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Topological K-theory

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3. Topological K-theory

In this lecture, we will discuss some of the machinery which makes topological K-theory both useful and computable. Not only does topological K-theory play a very important role in topology, but also it has played the most important guiding role in the development of algebraic K-theory.

3.1. **The Classifying space $BU \times \mathbb{Z}$.**

The following statements about topological vector bundles are not valid (in general) for algebraic vector bundles. These properties suggest that topological K-theory is better behaved than algebraic K-theory.

Proposition 3.1. *(cf. [?]) Let T be a compact Hausdorff space. If $p : E \to T$ is a topological vector bundle on T, then for some $N > 0$ there is a surjective map of bundles on T, $(\mathbb{C}^{N+1} \times T) \to E$.*

Any surjective map $E \to F$ of topological vector bundles on T admits a splitting over T.

The set of homotopy classes of maps $[T, BU(n)]$ is in natural 1-1 correspondence with the set of isomorphism classes of rank n topological vector bundles on T.

Proof. The first statement is proved using a partition of unity argument.

The proof of the second statement is proved by establishing a Hermetian metric on E (so that $E \simeq F \oplus F^{\perp}$), which is achieved by once again using a partition of unity argument.

To prove the last statement, one verifies that if $T \times I \to G$ is a homotopy relating continuous maps $f, g : T \to G$ and if E is a topological vector bundle on G, then $f^*E \simeq g^*E$ as topological vector bundles on T. Once again, a partition of unity argument is the key ingredient in the proof. □

Proposition 3.2. *For any space T, the set of homotopy classes of maps*

$$[T, BU \times \mathbb{Z}], \quad BU = \varinjlim_{n} BU_n$$

admits a natural structure of an abelian group induced by block sum of matrices $U_n \times U_m \to U_{n+m}$. We define

$$K^0_{top}(T) \equiv [T, BU \times \mathbb{Z}].$$

For any compact, Hausdorff space T, $K^0_{top}(T)$ is naturally isomorphic to the Grothendieck group of topological vector bundles on T:

$$K^0_{top}(T) \simeq \frac{\mathbb{Z}[\text{iso classes of top vector bundles on } T]}{[E] = [E_1] + [E_2], \ \text{whenever } E \simeq E_1 \oplus E_2}.$$

Proof. (External) direct sum of matrices gives a monoid structure on $\sqcup_n BU_n$ which determines a (homotopy associative and commutative) H-space structure on $BU \times Z$ which we view as the mapping telescope of the self map

$$\sqcup_n BU_n \to \sqcup_n BU_n, \quad BU_i \times \{\star \in BU_1\} \to BU_{i+1}.$$

The (abelian) group structure on $[T, BU \times \mathbb{Z}]$ is then determined.

To show that this mapping telescope is actually an H-space, one must verify that it has a 2-sided identity up to *pointed* homotopy: one must verify that product on the left with $\star \in BU_1$ gives a self map of $BU \times \mathbb{Z}$ which is related to the identity via a base-point preserving homotopy. (Such a verification is not difficult, but the analogous verification fails if we replace the topological groups U_n by discrete groups $GL_n(A)$ for some unital ring A.) $\square$

Example 3.3. Since the Lie groups U_n are connected, the spaces BU_n are simply connected and thus

$$K^0_{top}(S^1) = \pi_1(BU \times \mathbb{Z}) = 0.$$

It is useful to extend $K^0_{top}(-)$ to a relative theory which applies to pairs (T, A) of spaces (i.e., T is a topological space and $A \subset T$ is a closed subset). In the special case that $A = \emptyset$, then $T/\emptyset = T_+/\star$, the pointed space obtained by taking the disjoint union of T with a point $\star$ which we declare to be the basepoint.

Definition 3.4. If T is a pointed space with basepoint t_0, we define the reduced K-theory of T by

$$\tilde{K}^*_{top}(T) \equiv K^*_{top}(T, t_0).$$

For any pair (T, A), we define

$$K^0_{top}(T, A) \equiv \tilde{K}^0_{top}(T/A)$$

thereby extending our earlier definition of $K^0_{top}(T)$.

For any $n > 0$, we define

$$K^n_{top}(T, A) \equiv \tilde{K}^0_{top}(\Sigma^n(T/A)).$$

In particular, for any $n \geq 0$, we define

$$K^{-n}_{top}(T) \equiv K^{-n}_{top}(T, \emptyset) \equiv \tilde{K}^0_{top}(\Sigma^n(T_+)).$$

Observe that

$$\tilde{K}^0_{top}(S \wedge T) = ker\{K^0_{top}(S \times T) \to K^0_{top}(S) \oplus K^0_{top}(T)\},$$

so that (external) tensor product of bundles induces a natural pairing

$$K^{-i}_{top}(S) \otimes K^{-j}_{top}(T) \to K^{-i-j}_{top}(S \times T).$$

Just to get the notation somewhat straight, let us take T to be a single point $T = \{t\}$. Then $T_+ = \{t, \star\}$, the 2-point space with new point $\star$ as base-point. Then $\Sigma^2(T_+)$ is the 2-sphere S^2, and thus

$$K^{-2}_{top}(\{t\}) = ker\{K^0_{top}(S^2)) \to K^0_{top}(\star)\}.$$

We single out a special element, *the Bott element*

$$\beta = [\mathcal{O}_{\mathbb{P}^1}(1)] - [\mathcal{O}_{\mathbb{P}^1}] \in K^{-2}_{top}(pt)),$$

where we have abused notation by identifying $(\mathbb{P}^1)^{an}$ with S^2 and the images of algebraic vector bundles on $\mathbb{P}^1$ in $K^0_{top}((\mathbb{P}^1)^{an})$ have the same names as in $K_0(\mathbb{P}^1)$.

3.2. **Bott periodicity.** Of fundamental importance in the study of topological K-theory is the following theorem of Raoul Bott. Recall that if (X, x) is pointed space, then the **loop space** ΩX is the function complex (with the compact-open topology) of continuous maps from (S^1, ∞) to (X, x). The loop space functor $\Omega(-)$ on pointed spaces is adjoint to the suspension functor $\Sigma(-)$: there is a natural bijection

$$Maps(\Sigma(X), Y) \ \simeq \ MapsX, \Omega(Y))$$

of sets of continuous, pointed (i.e, base point preserving) maps.

Theorem 3.5. *(Bott Periodicity) There are the following homotopy equivalences.*

- *From $BO \times \mathbb{Z}$ to its 8-fold loop space:*

$$BO \times \mathbb{Z} \ \sim \ \Omega^8(BO \times \mathbb{Z})$$

  Moreover, the homotopy groups $\pi_i(BO \times \mathbb{Z})$ are given by

$$\mathbb{Z},\ \mathbb{Z}/2,\ \mathbb{Z}/2,\ 0,\ \mathbb{Z},\ 0,\ 0,\ 0$$

  depending upon whether i is congruent to $0, 1, 2, 3, 4, 5, 6, 7$ modulo 8.
- *From $BU \times \mathbb{Z}$ to its 2-fold loop space:*

$$BU \times \ \mathbb{Z} \sim \ \Omega^2(BU \times \mathbb{Z})$$

  Moreover, $\pi_i(BU \times \mathbb{Z})$ is $\mathbb{Z}$ if i is even and equals 0 if i is odd.

Atiyah interprets this 2-fold periodicity in terms of K-theory as follows.

Theorem 3.6. *(Bott Periodicity) For any space T and any $i \geq 0$, mutliplication by the Bott element induces a natural isomorphism*

$$\beta : K^{-i}_{top}(T) \to K^{-i-2}_{top}(T).$$

Using the above theorem, we define $K^i_{top}(X)$ for any topological space X and any integer i as $K^{\bar{i}}_{top}(X)$, where $\bar{i}$ is 0 if i is even and $\bar{i}$ is -1 if i is odd.

In particular, taking T to be a point, we conclude that $\tilde{K}^0_{top}(S^2) = \mathbb{Z}$, generated by the Bott element.

Example 3.7. Let S^0 denote $\{*, \star\} = *_+$. According to our definitions, the K-theory $K_{top}(*)$, of a point equals the reduced K-theory of S^0. In particular, for $n > 0$,

$$K^{-n}_{top}(*) = \tilde{K}^{-n}_{top}(S^0) = \tilde{K}^0_{top}(S^n) = \pi_n(BU).$$

Thus, we conclude

$$K^n_{top}(*) = \begin{cases} \mathbb{Z} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

We can reformulate this by writing

$$K^i_{top}(S^n) = \begin{cases} \mathbb{Z} & \text{if } i+n \text{ is even} \\ 0 & \text{if } i+n \text{ is odd} \end{cases}$$

3.3. **Spectra and Generalized Cohomology Theories.** Thus, both $BO\times\mathbb{Z}$ and $BU \times \mathbb{Z}$ are "infinite loop spaces" naturally determining Ω-spectra in the following sense.

Definition 3.8. A spectrum $\underline{\mathrm{E}}$ is a of pointed spaces $\{E^0, E^1, \ldots\}$, each of which has the homotopy type of a pointed C.W. complex, together with continuous structure maps $\Sigma(E^i) \to E^{i+1}$.

The spectrum $\underline{\mathrm{E}}$ is said to be an Ω-spectrum if the *adjoint* $E^i \to \Omega(E^{i+1})$ of each map is a homotopy equivalence; in other words, a sequence of pointed homotopy equivalences

$$E^0 \stackrel{\simeq}{\to} \Omega E^1 \stackrel{\simeq}{\to} \Omega^2 E^2 \stackrel{\simeq}{\to} \cdots \stackrel{\simeq}{\to} \Omega^n E^n \to \cdots$$

Each spectrum $\underline{\mathrm{E}}$ determines an Ω-spectrum $\underline{\tilde{\mathrm{E}}}$ defined by setting

$$\tilde{E}_n \;=\; \varinjlim_j \Omega^j \Sigma^{j-n}(E_n).$$

The importance of Ω-spectra is clear from the following theorem which asserts that an Ω-spectrum determines a "generalized cohomology theory"

Theorem 3.9. *(cf. [Spanier]) Let $\underline{E}$ be an Ω-spectrum. For any topological space X with closed subspace $A \subset X$, set*

$$h^n_{\underline{E}}(X, A) = [(X, A), E^n], \quad n \geq 0$$

Then $(X, a) \mapsto h^*_{\underline{E}}(X, A)$ is a generalized cohomology theory; namely, this satisfies all of the Eilenberg-Steenrod axioms except that its value at a point (i.e., $(*, \emptyset)$) may not be that of ordinary cohomology:

(a.) $h^*_{\underline{E}}(-)$ is a functor from the category of pairs of spaces to graded abelian groups.

(b.) for each $n \geq 0$ and each pair of spaces (X, A), there is a functorial connecting homomorphism $\partial : h^n_{\underline{E}}(A) \to h^{n+1}_{\underline{E}}(X, A)$.

(c.) the connecting homomorphisms of (b.) determine long exact sequences for every pair (X, A).

(d.) $h^*_{\underline{E}}(-)$ satisfies excision: i.e., for every pair (X, A) and every subspace $U \subset A$ whose closure lies in the interior of A, $h^*_{\underline{E}}(X, A) \simeq h^*_{\underline{E}}(X - U, A - U)$.

Observe that in the above definition we use the notation $h^*_{\underline{\mathrm{E}}}(X)$ for $h^*_{\underline{\mathrm{E}}}(X, \emptyset) = h^*_{\underline{\mathrm{E}}}(X_+, *)$, where X_+ is the disjoint union of X and a point $*$.

Definition 3.10. The (periodic) topological K-theories $KO^*_{top}(-)$, $K^*_{top}(-)$ are the generalized cohomology theories associated to the Ω-spectra given by $BO \times \mathbb{Z}$ and $BU \times \mathbb{Z}$ with their deloopings given by Bott periodicity.

In particular, whenever X is a finite dimensional C.W. complex,

$$K_{top}^{2j}(X) = [X, BU \times \mathbb{Z}], \quad K_{top}^{2j-1}(X) = [X, U],$$

so that we recover our definition of $K_{top}^0(X)$ (and similarly $KO_{top}^0(X)$).

Let us restrict attention to $K_{top}^*(X)$ which suffices to motivate our further discussion in algebraic K-theory. ($K0_{top}^*(X)$ motivates Hermetian algebraic K-theory.) There are also other interesting generalized cohomology theories (e.g., cobordism theory represented by the infinite loop space MU) which play a role in algebraic K-theory, and there are also more sophisticated equivariant K-theories, none of which will we discuss in these lectures.

Tensor product of vector bundles induces a multiplication

$$K_{top}^0(X) \otimes K_{top}^0(X) \to K_{top}^0(X)$$

for any finite dimensional C.W. complex X. This can be generalized by observing that tensor product induces group homomorphisms $U(m) \times U(n) \to U(n+m)$ and thereby maps of classifying spaces

$$BU(m) \times BU(n) \to BU(n+m).$$

With a little effort, one can show that these multiplication maps are compatible up to homotopy with the standard embeddings $U(m) \subset U(m+1), U(n) \subset U(n+1)$ and thereby give us a pairing

$$(BU \times \mathbb{Z}) \times (BU \times \mathbb{Z}) \to BU \times \mathbb{Z}$$

(factoring through the smash product). In this way, $BU \times \mathbb{Z}$ has the structure of an H-space which induces a pairing of spectra and thus a multiplication for the generalized cohomology theory $K_{top}^*(-)$. (A completely similar argument applies to $KO_{top}^*(-)$).

Remark: Each of the topological K-groups, $K_{top}^{-i}(X),\ i \in \mathbb{N}$, is given as $K_{top}^0(\Sigma^i X)$ where $\Sigma^i X$ is the i^{th} suspension of X. On the other hand, algebraic K-groups in non-zero degree are not easily related to the algebraic K_0 of some associated ring.

As an example of how topological K-theory inspired even the early (very algebraic) effort in algebraic K-theory we mention the following classical theorem of Hyman Bass. The analogous result in topological K-theory for rank e vector bundles over a finite dimension C.W. complex of dimension $d < e$ can be readily proved using the standard method of "obstruction theory".

Theorem 3.11. *(Bass stability theorem) Let A be a commutative, noetherian ring of Krull dimension d. Then for any two projective A-modules P, P' of rank $e > d$, if $[P] = [P'] \in K_0(A)$ then P must be isomorphic to P'.*

3.4. **Skeleta and Postnikov towers.** If X is a C.W. complex then we can define its **p-skeleton** $sk_p(X)$ for each $p \geq 0$ as the subspace of X consisting of the union of those cells of dimension $\leq p$. Thus, the C.W. complex can be written as the union (or colimit) of its skeleta,

$$X \ = \ \cup_p sk_p(X).$$

There is a standard way to "chop off" the bottom homotopy groups of a space (or an Ω-spectrum) using an analogue of the universal covering space of a space (which "chops off" the fundamental group).

Definition 3.12. Let X be a C.W. complex. For each $n \geq 0$, construct a map $X \to X[n]$ by attaching cells (proceeding by dimension) to kill all homotopy groups of X above dimension $n-1$. Define

$$X^{(n)} \ to \ X, \quad htyfib\{X \to X[n]\}.$$

So defined, $X^{(n)} \to X$ induces an isomorphism on homotopy groups $\pi_i, \ i \geq n$ and $\pi_j(X^{(n)}) = 0, \ j \leq n$.

The **Postinov tower** of X is the sequence of spaces

$$X \ \cdots \ \to X^{(n+1)} \ \to X^{(n)} \ \to \cdots$$

Thus, X can be viewed as the "homotopy inverse limit" of its Postnivkov tower.

Algebraic K-theory corresponds most closely the topological K-theory which is obtained by replacing the Ω-spectrum $\underline{\mathrm{K}} = \underline{\mathrm{BU}} \times \mathbb{Z}$ by $\underline{\mathrm{kU}} = \underline{\mathrm{bu}} \times \mathbb{Z}$ obtained by taking at stage i the i^{th} connected cover of $BU \times \mathbb{Z}$ starting at stage 0. The associated generalized cohomology theory is denoted $kU^*(-)$ and satisfies

$$kU^i(X) \ \simeq K^i_{top}(X), \quad i \leq 0.$$

3.5. **The Atiyah-Hirzebruch Spectral sequence.** The Atiyah-Hirzebruch spectral sequence for topological K-theory has been a strong motivating factor in recent developments in algebraic K-theory. Indeed, perhaps the fundamental criterion for *motivic cohomology* is should satisfy a relationship to algebraic K-theory strictly analogous to the relationship of singular cohomology to topological K-theory.

Theorem 3.13. *(Atiyah-Hirzebruch spectral sequence) For any generalized cohomology theory $h^*_{\underline{E}}(-)$ and any topological space X, there exists a right half-plane spectral sequence of cohomological type*

$$E_2^{p,q} = H^p(X, h^q(*)) \Rightarrow h^{p+q}_{\underline{E}}(X).$$

The filtration on $h^*_{\underline{E}}(X)$ is given by

$$F^pE^*_\infty = ker\{h^*_{\underline{E}}(X) \to h^*_{\underline{E}}(sk_p(X)\}.$$

In the special case of $K^*_{top}(-)$, this takes the following form

$$E_2^{p,q} = H^p(X, \mathbb{Z}(q/2)) \Rightarrow K^{p+q}_{top}(X)$$

where $\mathbb{Z}(q/2) = \mathbb{Z}$ *if* q *is even and 0 otherwise.*

In the special case of $kU^*(-)$*, this takes the following form*

$$E_2^{p,q} = H^p(X, \mathbb{Z}(q/2)) \Rightarrow kU^{p+q}(X)$$

where $\mathbb{Z}(q/2) = \mathbb{Z}$ *if* q *is an even non-positive integer and 0 otherwise.*

Proof. There are two basic approaches to proving this spectral sequence. The first is to assume T is a cell complex, then consider T as a filtered space with $T_n \subset T$ the union of cells of dimension $\leq n$. The properties of $K^*_{top}(-)$ stated in the previous theorem give us an exact couple associated to the long exact sequences

$$\cdots \to \oplus K^q_{top}(S^n) \simeq K^q_{top}(T_n/T_{n-1}) \to K^q_{top}(T_n) \to K^q_{top}(T_{n-1}) \to \oplus K^{q+1}_{top}(S^n) \to \cdots$$

where the direct sum is indexed by the n-cells of T.

The second approach applies to a general space T and uses the Postnikov tower of $BU \times \mathbb{Z}$. This is a tower of fibrations whose fibers are Eilenberg-MacLane spaces for the groups which occur as the homotopy groups of $BU \times \mathbb{Z}$.

What is a spectral sequence of cohomological type? This is the data of a 2-dimensional array $E_r^{p,q}$ of abelian groups for each $r \geq r_0$ (typically, r_0 equals 0, or 1 or 2; in our case $r_0 = 2$) and homomorphisms

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

such that the next array $E_{r+1}^{p,q}$ is given by the cohomology of these homomorphisms:

$$E_{r+1}^{p,q} = ker\{d_r^{p,q}\}/im\{d_r^{p-r,q+r-1}\}.$$

To say that the spectral sequence is "right half plane" is to say $E_r^{p,q} = 0$ whenever $p < 0$. We say that the spectral sequence **converges to the abutment** E^*_∞ (in our case $h^*_{\underline{\mathrm{E}}}(X)$) if at each spot (p, q) there are only finitely many non-zero homomorphisms going in and going out and if there exists a decreasing filtration $\{F^p E^n_\infty\}$ on each E^n_∞ so that

$$E^n_\infty = \bigcup_p F^p E^n_\infty, \; 0 = \bigcap_p F^p E^n_\infty,$$

$$F^p E^n_\infty / F^{p+1} E^n_\infty = E_R^{p,n-p}, \quad R >> 0.$$

□

The Postnikov tower argument together with a knowledge of the k-invariants of $BU \times \mathbb{Z}$ shows that after tenoring with $\mathbb{Q}$ this Atiyah-Hirzebruch spectral sequence collapses; in other words, that $E_2^{*,*} \otimes \mathbb{Q} = E_\infty^{*,*} \otimes \mathbb{Q}$.

Theorem 3.14. *(Atiyah-Hirzebruch) Let* X *be a C.W. complex. Then there are isomorphisms*

$$kU^0(X)) \otimes \mathbb{Q} \;\simeq\; H^{ev}(X, \mathbb{Q}), \quad kU^{-1}(X) \otimes Q \simeq H^{odd}(X, \mathbb{Q}).$$

These isomorphisms are induced by the *Chern character*

$$ch = \sum_i ch_i : K_0(-) \ \rightarrow \ H^{ev}(-, \mathbb{Q})$$

discussed in Lecture 4.

While we are discussing spectral sequences, we should mention the following:

Theorem 3.15. *(Serre spectral sequence) Let (B, b) be a connected, pointed C.W. complex. For any fibration $p : E \rightarrow B$ of topological spaces with fibre $F = p^{-1}(b)$ and for any abelian group A, there exists a convergent first quadrant spectral sequence of cohomological type*

$$E_2^{p,q} = H^p(B, H^q(F, A)) \Rightarrow H^{p+q}(E, A)$$

provided that $\pi_1(B, b)$ acts trivially on $H^*(F, A)$.

The non-existence of an analogue of the Serre spectral sequence in algebraic geometry (for cohomology theories based on algebraic cycles or algebraic K-theory) presents one of the most fundamental challenges to computations of algebraic K-groups.

3.6. **K-theory Operations.** There are several reasons why topological K-theory has sometimes proved to be a more useful computational tool than singular cohomology.

- $K^0_{top}(-)$ can be torsion free, even though $H^{ev}(-, \mathbb{Z})$ might have torsion. This is the case, for example, for compact Lie groups.
- $K^*_{top}(-)$ is essentially $\mathbb{Z}/2$-graded rather than graded by the natural numbers.
- $K^*_{top}(-)$ has interesting cohomology operations not seen in cohomology. These operations originate from the observation that the exterior products $\Lambda^i(P)$ of a projective module P are likewise projective modules and the exterior products $\Lambda^i(E)$ of a vector bundle E are likewise vector bundles.

Definition 3.16. Let X be a finite dimensional C.W. complex and $E \rightarrow X$ be a topological vector bundle of rank r. Define

$$\lambda_t(E) = \sum_{i=0}^{r} [\Lambda^i E] t^i \ \in \ K^0_{top}(X)[t],$$

a polynomial with constant term 1 and thus an invertible element in $K^0_{top}(X)[[t]]$. Extend this to a homomorphism

$$\lambda_t : K^0_{top}(X) \rightarrow (1 + K^0_{top}(X)[[t]])^*,$$

(using the fact that $\lambda_t(E \oplus F) = \lambda_t(E) \cdot \lambda_t(F)$) and define $\lambda^i : K^0_{top}(T) \rightarrow K^0_{top}(T)$ to be the coefficient of t^i of λ_t.

For a general topological space X, define these λ operations on $K^0_{top}(X)$ for by defining them first on the universal vector bundles over Grassmannians and using the functoriality of $K^0_{top}(-)$.

In particular, J. Frank Adams introduced operations

$$\psi^k(-) : K^0_{top}(-) \to K^0_{top}(-), \quad k > 0$$

(called **Adams operations**) which have many applications and which are similarly constructed for algebraic K-theory.

Definition 3.17. For any topological space T, define

$$\psi_t(x) = \sum_{i \geq 0} \psi^i(X) t^i \equiv rank(x) - t \cdot \frac{d}{dt}(log\lambda_{-t}(x))$$

for any $x \in K^0_{top}(T)$.

The *Adams operations* ψ^k satisfy many good properties, some of which we list below.

Proposition 3.18. *For any topological space T, any $x, y \in K^0_{top}(T)$, any $k > 0$*

- $\psi^k(x+y) = \psi^k(x) + \psi^k(y)$.
- $\psi^k(xy) = \psi^k(x)\psi^k(y)$.
- $\psi^k(\psi^\ell(x) = \psi^{k\ell}(x)$.
- $ch_q(\psi^k(x)) = k^q ch_q(x) \in H^{2q}(T, \mathbb{Q})$.
- *$\psi^p(x)$ is congruent modulo p to x^p if p is a prime number.*
- *$\psi^k(x) = x^k$ whenever x is a line bundle*

In particular, if E is a sum of line bundles $\oplus_i L_i$, then $\psi^k(E) = \oplus((L_i)^k)$, the k-th power sum. By the splitting principle, this property alone uniquely determines ψ^k.
We introduce further operations, the γ-operations on $K_0^{top}(T)$.

Definition 3.19. For any topological space T, define

$$\gamma_t(x) = \sum_{i \geq 0} \gamma^i(X) t^i \equiv \lambda_{t/1-t}(x)$$

for any $x \in K^0_{top}(T)$.

Basic properties of these γ-oerations include the following

(1) $\gamma_t(x+y) = \gamma_t(x)\gamma_t(y)$
(2) $\gamma([L]-1) = 1 + t([L]-1)$.
(3) $\lambda_s(x) = \gamma_{s/1+s}(x)$

Using these γ operations, we define the γ filtration on $K^0_{top}(T)$ as follows.

Definition 3.20. For any topological space T, define $K^{\gamma,1}_{top}(T)$ as the kernel of the rank map

$$K^{\gamma,1}_{top}(T) \equiv ker\{\text{rank} : K^0_{top}(T) \to K^0_{top}(\pi_0(T))\}.$$

For $n > 1$, define

$$K^0_{top}(T)^{\gamma,n} \subset K^{\gamma,0}_{top}(T) \equiv K^0_{top}(T)$$

to be the subgroup generated by monomials $\gamma^{i_1}(x_1)\cdots\gamma^{i_k}(x_k)$ with $\sum_j i_j \geq n, x_i \in K^{\gamma,1}_{top}(T)$.

3.7. **Applications.** We can use the Adams operations and the γ-filtration to describe in the following theorem the relationship between $K^0_{top}(T)$, a group which has no natural grading, and the graded group $H^{ev}(T, \mathbb{Q})$.

Theorem 3.21. *Let T be a finite cell complex. Then for any $k > 0$, ψ^k restricts to a self-map of each $K^{\gamma,n}_{top}(T)$ and satisfies the property that it induces multiplication by k^n on the quotient*

$$\psi^k(x) = k^n \cdot x, \quad x \in K^{\gamma,n}_{top}(T)/K^{\gamma,n+1}_{top}(T)).$$

Furthermore, the Chern character ch induces an isomorphism

$$ch_n : K^{\gamma,n}_{top}(T)/K^{\gamma,n+1}_{top}(T)) \otimes Q \simeq H^{2n}(T, \mathbb{Q}).$$

In particular, the preceding theorem gives us a K-theoretic way to define the grading on $K^0_{top}(T) \otimes \mathbb{Q}$ induced by the Chern character isomorphism. The graded piece of (the associated graded of) $K^0_{top}(T) \otimes \mathbb{Q}$ corresponding to $H^{2n}(T, \mathbb{Q})$ consists of those classes x for which $\psi^k(x) = k^n x$ for some (or all) $k > 0$.

Here is a short list of famous theorems of Adams using topological K-theory and Adams operations:

Application 3.22. *Adams used his operations in topological K-theory to solve fundamental problems in algebraic topology. Examples include:*

- *Determination of the number of linearly independent vector fields on the n-sphere S^n for all $n > 1$.*
- *Determination of the only dimensions (namely, $n = 1, 2, 4, 8$) for which $\mathbb{R}^n$ admits the structure of a division algebra. (The examples of the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions, and the Cayley numbers gives us structures in these dimensions.)*
- *Determination of those (now well understood) elements of the homotopy groups of spheres associated with $KO^0_{top}(S^n)$.*