The Abdus Salam
International Centre for Theoretical Physics

United Nations
Educational, Scientific and
Cultural Organization

IAEA
International Atomic Energy Agency

SMR/1840-13

School and Conference on Algebraic K-Theory and its Applications

14 May - 1 June, 2007

Linear algebraic groups over fields

Ulf Rehmann (rehmann@math.uni-bielefeld.de)
Universitaet Bielefeld, Germany

May 16: 2. Linear algebraic groups over fields

Def A linear algebraic k-group is an affine k-variety G, together with k-morphisms $(x,y) \longmapsto xy$ of $G \times G$ into G and $x \longmapsto x^{-1}$ of G into G, satisfying the usual group axioms, s.th. the unit element is rational! (k = field or comm ring)

This makes $G(k')$ a group for every k'/k

Notation: For k'/k (field or comm. ring extension) with $G_{k'} = G(k')$ for the k'-rational points of G.

Convention from now:

k-group, k-subgroup etc will always mean "alg. k-group", "alg k-subgroup" which are k-closed k-subvarieties

Homom of k-groups: k-morphism of the underlying k-varieties inducing group homom. etc.

Ex: – GL_n is a linear k-group, since

$GL_n(k) = \{(x_{ij}, y) \in k^{n^2+1} \mid \det(x_{ij}) \cdot y - 1 = 0\}$

$SL_n(k) = \{(x_{ij}) \mid \det(x_{ij}) - 1 = 0\}$

– $\mathbb{G}_a$ = additive group: $\mathbb{G}_a(k) = k^+$

– $\mathbb{G}_m$ = mult. group: $\mathbb{G}_m(k) = k^* = GL_1(k)$ $\{(x,y) \in k^2 \mid xy-1 = 0\}$

– $O(q)$, $SO(q)$ are linear k-group for quad form q: $O(q)$ …

OVER.

Remark: The connected e-component G^0 of a linear alg k group is a normal subgroup of finite Index.

Expl. $O(q) = SO(q) \cup O^-(q)$, $O^-(q) = \{x \in O(q) \mid \det(x) = -1\}$ (… is …)

Overall structure of linear alg. groups G:

k perfect:

i) G has a unique maximal connected linear solvable normal k-subgroup

G_1 = radical of G = rad G

G/G_1 is *semisimple*, i.e., connected, linear, with radical $= \{1\}$

ii) G_1 has a unique maximal connected unipotent k-subgroup G_2: "unipotent radical of G":

$G_2 = \mathrm{rad}_u G$

G_1/G_2 is a *torus*, i.e. $\simeq \mathbb{G}_{m,\bar{k}}^r$ over $\bar{k}$

General picture:

Quotient:

$$\begin{array}{lll} \text{con. linear} & G & \\ & \triangledown & \\ \text{con solvable} & \trianglerighteq G_1 = \mathrm{rad}\, G & \\ & \triangledown & \\ \text{con unipotent} & G_2 = \mathrm{rad}_u G & \\ & \triangledown & \\ & 1 & \end{array}$$

semisimple = almost direct product of almost simple factors

torus

reductive:

$G = T \cdot G'$ almost direct

T = central torus

G' semisimple

For char $k=0$:

$\mathrm{rad}_u G$ has a reductive complement H:

$G_1 = H \cdot \mathrm{rad}_u G$ (semidirect)

Examples:

i) $G = GL_n$: $\operatorname{rad} G = \operatorname{centre} G \simeq \mathbb{G}_m$,

$$G/\mathbb{G}_m = PGL_n \quad \text{simple}$$

$$\operatorname{rad}_u G = 1$$

ii) $$G = \left\{ \begin{array}{cc} \overbrace{}^{r} & \overbrace{}^{s} \\ \end{array} \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in GL_n \right\} = \operatorname{Stab}(ke_1 \oplus \cdots \oplus ke_r)$$

$$\operatorname{rad} G = \left\{ \left(\begin{array}{cc|cc} \alpha & 0 & \multicolumn{2}{c}{*} \\ 0 & \alpha & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \beta & \\ & & & \beta \end{array} \right) \right\}$$

$$\operatorname{rad}_u G = \left\{ \left(\begin{array}{cc|cc} 1 & 0 & \multicolumn{2}{c}{*} \\ 0 & 1 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 1 & 0 \\ & & 0 & 1 \end{array} \right) \right\}$$

$$G/\operatorname{rad}_u G \simeq GL_r \times GL_s$$

$$G/\operatorname{rad} G \simeq PGL_r \times PGL_s$$

$$\operatorname{rad} G/\operatorname{rad}_u G \simeq \mathbb{G}_m \times \mathbb{G}_m$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & g \end{pmatrix} \begin{pmatrix} a_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a a_1 \\ d a_1 \\ 0 \end{pmatrix}$$

<u>Tori</u>: T is a torus if, over some field ex. k'/k,
$T \times_k k' \simeq \prod \mathbb{G}_m$ (extend fields
If so, then $T \times_k k_{sep} \simeq \prod \mathbb{G}_m$

T is <u>split</u> if $T \simeq \prod \mathbb{G}_m$ (over k)

T is <u>anisotropic</u> if it does <u>not</u> contain any split subtorus

* Exp.: $T(\mathbb{R}) = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \in SL_2(\mathbb{R}) \;\middle|\; x^2+y^2 = 1 \right\} \simeq \bigcirc$ (circle.

→ base change $T \times_{\mathbb{R}} \mathbb{C} \cong \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \in SL_2(\mathbb{C}) \;\middle|\; \lambda \neq 0 \right\} \simeq \mathbb{C}^*$

$X(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ is a free $\mathbb{Z}$-module and a $\Gamma = \mathrm{Gal}(k_{sep}/k)$-module

T split $\iff$ Γ operates trivially on $X(T)$

T anisotropic $\iff$ $X(T)^{\Gamma} = \{0\}$ (see expl. above

T arbitrary: $T = T_a \cdot T_s$, $T_a \cap T_s$ finite

$$\left.\begin{array}{l} T_a = \text{max anisotropic } k\text{-subtorus} \\ T_s = \text{max split } k\text{-subtorus} \end{array}\right\} \text{both unique.}$$

~~*/ T(R)~~

* Expl 2. Let $k' : k$ be a finite field extension
Then the group of Norm-1-elements is a torus

Thm (Borel 1962) G connected linear alg k-group

i) All max tori in G are conjugate over $\bar{k}$.
Every semisimple element of G is contained in a torus; the centralizer of a torus in G is connected

ii) All max connected solvable subgroups (= Borel-subgp.) are conjugate. Every element of G is in such a group

iii) $P \subset G$ closed subgroup:
G/P projective $\Leftrightarrow$ P contains a Borel subgroup
Such groups P are called parabolic
P parabolic $\Rightarrow$ P connected and $N_G(P) = P$
P, Q parabolic $\supset B$, conjugate $\Rightarrow$ $P = Q$

Expl: $G = GL_n$

$T = \text{diag} = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$ max torus

$B = U = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$ upper triang = Borel

$P = \text{Stab}(ke_1) = \left\{ \begin{pmatrix} * & \cdots & * & \cdots \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{pmatrix} \right\} \supseteq B$,

$G/P \simeq \mathbb{P}^{n-1}$

<u>Def</u> G reductive $\Leftrightarrow$ $\mathrm{Rad}_u\, G = 1$

G semisimple $\Leftrightarrow$ $\mathrm{Rad}\, G = 1$

<u>Thm:</u> G alg grp. Equivalent:

i) G^0 reductive

ii) $G^0 = S \cdot G'$ almost direct, S = central torus, G' semisimple

(iii) G^0 has a locally faithful reducible rational representation

iv) char $k = 0$: all rational reps of G are fully reducible.)