The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization

IAEA International Atomic Energy Agency

SMR/1840-11

School and Conference on Algebraic K-Theory and its Applications

14 May - 1 June, 2007

Motives of quadrics

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Lecture 3
Motives of quadrics

We have a motivic functor

$$Sm.Proj./k \stackrel{M}{\to} \mathrm{Chow}^{eff}(k),$$

which provides each smooth projective variety with its invariant *the motive.*

In $\mathrm{Chow}^{eff}(k)$ we get new objects - the direct summands in the motives of smooth projective varieties. In particular, $M(\mathbb{P}^1)$ will be decomposable. Notice, that $M(\mathbb{P}^1)$ is given by the pair $([\mathbb{P}^1],[\Delta(\mathbb{P}^1)] \in \mathrm{CH}_1(\mathbb{P}^1 \times \mathbb{P}^1))$, but in $\mathrm{CH}_1(\mathbb{P}^1 \times \mathbb{P}^1)$ the class $[\Delta(\mathbb{P}^1)]$ is equal to the sum of two mutually orthogonal projectors (with respect to the composition operation $\circ$) $[pt \times \mathbb{P}^1] + [\mathbb{P}^1 \times pt]$, where pt is any k-rational point on $\mathbb{P}^1$. Thus, $M(\mathbb{P}^1) = ([\mathbb{P}^1],[\mathbb{P}^1 \times pt]) \oplus ([\mathbb{P}^1],[pt \times \mathbb{P}^1])$. The first summand here is isomorphic to $M(\mathrm{Spec}(k))$ and will be denoted $\mathbb{Z}(0)[0]$ (or, simply, $\mathbb{Z}$) - the *trivial Tate-motive*, and the second is denoted $\mathbb{Z}(1)[2]$ - the *Tate-motive.* $\mathrm{Chow}^{eff}(k)$ is tenzor additive category with $M(X) \oplus M(Y) = M(X \coprod Y)$, and $M(X) \otimes M(Y) = M(X \times Y)$. Can define Tate-motive $\mathbb{Z}(n)[2n]$ as $(\mathbb{Z}(1)[2])^{\otimes n}$. It is given as a direct summand in the motive of $(\mathbb{P}^1)^n$, but will be also a direct summand in the motive $M(X)$ of any smooth projective n-dimensional variety X which has a k-rational point - the respective projector is given by $[pt \times X]$ (in reality, you just need a zero-cycle of degree 1).

Inside $\mathrm{Chow}^{eff}(k)$ you will meet only Tate motives $\mathbb{Z}(n)[m]$ with $m = 2n$. But $\mathrm{Chow}^{eff}(k)$ is naturally a full additive subcategory of the bigger *triangulated category of motives* $DM_{-}^{eff}(k)$, and the latter category already contains Tate-motives $\mathbb{Z}(n)[m]$ with all possible m and n. This is why we will keep both numbers in the notation of Tate-motives, although, in our situation, these numbers are not independent. We get

$$M(\mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z}(1)[2].$$

In the same way,

$$M(\mathbb{P}^r) = \mathbb{Z} \oplus \mathbb{Z}(1)[2] \oplus \ldots \oplus \mathbb{Z}(r)[2r].$$

with the projectors $[\mathbb{P}^s \times \mathbb{P}^{r-s}]$, for $0 \leqslant s \leqslant r$.

<u>Connection to Chow groups and motivic cohomology</u>

For smooth projective varieties one can naturally identify:

$$\mathrm{CH}^n(X) = \mathrm{Hom}_{\mathrm{Chow}^{eff}(k)}(M(X), \mathbb{Z}(n)[2n]);$$
$$\mathrm{CH}_n(X) = \mathrm{Hom}_{\mathrm{Chow}^{eff}(k)}(\mathbb{Z}(n)[2n], M(X)).$$

and since $\mathrm{Chow}^{eff}(k)$ is a full subcategory of $DM_{-}^{eff}(k)$, the former group can be identified with $\mathrm{Hom}_{DM_{-}^{eff}(k)}(M(X), \mathbb{Z}(n)[2n]) = \mathrm{H}_{\mathcal{M}}^{2n,n}(X, \mathbb{Z})$ - the *motivic cohomology*. Thus we see that

$$\mathrm{CH}^n(X) = \mathrm{H}_{\mathcal{M}}^{2n,n}(X, \mathbb{Z}).$$

Quadrics

The motive of a quadric is the simplest when the quadric is completely split. In this case, it can be decomposed into the direct sum of Tate-motives.

$$M(Q) = \oplus_{i=0}^{[\frac{\dim(Q)}{2}]}(\mathbb{Z}(i)[2i] \oplus \mathbb{Z}(\dim(Q) - i)[2\dim(Q) - 2i])$$

The respective projectors have the form $[l_i \times h^i]$ and $[h^i \times l_i]$, where h^i is a plane section of codimension i on Q, and l_i is a projective subspace of dimension i on Q (which exists since Q is completely split). In particular, one can observe that the motive of odd-dimensional split quadric coincides with the motive of the projective space of the same dimension, although, as algebraic varieties they are not isomorphic (when dimension > 1). This shows that the variety can not be reconstructed from its motive, in general.

Using the fact that

$$\mathrm{Hom}_{\mathrm{Chow}^{eff}(k)}(\mathbb{Z}(i)[2i], \mathbb{Z}(j)[2j]) = \begin{cases} 0, i \neq j; \\ \mathbf{Z}, i = j. \end{cases}$$

we can compute Chow groups of Q:

$$\mathrm{CH}^i(Q) = \begin{cases} \mathbf{Z}, 0 \leqslant i \leqslant \dim(Q), i \neq \dim(Q)/2; \\ \mathbf{Z} \oplus \mathbf{Z}, i = \dim(Q)/2; \\ 0, \text{ otherwise}. \end{cases}$$

Examples:

1) C - split conic, $M(C) = \mathbb{Z} \oplus \mathbb{Z}(1)[2]$;

2) Q - split 2-dimensional quadric, $M(Q) = \mathbb{Z} \oplus \mathbb{Z}(1)[2] \oplus \mathbb{Z}(1)[2] \oplus \mathbb{Z}(2)[4]$.

What if quadric is not completely split, but just isotropic? Let $q = \mathbb{H} \perp q'$. Then

$$M(Q) = \mathbb{Z} \oplus M(Q')(1)[2] \oplus \mathbb{Z}(\dim(Q))[2\dim(Q)]$$

Applying inductively this fact one gets the case of the split quadric above. Also, this shows that the motive of a quadric can be expressed in terms of the Tate-motives and the motive of the anisotropic part of it.

But what if the quadric is anisotropic, can we still say something about its motive?

Consider the case of a conic C. First of all we observe the following simple fact:

$$C \text{ has a } k\text{- rational point} \quad \Leftrightarrow \quad C \cong \mathbb{P}^1$$

Indeed, the ($\Leftarrow$) conclusion is obvious, since $\mathbb{P}^1$ has plenty of k-rational points. Conversely, let $x \in C$ be some k-rational point. Then C is naturally identified with the $\mathbb{P}^1$ of projective lines on $\mathbb{P}^2$ passing through $x \subset C \subset \mathbb{P}^2$. Thus, if conic is somewhat interesting (do not coincide with the projective line, at least), then it has no rational points.

Suppose that C is arbitrary conic given by some equation $Ax_o^2 + Bx_1^2 + Cx_2^2$. We can divide it by A and get $x_0^2 - ax_1^2 - bx_2^2$ ($a = -B/A, b = -C/A$), so that our form is $\langle 1, -a, -b \rangle$. Then it is a subform of a Pfister form $\langle\langle a, b \rangle\rangle$. By the Main property of Pfister forms, for arbitrary field extension E/k,

$$\langle\langle a, b \rangle\rangle|_E \text{ is isotropic } \Leftrightarrow \langle\langle a, b \rangle\rangle|_E \text{ is completely split.}$$

Hence, this condition is also equivalent to: $\langle 1, -a, -b \rangle|_E$ is isotropic. Really, isotopity of $\langle 1, -a, -b \rangle|_E$ implies isotropity of $\langle\langle a, b \rangle\rangle|_E$ since the former is a subform of the latter. In the other direction, if $\langle\langle a, b \rangle\rangle|_E$ isotropic, then it is completely split, that is, has a totally isotropic subspace of dimension 2, but then such subspace should intersect nontrivially with the 1-codimensional subform $\langle 1, -a, -b \rangle|_E$ to produce isotropic vector for the latter.

Now, we can also remind, that for arbitrary field extension E/k,

$$\langle\langle a, b \rangle\rangle|_E \text{ is completely split } \Leftrightarrow \{a, b\}|_E = 0.$$

This shows that our conic $C_{\{a,b\}}$ and the Pfister quadric $Q_{\{a,b\}}$ are the *norm-varieties* for the pure symbol $\{a, b\} \in \mathrm{K}_2^M(k)/2$. A variety X is called a norm-variety for $\alpha \in \mathrm{K}_n^M(k)/r$ if for arbitrary field extension E/k, $X|_E$ has E-rational point if and only if $\alpha|_E = 0 \in \mathrm{K}_n^M(E)/r$.

Exactly the same considerations show that for arbitrary subform p of $\langle\langle a_1, \ldots, a_n \rangle\rangle$ of dimension $> \frac{1}{2}\dim(\langle\langle a_1, \ldots, a_n \rangle\rangle) = 2^{n-1}$, the respective projective quadric will be a norm-variety for the symbol $\{a_1, \ldots, a_n\} \in \mathrm{K}_n^M(k)/2$. Notice that we have many different varieties corresponding to the same symbol. It is clear that all of them have something in common. And this something appears to be certain direct summand in their motives.

Consider again the case of 2-dimensional 2-fold Pfister quadric $Q_{\{a,b\}}$. Since determinant of $\langle\langle a, b \rangle\rangle$ is 1, the projective lines on $Q_{\{a,b\}}$ split into two families, each of which is naturally identified with $C_{\{a,b\}}$ (each line intersects $C_{\{a,b\}}$ in a unique point - this defines the identification). This simultaneously shows that $Q_{\{a,b\}} = C_{\{a,b\}} \times C_{\{a,b\}}$ (since each point on $Q_{\{a,b\}}$ is determined uniquely by the pair of projective lines on $Q_{\{a,b\}}$ (one from each of the two families) passing through it), and identifies it with $\mathbb{P}_{C_{\{a,b\}}}(\mathcal{V})$ - the projectivisation of certain 2-dimensional vector bundle on $C_{\{a,b\}}$ (since there is a natural projection $Q_{\{a,b\}} \to C_{\{a,b\}}$ given by the lines of one of the families, with the fibers - those lines). It follows from the general theory that the motive of the projective bundle is a direct summand of several copies of the motive of the base with various Tate-twists (for $U \in Ob(\mathrm{Chow}^{eff}(k))$ we call $U(n)[2n] := U \otimes \mathbb{Z}(n)[2n]$ - the *Tate-twist* of U). In our situation, we get:

$$M(Q_{\{a,b\}}) = M(C_{\{a,b\}}) \oplus M(C_{\{a,b\}})(1)[2].$$

This is the first example of the following general result obtained by M.Rost:

Theorem 0.1 (M.Rost) *Let $\alpha \in \mathrm{K}_n^M(k)/2$ be the pure symbol, and Q_α be the respective Pfister form. Then there exists such motive $M_\alpha \in Ob(\mathrm{Chow}^{eff}(k))$ that*

$$M(Q_\alpha) = \oplus_{i=0}^{2^{n-1}-1} M_\alpha(i)[2i],$$

(then it is easy to see that $M_\alpha|_{\overline{k}} = \mathbb{Z} \oplus \mathbb{Z}(2^{n-1}-1)[2^n-2]$), and M_α splits into the sum of Tate-motives if and only if $\alpha = 0$.

The motive M_α is called the *Rost motive*.

Examples:

1) $n = 1$, then $M_{\{a\}} = M(\mathrm{Spec}(k\sqrt{a}))$;

2) $n = 2$, then $M_{\{a,b\}} = M(C_{\{a,b\}})$.

3) For $n > 3$, M_α is no longer represented by the motive of any algebraic variety, but only by a direct summand in such.

M.Rost also had shown that M_α is also a direct summand in the motives of any subquadrics of Q_α of codimension $< 2^{n-1}$ (such subquadrics are called *Pfister neighbours*). Let q_α be n-fold Pfister form, $p \subset q_\alpha$ a subform of dimension $2^{n-1} + m$, $m > 0$, and $p^\perp$ be the *complimentary form* ($q_\alpha = p \perp p^\perp$). Then

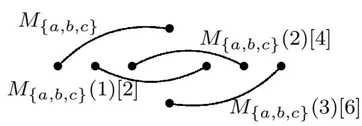
$$M(P) = \oplus_{i=0}^{m-1} M_\alpha(i)[2i] \oplus M(P^\perp)(m)[2m].$$

And the appearence of M_α in this decomposition explains why all such quadrics are the norm-varieties for the pure symbol α. Namely, the existence of a rational point on P is equivalent to $M(P)$ containing Tate-motive $\mathbb{Z}$ as a direct summand (follows from the Theorem of Springer), and is further equivalent to M_α containing such a summand - equivalent to M_α being split, which happens if and only if $\alpha = 0$.

Applying the above statement inductively, one gets that the motive of an excellent quadric is a sum of Rost-motives (of different foldness).

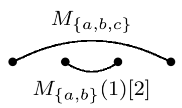
Examples:

1) The motive of 3-fold Pfister form $Q_{\{a,b,c\}}$ can be visualized as

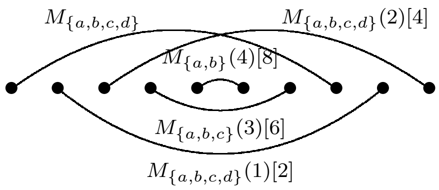


where each $\bullet$ represents a Tate-motive over $\overline{k}$, ranging from $\mathbb{Z}$ on the left to $\mathbb{Z}(6)[12]$ on the right, and each pair of connected $\bullet$'s represents the copy of the Rost-motive $M_{\{a,b,c\}}(i)[2i]$.

2) Let q be 5-dimensional excellent form $\langle 1, -c, ac, bc, -abc \rangle$, then $M(Q)$ can be visualized as



3) Let q be 11-dimensional excellent form $(\langle\langle a,b,c,d\rangle\rangle \perp -\langle\langle a,b,c\rangle\rangle \perp \langle\langle a,b\rangle\rangle \perp -\langle 1\rangle)_{an.}$ (we assume a,b,c,d algebraically independent). Then $M(Q)$ looks as



Hypothetically, the Rost-motives are the only possible *binary direct summands* (that is, motives, which split into the direct sum of exactly 2 Tate-motives over $\overline{k}$) in the motives of quadrics, and the excellent forms are the only forms whose motives split into binary direct summands.

Motivic decomposition type

Definition 0.2 *For the quadric Q let us denote as $\Lambda(Q)$ the set of Tate-motives in the decomposition of its motive over $\overline{k}$:*

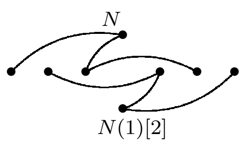
$$M(Q|_{\overline{k}}) = \oplus_{\lambda\in\Lambda(Q)}\mathbb{Z}(i_\lambda)[2i_\lambda].$$

Then for any direct summand N of $M(Q)$ we can identify the set $\Lambda(N)$ of Tate-motives in the decomposition of $N|_{\overline{k}}$ with the subset of $\Lambda(Q)$. We say that $\lambda\in\Lambda(Q)$ and $\mu\in\Lambda(Q)$ are connected, if for any direct summand N of $M(Q)$, $\lambda\in\Lambda(N)\Leftrightarrow\mu\in\Lambda(N)$. The presentation of $\Lambda(Q)$ as the disjoint union of its connected components is called motivic decomposition type *of Q - $MDT(Q)$.*

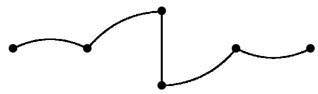
The *motivic decomposition type* can be visualized as a picture of the same sort as above.

Examples:

1) Let $q = \langle\langle a\rangle\rangle\cdot\langle b,c,d,e\rangle$, where a,b,c,d are algebraically independent. Then $M(Q)$ splits into the sum of two (isomorphic up to Tate-shift) indecomposable direct summands, and $MDT(Q)$ looks as



2) Let q be *Albert form* $\langle a, b, -ab, -c, -d, cd\rangle$. Then $M(Q)$ is indecomposable, and $MDT(Q)$ consists of one connected component:



3) Let q be 9-dimensional form $(\langle\langle a, b, c\rangle\rangle \perp -\langle 1, -e, -f\rangle)_{an}$, where a, b, c, d, e, f are algebraically independent. Then $MDT(Q)$ looks as:

4) Let q be 9-dimensional form $\langle\langle a\rangle\rangle \cdot \langle b, c, d, e\rangle \perp \langle 1\rangle$, where a, b, c, d, e are algebraically independent. Then $MDT(Q)$ looks as:

Splitting pattern

Another discrete invariant of quadrics is the *splitting pattern* invariant. Introduced by M.Knebusch, U.Rehmann and J.Hurrelbrink, it measures what are possible *Witt-indices* $i_W(q|_E)$ of our form over all possible field extensions E/k. One gets the increasing sequence of natural numbers $j_0 < j_1 < j_2 < \ldots < j_h$ - the possible values of $i_W(q|_E)$. The numbers $i_l := j_l - j_{l-1}, l \geqslant 1$ are called the *higher Witt indices.* Assuming q-anisotropic ($j_0 = 0$), the sequence $(i_1, i_2, \ldots, i_h)$ is called the *splitting pattern* $SP(Q)$. The number h is called the *height* of Q.

Examples:

1) For the n-fold Pfister form q_α, $SP(Q_\alpha) = (2^{n-1})$, and the height is 1, since the Pfister form becomes complitely split as soon as it is isotropic. The Pfister quadrics and the subquadrics of codimension 1 in them are the only examples of (anisotropic) quadrics of height 1.

2) For Albert form $q = \langle a, b, -ab, -c, -d, cd\rangle$, we have $SP(Q) = (1, 2)$, and $h(Q) = 2$.

3) For the generic form $q = \langle b_1, \ldots, b_m\rangle$, where $b_1, \ldots, b_m$ are algebraically independent, $SP(Q) = (1, 1, \ldots, 1)$, and $h(Q) = [m/2]$.

4) For the form $q = \langle\langle a_1, \ldots, a_n \rangle\rangle \cdot \langle b_1, \ldots, b_{2r} \rangle$, where $a_1, \ldots, a_n, b_1, \ldots, b_{2r}$ are algebraically independent, $SP(Q) = (2^n, 2^n, \ldots, 2^n)$, and $h(Q) = r$.

5) An (anisotropic) excellent form q of dimension 19 has the splitting pattern $(3, 5, 1)$ and height 3.

It is an important problem in the theory of quadratic forms to find all the possible values of the invariants $MDT(Q)$ and $SP(Q)$. Among the partial results I should mention the Theorem of N.Karpenko, which claims that $(i_1(q) - 1)$ should always be a remainder of the division of $(\dim(q) - 1)$ by some power of 2. Although, we understand MDT and SP to some extent, there is no even hypothetical description of the possible answer. Nevertheless, the interaction between the splitting pattern and motivic decomposition type invariants provides a lot of information about both of them. This suggests that one should try to embed them as faces into some larger invariant, where one can expect to have more structure. In the next lecture we will introduce such big invariant of geometric origin, called *Generic discrete invariant of* Q.