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The group structure of SL_n over a field

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May 16: 1. The group structure of SL_n over a field

k any field: $GL_n(k) = \{ (a_{ke})_{1\le k,e\le n} \mid \det(a_{ke}) \neq 0 \}$

$$G = SL_n(k) = \{ (a_{ke}) \mid \det(a_{ke}) = 1 \}$$

Exact: $1 \to SL_n(k) \to GL_n(k) \to k^* \to 1$

$x \in k$: $u_{ij}(x) := 1 + x e_{ij}$ w. $e_{ij} = (a_{ke})$ with $a_{ke} = \begin{cases} 1 & ij = ke \\ 0 & ij \neq ke \end{cases}$
(unipotent)

$x \neq 0$: $w_{ij}(x) = u_{ij}(x)\, u_{ji}(-x^{-1})\, u_{ij}(x)$

$$h_{ij}(x) = w_{ij}(x)\, w_{ij}(-1)$$

These are monomial matrices

$h_{ij}(x)$ are diagonal

Expl: $n = 2$:

$w_{12}(x) = \begin{pmatrix} 0 & x \\ -x^{-1} & 0 \end{pmatrix}$, $h_{12}(x) = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$

$w_{21}(x) = \begin{pmatrix} 0 & -x^{-1} \\ x & 0 \end{pmatrix}$, $h_{21}(x) = \begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix}$

Straightforward:

$SL_n(k) = \langle u_{ij}(x) \mid 1 \le i,j \le n,\ i \neq j,\ x \in k \rangle$

$N := \langle w_{ij}(x) \mid 1 \le i,j \le n,\ i \neq j,\ x \in k^* \rangle$ = group of all monomial matrices

$T := \langle h_{ij}(x) \mid$ — " — $\rangle$ = group of all diagonal matr. in

Easy: $N = \mathrm{Norm}_G(T)$, $W := N/T \simeq \mathfrak{S}_n$

$w_{ij}(x) \mapsto (ij)$

<u>Relations</u>:

(A) $\qquad u_{ij}(x+y) = u_{ij}(x)\, u_{ij}(y)$

(B)$_{n\geq 3}$ $\qquad [u_{ij}(x), u_{kl}(y)] = \begin{cases} u_{il}(xy) & \text{if } j=k \\ u_{kj}(-xy) & \text{if } i=l \\ 1 & \text{otherwise} \end{cases} \quad (j,i) \neq (k,l)$

(B') $\qquad w_{ij}(t)\, u_{ij}(x)\, w_{ij}(t)^{-1} = u_{ji}(-t^{-2}x) \qquad t \in k^*,\ x \in k$

(C) $\qquad h_{ij}(xy) = h_{ij}(x)\, h_{ij}(y)$

<u>Remark</u>: B is void if $n=2$;
If $n \geq 3$: $\quad$ (A), (B) $\Longrightarrow$ (B')

Let $B := \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \;\middle|\; \text{upper triangular} \right\} \subseteq G$

$U := \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \;\middle|\; \text{unipotent upper triangular} \right\} \subseteq G$

Then: $\quad B = T \ltimes U = \langle h_{ij}(x), u_{kl}(y) \mid x \in k^*, y \in k, k<l \rangle$

- B is a maximal connected solvable subgroup of G
- B is the stabiliser of the "canonical" maximal flag of k^n: $0 \subset k \subset k^2 \subset \cdots \subset k^{n-1} \subset k^n$
- G/B is a projective variety
- Any subgroup $P \subset G$ containing B stabilises a <u>subflag</u> of the canonical max flag $0 \subset k \subset k^2 \subset \cdots \subset k^{n-1} \subset k^n$. In this case G/P is projective v. <u>and vice versa</u>.
- Every subgroup of G stabilizing some flag is conjugate to such a P.

$$-\; G = BNB = \bigcup_{w\in W} B\tilde{w}B \quad \text{disjoint!}$$

($\tilde{w}$ pre-image of w)

$$= \bigcup_{w\in W} BwB \quad \text{"Bruhat-decomposition"}$$

Expl: $n=2$: $W = \mathfrak{S}_2 = \mathbb{Z}/2\mathbb{Z}$

$$G = B \cup B\begin{pmatrix}0&1\\-1&0\end{pmatrix}B$$

$n=3$: $W = \mathfrak{S}_3$ generated by images of

$$\underbrace{w_{12}}_{\mapsto (12)} = \begin{pmatrix}0&1&0\\-1&0&0\\0&0&1\end{pmatrix}, \quad \underbrace{w_{23}}_{\mapsto (23)} = \begin{pmatrix}1&0&0\\0&0&1\\0&-1&0\end{pmatrix}$$

$$\Rightarrow \underbrace{w_{12}w_{13}}_{\mapsto (132)} = \begin{pmatrix}0&0&1\\-1&0&0\\0&-1&0\end{pmatrix}, \quad \underbrace{(w_{12}w_{13})^2}_{\mapsto (123)} = \begin{pmatrix}0&-1&0\\0&0&-1\\1&0&0\end{pmatrix}$$

Moreover: $\underbrace{w_{13}}_{\mapsto (13)} = \begin{pmatrix}0&0&1\\0&1&0\\-1&0&0\end{pmatrix}$

$$\Rightarrow G = B \cup Bw_{12}B \cup Bw_{23}B \cup Bw_{13}B \cup Bw_{12}w_{13}B \cup B(w_{12}w_{13})^2B$$

Characters on $T \subset \mathrm{Diag}_n \subset GL_n(k)$:

Let $\varepsilon_i : \mathrm{Diag}_n \to k^*$ denote the hom $\varepsilon_i\begin{pmatrix}d_1&&\\&\ddots&\\&&d_n\end{pmatrix} = d_i$

$\Rightarrow \bigoplus \mathbb{Z}\varepsilon_i = \mathrm{Hom}(\mathrm{Diag}_n, \mathbb{G}_m) \qquad \mathbb{G}_m(k) = k^*$

This has scalar product with $\{\varepsilon_i\}$ as ortho-basis, invariant under W.

$\Phi = \{\varepsilon_{ij} := \varepsilon_i - \varepsilon_j \mid 1\le i,j\le n,\ i\neq j\}$ stays invariant under W

$\varepsilon_{12}, \varepsilon_{23}, \dots, \varepsilon_{n-1\,n}$ is a basis of $\mathrm{Hom}(T, \mathbb{G}_m)$

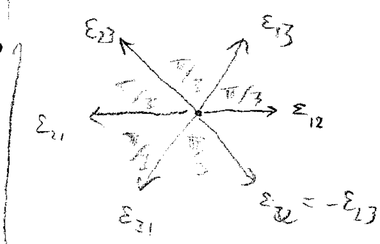
$(\varepsilon_{ij}, \varepsilon_{ij}) = (\varepsilon_i, \varepsilon_i) + (\varepsilon_j, \varepsilon_j) = 2.$

$(\varepsilon_{ij}, \varepsilon_{jk}) = -(\varepsilon_j, \varepsilon_j) = -1$ if $i \neq k$. $\Rightarrow$

$(\varepsilon_{ij}, \varepsilon_{kl}) = 0$ if $\{i,j\}\cap\{k,l\} = \emptyset$

Φ = root system of $SL_n(k)$

$n=3$:



Each is sum of its two neighbours

<u>Thm 1</u> (Dickson, Steinberg)

i) A presentation of $G = SL_n(k)$ is given by

(A), (B), (C) if $n \geq 3$

(A), (B'), (C) if $n = 2$

ii) Denote by $\tilde{G}$ the group given by the presentation

(A), (B) if $n \geq 3$

(A), (B') if $n = 2$.

Assume $|k| > 4$ if $n \geq 3$ and $|k| \neq 4, 9$ if $n = 2$.

Then, $\tilde{G} \xrightarrow{\pi} G$ (canonical) is central (i.e. $\ker \pi \subset Z(\tilde{G})$)

and every central extension $G_1 \xrightarrow{\pi_1} G$ factors from π,

i.e., $\exists\, \varphi : \tilde{G} \to G_1$ s. th. $\pi = \pi_1 \varphi$.

$$\begin{tikzcd} \tilde{G} \arrow[dr, "\pi"] \arrow[d, "\varphi"'] & \\ G_1 \arrow[r, "\pi_1"'] & G \end{tikzcd}$$

<u>Remark 1.)</u> as $G, \tilde{G}$ are perfect, $\tilde{G}$ is <u>the</u> universal central extension of G, unique up to isomorphism

$St_n(k) := \tilde{G}$ is the "Steinberg group" of $SL_n(k)$ (not algebraic)

<u>Remark 2)</u> If k is algebraic over a finite field, then $\tilde{G} = G$.

Remark 3) By definition of $\tilde{G}, G$, the elements

$$h_{ij}(xy)\, h_{ij}(x)^{-1}\, h_{ij}(y)^{-1}$$

generate the kernel of $\pi : \tilde{G} \to G$.

Thm 2 (Moore/Matsumoto): Ass. as in Thm 1.

The elements $c_{ij}(x,y) := h_{ij}(x) h_{ij}(y) h_{ij}(xy)^{-1}$ yield $c_{ij}(x,y) = c_{ji}(y,x)^{-1}$ and are independent of i,j if $n \geq 3$.
Let $c(x,y) := c_{12}(x,y)$. These fulfill the following relations:

$n = 2$:

$$\left.\begin{array}{ll} (S1) & c(x,y)\,c(xy,z) = c(x,yz)\,c(y,z) \\ (S2) & c(1,1) = 1, \quad c(x,y) = c(x^{-1},y^{-1}) \\ (S3) & c(x,y) = c(x,(1-x)y) \quad \text{if } x \neq 1 \end{array}\right\} \text{Steinberg cocycle}$$

$n \geq 3$:

$$\left.\begin{array}{ll} (S^\circ 1) & c(x,yz) = c(x,y)\,c(x,z) \\ (S^\circ 2) & c(xy,z) = c(x,z)\,c(y,z) \\ (S^\circ 3) & c(x,1-x) = 1 \quad \text{if } x \neq 1 \end{array}\right\} \text{Steinberg "symbol" } \{x,y\}$$

$k^* \otimes_{\mathbb{Z}} k^* / \langle x \otimes (1-x) \mid x \in k^* - 1 \rangle$

In case $n = 2$, let

$$c^\natural(x,y) := c(x,y^2)$$

Then $c^\natural(x,y)$ fulfills the relations $(S^\circ 1), (S^\circ 2), (S^\circ 3)$.
Hence is a Steinberg symbol

Moreover, $\ker \pi : \tilde{G} \to G$ is isomorphic to the abelian group presented by
$(S1), (S2), S(3)$ in case $n = 2$
$(S^\circ 1), (S^\circ 2), (S^\circ 3)$ in case $n \geq 3$

Remark: $\ker \pi$ is denoted by $K_2(n,k)$. Clearly we have hom's

$$K_2(2,k) \twoheadrightarrow K_2(3,k) \xrightarrow{\sim} K_2(4,k) \xrightarrow{\sim} K_2(n,k) \; (n \geq 3)$$

Hence $K_2(k) := \varinjlim K_2(n,k) \simeq K_2(n,k) \; (n \geq 3)$