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Algebraic K-theory and algebraic geometry

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4. Algebraic K-theory and Algebraic Geometry

4.1. **Schemes.** Although our primary interest will be in the K-theory of smooth, quasi-projective algebraic varieties, for completeness we briefly recall the more general context of schemes. A quasi-projective variety corresponds to a globalization of a finitely generated commutative algebra over a field; a scheme similarly corresponds to the globalization of a general commutative ring.

Recall that if A is a commutative ring we denote by $\operatorname{Spec} A$ the set of prime ideals of A. The set $X = SpecA$ is provided with a topology, the **Zariski topology** defined as follows: a subset $Y \subset X$ is closed if and only if there exists some ideal $I \subset A$ such that $Y = \{p \in X; I \subset p\}$. We define the **structure sheaf** $\mathcal{O}_X$ of commutative rings on $X = \operatorname{Spec} A$ by specifying its value on the basic open set $X_f = \{p \in SpecA, f \notin p\}$ for some $f \in A$ to be the ring A_f obtained from A by adjoining the inverse to f. (Recall that $A \to A_f$ sends to 0 any element $a \in A$ such that $f^n \cdot a = 0$ for some n). We now use the sheaf axiom to determine the value of $\mathcal{O}_X$ on any arbitrary open set $U \subset X$, for any such U is a finite union of basic open subsets. The stalk $\mathcal{O}_{X,p}$ of the structure sheaf at a prime ideal $p \subset A$ is easily computed to be the local ring $A_p = \{f \notin p\}^{-1}A$.

Thus, $(X = \operatorname{Spec} A, \mathcal{O}_X)$ has the structure of a **local ringed space**: a topological space with a sheaf of commutative rings each of whose stalks is a local ring. A map of local ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is the data of a continuous map $f : X \to Y$ of topological spaces and a map of sheaves $O_Y \to f_*O_X$ on Y, where $f_*O_X(V) = O_X(f^{-1}(V))$ for any open $V \subset Y$.

If M is an A-module for a commutative ring A, then M defines a sheaf $\tilde{M}$ of $\mathcal{O}_X$-modules on $X = \operatorname{Spec} A$. Namely, for each basic open subset $X_f \subset X$, we define $\tilde{M}(X_f) \equiv A_f \otimes_A M$. This is easily seen to determine a sheaf of abelian groups on X with the additional property that for every open $U \subset X$, $\tilde{M}(U)$ is a sheaf of $\mathcal{O}_X(U)$-modules with structure compatible with restriction to smaller open subsets $U' \subset U$.

Definition 4.1. A local ringed space $(X, \mathcal{O}_X)$ is said to be an **affine scheme** if it is isomorphic (as local ringed spaces) to $(X = \operatorname{Spec} A, \mathcal{O}_X)$ as defined above. A **scheme** $(X, \mathcal{O}_X)$ is a local ringed space for which there exists a finite open covering $\{U_i\}_{i \in I}$ of X such that each $(U_i, \mathcal{O}_{X|U_i})$ is an affine scheme.

If k is a field, a k-**variety** is a scheme $(X, \mathcal{O}_X)$ with the property there is a finite open covering $\{U_i\}_{i \in I}$ by affine schemes with the property that each $(U_i, \mathcal{O}_{X|U_i}) \simeq (SpecA_i, \mathcal{O}_{SpecA_i})$ with A_i a finitely generated k-algebra without nilpotents. The $(\operatorname{Spec} A_i, \mathcal{O}_{\operatorname{Spec} A_i})$ are affine varieties admitting a locally closed embedding in $\mathbb{P}^N$, where $N + 1$ is the cardinality of some set of generators of A_i over k.

Example 4.2. The scheme $\mathbb{P}^1_{\mathbb{Z}}$ is a non-affine scheme defined by patching together two copies of the affine scheme $Spec\mathbb{Z}[t]$. So $\mathbb{P}^1_{\mathbb{Z}}$ has a covering $\{U_1, U_2\}$ corresponding to rings $A_1 = \mathbb{Z}[u], A_2 = \mathbb{Z}[v]$. These are "patched together" by identifying the open subschemes $Spec(A_1)_u \subset SpecA_1$, $Spec(A_2)_v \subset SpecA_2$ via the isomorphism of rings $(A_1)_u \simeq (A_2)_v$ which sends u to v^{-1}.

Note that we have used $SpecR$ to denote the local ringed space $(SpecR, \mathcal{O}_{SpecR})$; we will continue to use this abbreviated notation.

Definition 4.3. Let $(X, \mathcal{O}_X)$ be a scheme. We denote by $\mathcal{V}ect(X)$ the exact category of sheaves F of $\mathcal{O}_X$-modules with the property that there exists an open covering $\{U_i\}$ of X by affine schemes $U_i = SpecA_i$ and free, finitely generated A_i-modules M_i such that the restriction $F_{|U_i}$ of F to U_i is isomorphic to the sheaf $\tilde{M}_i$ on $SpecA_i$. In other words, $\mathcal{V}ect(X)$ is the exact category of coherent, locally free $\mathcal{O}_X$-modules (i.e., of vector bundles over X).

We define the algebraic K-theory of the scheme X by setting

$$K_*(X) = K_*(\mathcal{V}ect(X)).$$

4.2. **Algebraic cycles.** For simplicity, we shall typically restrict our attention to quasi-projective varieties. In some sense, the most intrinsic objects associated to an algebraic variety are the (algebraic) vector bundles $E \to X$ and the algebraic cycles $Z \subset X$ on X. As we shall see, these are closely related.

Definition 4.4. Let X be a scheme. An *algebraic r-cycle* on X if a formal sum

$$\sum_Y n_Y[Y], \quad Y \text{ irreducible of dimension } r, \quad n_Y \in \mathbb{Z}$$

with all but finitely many n_Y equal to 0.

Equivalently, an algebraic r-cycle is a finite integer combination of (not necessarily closed) points of X of dimension r. (This is a good definition even for X a quite general scheme.)

If $Y \subset X$ is a reduced subscheme each of whose irredicible components $Y_1, \ldots, Y_m$ is r-dimensional, then the algebraic r-cycle

$$Z = \sum_{i=1}^m [Y_i]$$

is called the *cycle associated* to Y.

The group of (algebraic) r-cycles on X will be denoted $Z_r(X)$.

For example, if X is an integral variety of dimension d (i.e., the field of fractions of X has transcendence d over k), then a Weil divisor is an algebraic $d-1$-cycle. In the following definition, we extend to r-cycles the equivalence relation we impose on locally principal divisor when we consider these modulo principal divisors. As motivation, observe that if C is a smooth curve and $f \in frac(C)$, then f determines a morphism $f : C \to \mathbb{P}^1$ and

$$(f) = f^{-1}(0) - f^{-1}(\infty),$$

where $f^{-1}(0), f^{-1}(\infty)$ are the scheme-theoretic fibres of f above $0, \infty$.

Definition 4.5. Two r-cycles Z, Z' on a quasi-projective variety X are said to be *rationally equivalent* if there exist algebraic $r+1$-cycles $W_0, \ldots, W_n$ on $X \times \mathbb{P}^1$ for some $n > 0$ with the property that each component of each W_i projects onto an open subvariety of $\mathbb{P}^1$ and that $Z = W_0[0], Z' = W_n[\infty]$, and $W_i[\infty] = W_{i+1}[0]$ for $0 \leq i < n$. Here, $W_i[0]$ (respectively, $W_i[\infty]$ denotes the cycle associated to the scheme theoretic fibre above $0 \in \mathbb{P}^1$ (resp., $\infty \in \mathbb{P}^1$) of the restriction of the projection $X \times \mathbb{P}^1 \to \mathbb{P}^1$ to (the components of) W_i.

The *Chow group* $CH_r(X)$ is the group of r-cycles modulo rational equivalence.

Observe that in the above definition we can replace the role of $r+1$-cycles on $X \times \mathbb{P}^1$ and their geometric fibres over $0, \infty$ by $r+1$-cycles on $X \times U$ for any non-empty Zaristik open $U \subset X$ and geometric fibres over any two k-rational points $p, q \in U$.

Remark 4.6. Given some $r+1$ dimensional irreducible subvariety $V \subset X$ together with some $f \in k(V)$, we may define $(f) = \sum_S ord_S(f)[S]$ where S runs through the codimension 1 irreducible subvarieties of V. Here, $ord_S(-)$ is the valuation centered on S if V is regular at the codimension 1 point corresponding to S; more generally, $ord_S(f)$ is defined to be the length of the $O_{V,S}$-module $O_{V,S}/(f)$.

We readily check that (f) is rationally equivalent to 0: namely, we associate to (V, f) the closure $W = \Gamma_f \subset X \times \mathbb{P}^1$ of the graph of the rational map $V \dashrightarrow \mathbb{P}^1$ determined by f. Then $(f) = W[0] - W[\infty]$.

Conversely, given an $r+1$-dimensional irreducible subvariety W on $X \times \mathbb{P}^1$ which maps onto $\mathbb{P}^1$, the composition $W \subset X \times \mathbb{P}^1 \stackrel{pr_2}{\to} \mathbb{P}^1$ determines $f \in frac(W)$ such that

$$(f) \ = \ W[0] - W[\infty].$$

Thus, the definition of rational equivalence on r-cycles of X can be given in terms of the equivalence relation generated by

$$\{(f), f \in frac(W); W \text{ irreducible of dimension } r+1\}$$

In particular, we conclude that the subgroup of principal divisors inside the group of all locally principal divisors consists precisely of those locally principal divisors which are rationally equivalent to 0.

4.3. **Chow Groups.** One should view $CH_*(X)$ as a homology/cohomology theory. These groups are covariantly functorial for proper maps $f : X \to Y$ and contravariantly functiorial for flat maps $W \to X$, so that they might best be viewed as some sort of "Borel-Moore homology theory.

Construction 1. Assume that X is integral and regular in codimension 1. Let $\mathcal{L} \in Pic(X)$ be a locally free sheaf of rank 1 (i.e., a "line bundle" or "invertible sheaf") and assume that $\Gamma(\mathcal{L}) \neq 0$. Then any $0 \neq s \in \Gamma(\mathcal{L})$ determines a well defined locally principal divisor on X, $Z(s) \subset X$. Namely, if $\mathcal{L}_{|U} \simeq \mathcal{O}_{X|U}$ is trivial when restricted to some open $U \subset X$, then $s_U \in \mathcal{L}(U)$ determines an element of $\mathcal{O}_X(U)$ well defined up to a unit in $\mathcal{O}_X(U)$ (i.e., an element of $\mathcal{O}_X^*(U)$) so that the valuation $v_x(s)$ is well defined for every $x \in U^{(1)}$. We define $Z(s)$ by the property that $Z(s)_U = (s_U)_{|U}$ for any open $U \subset X$ restricted to which $\mathcal{L}$ is trivial, and where (s_U) denotes the divisor of an element of $\mathcal{O}_X(U)$ corresponding to s_U under any $(\mathcal{O}_X)_{|U}$ isomorphism $\mathcal{L}_{|U} \simeq (\mathcal{O}_X)_{|U}$.

Theorem 4.7. *Assume that X is an integral variety regular in codimension 1. Let $\mathcal{D}(X)$ denote the group of locally principal divisors on X modulo principal divisors. Then the above construction determines a well defined isomoprhism*

$$Pic(X) \;\simeq\; \mathcal{D}(X).$$

Moreover, if $\mathcal{O}_{X,x}$ is a unique factorization domain for every $x \in X$, then $D(X)$ equals the group $CH^1(X)$ of codimension 1 cycles modulo rational equivalence.

Proof. If $s, s' \in \Gamma(\mathcal{L})$ are non-zero global sections, then there exists some $f \in K = frac(\mathcal{O}_X)$ such that with respect to any trivialization of $\mathcal{L}$ on some open covering $\{U_i \subset X\}$ of X the quotient of the images of s, s' in $\mathcal{O}_X(U_i)$ equals f. A line bundle $\mathcal{L}$ is trivial if and only if it is isomorphic to $\mathcal{O}_X$ which is the case if and only if it has a global section $s \in \Gamma(X)$ which never vanishes if and only if $(s) = 0$. If $\mathcal{L}_1, \mathcal{L}_2$ are two such line bundles with non-zero global sections s_1, s_2, then $(s_1 \otimes s_2) = (s_1) + (s_2)$.

Thus, the map is a well defined homomorphism on the monoid of those line bundles with a non-zero global section. By Serre's theorem concerning coherent sheaves generated by global sections, for any line bundle $\mathcal{L}$ there exists a positive integer n such that $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$ is generated by global sections (and in particular, has non-zero global sections), where we have implicitly chosen a locally closed embedding $X \subset \mathbb{P}^M$ and taken $\mathcal{O}_X(n)$ to be the pull-back via this embedding of $\mathcal{O}_{\mathbb{P}^M}(n)$. Thus, we can send such an $\mathcal{L} \in Pic(X)$ to $(s) - (w)$, where $s \in \Gamma(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n))$ and $w \in \Gamma(\mathcal{O}_X(n))$.

The fact that $Pic(X) \to \mathcal{D}(X)$ is an isomorphism is an exercise in unravelling the formulation of the definition of line bundle in terms of local data.

Recall that a domain A is a unique factorization domain if and only every prime of height 1 is principal. Whenever $\mathcal{O}_{X,x}$ is a unique factorization domain for every $x \in X$, every codimension 1 subvariety $Y \subset X$ is thus locally principal, so that the natural inclusion $D(X) \subset CH^1(X)$ is an equality. □

Remark 4.8. This is a first example of relating bundles to cycles, and moreover a first example of duality. Namely, $Pic(X)$ is the group of rank 1 vector bundles; the group $CH^1(X)$ of is a group of cycles. Moreover, $Pic(X)$ is contravariant with respect X whereas $Z^1(X)$ is covariant with respect to equidimensional maps. To relate the two as in the above theorem, some smoothness conditions are required .

Example 4.9. Let $X = \mathbb{A}^N$. Then any $N-1$-cycle (i.e., Weil divisor) $Z \in CH_{N-1}(\mathbb{A}^N)$ is principal, so that $CH_{N-1}(\mathbb{A}^N) = 0$.

More generally, consider the map $\mu : \mathbb{A}^N \times \mathbb{A}^1 \to \mathbb{P}^N \times \mathbb{A}^1$ which sends $(x_1, \ldots, x_n), t$ to $\langle t \cdot x_1, \ldots, t \cdot x_n, 1 \rangle, t$. Consider an ireducible subvariety $Z \subset \mathbb{A}^N$ of dimension $r > N$ not containing the origin and $\overline{Z} \subset \mathbb{P}^N$ be its closure. Let $W = \mu^{-1}(\overline{Z} \times \mathbb{A}^1)$. Then $W[0] = \emptyset$ whereas $W[1] = Z$. Thus, $CH_r(\mathbb{A}^N) = 0$ for any $r < N$.

Example 4.10. Arguing in a similar geometric fashion, we see that the inclusion of a linear plane $\mathbb{P}^{N-1} \subset \mathbb{P}^N$ induces an isomorphism $CH_r(\mathbb{P}^{N-1}) = CH_r(\mathbb{P}^N)$ provided that $r < N$ and thus we conclude by induction that $CH_r(\mathbb{P}^N) = \mathbb{Z}$ if $r \leq N$. Namely, consider $\mu : \mathbb{P}^N \times \mathbb{A}^1 \to \mathbb{P}^N \times \mathbb{A}^1$ sending $\langle x_0, \ldots, x_N \rangle, t$ to $\langle x_), \ldots, x_{N-1}, t{\cdot}x_N \rangle, t$ and set $W = \mu^{-1}(Z \times \mathbb{A}^1)$ for any Z not containing $\langle 0, \ldots, 0, 1 \rangle$. Then $W[0] = pr_{N*}(Z), W[1] = Z$.

Example 4.11. Let C be a smooth curve. Then $Pic(C) \simeq CH_0(X)$.

Definition 4.12. If $f : X \to Y$ is a proper map of quasi-projective varieties, then the proper push-forward of cycles determines a well defined homomorphism

$$f_* : CH_r(X) \ \to \ CH_r(Y), \quad r \geq 0.$$

Namely, if $Z \subset X$ is an irreducible subvariety of X of dimension r, then $[Z]$ is sent to $d \cdot [f(Z)] \in CH_r(Y)$ where $[k(Z) : k(f(Z))] = d$ if $dim\ Z = dim\ f(Z)$ and is sent to 0 otherwise.

If $g : W \to X$ is a flat map of quasi-projective varieties of relative dimension e, then the flat pull-back of cycles induces a well defined homomorphism

$$g^* : CH_r(X) \ \to \ CH_{r+e}(W), \quad r \geq 0.$$

Namely, if $Z \subset X$ is an irreducible subvariety of X of dimension r, then $[Z]$ is sent to the cycle on W associated to $Z \times_X W \subset W$.

Proposition 4.13. *Let Y be a closed subvariety of X and let $U = X \backslash Y$. Let $i : Y \to X, j : U \to X$ be the inclusions. Then the sequence*

$$CH_r(Y) \xrightarrow{i_*} CH_r(X) \xrightarrow{j^*} CH_r(U) \to 0$$

is exact for any $r \geq 0$.

Proof. If $V \subset U$ is an irreducible subvariety of U of dimension r, then the closure of V in X, $\overline{V} \subset X$, is an irreducible subvariety of X of dimension r with the property that $j^*([\overline{V}]) = [V]$. Thus, we have an exact sequence

$$Z_r(Y) \xrightarrow{i_*} Z_r(X) \xrightarrow{j^*} Z_r(U) \to 0.$$

If $Z = \sum_i n_i [Y_i]$ is a cycle on X with $j^*(Z) = 0 \in CH_r(U)$, then $j^* Z = \sum_{W,f} (f)$ where each $W \subset U$ is an irreducible subvarieties of U of dimension $r+1$ and $f \in k(W)$. Thus, $Z' = \sum_i n_i [\overline{Y}_i] - \sum_{\overline{W},f} (f)$ is an r-cycle on Y with the property that $i_*(Z')$ is rationally equivalent to Z. Exactness of the asserted sequence of Chow groups is now clear.

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Corollary 4.14. *Let $H \subset \mathbb{P}^N$ be a hypersurface of degree d. Then $CH_{N-1}(\mathbb{P}^N \backslash H) = \mathbb{Z}/d\mathbb{Z}$.*

The following "examples" presuppose an understanding of "smoothness" briefly discussed in the next section.

Example 4.15. Mumford shows that if S is a projective smooth surface with a non-zerol global algebraic 2-form (i.e., $H^0(S, \Lambda^2(\Omega_S)) \neq 0$), then $CH_0(S)$ is *not finite dimensional* (i.e., must be very large).

Bloch's Conjecture predicts that if S is a projective, smooth surface with geometric genus equal to 0 (i.e., $H^0(S, \Lambda^2(\Omega_S)) = 0$), then the natural map from $CH_0(S)$ to the (finite dimensional) Albanese variety is injective.

4.4. **Smooth Varieties.** We restrict our attention to quasi-projective varieties over a field k.

Definition 4.16. A quasi-projective variety X is smooth of dimension n at some point $x \in X$ if there exists an open neighborhood $x \in U \subset X$ and k polynomials $f_1, \ldots, f_k$ in $n+k$ variables (viewed as regular functions on $\mathbb{A}^{n+k}$) vanishing at $0 \in \mathbb{A}^{n+k}$ with Jacobian $|\frac{\partial f_i}{\partial x_j}|(0)$ of rank k and an isomorphism of U with $Z(f_1, \ldots, f_k) \subset \mathbb{A}^{n+k}$ sending x to 0.

In more algebraic terms, a point $x \in X$ is smooth if there exists an open neighborhood $x \in U \subset X$ and a map $p : U \to \mathbb{A}^n$ sending x to 0 which is flat and unramified at x.

Definition 4.17. Let X be a quasi-projective variety. Then $K'_0(X)$ is the Grothendieck group of isomorphism classes of coherent sheaves on X, where the equivalence relation is generated pairs $([\mathcal{E}], [\mathcal{E}_1] + [\mathcal{E}_2])$ for short exact sequences $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$ of $\mathcal{O}_X$-modules.

Example 4.18. Let $A = k[x]/x^2$. Consider the short exact sequence of A-modules

$$0 \to k \to A \to k \to 0$$

where k is an A-module via the agumentation map (i.e., x acts as multiplication by 0), where the first map sends $a \in k$ to $ax \in A$, and the second map sends x to 0. We conclude that the class $[A]$ of the rank 1 free module equals $2[k]$.

On the other hand, because A is a local ring, $K_0(A) = \mathbb{Z}$, generated by the class $[A]$. Thus, the natural map $K_0(\operatorname{Spec} A) \to K'_0(\operatorname{Spec} A)$ is not surjective. The map is, however, injective, as can be seen by observing that $dim_k(-) : K'_0(\operatorname{Spec} A) \to \mathbb{Z}$ is well defined.

Theorem 4.19. *If X is smooth, then the natural map $K_0(X) \to K'_0(X)$ is an isomorphism.*

Proof. Smoothness implies that every coherent sheaf has a finite resolution by vector bundles, This enables us to define a map

$$K_0'(X) \ \to \ K_0(X)$$

by sending a coherent sheaf $\mathcal{F}$ to the alternating sum $\Sigma_{i=1}^N(-1)^i\mathcal{E}_i$, where $0 \to \mathcal{E}_N \to \cdots \mathcal{E}_0 \to \mathcal{F} \to 0$ is a resolution of $\mathcal{F}$ by vector bundles.

Injectivity follows from the observation that the composition

$$K_0(X) \ \to \ K_0'(X) \ \to \ K_0(X)$$

is the identity. Surjectivity follows from the observation that $\mathcal{F} = \Sigma_{i=1}^N(-1)^i\mathcal{E}_i$ so that the composition

$$K_0'(X) \ \to \ K_0(X) \ \to \ K_0'(X)$$

is also the identity. □

Perhaps the most important consequence of this is the following observation. Grothendieck explained to us how we can make $K_0'(-)$ a *covariant* functor with respect to proper maps. (Every morphism between projective varieties is proper.) Consequently, restricted to smooth schemes, $K_0(-)$ is not only a contravariant functor but also a covariant functor for proper maps.

"Chow's Moving Lemma" is used to give a ring structure on $CH^*(X)$ on smooth varieties as made explicit in the following theorem. The role of the moving lemma is to verify for an r-cycle Z on X and an s-cycle W on X that Z can be moved within its rational equivalence class to some $Z^|prime$ such that Z' meets W "properly". This means that the intersection of any irreducible component of Z' with any irreducible component of W is either empty or of codimension $d - r - s$, where $d = dim(X)$.

Theorem 4.20. *Let X be a smooth quasi-projective variety of dimension d. Then there exists a pairing*

$$CH_r(X) \otimes CH_s(X) \xrightarrow{\bullet} CH_{d-r-s}(X), \quad d \geq r + s,$$

with the property that if $Z = [Y], Z' = [W]$ are irreducible cycles of dimension r, s respectively and if $Y \cap W$ has dimension $\leq d - r - s$, then $Z \bullet Z'$ is a cycle which is a sum with positive coefficients (determined by local data) indexed by the irreducible subvarieties of $Y \cap W$ of dimension $d - r - s$.

Write $CH^s(X)$ for $CH_{d-s}(X)$. With this indexing convention, the intersection pairing has the form

$$CH^s(X) \otimes CH^t(X) \ \xrightarrow{\bullet} \ CH^{s+t}(X).$$

4.5. **Chern classes and Chern character.** The following construction of Chern classes is due to Grothendieck; it applies equally well to topological vector bundles (in which case the Chern classes of a topological vector bundle over a topological space T are elements of the singular cohomology of T).

If $\mathcal{E}$ is a rank $r+1$ vector bundle on a quasi-projective variety X, we define $\mathbb{P}(\mathcal{E}) = \text{Proj } (Sym_{O_X}\mathcal{E}) \to X$ to be the projective bundle of lines in $\mathcal{E}$. Then $\mathbb{P}(\mathcal{E})$ comes equipped with a canonical line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$; for X a point, $\mathbb{P}(\mathcal{E}) = \mathbb{P}^r$ and $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = \mathcal{O}_{\mathbb{P}^r}(1)$.

Construction 2. Let $\mathcal{E}$ be a rank r vector bundle on a smooth, quasi-projective variety X of dimension d. Then $CH^*(\mathbb{P}(\mathcal{E}))$ is a free module over $CH^*(X)$ with generators $1, \zeta, \zeta^2, \ldots, \zeta^{r-1}$, where $\zeta \in CH^1(\mathbb{P}(\mathcal{E}))$ denotes the divisor class associated to $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. .

We define the i-th Chern class $c_i(\mathcal{E}) \in CH^i(X)$ of $\mathcal{E}$ by the formula

$$CH^*(\mathbb{P}(\mathcal{E})) \;=\; CH^*(X)[\zeta]/\sum_{i=0}^{r}(-1)^i\pi^*(c_i(\mathcal{E}))\cdot\zeta^{r-i}.$$

We define the total Chern class $c(\mathcal{E})$ by the formula

$$c(\mathcal{E}) \;=\; \sum_{i=0}^{r} c_i(\mathcal{E})$$

and set $c_t(\mathcal{E}) = \sum_{i=0}^{r} c_i(\mathcal{E})t^i$. Then the *Whitney sum* formula asserts that $c_t(\mathcal{E}\oplus\mathcal{F}) = c_t(\mathcal{E})\cdot c_t(\mathcal{F})$.

We define the *Chern roots*, $\alpha_1, \ldots, \alpha_r$ of $\mathcal{E}$ by the formula

$$c_t(\mathcal{E}) \;=\; \prod_{i=1}^{r}(1+\alpha_i t)$$

where the factorization can be viewed either as purely formal or as occurring in $\mathbb{F}(\mathcal{E})$. Observe that $c_k(\mathcal{E})$ is the k-th elementary symmetric function of these Chern roots.

In other words, the Chern classes of the rank r vector bundle $\mathcal{E}$ are given by the expression for $\zeta^r \in CH^r(\mathbb{P}(\mathcal{E}))$ in terms of the generators $1, \zeta, \ldots, \zeta^{r-1}$. Thus, the Chern classes depend critically on the identification of the first Chern class ζ of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ and the multiplicative structure on $CH^*(X)$. The necessary structure for such a definition of Chern classes is called an *oriented multiplicative cohomology theory.* The splitting principle guarantees that Chern classes are uniquely determined by the assignment of first Chen classes to line bundles.

We refer the interested reader to [**?**] for the definition of "operational Chern classes" defined for bundles on a non necessarily smooth variety.

Grothendieck introduced many basic techniques which we now use as a matter of course when working with bundles. The following *splitting principle* is one such technique, a technique which enable one to frequently reduce constructions for arbitrary vector bundles to those which are a sum of line bundles.

Proposition 4.21. *(Splitting Principle) Let $\mathcal{E}$ be a rank $r+1$ vector bundle on a quasi-projective variety X. Then $p_1^* : CH_*(X) \to CH_{*+r}(\mathbb{P}(\mathcal{E}))$ is split injective and $p_1^*(\mathcal{E}) = \mathcal{E}_1$ is a direct sum of a rank r bundle and a line bundle.*

Applying this construction to $\mathcal{E}_1$ on $\mathbb{P}(\mathcal{E})$, we obtain $p_2 : \mathbb{P}(\mathcal{E}_1) \to \mathbb{P}(\mathcal{E})$; proceeding inductively, we obtain

$$p = p_r \circ \cdots \circ p_1 : \mathbb{F}(\mathcal{E}) = \mathbb{P}(\mathcal{E}_{r-1}) \to X$$

with the property that $p^* : K_0(X) \to K_0(\mathbb{F}(\mathcal{E}))$ is split injective and $p^*(\mathcal{E})$ is a direct sum of line bundles.

One application of the preceding proposition is the following definition (due to Grothendieck) of the Chern character.

Construction 3. Let X be a smooth, quasi-projective variety, let $\mathcal{E}$ be a rank r vector bundle over X, and let $\pi : \mathbb{F}(\mathcal{E}) \to X$ be the associated bundle of flags of $\mathcal{E}$. Write $\pi^*(\mathcal{E}) = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r$, where each $\mathcal{L}_i$ is a line bundle on $\mathbb{F}(\mathcal{E})$. Then $c_t(\pi^*(\mathcal{E})) = \prod_{i=1}^r (1 \oplus c_1(\mathcal{L}_i))t$.

We define the Chern character of $\mathcal{E}$ as

$$ch(\mathcal{E}) = \sum_{i=1}^r \{1 + c_1(\mathcal{L}_i) + \frac{1}{2}c_1(\mathcal{L}_i)^2 + \frac{1}{3!}c_1(\mathcal{L}_i)^3 + \cdots\} = \sum_{i=1}^r exp(c_t(\mathcal{L}_i)),$$

where this expression is verified to lie in the image of the injective map $CH^*(X) \otimes \mathbb{Q} \to CH^*(\mathbb{F}(\mathcal{E})) \otimes \mathbb{Q}$. (Namely, one can identify $ch_k(\mathcal{E})$ as the k-th power sum of the Chern roots, and therefore expressible in terms of the Chern classes using Newton polynomials.)

Since $\pi^* : K_0(X) \to K_0(\mathbb{F}(\mathcal{E}))$, $\quad \pi^* : CH^*(X) \to CH^*(\mathbb{F}(\mathcal{E}))$ are ring homomorphisms, the splitting principle enables us to immediately verify that ch is also a ring homomoprhism (i.e., sends the direct sum of bundles to the sum in $CH^*(X)$ of Chern characters, sends the tensor product of bundles to the product in $CH^*(X)$ of Chern characters).

4.6. **Riemann-Roch.** Grothendieck's formulation of the Riemann-Roch theorem is an assertion of the behaviour of the Chern character ch with respect to push-forward maps induced by a proper smooth map $f : X \to Y$ of smooth varieties. It is not the case that ch commutes with the these push-forward maps; one must modify the push forward map in K-theory by multiplication by the Todd class.

This modification by multiplication by the Todd class is necessary even when consideration of the push-forward of a divisor. Indeed, the Todd class

$$td : K_0(X) \to CH^*(X)$$

is characterized by the properties that

- i. $td(L) \;=\; c_1(L)/(1 - exp(-c_1(L)) \;=\; 1 + \frac{1}{2}c_1(L) + \cdots$;
- ii. $td(E_1 \oplus E_2) = td(E_1) \cdot td(E_2)$; and

- iii. $td \circ f^* \; = \; f^* \circ td$.

The reader is recommended to consult [?] for a very nice overview of Grothendieck's Riemann-Roch Theorem.

Theorem 4.22. *(Grothendieck's Riemann-Roch Theorem)*
Let $f : X \to Y$ be a projective map of smooth varieties. Then for any $x \in K_0(X)$, we have the equality

$$ch(f_!(x)) \cdot td(T_Y) \; = \; f_*(ch(x) \cdot td(T_X))$$

where T_X, T_Y are the tangent bundles of X, Y and $td(T_X), td(T_Y)$ are their Todd classes.
Here, $f_! : K_0(X) \to K_0(Y)$ is defined by identifying $K_0(X)$ with $K_0'(X)$, $K_0(Y)$ with $K_0'(Y)$, and defining $f_! : K_0'(X) \to K_0'(Y)$ by sending a coherent sheaf $\mathcal{F}$ on X to $\sum_i (-1)^i R^i f_*(F)$. The map $f_* : CH_*(X) \to CH_*(Y)$ is proper push-forward of cycles.

Just to make this more concrete and more familiar, let us consider a very special case in which X is a projective, smooth curve, Y is a point, and $x \in K_0(X)$ is the class of a line bundle $\mathcal{L}$. (Hirzebruch had earlier proved a version of Grothendieck's theorem in which the target Y was a point.)

Example 4.23. Let C be a projective, smooth curve of genus g and let $f : C \to Spec\mathbb{C}$ be the projection to a point. Let $\mathcal{L}$ be a line bundle on C with first Chern class $D \in CH^1(C)$. Then

$$f_!([\mathcal{L}]) = dim\mathcal{L}(C) - dimH^1(C, \mathcal{L}) \in \mathbb{Z},$$

and $ch : K_0(Spec\mathbb{C}) = \mathbb{Z} \to A^*(Spec\mathbb{C}) = \mathbb{Z}$ is an isomorphism. Let $K \in CH^1(C)$ denote the "canonical divisor", the first Chern class of the line bundle Ω_C, the dual of T_C. Then

$$td(T_C) = \frac{-K}{1 - (1 + K + \frac{1}{2}K^2)} = 1 - \frac{1}{2}K.$$

Recall that $deg(K) = 2g - 2$. Since $ch([\mathcal{L}]) = 1 + D$, we conclude that

$$f_*(ch([\mathcal{L}]) \cdot td(T_C)) = f_*((1 + D) \cdot (1 - \frac{1}{2}K)) = deg(D) - \frac{1}{2}deg(K).$$

Thus, in this case, Riemann-Roch tell us that

$$dim\mathcal{L}(C) - dimH^1(C, \mathcal{L}) = deg(D) + 1 - g.$$

For our purpose, Riemann-Roch is especially important because of the following consequence.

Corollary 4.24. *Let X be a smooth quasi-projective variety. Then*

$$ch : K_0(X) \otimes \mathbb{Q} \to CH^*(X) \otimes \mathbb{Q}$$

is a ring isomorphism.

Proof. The essential ingredient is the Riemann-Roch theorem. Namely, we have a natural map $CH^*(X) \to K_0'(X)$ sending an irreducible subvariety W to the $\mathcal{O}_X$-module $\mathcal{O}_W$. We show that the composition with the Chern character is an isomorphism by applying Riemann-Roch to the closed immersion $W \backslash W_{sing} \to X \backslash W_{sing}$. □

References

A. Borel and J-P Serre, Le Th'eorème de Riemann-Roch, Bull Soc. Math France 36 1958), 97 - 136.