langle -\beta|\rho|\beta\rangle &= \langle -\beta|\alpha_\circ\rangle\langle\alpha_\circ|\beta\rangle \\
&= e^{(-|\alpha_\circ|^2-|\beta|^2-\alpha_\circ\beta^*+\beta\alpha_\circ^*)}
\end{aligned}$$

Putting this in equation for $P(\alpha,\alpha^*)$ we get

$$\begin{aligned}
P(\alpha,\alpha^*) &= \frac{1}{\pi^2} e^{|\alpha|^2-|\alpha_\circ|^2} \int e^{-\beta(\alpha^*-\alpha_\circ^*)+\beta^*(\alpha-\alpha_\circ)} d^2\beta \\
&= \delta^{(2)}(\alpha-\alpha_\circ) \\
&= \delta(\alpha^*-\alpha_\circ^*)\delta(\alpha-\alpha_\circ)
\end{aligned}$$

Thus the P-representation of a coherent state is a two-dimensional delta function.

2.5.3 P-representation for Fock state

The photon distribution function $P(n)$ for a Fock state is obtain by using

$$\begin{aligned}
\rho &= |n_\circ\rangle\langle n_\circ| \\
\langle n|\rho|m\rangle &= \langle n|n_\circ\rangle\langle n_\circ|m\rangle \\
\rho_{nm} &= \delta_{nn_\circ}\delta_{mn_\circ} \\
&= \delta_{nm}
\end{aligned}$$

For $n = m$

$$P(n) = \rho_{nn} = 1$$

$$\begin{aligned}
\Longrightarrow \quad & \langle -\beta|\rho|\beta\rangle = \langle -\beta|n\rangle\langle n|\beta\rangle \\
= \quad & e^{-|\beta|^2}\frac{(-1)^n|\beta|^{2n}}{n!}
\end{aligned}$$

The corresponding P-representation is therefore given by

$$\begin{aligned}
P(\alpha,\alpha^*) &= \frac{(-1)^n e^{|\alpha|^2}}{\pi^2 n!}\int |\beta|^{2n} e^{-\beta\alpha^*+\beta^*\alpha} d^2\beta \\
&= \frac{e^{|\alpha|^2}}{\pi^2 n!}\int (-\beta^*)^n (\beta)^n e^{-\beta\alpha^*+\beta^*\alpha} d^2\beta \\
&= \frac{e^{|\alpha|^2}}{\pi^2 n!}\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*n}}\int e^{-(\beta\alpha^*-\beta^*\alpha)} d^2\beta \\
&= \frac{e^{|\alpha|^2}}{n!}\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*n}}\delta^2(\alpha) \\
&= \frac{e^{|\alpha|^2}}{n!}\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*n}}\delta(\alpha-0)\,\delta(\alpha^*-0)
\end{aligned}$$

For $n>0$ this is clearly not a non-negative definite function and, therefore, a number state does not have a well-defined P-representation. If the photon distribution $P(n)=\rho_{nn}$ is narrower than the Poisson distribution as in the case of Fock state $|n\rangle$, the $P(\alpha,\alpha^*)$ becomes badly behaved. The $P(\alpha,\alpha^*)$ for a Fock state is a $2\mathrm{n}^{\mathrm{th}}$ deviation of 2-D delta function.

3 Non-classical state

If $F(\alpha,\alpha^*)$ is a non-negative definite function

$$\int F(\alpha,\alpha^*)P(\alpha,\alpha^*)d^2\alpha<0$$

Then $P(\alpha,\alpha^*)$ will take negative values. A state of the field is non-classical if

$$\langle : F(a,a^\dagger) : \rangle = \int F(\alpha,\alpha^*)P(\alpha,\alpha^*)d^2\alpha<0$$

Example:

$$\triangle n^2 = \left\langle \left(a^\dagger a - \langle n\rangle\right)^2 \right\rangle$$

or

$$\triangle n^2 = \langle n^2\rangle - \langle n\rangle^2$$

In normal order

$$\begin{aligned}
: \quad \triangle n^2 : &= \left\langle : \left(a^\dagger a - \langle n\rangle\right)^2 : \right\rangle \\
: \quad \triangle n^2 : \ &= \langle : n^2 : \rangle - \langle : n : \rangle^2
\end{aligned}$$

In terms of P-representation

$$:\triangle n^2: \;=\int\left(|\alpha|^2-\langle n\rangle\right)^2 P(\alpha,\alpha^*)d^2\alpha$$

If $:\triangle n^2:<0$, state of the field is non-classical.
Now consider Fock state of the field,

$$\begin{aligned}
:\;\;\triangle n^2: \;&=\left\langle a^\dagger a^\dagger aa\right\rangle-\left\langle a^\dagger a\right\rangle^2\\
:\;\;\triangle n^2: \;&=\langle n|\,a^\dagger a^\dagger aa\,|n\rangle-\langle n|\,a^\dagger a\,|n\rangle^2\\
:\;\;\triangle n^2: \;&=n\,(n-1)-n^2\\
:\;\;\triangle n^2: \;&=-n
\end{aligned}$$

This is a non-classical state of the field. For coherent state

$$:\triangle n^2: \;=|\alpha|^4-|\alpha|^4=0$$

Therefore, a coherent state is a classical state.

3.1 Q-Representation

Another field representation in terms of coherent state is Q-representation and is defined as

$$Q(\alpha,\alpha^*)=\frac{1}{\pi}\left\langle\alpha\left|\rho\right|\alpha\right\rangle$$

This distribution function helps in determining the antinormally ordered correlation functions.

$$\left\langle\hat a\hat a^\dagger\right\rangle=Tr\left(\rho\hat a\hat a^\dagger\right)$$

Inserting unity

$$\begin{aligned}
\left\langle\hat a\hat a^\dagger\right\rangle &= \frac{1}{\pi}\int d^2\alpha Tr\left(\rho\hat a\left|\alpha\right\rangle\left\langle\alpha\right|\hat a^\dagger\right)\\
&= \frac{1}{\pi}\int d^2\alpha Tr\left(\rho\left|\alpha\right\rangle\left\langle\alpha\right|\right)\alpha\alpha^*\\
&= \frac{1}{\pi}\int d^2\alpha\left\langle\alpha\right|\rho\left|\alpha\right\rangle\alpha\alpha^*\\
&= \int d^2\alpha\left(\alpha,\alpha^*\right)Q\left(\alpha,\alpha^*\right)
\end{aligned}$$

This is just like P-representation for normal ordering.

3.1.1 Properties

1)$Q(\alpha,\alpha^*)$ is real because ρ is hermitian.

2) $Q(\alpha,\alpha^*)$ is normalized i.e

$$\int Q(\alpha,\alpha^*)\,d^2\alpha = 1$$

$$\begin{aligned} Q(\alpha,\alpha^*) &= \frac{1}{\pi}Tr(\rho\,|\alpha\rangle\langle\alpha|) \\ \int Q(\alpha,\alpha^*)\,d^2\alpha &= \frac{1}{\pi}\int d^2\alpha Tr(\rho\,|\alpha\rangle\langle\alpha|) \end{aligned}$$

as

$$\begin{aligned} \frac{1}{\pi}\int d^2\alpha\,|\alpha\rangle\langle\alpha| &= 1 \\ \int Q(\alpha,\alpha^*)\,d^2\alpha &= Tr(\rho) = 1 \end{aligned}$$

3)$Q(\alpha,\alpha^*)$ is non-negative definite

$$\begin{aligned} Q(\alpha,\alpha^*) &= \frac{1}{\pi}\langle\alpha|\,\rho\,|\alpha\rangle \\ \rho &= \sum_{\Psi} P_\Psi\,|\Psi\rangle\langle\Psi| \\ &\Longrightarrow\ Q(\alpha,\alpha^*) = \frac{1}{\pi}\sum_{\psi} P_\psi\,|\langle\psi|\,\alpha\rangle|^2 \end{aligned}$$

As $P_\psi > 0$ and $|\langle\psi|\,\alpha\rangle|^2$ is always positive ,therefore, $Q(\alpha,\alpha^*)$ is always positive.

3.1.2 For Coherent state

$$\begin{aligned} Q(\alpha,\alpha^*) &= \frac{1}{\pi}\langle\alpha|\,\alpha_\circ\rangle\langle\alpha_\circ|\,\alpha\rangle \\ &= \frac{1}{\pi}e^{-|\alpha-\alpha_\circ|^2} \end{aligned}$$

3.1.3 For Thermal field

$$Q(\alpha,\alpha^*) = \frac{1}{\pi(1+\langle n\rangle)} e^{-\frac{|\alpha|^2}{1+\langle n\rangle}}$$

where

$$\rho = \sum_n \frac{\langle n\rangle^n}{(1+\langle n\rangle)^{n+1}} |n\rangle \langle n|$$

3.1.4 For Number state

$$Q(\alpha,\alpha^*) = \frac{1}{\pi} \langle\alpha| n_\circ\rangle \langle n_\circ| \alpha\rangle$$

$$Q(\alpha,\alpha^*) = \frac{|\alpha|^{2n} e^{-|\alpha|^2}}{\pi n!}$$