<u>Introduction to the Hodge Conjecture</u>

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On a complex projective non singular algebraic variety, any Hodge class is a rational linear combination of fundamental classes of algebraic cycles.

Atiyah-Hirzebruch' correction to a question of Hodge (cf. Deligne in www.claymath.org/millenium)

Transcendental *(Hodge class)* $\leftrightarrow$ Algebraic-Topological *(algebraic cycles)*

Talk: explain terms terms and origins:

- *Algebraic Integrals*
- *Hodge Theory*
- *The Conjecture*
- *Current Status*

Algebraic Integrals

$$\int \frac{dx}{\sqrt{1-x^2}}, \qquad \int \frac{dx}{\sqrt{1-x^4}}, \qquad \int \frac{dx}{\sqrt{x^3+ax+1}}$$

yield transcendental functions ... but very special, e.g., satisfy addition formulas:

$$\int_0^a \frac{dx}{\sqrt{1-x^2}} + \int_0^b \frac{dx}{\sqrt{1-x^2}} \quad = \int_0^{a\sqrt{1-b^2}+b\sqrt{1-a^2}} \frac{dx}{\sqrt{1-x^2}}$$

This is $\Leftrightarrow \sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$. Less elementary:

$$\int_0^a \frac{dx}{\sqrt{1-x^4}} + \int_0^b \frac{dx}{\sqrt{1-x^4}} \; = \int_0^{\frac{a\sqrt{1-b^4}+b\sqrt{1-a^4}}{1+a^2b^2}} \frac{dx}{\sqrt{1-x^4}}$$

Once guessed, proof is elementary. Yet ... Fagnano 1720 - for $a=b$; Euler 1752 - for all a,b; ...; Abel 1825 - for all Algebraic Integrals:

$$\int R(x, A(x))dx$$

$R(x,y)$ rational, $A(x)$ algebraic. Abel's addition formula ... another talk. Here we're interested in:

....

<u>Riemann's General Theory</u> (1850): In order to describe *all* the properties of an algebraic integral, proceed as follows.

1st. step: allow complex values for the variable:

$$\int_\gamma R(z, A(z))dz$$

γ a curve in the 'Riemann surface' of $A(z)$. Explanation:

$$A(z) \text{ algebraic } \Leftrightarrow P(z, A(z)) \equiv 0$$

for some polynomial $P(z, w) = \sum a_{kl} z^k w^l$ with $a_{kl} \in \mathbb{C}$. Pre-Riemann surface of $A(z)$:

$$S_A = \{(z, w) \in \mathbb{C}^2 : \ P(z, w) = 0\}$$

2nd. step: compactify, using $S_A \subset \mathbb{C}^2 \subset \mathbb{C}P^2$, and **algebraicity of** A: Let

$$z = \frac{Z_1}{Z_0} \qquad w = \frac{Z_2}{Z_0}$$

in $\mathbb{C}^3$, so

$$P(z, w) = P(\frac{Z_1}{Z_0}, \frac{Z_1}{Z_0}) = Z_0^{-K} Q(Z_0, Z_1, Z_2)$$

with $Q(Z_0, Z_1, Z_2)$ **homogeneous**. Then

$$X_A := \{[Z_0, Z_1, Z_2] \in \mathbb{C}P^2 : Q(Z_0, Z_1, Z_2) = 0\}$$

makes sense. Complex curve in projective plane

$$S_A \subset X_A \subset \mathbb{C}P^2$$

If non-singular, X_A = Riemann surface of $A(z)$. Otherwise desingularize

Example: For $A(z) = \sqrt{1 - z^3}$, $P(z, w) = z^3 + w^2 - 1$, $S_A = \{(z, w) \in \mathbb{C}^2 : \ z^3 + w^2 = 1\}$, $X_A = \{[Z_0, Z_1, Z_2] \in \mathbb{C}P^2 : \ Z_1^3 + Z_0 Z_2^2 - Z_0^3 = 0\}$.

Gains: the function $A(z)$ is univalued on S_A (e.g., $A(z) = \sqrt{1 - z^3}$ is $= 1$ at $(0, 1)$ and $= -1$ at $(0, -1)$). It may not be well defined on all of X_A, but the 1-form $A(z)dz$ is (except for possible poles).

3d. step: Study

$$\int_\gamma R(z, w)dz = \int_\gamma \omega$$

ω rational (meromorphic) 1-form on X.

Main tool: potential theory (= harmonic forms = Laplace's equation on Riemann surfaces) $\mapsto$ good basis for the ω's ...

End result: Reduction to Algebraic Geometry + Potential Theory in 1 complex dimension.

For

Multiple Algebraic Integrals

like, say, $\int\int \frac{\sqrt{x^3-y^4+1}}{\sqrt{x^4+y^3-1}}\ dx\ dy$, setup is similar, Lefschetz (1920) - ... - Hodge (1940).

General Algebraic Integral:

$$\int_M R(x_1, ..., x_n; A_1(x_1, ..., x_n), ..., A_m(x_1, ..., x_n))\ dx_1...dx_n$$

M an n-dimensional domain, A_i algebraic functions: $\exists$ polynomials $P_i(x_1, ..., x_n, w)$ such that

$$P_i(x_1, ..., x_n, A_i(x_1, ..., x_n)) = 0$$

identically. Set

$$S = \{(z_1, ..., z_n; w_1, ..., w_m) :\ P_i(z_1, ..., z_n, w_i) = 0\}$$

introduce $z_j = Z_j/Z_0$, $w_i = W_i/Z_0$, homogenize the P's to $Q_i(Z_0, Z_1, ..., Z_n, W_1, ..., W_m)$'s, and let

$$X = \{[Z_0, Z_1, ..., Z_n, W_1, ..., W_m] \in \mathbb{C}P^{n+m} :\ Q_1 = 0, \cdots, Q_m = 0\}$$

projective algebraic variety. Now (if smooth ...) study

$$\int_M \omega$$

where $\dim_{\mathbb{C}} X = n$ (so $\dim_{\mathbb{R}} X = 2n$), $\dim_{\mathbb{R}} M = n$, $\omega \in \Lambda^n(X)$

Hodge: middle dimension is not enough, one must fix X complex projective and study $\int_M \omega$ with $M \subset X$ of any real dimension k and $\omega \in \Lambda^k(X)$

Integration, Homology, Cohomology, Fundamental classes

<u>Stokes</u>:

$$\int_M d\omega = \int_{\partial M} \omega$$

implies

$$d\omega = 0 \ \Leftrightarrow\ \int_{\text{boundary}} \omega = 0 \qquad \text{and} \qquad \partial M = 0 \ \Leftrightarrow\ \int_M \text{exact} = 0$$

so , defining

$$H^k(X) = \frac{\{\omega \in \Lambda^k :\ d\omega = 0\}}{\{\omega \in \Lambda^k :\ \omega = d\phi\}} = \frac{\text{closed}}{\text{exact}}, \qquad H_k(X) = \frac{\{M \subset X :\ \partial M = 0\}}{\{M \subset X :\ M = \partial N\}} = \frac{\text{cycles}}{\text{boundaries}}$$

the bilinear

$$H^k(X) \times H_k(X) \longrightarrow \mathbb{R}, \qquad ([\omega], [M]) \quad \longmapsto \quad \int_M \omega$$

(= flux of ω through M) is well-defined.

Now, de Rham: it is *non-degenerate*. Hence

$$(H_k)^* = H^k$$

Poincaré:

$$H^k \times H^{d-k} \to \mathbb{R}$$

$$([\alpha], [\beta]) \mapsto \int \alpha \wedge \beta$$

is also non-degenerate. Consequently

$$H_{d-k} = H^k$$

(classes of) Closed Submanifolds $\leftrightarrow$ (classes of) Closed Forms

$$Z \mapsto cl(Z)$$

the fundamental class of Z.

....

On **Algebraic/Analytic Cycles** = Homology classes representable by Algebraic/Analytic subvarieties, and their fundamental classes.

- Such cycles reflect algebraic/analytic part of topology

- Complex projective manifolds have few analytic submanifolds - as opposed to smooth ones (or even 'none', if not projective)

- They are *very* hard to find (Mirror Symmetry counts number of complex curves in C-Y 3-folds)

The Hodge Conjecture characterizes the fundamental classes $cl(Z)$ **for** $Z \subset X$ **algebraic**, as certain 'Hodge classes' - also hard to find, but less so.

Hodge Theory

X complex structure $\Rightarrow$ decomposition of forms into types

$$\Lambda^k(X, \mathbb{C}) = \Lambda^{k,0}(X) \oplus \Lambda^{k-1,1}(X) \oplus ... \oplus \Lambda^{0,k}(X)$$

$\Lambda^{p,q}$ = locally combinations of monomials

$$dz_{i_1} \wedge ... \wedge dz_{i_{\mathbf{p}}} \wedge d\bar{z}_{i_1} \wedge ... \wedge d\bar{z}_{i_{\mathbf{q}}}$$

with smooth $\mathbb{C}$-valued coefficients.

Does this $\Rightarrow H^k(X, \mathbb{C}) = H^{k,0}(X) \oplus H^{k-1,1}(X) \oplus ... \oplus H^{0,k}(X)$?. Not in general ... $d : \Lambda^{p,q} \to \Lambda^{p+1,q} \oplus \Lambda^{p,q+1}$. But

Theorem (Hodge)

(a) *If X is a compact analytic submanifold of $\mathbb{C}P^N$, then*

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{(p,q)}(X)$$

(b) $H^k(X,\mathbb{C}) = \mathcal{H}^k(X)$ *= space of harmonic k-forms on X*

'Proof': As Riemann, use potential theory. Statement (b) makes sense after Riemannian-Hermitian metric is chosen, and that is enough to obtain $H^k(X,\mathbb{C}) = \mathcal{H}^k(X)$ (but hard: existence, uniqueness and regularity of elliptic PDE). Choose Kahler metric, e.g., induced by Fubini-Study metric in $\mathbb{C}P^N$. Next, "Kahler identities" (non-trivial, explain ...) imply that the Laplacian

$$\Delta = dd^* + d^*d \qquad\qquad (d^*\phi, \psi)_{L^2(X)} = (\phi, d\psi)_{L^2(X)}$$

preserves types, so the (p,q)-components of a harmonic form are harmonic. This implies (a), QED.

Topological meaning of $H^{(p,q)}(X)$?

Clarify: fundamental classes defined by submanifolds of X, live in $H^k(X,\mathbb{Z}) \to H^{2p}(X,\mathbb{R})$, while $H^{(p,q)} = \overline{H^{(q,p)}}$: the Hodge decomposition of $H^k(X,\mathbb{C})$ is transversal to the real form $H^k(X,\mathbb{R})$. But $H^{(p,p)}(X)$, (or $H^{(p,q)} \oplus H^{(q,p)}$, or ... Grothendieck' generalized Hodge conjecture) can intersect (the image of) $H^k(X,\mathbb{Z})$.

Theorem:

$$H^{(p,p)}(X) \cap H^{2p}(X,\mathbb{Z}) \supset H^{2p}_{ALG}(X,\mathbb{Z})$$

Proof: By definition, $cl(Z) \in H^{2p}(X,\mathbb{Z})$. Type? Since $\dim_{\mathbb{C}} Z = n-p \Rightarrow$ locally

$$Z = \{(0,...,0,z_{p+1},...,z_n)\}$$

so $dz_1, d\bar{z}_1, ..., dz_p, d\bar{z}_p$ all vanish on Z. Then

$$\int_X cl(Z)\wedge\mu \;=\; \sum_{a+b=2n-2p} \int_X cl(Z)\wedge\mu^{(a,b)} = \sum_{a+b=2n-2p} \int_Z \mu^{a,b} \;=\; \int_Z \mu^{(n-p,n-p)}$$

$$= \int_X cl(Z)\wedge\mu^{(n-p,n-p)} = \int_X cl(Z)^{(p,p)}\wedge\mu^{(n-p,n-p)} = \int_X cl(Z)^{(p,p)}\wedge\mu$$

Non-degeneracy of Poincaré duality $\Rightarrow cl(Z) = cl(Z)^{(p,p)} \in H^{(p,p)}$, QED.

Hodge Question (1940): is the converse $H^{(p,p)}(X) \cap H^{2p}(X,\mathbb{Z}) \subset H^{2p}_{ALG}(X,\mathbb{Z})$ true? Atiyah-Hirzebruch (1965): must replace $\mathbb{Z}$ by $\mathbb{Q}$. Equivalently,

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REMARK: H.C. pertains to *Projective Algebraic* objects. For example, the Hodge decomposition holds for any compact Kahler manifold, but $\exists$ *Kahler X with Hodge classes that don't come from analytic cycles.* On the other hand, if X is projective algebraic, then analytic $\Leftrightarrow$ algebraic.

Current Status

Proved for:

- Any $X^{(n)}$, $p = 1$ and $p = n - 1$. Therefore,
- Any $X^{(2)}$ and $X^{(3)}$, all p.

Also for

- Any flag manifold, $\forall p$.
- Any hypersurface deg ≤ 2, $\forall p$.

as well as

- some Hypersurfaces of higher degree
- some special fibrations
- some Abelian Varieties.

Some recent evidence for validity: The following statements are implied by the Hodge Conjecture. Conversely, some experts ... hope that "enough" theorems like these $\Rightarrow$ Hodge Conjecture.

In an algebraic family, generically smooth, the condition for an integral cycle to be (p, p) is algebraic (Weil's question (1979), proof by Cattani-Deligne-Kaplan (1996))

If the family is defined $/k \subset \mathbb{C}$ alg. closed, then the (p, p) locus of the integral cycle of above, is also defined $/k$ (Deligne's question (1995), explored, and proved in some cases, by Voisin (2006))

The zero locus of a horizontal normal function is algebraic. (Green-Griffiths Conjecture (2001), proved on curves by Brosnan-Pearlstein (2006))